

Center-stabilized Yang-Mills theory: confinement and large N volume independence

Mithat Ünsal^{1*} and Laurence G. Yaffe^{2†}

¹*SLAC and Physics Department, Stanford University, Stanford, CA 94305*

²*Department of Physics, University of Washington, Seattle, Washington 98195-1560*

ABSTRACT: We examine a double trace deformation of $SU(N)$ Yang-Mills theory which, for large N and large volume, is equivalent to unmodified Yang-Mills theory up to $O(1/N^2)$ corrections. In contrast to the unmodified theory, large N volume independence is valid in the deformed theory down to arbitrarily small volumes. The double trace deformation prevents the spontaneous breaking of center symmetry which would otherwise disrupt large N volume independence in small volumes. For small values of N , if the theory is formulated on $\mathbb{R}^3 \times S^1$ with a sufficiently small compactification size L , then an analytic treatment of the non-perturbative dynamics of the deformed theory is possible. In this regime, we show that the deformed Yang-Mills theory has a mass gap and exhibits linear confinement. Increasing the circumference L or number of colors N decreases the separation of scales on which the analytic treatment relies. However, there are no order parameters which distinguish the small and large radius regimes. Consequently, for small N the deformed theory provides a novel example of a locally four-dimensional pure gauge theory in which one has analytic control over confinement, while for large N it provides a simple fully reduced model for Yang-Mills theory. The construction is easily generalized to QCD and other QCD-like theories.

KEYWORDS: [nonperturbative QCD](#).

*unsal@slac.stanford.edu

†yaffe@phys.washington.edu

Contents

1. Large N volume independence	1
2. Deformed Yang-Mills theory	3
2.1 Stabilization of center symmetry	6
2.2 Large N equivalence between ordinary and deformed YM	7
2.3 Large N volume independence of deformed YM theory	9
2.4 Addition of matter fields	10
3. Confinement at small radius and small N	11
3.1 Mass gap	17
3.2 String tensions	18
3.3 Larger size or larger N	20
3.4 Connection to integrable Toda theory	21
A. Monopole measure	22

1. Large N volume independence

The large N limit of $SU(N)$ Yang-Mills theories, when formulated on toroidal compactifications of \mathbb{R}^d , are independent of volume provided the $(\mathbb{Z}_N)^d$ center symmetry is not spontaneously broken [1, 2, 3].¹ However, above two dimensions, center symmetry *does* break spontaneously when the (smallest) compactification circumference L is less than a critical size L_c [4]. (If just one dimension is compactified, then this center-symmetry breaking transition is the usual thermally induced deconfinement transition.) In four dimensions, the critical size L_c is approximately Λ^{-1} where Λ is the \overline{MS} strong scale of the theory [4, 5].

Notwithstanding the limitation to $L \geq L_c$, the volume independence of large N Yang-Mills theory (“partial reduction”) has practical utility for lattice studies, because simulations on lattices of size $(L_c)^d$ are sufficient to extract infinite volume properties of large N Yang-Mills theory [4, 6, 7, 8]. But it would be even more helpful to have a formulation of the theory in which volume independence holds for arbitrarily small volumes — since this allows one to reduce the lattice all the way down to a single site.

Several schemes for preserving volume independence in arbitrarily small volumes have been proposed. In so-called quenched reduced models, one constrains the eigenvalues of link

¹Center symmetry transformations are gauge transformations which are periodic only up to an element of the center of the gauge group. Volume independence applies to the leading large N behavior of expectation values and connected correlators of topologically trivial Wilson loops.

variables (or Wilson lines) in a manner which prevents Wilson lines from acquiring expectation values [3].² In the $N \rightarrow \infty$ limit, quenched reduced models correctly reproduce properties of infinite volume Yang-Mills theory. However, corrections to the $N = \infty$ limit scale as $1/N$, not $1/N^2$, in quenched reduced models. This makes extracting large N properties from numerical simulations of quenched reduced models quite challenging. An alternative proposal, known as twisted reduced models, involves modifying the Wilson action of a single-site model so that the action explicitly disfavors configurations in which Wilson lines in different directions mutually commute [10, 11]. Unfortunately, this clever scheme fails to work sufficiently close to the continuum limit [12, 13]. In essence, the penalty imposed by the twisting of the action is insufficient to overcome entropic effects which favor breaking of the center symmetry.

If light adjoint representation fermions are added to an $SU(N)$ Yang-Mills theory, and periodic (not anti-periodic) boundary conditions imposed on the fermions, then the fermion contribution to the Wilson line effective potential stabilizes the unbroken center symmetry phase. Hence these QCD-like theories satisfy large- N volume independence for arbitrarily small volumes [14]. (In addition, in the large- N limit \mathcal{C} -even observables coincide between these theories and corresponding theories, in sufficiently large volume, with fermions in the rank-two symmetric or antisymmetric tensor representations [15, 16, 17].)

Motivated by this fermion-induced stabilization of center symmetry, in this paper we introduce a simple scheme for preserving volume independence in pure Yang-Mills theory. We add double trace terms to the action which prevent spontaneous breaking of center symmetry, while simultaneously perturbing the dynamics of the unbroken symmetry phase only by $O(1/N^2)$ corrections. This leads to a “stabilized reduced model” which reproduces the dynamics of infinite volume Yang-Mills theory up to corrections which scale as $1/N^2$. The construction may be easily generalized to other QCD-like theories with matter fields in rank-one or rank-two representations.

In addition to providing a simple large N reduced model, the deformed Yang-Mills theory is interesting in its own right when N is not large.³ When formulated on $\mathbb{R}^3 \times S^1$, we show that the large distance dynamics of the theory is analytically tractable provided $NAL \ll 1$. In this regime, a semiclassical analysis (closely related to Polyakov’s classic treatment of $3d$ $SU(2)$ adjoint Higgs theory [22]) reveals the existence of a mass gap and area law behavior of spatial Wilson loops. It is noteworthy that our compactified, deformed Yang-Mills theory is an analytically tractable confining theory with no fundamental scalar fields or supersymmetry, in contrast to other instructive models of confinement [22, 25, 26, 27]. The confinement mechanism involves the formation of a dilute plasma of magnetic monopoles (and antimonopoles) carrying topological charge $\pm 1/N$.⁴

²See Ref. [9] for an extended discussion of quenched and twisted reduced models.

³See also related recent work in Refs. [18, 19, 20, 21].

⁴Confinement due to such topological objects has been previously discussed in, for example, Refs. [23, 24] and references therein. What is novel about our deformed theory at small NAL is that this confinement mechanism operates in a regime in which one has analytic control over the long distance dynamics.

2. Deformed Yang-Mills theory

We consider pure Yang-Mills (YM) theory with gauge group $SU(N)$ defined on the four manifold $\mathbb{R}^3 \times S^1$, with the S^1 having circumference L . The extension to multiple compactified dimensions is straightforward, but we will stick to a single compactified dimension to simplify the exposition. We start with the usual continuum action,

$$S^{\text{YM}} = \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu}^2(x), \quad (2.1)$$

or a lattice formulation with the Wilson action,

$$S^{\text{YM}} = \frac{\beta}{2} \sum_{p \in \Lambda_4} \text{tr} (U[\partial p] + U[\partial p]^\dagger), \quad (2.2)$$

where the sum is over all oriented plaquettes, Λ_4 is the four dimensional spacetime lattice, and $U[\partial p]$ denotes the usual product of link matrices around the boundary of plaquette p . The lattice coupling $\beta \equiv 2/g^2$. In our discussion, we will use both continuum and lattice formulations, and benefit from both perspectives. As usual, a key virtue of the lattice formulation is that it provides an explicit non-perturbative definition of the theory.

Let $\Omega(\mathbf{x}) \equiv \mathcal{P} (e^{i \int dx_4 A_4(\mathbf{x}, x_4)})$ denote the Wilson line (or Polyakov loop) operator — the holonomy of the gauge field around a circle wrapping the S^1 and sitting at the point $\mathbf{x} \in \mathbb{R}^3$. We will construct a deformation of the Yang-Mills action on our compactified geometry by adding terms, respecting all symmetries of the unmodified theory, built from the Wilson line operator. The deformed action is given by

$$S^{\text{deformed}} = S^{\text{YM}} + \Delta S, \quad (2.3)$$

with

$$\Delta S \equiv \int_{\mathbb{R}^3} \frac{1}{L^3} P[\Omega(\mathbf{x})] \quad (2.4a)$$

in the continuum, or

$$\Delta S \equiv \frac{1}{N_t^3} \sum_{\mathbf{x} \in \Lambda_3} P[\Omega(\mathbf{x})] \quad (2.4b)$$

on the lattice. In the lattice form, $N_t \equiv L/a$ denotes the size of the lattice in the compactified direction and $\Lambda_3 \subset \Lambda_4$ is a three dimensional sublattice of the four dimensional lattice corresponding to a fixed Euclidean time-slice. We want the deformation potential $P[\Omega]$ to guarantee the stability of the phase with unbroken center symmetry (at small volume). It will be chosen to have the form

$$P[\Omega] \equiv \sum_{n=1}^{\lfloor N/2 \rfloor} a_n |\text{tr}(\Omega^n)|^2, \quad (2.5)$$

with positive coefficients $\{a_n\}$ (and $\lfloor N/2 \rfloor$ denoting the integer part of $N/2$). In other words, $P[\Omega]$ is a sum of the double trace operators $\text{tr}(\Omega^n) \text{tr}(\Omega^n)^\dagger$. When considering the large N limit, the coefficients $\{a_n\}$ will be held fixed as $N \rightarrow \infty$.

Under a center symmetry transformation by some element $z \in \mathbb{Z}_N$, the Wilson loop $\text{tr}(\Omega^p)$ is multiplied by z^p . The value of $p \bmod N$, which determines the \mathbb{Z}_N representation, is referred to as the N -ality. It will be important that the deformation potential (2.5) only contains, by construction, absolute squares of Wilson loops with non-zero N -ality.

If $P[\Omega]$ were simply proportional to $|\text{tr} \Omega|^2$, with a sufficiently large positive coefficient, then this would prevent breaking of the center symmetry with $\langle \text{tr}(\Omega) \rangle$ as an order parameter. But if $N > 3$ then this single term is not sufficient to prevent any spontaneous breaking of center symmetry, as this term alone does nothing to prevent $\text{tr}(\Omega^2)$ from developing an expectation value. In other words, a stabilizing term proportional to $|\text{tr}(\Omega)|^2$ cannot prevent \mathbb{Z}_N breaking to \mathbb{Z}_2 (assuming N is even) with $\text{tr}(\Omega^2)$ as an order parameter. Adding an additional stabilizing term proportional to $|\text{tr}(\Omega^2)|^2$ could prevent such a breaking to \mathbb{Z}_2 , but does not prevent breaking to \mathbb{Z}_3 (if $N \bmod 3 = 0$), or to any larger discrete subgroup of \mathbb{Z}_N . This is why we have allowed $P[\Omega]$ to include terms up to $|\text{tr} \Omega^{\lfloor N/2 \rfloor}|^2$.

We will argue that the deformed theory satisfies the following:

- i)* For suitable choices of the deformation parameters a_n (*i.e.*, each coefficient sufficiently large and positive) the stabilizing potential (2.5) will prevent the \mathbb{Z}_N center symmetry from breaking to any subgroup.
- ii)* In the $N \rightarrow \infty$ limit, pure Yang-Mills theory on \mathbb{R}^4 is equivalent to the deformed theory formulated on any $\mathbb{R}^3 \times S^1$ (for choices of the $\{a_n\}$ satisfying point *i*). This equivalence applies to expectation values of Wilson loops (on \mathbb{R}^4), or the leading large N behavior of their connected correlators. In the lattice formulation, the number of sites in the compactified direction may be reduced to one.
- iii)* When $N\Lambda L \ll 1$ limit, the deformed Yang-Mills theory is solvable in the same sense as the Polyakov model. The existence of a mass gap and linear confinement can be shown analytically. One can regard this regime as having spontaneous breaking of the $SU(N)$ gauge symmetry down to $U(1)^{N-1}$, but this is a perturbative gauge dependent description with no well-defined invariant content.
- iv)* There exist no order parameters which can distinguish the $N\Lambda L \ll 1$ ‘‘Higgs’’ regime from the $N\Lambda L \gg 1$ regime in which gauge symmetry is ‘‘restored’’.

As noted earlier, pure Yang-Mills theory on $\mathbb{R}^3 \times S^1$ satisfies volume independence (in the large N limit) so long as the \mathbb{Z}_N center symmetry remains unbroken [2, 3, 4]. We will show that this is equally true for the deformed theory; the additional terms in the action do not affect the proof that volume independence, at $N = \infty$, is an automatic consequence of unbroken center (and translation) symmetry. In the undeformed theory, the unbroken center symmetry phase is the low temperature confined phase, $L > L_c$ with $L_c \sim \Lambda^{-1}$ the inverse deconfinement temperature. But in the deformed Yang-Mills theory, the stability of the unbroken center symmetry phase is enforced by hand for all values of L .

Large N volume independence, and the large N equivalence between ordinary Yang-Mills theory on \mathbb{R}^4 and the deformed theory on $\mathbb{R}^3 \times S^1$, may be demonstrated by comparing Dyson-Schwinger equations (*i.e.*, Migdal-Makeenko loop equations) for expectation values

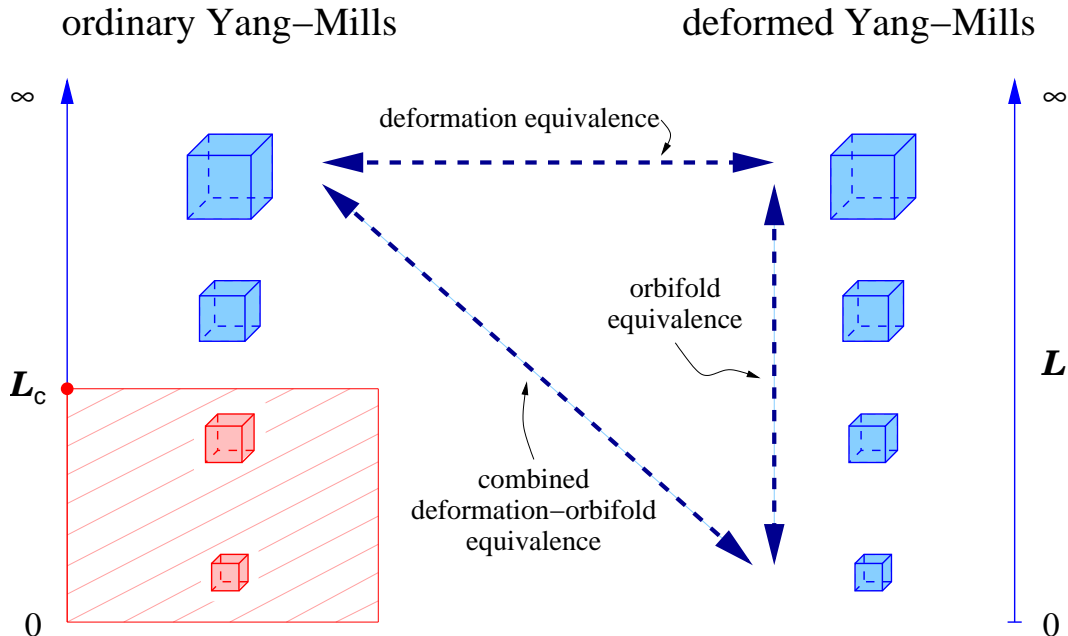


Figure 1: Large N equivalences relating ordinary and deformed $SU(N)$ Yang-Mills theories, as a function of the size L of the periodic volume in which the theories are defined. In deformed YM volume independence holds for all L , while in ordinary YM volume independence fails below a critical size, $L < L_c$, (shaded region) due to spontaneous breaking of center symmetry. This prevents reduction all the way down to a single-site matrix model for ordinary YM. Large N equivalence holds between ordinary and deformed YM theories as long as center symmetry is unbroken in ordinary YM. Large N volume independence is a type of orbifold equivalence, as discussed in Ref. [14]. The combination of volume changing orbifold projections in the deformed theory, along with the deformation equivalence in sufficiently large volume, provides a useful equivalence between deformed YM in small volume and ordinary YM in large volumes. In particular, a single-site matrix model of the deformed theory will reproduce properties of ordinary Yang-Mills theory in infinite volume. The construction can be generalized to QCD, with the deformed theory providing a fully reduced matrix model for QCD.

and correlators of Wilson loops, or alternatively by comparing the large N classical dynamics that may be derived by using appropriate large N coherent states [2, 30, 31, 29]. Figure 1 summarizes the relation between the large N limits of ordinary and deformed Yang-Mills theories. As long as the \mathbb{Z}_N center symmetry is not spontaneously broken, the dynamics of the theories defined by S^{YM} and S^{deformed} are indistinguishable at leading order in the $1/N$ expansion. In particular, glueball spectra of the two theories can differ only by order $1/N^2$ effects. Agreement up to $O(1/N^2)$ terms also applies to the string tension which characterizes the area law behavior of large Wilson loops.

The large N equivalence between ordinary Yang-Mills theory and the deformed theory in large volume, combined with the large N volume independence of the deformed theory, circumvents the problems with previous formulations of reduced models for pure Yang-Mills theory. Unlike the original Eguchi-Kawai model [1], its twisted variant [10, 11], and the partial

reduction of Refs. [4, 5], the equivalence to deformed YM theory remains valid in the limit of zero compactification radius, irrespective of the value of the (bare) gauge coupling. And for finite N , corrections to the large N limit scale as $1/N^2$, not $1/N$ as in quenched reduced models [3]. Consequently, it should be possible to study the deformed theory, for relatively modest values of N and vanishingly small volume, and obtain accurate results for properties of ordinary Yang-Mills theory, in the large N limit, on \mathbb{R}^4 . As we discuss below, it is also instructive to study the deformed theory for small values of N . In this regime, it will be seen to provide a novel example of a confining theory, only involving an $SU(N)$ gauge field, which is analytically soluble.

2.1 Stabilization of center symmetry

The possibility of preventing spontaneous breaking of center symmetry through the addition of a deformation potential of the form (2.5) is largely self-evident. A positive coefficient a_n suppresses configurations in which $\text{tr}(\Omega^n)$ is non-zero. Although the pure-gauge dynamics of the undeformed theory, in small volume, leads to an effective potential for the Wilson line which is minimized when Ω is an element of \mathbb{Z}_N , adding the deformation potential $P[\Omega]$ changes the shape of the Wilson line effective potential. For sufficiently large values of the coefficients $\{a_n\}$, the effective potential will be minimized by configurations in which $\text{tr}(\Omega^n) = 0$ for $1 \leq n \leq \lfloor N/2 \rfloor$. This implies that $\text{tr}(\Omega^n) = 0$ for any integer n which is non-zero modulo N because, for $SU(N)$ -valued matrices, $\text{tr}(\Omega^n)$ is not independent of lower order traces when $n > \lfloor N/2 \rfloor$. Vanishing of these traces implies that the eigenvalues of Ω are uniformly spaced around the unit circle, so that the set of eigenvalues is invariant under \mathbb{Z}_N transformations (which multiply every eigenvalue by $e^{2\pi i/N}$). This shows that the center symmetry is not spontaneously broken. Henceforth, we assume that the coefficients $\{a_n\}$ of the deformation potential $P[\Omega]$ are suitably chosen so as to enforce unbroken center symmetry for all compactification radii.

This argument may be made much more explicit if one considers small compactifications, $L \ll \Lambda^{-1}$, so that (due to asymptotic freedom) the gauge coupling at the scale of the compactification is small and the theory is amenable to a perturbative treatment. Quantum fluctuations generate a nontrivial potential for the Wilson line [32]. In ordinary Yang-Mills theory, integrating out the gauge field (and Faddeev-Popov ghosts) produces the functional determinants

$$[\det_+(-D_{\text{adj}}^2 \delta_{\mu\nu})]^{-1/2} [\det_+(-D_{\text{adj}}^2)] = [\det_+(-D_{\text{adj}}^2)]^{-1}, \quad (2.6)$$

where \det_+ denotes a determinant in the space of periodic functions with period L . Therefore, the effective potential for the Wilson line is

$$V[\Omega] = L^{-1} \ln \det_+(-D_{\text{adj}}^2). \quad (2.7)$$

For constant (or slowly varying) configurations, the evaluation of the functional determinant is straightforward and yields [32]

$$V[\Omega] = \int_{\mathbb{R}^3} \frac{1}{L^4} \mathcal{V}[\Omega(x)], \quad (2.8)$$

with

$$\mathcal{V}[\Omega] \equiv -\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} |\text{tr} \Omega(x)^n|^2. \quad (2.9)$$

For sufficiently small L , corrections to this one-loop result are negligible. The effective potential (2.9) is minimized when the Wilson line is an element of the center, $\Omega = e^{2\pi i k/N}$, $k = 0, \dots, N-1$, demonstrating the spontaneous breaking of \mathbb{Z}_N symmetry, in ordinary Yang-Mills theory, for sufficiently small compactifications.

To force unbroken center symmetry in the deformed theory, the deformation potential $P[\Omega]$ must overcome the effect of the one-loop potential (2.9). A simple specific choice for the deformation coefficients which, in the continuum limit and for sufficiently large N , accomplishes this is $a_n = 4/(\pi^2 n^4)$. For this choice, the deformation potential $\int_{\mathbb{R}^3} L^{-4} P[\Omega]$ is minus twice the one-loop Wilson line effective potential (2.9) of the undeformed theory, so the net effect of the deformation is to flip the sign of the effective potential for the Wilson line. The resulting combined potential is minimized when $\text{tr} \Omega^n = 0$ for all n which are non-zero modulo N , indicating unbroken center symmetry.

2.2 Large N equivalence between ordinary and deformed YM

The most direct way to demonstrate equivalence between ordinary Yang-Mills theory and our deformed theory is to compare the Schwinger-Dyson (or loop) equations for gauge invariant observables. As usual, for a rigorous treatment it is appropriate (and convenient) to work with lattice regulated formulations of both theories. It is also convenient to consider $U(N)$ gauge theories instead of $SU(N)$; the difference in gauge groups only affects subleading $O(1/N^2)$ relative corrections to Wilson loop expectation values or connected correlators.

Let δ_ℓ^a denote an operator which varies individual link fields according to $\delta_\ell^a(U[\ell']) \equiv \delta_{\ell\ell'} t^a U[\ell]$, where $\{t^a\}$ is a set of $U(N)$ Lie algebra basis matrices satisfying $\text{tr} t^a t^b = \frac{1}{2} \delta^{ab}$. Invariance of the Haar measure implies that the integral of any variation vanishes,

$$\int d\mu_0 \delta_\ell^a (\text{anything}) = 0, \quad (2.10)$$

where $d\mu_0 \equiv \prod_{\ell'} dU[\ell']$. Choose (anything) to be $e^S \delta_\ell^a W[C]$, where $W[C] \equiv \frac{1}{N} \text{tr} U[C]$ is the Wilson loop around some closed contour C . Summing over the Lie algebra index a and the link ℓ yields

$$\int d\mu_0 e^S (\delta S \cdot \delta W[C] + \delta^2 W[C]) = 0, \quad (2.11)$$

where the dot product is shorthand for the sum over a and ℓ . Dividing by the partition function $Z \equiv \int d\mu_0 e^S$ yields relations among expectation values,

$$\langle \delta S \cdot \delta W[C] \rangle + \langle \delta^2 W[C] \rangle = 0. \quad (2.12)$$

These are Schwinger-Dyson equations for Wilson loop expectation values.

After working out the action of the variations, the result may be expressed in a purely geometric form. For lattice gauge theory with the Wilson action one finds [33],

$$\begin{aligned} \frac{1}{2}|C|\langle W[C] \rangle &= \sum_{\ell \subset C} \sum_{p|\ell \subset \partial p} \frac{\beta}{4N} \left[\langle W[(\overline{\partial p})C] \rangle - \langle W[(\partial p)C] \rangle \right] \\ &+ \sum_{\text{self-intersections}} \mp \langle W[C']W[C''] \rangle. \end{aligned} \quad (2.13)$$

Here $|C|$ is the length of the loop C (*i.e.*, the number of links in the loop), $W[(\partial p)C]$ denotes a Wilson loop which goes around the boundary of plaquette p (which contains a link contained in the contour C) and then around the contour C , and $\overline{\partial p}$ denotes the oppositely oriented plaquette boundary. The sum over self-intersections runs over all ways of decomposing a loop C which multiply traverses some link ℓ into two separate loops, $C = C'C''$, with the associated sign determined by whether C' and C'' traverse the link ℓ in the same or opposite directions. See Ref. [28] for more detailed discussion.

In the large N limit, with $\tilde{\beta} \equiv \frac{\beta}{N} = \frac{\lambda}{g^2}$ held fixed (where $\lambda \equiv g^2 N$ is the 't Hooft coupling), all N dependence disappears. Fluctuations in the values of Wilson loops vanish in this limit (their distributions become arbitrarily sharply peaked). This is a reflection of the classical nature of the large N limit [2], and implies that the expectation value of a product of loops factorizes, up to $1/N^2$ corrections,

$$\langle W[C']\tilde{W}[C''] \rangle = \langle W[C'] \rangle \langle W[C''] \rangle + O(1/N^2). \quad (2.14)$$

(The $O(1/N^2)$ remainder is the connected correlator.) Consequently, in the large N limit Wilson loop expectation values satisfy a closed set of nonlinear algebraic equations,

$$\begin{aligned} \frac{1}{2}|C|\langle W[C] \rangle &= \sum_{\ell \subset C} \sum_{p|\ell \subset \partial p} \frac{\tilde{\beta}}{4} \left[\langle W[(\overline{\partial p})C] \rangle - \langle W[(\partial p)C] \rangle \right] \\ &+ \sum_{\text{self-intersections}} \mp \langle W[C'] \rangle \langle W[C''] \rangle. \end{aligned} \quad (2.15)$$

The loop equations in the concise form (2.12) are equally valid for the deformed theory. The only difference is that S now includes the double trace deformation ΔS , and this generates new terms in the loop equations given by $\langle \delta(\Delta S) \cdot \delta W[C] \rangle$. Just as the usual Wilson action leads to terms in which a plaquette is inserted into the loop C , the piece of ΔS proportional to $|\text{tr } \Omega^k|^2$ generates terms in the loop equation in which the topologically non-trivial loop Ω^k (or its inverse) is “sewn” into the loop C (if C contains links pointing in the compactified direction). But because ΔS contains absolute squares of traces, each such term is multiplied by the complex conjugate of the trace of the inserted loop. Hence,

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = \sum_{k \neq 0} b_k[C] \langle W[\Omega^k C] W[\Omega^{-k}] \rangle, \quad (2.16)$$

where $\Omega^k C$ denotes a loop obtained by concatenating Ω^k and C at their intersection links, and the coefficients $b_k[C]$ are proportional to $a_{|k|}$ but also depend on the number of links in

C which point in the compactified direction. The essential point is that the variation acts on one of the two traces comprising the double trace deformation, leaving the other trace unchanged. In the large N limit, due to factorization,

$$\langle W[\Omega^k C] W[\Omega^{-k}] \rangle = \langle W[\Omega^k C] \rangle \langle W[\Omega^{-k}] \rangle + O(1/N^2). \quad (2.17)$$

But for any k which is non-zero modulo N , $W[\Omega^{-k}]$ transforms non-trivially (acquiring a phase $e^{-2\pi i k/N}$) under a \mathbb{Z}_N center symmetry transformation. Hence its expectation value is an order parameter for the center symmetry and $\langle W[\Omega^{-k}] \rangle$ must vanish in any phase with unbroken center symmetry. As discussed above, the deformed theory is constructed so as to ensure unbroken center symmetry for all compactification radii. Consequently, all additional terms in the loop equations generated by the deformation of the action vanish in the large N limit,

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = O(1/N^2), \quad (2.18)$$

implying that Wilson loops in the original and deformed Yang-Mills theory satisfy identical large- N Schwinger-Dyson equations.

Ordinary Yang-Mills theory has unbroken center symmetry only for sufficiently large compactifications, $L > L_c$. The coinciding large N loop equations in ordinary and deformed Yang-Mills theories imply that Wilson loop expectation values in these two theories have identical large N limits when $L > L_c$.⁵

The same approach may be used to compare the Schwinger-Dyson equations satisfied by connected correlators of two or more Wilson loops, with exactly the same conclusion: the leading large N behavior of connected correlators coincide between ordinary and deformed Yang-Mills theories, provided $L > L_c$. Thus, in sufficiently large volume the net effect of the double trace deformation on the dynamics of the theory is $\mathcal{O}(1/N^2)$, and vanishes in the large N limit. In another words, the physics of the deformed Yang-Mills theory depends on the deformation parameters $\{a_i\}$ only in the combination a_i/N^2 which vanishes at $N = \infty$. This demonstrates the nonperturbative equivalence of ordinary Yang-Mills theory and the deformed YM theory, formulated on $\mathbb{R}^3 \times S^1$ (or more generally, on any toroidal compactification of flat space), provided the compactification size is above the critical size for center symmetry breaking in the undeformed theory.

2.3 Large N volume independence of deformed YM theory

Unbroken center symmetry is necessary and sufficient for the validity of the large N volume independence of Yang-Mills theory (or more general gauge theories containing adjoint representation matter fields). This may be demonstrated by comparing large N loop equations, or the $N = \infty$ classical dynamics generated by suitable coherent states [14]. Corrections to

⁵This argument, that coinciding loop equations imply coinciding expectation values, glosses over the possibility that the infinite set of loop equations may have multiple solutions which respect center symmetry, with different theories potentially corresponding to different solutions of the same set of equations. The alternative approach of comparing the $N = \infty$ classical dynamics generated by appropriate large N coherent states, discussed in Ref. [29], eliminates this loophole and demonstrates equivalence in any phase of the theories which satisfy the necessary and sufficient symmetry realization conditions.

this equivalence for finite N scale as $1/N^2$. The loop equation analysis is very similar to that sketched above. In the large N loop equations for topologically trivial Wilson loops, one finds that the only volume-dependent terms (arising from self-intersections) automatically vanish as long as the center symmetry is not spontaneously broken.

The analysis of large N volume independence in Ref. [14] applies equally well to the deformed theory which, by construction, has unbroken center symmetry for any compactification radius. Because the double trace operators in $P[\Omega]$ are squares of loops with non-zero N -ality, the presence of the deformation potential $P[\Omega]$ has no effect on the large N classical dynamics of center-symmetry symmetric states. Consequently, deformed Yang-Mills theory, in the large N limit, is completely volume independent.

In the lattice formulation, if one compactifies all directions then one may reduce the lattice size all the way down to a single site, in which case the theory becomes a simple matrix model of Wilson lines $\{\Omega_i\}$ running in each lattice direction with action,

$$S_{\text{single-site}}^{\text{deformed}} = \frac{\beta}{2} \sum_{i>j=1}^d \text{tr} (\Omega_i \Omega_j \Omega_i^\dagger \Omega_j^\dagger + \Omega_j \Omega_i \Omega_j^\dagger \Omega_i^\dagger) + \sum_{i=1}^d P[\Omega_i]. \quad (2.19)$$

The large N limit of this matrix model must reproduce the leading large N behavior of expectation values and connected correlators of Wilson loops in uncompactified Yang-Mills theory. As discussed in the Introduction, the single-site deformed Yang-Mills theory (2.19) provides a simple generalization of Eguchi-Kawai reduction which is valid for any value of the lattice coupling β .

2.4 Addition of matter fields

Consider adding N_f species of matter fields (either fermions or scalars) in the fundamental representation to $SU(N)$ Yang-Mills theory, either ordinary or deformed, with one dimension compactified. The addition of fundamental representation matter explicitly breaks the \mathbb{Z}_N center symmetry. However, if N_f is held fixed as $N \rightarrow \infty$, then the fundamental representation matter fields have only a subleading $O(N_f/N)$ effect on the gauge field dynamics. As a result, everything discussed above remains valid. That is, the leading large N behavior of expectation values or connected correlators of Wilson loops in the undeformed theory, in sufficiently large volume, coincide with the corresponding observables in the theory, in arbitrary volume, deformed by the addition of the stabilizing potential $P[\Omega]$. In addition, one may also show that the same equivalence applies to the leading large N behavior of mesonic expectation values and connected correlators.⁶ Note, however, that these large N equivalences cease to apply if N_f/N is held fixed as $N \rightarrow \infty$ [14].

If adjoint representation matter fields are added to the theory (ordinary or deformed), then the large N equivalences discussed above also remain valid. Adding adjoint representation fields enlarges the natural set of gauge invariant observables from simple Wilson loops to

⁶This is easiest to understand by considering the equivalent gluonic observables produced by integrating out the matter fields. For the case of large N volume independence in the undeformed theory, see Ref. [14] for details. The presence of the deformation potential does not affect this analysis.

Wilson loops decorated by arbitrary numbers of insertions of adjoint matter fields. But the presence of adjoint matter fields preserves the center symmetry of the underlying Yang-Mills theory. As a result, the above-described comparison of large N loop equations (or large N classical dynamics) between the ordinary and deformed theories immediately generalizes to the case of Yang-Mills theories with adjoint matter, with exactly the same conclusions.⁷

Finally, one may also consider the addition of matter fields in rank-two antisymmetric or symmetric tensor representations (yielding theories we will refer to as QCD(AS) or QCD(S), respectively). The presence of fields in these representations reduces the $U(1)$ center symmetry of $U(N)$ Yang-Mills theory down to \mathbb{Z}_2 . Given the central role the center symmetry played in the above large N equivalences, one might think this reduction in center symmetry would destroy these large N equivalences. This is not the case. One way to see this is to note, as discussed in Ref. [14], that volume-dependent terms in the $N = \infty$ loop equations only appear if loops with non-zero winding number around the compactified direction acquire non-zero expectation values. The addition of the deformation potential $P[\Omega]$ prevents topologically non-trivial Wilson loops from acquiring non-zero large N expectation values, even in small volumes. Consequently, the situation is analogous to the pure gauge case: the leading large N behavior of expectation values or connected correlators of single trace observables in QCD(AS/S) in sufficiently large volume coincide with the corresponding observables in the theory modified by the addition of the deformation potential in arbitrary volume, as depicted in Fig.1.⁸

3. Confinement at small radius and small N

When compactified on a small circle, $L \ll \Lambda^{-1}$, the gauge coupling of the deformed theory is small at the compactification scale, $g^2(1/L) \ll 1$. As discussed earlier, the combined potential $\mathcal{V}[\Omega] + P[\Omega]$ is minimized when

$$\Omega = \text{Diag} \left(1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2\pi i(N-1)/N} \right), \quad (3.1)$$

⁷For a detailed discussion of loop equations in theories with adjoint matter, see Ref. [28].

⁸Another way to understand this is to note the existence of a large N equivalence (so-called “orientifold equivalence”) between theories with rank-two symmetric or antisymmetric representation matter and corresponding theories with adjoint representation matter [“QCD(adj)”] [15, 16, 17]. This large N equivalence applies to the charge-conjugation even sectors of the two theories, and only holds if charge conjugation symmetry is not spontaneously broken. When, for example, the matter fields are fermions with periodic boundary conditions, examination of the Wilson line effective potential shows that QCD(AS/S) *does* spontaneously break both charge conjugation and center symmetry when compactified with sufficiently small size [15]. But the addition of a deformation potential of the form (2.5) (with sufficiently positive coefficients) will prevent this spontaneous symmetry breaking, just as it does in the pure Yang-Mills case. Since QCD(adj) satisfies large N volume independence (as long as its center symmetry is not spontaneously broken), the same large N volume independence must also apply to QCD(AS/S) (as long as charge conjugation is not broken). In sufficiently large volumes, there is no reason to believe that charge conjugation symmetry breaks spontaneously in QCD-like theories with rank-two tensor representation matter. Therefore, large N orientifold equivalence combines with large N volume independence of QCD(adj) to imply volume independence in QCD(AS/S) as long as center and charge conjugation symmetries are not spontaneously broken — which is what the deformation potential ensures.

up to conjugation by an arbitrary $SU(N)$ matrix. Working in a gauge in which Ω is diagonal, and using (temporarily) gauge-dependent language, this configuration may be regarded as breaking the gauge symmetry down to the maximal Abelian subgroup,

$$SU(N) \rightarrow U(1)^{(N-1)}. \quad (3.2)$$

Modes of the diagonal components of the $SU(N)$ gauge field with no momentum along the compactified \hat{x}_4 direction describe photons associated with the Cartan subgroup of $SU(N)$. Modes of the diagonal components of the gauge field with non-zero momentum in the compactified direction form a Kaluza-Klein tower and receive masses which are integer multiples of $2\pi/L$. The off-diagonal components of the $SU(N)$ gauge field describe Kaluza-Klein towers of W -bosons which are charged under the unbroken $U(1)^{N-1}$ gauge group. The non-zero value of $A_4 \equiv -(i/L) \ln \Omega$ shifts the masses of these off-diagonal components by multiples of $2\pi/(NL)$. The net effect is that there are charged W -bosons with masses

$$m_{W_k} = \frac{2\pi k}{NL}, \quad k = 1, 2, \dots, . \quad (3.3)$$

For later convenience, we define m_W to be the mass of the lightest W bosons,

$$m_W \equiv \frac{2\pi}{NL}. \quad (3.4)$$

This is the mass scale below which the dynamics is effectively Abelian.⁹ Note that, at fixed L , the lightest W bosons have masses which become small when $N \rightarrow \infty$. This will be important in the discussion of the large N behavior of the deformed YM theory. But first, in this section, we consider the dynamics of the deformed theory when N is fixed and small.

The $N-1$ photons of the Cartan subgroup do not couple (directly) to the Wilson line and remain massless to all orders in perturbation theory. Thus, a strictly perturbative analysis would lead one to expect that the deformed theory, for sufficiently small L , would have a non-confining Coulomb phase. We will see that this is incorrect — nonperturbative effects lead to the generation of a mass gap and produce confining long distance physics.

The analysis of non-perturbative properties in our compactified deformed Yang-Mills theory is very similar to Polyakov’s treatment of the $3d$ Georgi-Glashow model [22]. But instead of a three-dimensional theory with a non-compact Higgs field, we have a compactified four-dimensional theory with the group-valued Wilson line Ω serving as a compact Higgs field. For theories involving massless complex fermions, the difference between compact and noncompact Higgs systems can be major [34]. However, in our case, the differences relative to Polyakov’s classic discussion are rather minimal.

Due to the $SU(N) \rightarrow U(1)^{N-1}$ gauge symmetry “breaking”, there exist topologically stable, semiclassical field configurations, namely monopoles [32]. At the center of a monopole,

⁹Fluctuations in the eigenvalues of Ω away from the minimum (3.1) correspond to neutral “Higgs bosons”. The masses of these fluctuations depend on the coefficients $\{a_n\}$ of the deformation potential but parametrically are of order $\sqrt{\lambda}/L \sim \sqrt{\lambda} N m_W$. This scale will be large compared to the mass scale of the non-perturbative $3d$ dynamics, and these fluctuations will play no role in the following discussion.

one pair of eigenvalues of the Wilson line become degenerate. For fundamental (*i.e.*, minimal action) monopoles, this will be a pair of eigenvalues which are nearest-neighbors at infinity. If the adjoint Higgs field was noncompact, then there would be $N-1$ species of fundamental monopoles. This follows from the topological considerations: the second homotopy group $\pi_2[SU(N)/U(1)^{N-1}] = \pi_1[U(1)^{N-1}] = \mathbb{Z}^{N-1}$, implying that fundamental monopoles come in $N-1$ varieties. However, with a compact Higgs field there is an extra fundamental (“Kaluza-Klein”) monopole which arises due to the fact that the underlying theory is formulated on a cylinder, $\mathbb{R}^3 \times S^1$, or equivalently that the configuration space of Ω is compact.

The monopoles may be characterized by their magnetic charges, topological charge, and their action. The magnetic charges of the N different types of fundamental monopoles are proportional to the simple roots and affine root of the Lie algebra of the unbroken $U(1)^N$ gauge group.¹⁰ The simple roots are given by¹¹

$$\alpha_1 = (1, -1, 0, \dots, 0) = \hat{e}_1 - \hat{e}_2, \quad (3.5a)$$

$$\alpha_2 = (0, 1, -1, \dots, 0) = \hat{e}_2 - \hat{e}_3, \quad (3.5b)$$

\vdots

$$\alpha_{N-1} = (0, \dots, 0, 1, -1) = \hat{e}_{N-1} - \hat{e}_N, \quad (3.5c)$$

and the affine root is

$$\alpha_N \equiv - \sum_{j=1}^{N-1} \alpha_j = (-1, 0, 0, \dots, 1) = \hat{e}_N - \hat{e}_1. \quad (3.6)$$

For later convenience, let Δ_{aff}^0 denote the affine (extended) root system of the the associated Lie algebra,

$$\Delta_{\text{aff}}^0 \equiv \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}, \alpha_N\}. \quad (3.7)$$

It is the affine root system which is relevant for compact Yang-Mills Higgs systems. The roots $\alpha_i \in \Delta_{\text{aff}}^0$ obey

$$\alpha_i \cdot \alpha_j = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}, \quad i, j = 1, \dots, N. \quad (3.8)$$

The form (3.8) of these inner products will translate into self and nearest neighbor interactions between monopoles in the Dynkin space. The above choice of basis is natural due its visual simplicity, but the inner products (3.8) of the roots of the associated Lie algebra are basis independent.

Let $F^\mu \equiv \frac{1}{2g} \epsilon^{\mu\nu\lambda 4} F_{\nu\lambda}$ denote the $U(1)^N$ valued $3d$ magnetic field, with conventional perturbative normalization. (In a gauge where Ω is diagonal, F^μ is just the list of diagonal

¹⁰Even though the gauge symmetry “breaking” is $SU(N) \rightarrow U(1)^{N-1}$, for ease of presentation it is convenient to add an extra photon to the original theory and discuss $U(N) \rightarrow U(1)^N$. This simplifies the discussion of charge assignments of monopoles, and the affine roots of the associated Lie algebra. In the continuum limit of the theory this extra photon completely decouples from the other degrees of freedom and may simply be ignored. It should not be confused with the $N-1$ photons which have non-trivial nonperturbative dynamics.

¹¹This set of simple roots corresponds to choosing Lie algebra generators normalized to satisfy $\text{tr } t^a t^b = \delta^{ab}$, instead of $\frac{1}{2} \delta^{ab}$ as in the previous section.

elements of the original non-Abelian field strength, multiplied by $1/g$.) The magnetic charges of a monopole of type $i = 1, \dots, N$ are given by the root α_i (up to a factor of $2\pi/g$),

$$\int_{S^2} d\Sigma \cdot \mathbf{F} = \frac{2\pi}{g} \alpha_i \quad [\text{type } (i) \text{ monopole}]. \quad (3.9)$$

(The S^2 is an arbitrarily large sphere in \mathbb{R}^3 . The flux is independent of the value of x^4 at which the integral is performed, as the long distance monopole fields are independent of x^4 .)

The topological charge is correlated with the magnetic charge of the monopole. For fundamental monopoles with magnetic charges $\alpha_i \in \Delta_{\text{aff}}^0$, the topological charge is

$$\nu \equiv \int_{\mathbb{R}^3 \times S^1} \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{N}. \quad (3.10)$$

For antimonopoles with magnetic charges $-\alpha_i$, the topological charge $\nu = -1/N$.

The electric charges of W -bosons may also be simply expressed in terms of the affine roots Δ_{aff}^0 . The lightest W bosons, with mass m_W , may be labeled by a single root which gives their electric charges (up to a factor of g),

$$Q_{W_{\alpha_i}} = g \alpha_i. \quad (3.11)$$

W -bosons in the next heavier multiplet are labeled by a pair of neighboring roots, and have charges

$$Q_{W_{\alpha_i + \alpha_{i+1}}} = g(\alpha_i + \alpha_{i+1}), \quad (3.12)$$

etc. Dot products of the W -boson charges and monopole charges obey the Dirac quantization condition,

$$Q_{W_{\alpha_i}} \cdot Q_{M_{\alpha_j}} = g \alpha_i \cdot \frac{2\pi}{g} \alpha_j = 2\pi(2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}) = \begin{cases} 4\pi, & \text{for } i = j; \\ -2\pi, & \text{for } i = j \pm 1; \\ 0, & \text{otherwise.} \end{cases} \quad (3.13)$$

Conjugation by a \mathbb{Z}_N “shift” matrix, which is part of the global gauge symmetry, cyclically permutes the Wilson line eigenvalues and hence cyclically permutes the N different species of fundamental monopoles. The presence of this symmetry (which is one of the features which distinguishes compact and non-compact Higgs systems) guarantees that the N different types of fundamental monopoles have identical values of the action. Monopole solutions are self-dual,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}, \quad (3.14)$$

and hence the Yang-Mills action of a fundamental monopole (or antimonopole) is

$$S_{\text{YM}} = \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu}^2 = \left| \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right| = \frac{8\pi^2}{g^2} |\nu| = \frac{8\pi^2}{g^2 N}. \quad (3.15)$$

After adding the contributions of the deformation potential $P[\Omega]$ and the induced one-loop effective potential $\mathcal{V}[\Omega]$, the complete monopole action will differ from this value. But the deviation is perturbative in g^2 , so the monopole action

$$S_0 \equiv S_{\text{YM}} + \Delta S + S_{1\text{-loop}} = \frac{8\pi^2}{g^2 N} + O(1). \quad (3.16)$$

The correct infrared description of the deformed Yang-Mills theory on $\mathbb{R}^3 \times S^1$ at small radius is generated by a dilute gas of monopoles (and antimonopoles) of N different types, interacting via the species-dependent long range Coulomb potential,

$$V_{\pm i, \pm j}(\mathbf{r}) = L \left(\frac{2\pi}{g} \right)^2 \frac{(\pm\alpha_i) \cdot (\pm\alpha_j)}{4\pi|\mathbf{r}|} = \pm L \left(\frac{2\pi}{g} \right)^2 \frac{2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}}{4\pi|\mathbf{r}|}. \quad (3.17)$$

(The overall sign is plus for monopole-monopole, and minus for monopole-antimonopole.) Hence, we are dealing with a multi-component classical plasma, with nearest-neighbor interactions in the Dynkin space. As with any classical plasma, this system will exhibit Debye screening. The field due to a static external magnetic charge will fall exponentially with distance, $|\mathbf{F}| \sim e^{-m_D r}/r$, with m_D^{-1} the characteristic Debye screening length. This implies that external fields cannot propagate coherently over distances large compared to the Debye length, which will be the longest correlation length in the system. The Debye mass m_D will appear as a dynamically generated photon mass. This will be shown explicitly.

For momentum scales small compared to the lightest W mass, the equilibrium dynamics is correctly represented by a grand canonical ensemble of all types of monopoles and antimonopoles. Consider a configuration in which there are $n^{(i)}$ monopoles and $\bar{n}^{(i)}$ antimonopoles of types $i = 1, \dots, N$, located at positions $\mathbf{r}_k^{(i)}$, $k = 1, \dots, n^{(i)}$ and $\bar{\mathbf{r}}_l^{(i)}$, $l = 1, \dots, \bar{n}^{(i)}$, respectively. The magnetic field generated by this ensemble of magnetic charges is

$$\mathbf{B}(\mathbf{x}) = \sum_{i=1}^N \frac{2\pi}{g} \alpha_i \left[\sum_{k=1}^{n^{(i)}} \frac{\mathbf{x} - \mathbf{r}_k^{(i)}}{4\pi|\mathbf{x} - \mathbf{r}_k^{(i)}|^3} - \sum_{l=1}^{\bar{n}^{(i)}} \frac{\mathbf{x} - \bar{\mathbf{r}}_l^{(i)}}{4\pi|\mathbf{x} - \bar{\mathbf{r}}_l^{(i)}|^3} \right]. \quad (3.18)$$

The action of such a monopole configuration is the sum of the monopole self-energies plus their potential energy due to Coulomb interactions,

$$S_{\text{monopole-gas}} = S_0 \sum_{i=1}^N \left(n^{(i)} + \bar{n}^{(i)} \right) + S_{\text{int}}, \quad (3.19)$$

with

$$S_{\text{int}} = \frac{2\pi^2 L}{g^2} \sum_{i,j=1}^N \alpha_i \cdot \alpha_j \left[\sum_{k=1}^{n^{(i)}} \sum_{l=1}^{n^{(j)}} G(\mathbf{r}_k^{(i)} - \mathbf{r}_l^{(j)}) + \sum_{k=1}^{\bar{n}^{(i)}} \sum_{l=1}^{\bar{n}^{(j)}} G(\bar{\mathbf{r}}_k^{(i)} - \bar{\mathbf{r}}_l^{(j)}) - 2 \sum_{k=1}^{n^{(i)}} \sum_{l=1}^{\bar{n}^{(j)}} G(\mathbf{r}_k^{(i)} - \bar{\mathbf{r}}_l^{(j)}) \right], \quad (3.20)$$

and

$$G(\mathbf{r}) \equiv \frac{1}{4\pi|\mathbf{r}|}. \quad (3.21)$$

The grand canonical partition function of this multi-component Coulomb gas is

$$Z = \prod_{i=1}^N \left\{ \sum_{n^{(i)}=0}^{\infty} \frac{\zeta^{n^{(i)}}}{n^{(i)}!} \sum_{\bar{n}^{(i)}=0}^{\infty} \frac{\zeta^{\bar{n}^{(i)}}}{\bar{n}^{(i)}!} \int_{\mathbb{R}^3} \prod_{k=1}^{n^{(i)}} d\mathbf{r}_k^{(i)} \int_{\mathbb{R}^3} \prod_{l=1}^{\bar{n}^{(i)}} d\bar{\mathbf{r}}_l^{(i)} \right\} e^{-S_{\text{int}}}, \quad (3.22)$$

where

$$\zeta \equiv C e^{-S_0} = A m_W^3 (g^2 N)^{-2} e^{-\Delta S} e^{-8\pi^2/Ng^2(m_W)} \quad (3.23)$$

is the monopole fugacity. The prefactor C represents the one-loop functional determinant in the monopole background. Extracting the zero-modes of the small fluctuation operator via the usual collective coordinate procedure leads to factors of $(g^2 N)^{-2} m_W^3$. (See the appendix for details.) If the coupling is evaluated at the scale m_W , which is natural for this problem, then the non-zero mode part of the one-loop determinant merely gives rise to an overall dimensionless (and N independent) coefficient A . In the final form of (3.23), ΔS denotes the deformation term in the action (2.4) evaluated in the background of a fundamental monopole. This is an $O(1)$ number, independent of the coupling g^2 , whose explicit value depends, of course, on the deformation parameters $\{a_n\}$. [For large N , ΔS scales as $O(1/N)$.]

Using the fact that $G(\mathbf{r})$ is the Green's function for the $3d$ Laplacian, this partition function can be exactly transformed into a $3d$ scalar field theory with an N -component real scalar field,

$$Z = \int \prod_{i=1}^N \mathcal{D}\sigma_i e^{-S_{\text{dual}}[\boldsymbol{\sigma}]}, \quad (3.24)$$

where

$$S^{\text{dual}} = \int_{\mathbb{R}^3} \left[\frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \boldsymbol{\sigma})^2 - \zeta \sum_{i=1}^N \cos(\alpha_i \cdot \boldsymbol{\sigma}) \right]. \quad (3.25)$$

To verify this, it is easiest to start with the functional integral (3.24), rewrite the cosines in terms of exponentials of $\boldsymbol{\sigma}$, expand the exponential of each of the resulting interaction terms in a power-series in $e^{-S_0} e^{\pm i\alpha_i \cdot \boldsymbol{\sigma}}$, and then perform the functional integral over $\boldsymbol{\sigma}$. The scalar fields σ_i appearing in this representation are dual fields for the $3d$ Abelian gauge fields A_μ^i .¹²

The fields $\{\sigma_i\}$ should be regarded as compact scalar fields defined modulo 2π . In addition to invariance under 2π shifts in any component of $\boldsymbol{\sigma}$, note that the monopole induced interaction vertex has the additional shift symmetry

$$\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + 2\pi\boldsymbol{\mu}_i, \quad i = 1, \dots, N-1 \quad (3.26)$$

where $\{\boldsymbol{\mu}_i\}$ are the $N-1$ fundamental weights of the $SU(N)$ algebra. These are defined by the reciprocity relation with the simple roots,

$$\boldsymbol{\mu}_i \cdot \boldsymbol{\alpha}_j = \frac{1}{2} \delta_{ij} \boldsymbol{\alpha}_j^2 = \delta_{ij} \quad (3.27)$$

¹²In three dimensions, Abelian duality relates a photon to a compact scalar. With $\sigma^j(\mathbf{x})$ the compact scalar dual to the photon $A_\mu^{(j)}(\mathbf{x})$ of the j 'th $U(1)$ subgroup, the Abelian duality relations are

$$*d\sigma^j = \frac{1}{2} L \text{Im}(\tau) F^{(j)}, \quad \tau(L^{-1}) = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \quad F_{\mu\nu}^{(j)} = \frac{g^2}{2\pi L} \epsilon_{\mu\nu\rho} \partial_\rho \sigma^j.$$

The $3d$ Maxwell action becomes $\frac{L}{4g^2} (F_{\mu\nu}^{(j)})^2 = \frac{L \text{Im}(\tau)}{16\pi} (F_{\mu\nu}^{(j)})^2 = \frac{1}{2\pi L \text{Im}(\tau)} (\partial_\mu \sigma^j)^2$. The path integral of the Abelian gauge theory in the presence of a monopole with charge $\pm\alpha_j$ located at position \mathbf{x} is equivalent to the insertion of $e^{\pm i\alpha_j \cdot \boldsymbol{\sigma}(\mathbf{x})}$ into the path integral over the dual scalar fields [22, 27]. The complete partition function of the long distance effective theory is a sum over all topological sectors, each of which may contain an arbitrary number of monopoles and antimonopoles (whose charges sum to give the appropriate topological class). Summing over all numbers and locations of monopoles (and antimonopoles), weighted with the appropriate fugacity, directly yields the result (3.25).

for $i = 1, \dots, N-1$, which implies that the fundamental weights $\{\mu_i\}$ form a basis which is dual to the fundamental roots $\{\alpha_j\}$. The presence of the symmetry (3.26) is related to the fact that the vacuum of the original theory can be probed by $N-1$ different types of external charges, distinguished by their (non-zero) values of N -ality. This will be discussed below.

Including a non-zero theta parameter in the original Yang-Mills action,

$$S^{\text{YM}} \rightarrow S^{\text{YM}} + i\theta \int_{\mathbb{R}^3 \times S^1} \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (3.28)$$

has the effect, in the grand canonical partition function (3.22), of multiplying the monopole fugacity by $e^{i\theta/N}$ and antimonopole fugacity by $e^{-i\theta/N}$. In the dual representation (3.25), this amounts to shifting the argument of the cosine by θ/N , so that the interaction becomes

$$-\zeta \sum_{i=1}^N \cos(\alpha_i \cdot \boldsymbol{\sigma} + i\theta/N). \quad (3.29)$$

In this form, 2π periodicity of the theory with respect to θ is not manifest. However, a shift of the dual scalar fields, $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + (\theta/N)\boldsymbol{\beta}$ with $\boldsymbol{\beta} \equiv (0, 1, 2, \dots, N-1)$, converts the interaction term to the manifestly 2π periodic form

$$-\zeta \left[\sum_{i=1}^{N-1} \cos(\alpha_i \cdot \boldsymbol{\sigma}) + \cos(\alpha_N \cdot \boldsymbol{\sigma} + \theta) \right], \quad (3.30)$$

in which theta dependence only appears in the term involving the affine root. For simplicity, in the following subsections we will focus on the case of $\theta = 0$.

3.1 Mass gap

The cosine potential in the dual action (3.25) generates a mass term for the photons. Rescaling $\boldsymbol{\sigma}$ to put the kinetic term into canonical form and expanding the potential to quadratic order around the minimum at $\boldsymbol{\sigma} = 0$ gives

$$V(\sigma_i) = (\text{const.}) + \frac{1}{2} m_\gamma^2 \sum_{i=1}^N (\sigma_{i+1} - \sigma_i)^2, \quad (3.31)$$

with $\sigma_{N+1} \equiv \sigma_1$ and

$$m_\gamma^2 \equiv \frac{(2\pi)^2}{g^2} L \zeta = A m_W^2 \left(\frac{2\pi}{g^2 N} \right)^3 e^{-\Delta S} e^{-8\pi^2/(Ng^2(m_W))}. \quad (3.32)$$

A \mathbb{Z}_N Fourier transform,

$$\tilde{\sigma}_p \equiv \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-2\pi i p j/N} \sigma_j, \quad p = 0, \dots, N-1, \quad (3.33)$$

diagonalizes this mass term and yields

$$V(\sigma_i) = (\text{const.}) + \frac{1}{2} \sum_{p=0}^{N-1} m_p^2 |\tilde{\sigma}_p|^2, \quad (3.34)$$

with

$$m_p \equiv m_\gamma \sin \frac{\pi p}{N}. \quad (3.35)$$

Expressing m_γ in terms of the renormalization group invariant scale Λ , defined by

$$\Lambda^{b_0} = \mu^{b_0} (Ng^2(\mu))^{-b_1/b_0} e^{-8\pi^2/(Ng^2(\mu))} \quad (3.36)$$

(with $b_0 = 11/3$ and $b_1 = 17/3$), yields

$$m_\gamma = \tilde{A} \Lambda (\Lambda N L)^{5/6} |\ln N \Lambda L|^{9/11}, \quad (3.37)$$

where \tilde{A} is an $O(1)$ coefficient. Relative corrections suppressed by powers of $g^2(m_W) \sim 1/|\ln N \Lambda L|$ have, of course, been neglected. The result (3.35), for $p = 1, \dots, N-1$, shows that the $N-1$ photons of the “unbroken” $U(1)^{N-1}$ gauge group receive non-zero masses due to nonperturbative effects.¹³

3.2 String tensions

Let us first examine the vacuum structure of the dual theory in more detail. The dual scalars are defined to be periodic with period 2π . This implies that shifting $\boldsymbol{\sigma}$ by 2π times any root vector is an identity, $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma} + 2\pi\alpha_i$ for all $\alpha_i \in \Delta_{\text{aff}}^0$. As noted earlier, the dual action (3.25) is also invariant under $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + 2\pi\mu_i$, where $\{\mu_i\}$ are the fundamental weights of the $SU(N)$ gauge group, defined by the reciprocity relations (3.27). The simple roots $\{\alpha_i\}$ generate the root lattice Λ_r . Its dual, the weight lattice Λ_w is generated by the fundamental weights $\{\mu_i\}$. The root lattice is a sublattice of the weight lattice and their quotient is

$$\Lambda_w/\Lambda_r = \mathbb{Z}_N. \quad (3.38)$$

This implies that the dual theory potential, $V(\boldsymbol{\sigma}) \equiv -\zeta \sum_i \cos(\alpha_i \cdot \boldsymbol{\sigma})$, has N isolated minima lying within the unit cell of Λ_r . These minima are located at $\boldsymbol{\sigma} = 0$ and

$$\boldsymbol{\sigma} = 2\pi\mu_j, \quad j = 1, \dots, N-1. \quad (3.39)$$

(Equivalently, one may describe the minima as lying at $2\pi j \mu_1$, for $j = 0, 1, \dots, N$, since $\mu_j = j \mu_1 + \alpha$ for some $\alpha \in \Lambda_r$.)

Let \mathcal{R} be some chosen irreducible representation of $SU(N)$. The expectation value of the Wilson loop $W_{\mathcal{R}}(C)$ characterizes the response of the system to external test charges in the representation \mathcal{R} . In a confining phase with a non-zero mass gap, if external charges in representation \mathcal{R} cannot be screened by gluons, then expectation values of large Wilson loops

¹³The vanishing $p = 0$ mass corresponds to the extra decoupled photon which was added to simplify the duality transformation but is not present in the original theory. It should be ignored.

in this representation are expected to decrease exponentially with the area of the minimal spanning surface,

$$\langle W_{\mathcal{R}}(C) \rangle \sim e^{-T(\mathcal{R}) \text{Area}(\Sigma)}. \quad (3.40)$$

Here Σ denotes the minimal surface with boundary C , and $T(\mathcal{R})$ is the string tension for representation \mathcal{R} . Such area law behavior implies the presence of an asymptotically linear confining potential between static charges in representation \mathcal{R} and anti-charges in representation $\overline{\mathcal{R}}$, $V_{\mathcal{R}}(\mathbf{x}) \sim T(\mathcal{R}) |\mathbf{x}|$ as $|\mathbf{x}| \rightarrow \infty$.

The irreducible representation \mathcal{R} may be associated with its highest weight vector $w \in \Lambda_w$. Identifying weight vectors which differ by elements of the root lattice produces a \mathbb{Z}_N grading of representations which corresponds to their N -ality (the charge of the representation under the \mathbb{Z}_N center). In particular, if this equivalence associates the highest weight vector w with k times the fundamental weight μ_1 ,

$$w = k \mu_1 + \alpha, \quad \text{for some } \alpha \in \Lambda_r, \quad (3.41)$$

then the representation \mathcal{R} has N -ality k .¹⁴

As discussed in Refs. [27, 22], the insertion of a Wilson loop $W_{\mathcal{R}}(C)$ in a representation \mathcal{R} with non-zero N -ality k corresponds, in the low-energy dual theory, to the requirement that the dual scalar fields have non-trivial monodromy,

$$\int_{C'} d\sigma = 2\pi\mu_k, \quad (3.42)$$

where C' is any closed curve whose linking number with C is one. In other words, in the presence of the Wilson loop $W_{\mathcal{R}}(C)$ the dual scalar fields must have a discontinuity of $2\pi\mu_k$ across some surface Σ which spans the loop C . One way to see this is to go back to the duality relation. For simplicity, consider the case of a large planar loop lying in the xy -plane. As the size of the loop grows, the spanning surface Σ approaches an infinite flat plane. In the presence of the Wilson loop, the Abelian duality relation $F \sim *d\sigma$ is replaced by $F \sim *d\sigma + \mu_k \delta(z) dx \wedge dy$. Therefore the dual scalars σ must be discontinuous across Σ in order for the field strength F to be continuous.

The fact that dual low energy theory depends on the representation \mathcal{R} of the Wilson loop only through its N -ality k shows that there are only $N-1$ distinct string tensions, referred to as k -string tensions, $\{T_k\}$. (Charge conjugation symmetry implies that $T_k = T_{N-k}$.) The dual theory representation of Wilson loops also shows that external charges in representations with zero N -ality will not be confined. These are precisely the representations which can be screened by adjoint representation gluons.

To evaluate a Wilson loop expectation value, one must minimize the dual action in the space of field configurations satisfying the monodromy condition (3.42). To extract the string tension,

$$T_k \equiv - \lim_{\text{area}(\Sigma) \rightarrow \infty} \frac{\ln \langle W_{\mathcal{R}}(C) \rangle}{\text{area}(\Sigma)}, \quad (3.43)$$

¹⁴Representations contained in the product of m powers of the fundamental representation with n powers of the antifundamental have N -ality $m-n$.

it is sufficient to consider the limit where Σ fills the xy -plane. In this case, the field $\sigma(\mathbf{x})$ will only depend on z . It must approach some minimum of the dual potential at infinity, $\lim_{z \rightarrow \pm\infty} \sigma(z) = 2\pi\mu_l$, and must be discontinuous across $z = 0$ with a jump given by the prescribed fundamental weight, $\lim_{z \rightarrow 0^+} \sigma(z) - \lim_{z \rightarrow 0^-} \sigma(z) = 2\pi\mu_k \pmod{2\pi}$. Because shifts by $2\pi\mu_k$ are an invariance of the dual potential $V(\sigma)$, one may equally well minimize the action for field configurations $\sigma(z)$ which are continuous but whose asymptotic values differ,

$$T_k = \min_{\sigma(z)} \frac{\Delta S(\sigma)}{\text{area}(\mathbb{R}^2)} \Big|_{\Delta\sigma=2\pi\mu_k \pmod{2\pi}}, \quad (3.44)$$

where $\Delta\sigma \equiv \sigma(\infty) - \sigma(-\infty)$, and $\Delta S(\sigma)$ is the dual action minus its vacuum value. Explicitly,

$$T_k = \min_{\sigma(z)} \int dz \left\{ \frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 \left(\frac{\partial\sigma}{\partial z} \right)^2 + \zeta \sum_i [1 - \cos(\sigma_i - \sigma_{i+1})] \right\} \Big|_{\Delta\sigma=2\pi\mu_k \pmod{2\pi}}. \quad (3.45)$$

In other words, the k -string tension T_k equals the action of a kink solution with topological charge k in this one dimensional theory.

The width of the kink solution must be of order of the inverse photon mass m_γ^{-1} . Consequently, the k -string tension will have the form $T_k = f_k T$, where

$$T \equiv \zeta/m_\gamma \sim \Lambda^2 (\Lambda LN)^{-1/6} |\log(\Lambda LN)|^{-3/11}, \quad (3.46)$$

and f_k is an $O(1)$ coefficient. Even without finding the minimizing kink solutions explicitly, it is apparent that the resulting k -string tension T_k will be non-zero (for $k = 1, \dots, N-1$), and must satisfy the convexity relation $T_{k+l} \leq T_k + T_l$.

We were unable to solve the kink equations of motions analytically for general N , but when $N = 2$ the equations of motion reduce to Sine-Gordon model. In this case, one finds

$$T \equiv T_1 = 4\sqrt{2}\zeta/m_\gamma. \quad (3.47)$$

3.3 Larger size or larger N

The above semiclassical analysis of the deformed Yang-Mills theory is reliable provided there is a parametrically large separation of scales between the lightest W -boson mass, $m_W = 2\pi/(NL)$, and the nonperturbatively induced dual photon mass m_γ . Their ratio scales as

$$\frac{m_W}{m_\gamma} \sim (LN\Lambda)^{-11/6} |\log(LN\Lambda)|^{-9/11}, \quad (3.48)$$

and hence there is a large separation of mass scales provided $LN\Lambda \ll 1$. In this regime, the monopole gas is highly dilute and a semiclassical analysis is justified. Increasing $LN\Lambda$, by increasing N , L , or both, decreases the separation of scales; the heaviest photon mass, m_γ , grows while the lightest W mass, m_W , drops. When $LN\Lambda \approx 1$, the scale separation is entirely lost, the effective 't Hooft coupling $\lambda \equiv g^2 N$ at the scale of m_W ceases to be small, and the long distance dynamics can no longer be described by a weakly coupled $U(1)^{N-1}$ effective theory.

One can consider sending N to infinity while staying within the analytically tractable regime. This is a double scaling limit in which $g^2 N$ and $L N \Lambda$ are both held fixed (and both are much less than unity) as $N \rightarrow \infty$. Taking a large N limit in this fashion allows monopole effects to survive and to continue dictating the nonperturbative physics of the deformed Yang-Mills theory. However, this region shrinks to a vanishingly small window in the large N limit, since the double scaling implies that $0 < L \ll L_{\max}$ with $L_{\max} \Lambda \sim 1/N$. For any fixed compactification size L , if one sends $N \rightarrow \infty$ the deformed YM theory ceases to possess a monopole dominated, Abelian long distance regime.¹⁵

3.4 Connection to integrable Toda theory

The \mathbb{Z}_N symmetric model (3.25) is a deformation of a complex affine Toda theory with action

$$S^{\text{affine Toda}} = \int_{\mathbb{R}^3} \left[\frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \sigma)^2 - \zeta \sum_{i=1}^N e^{i(\sigma_i - \sigma_{i+1})} \right]. \quad (3.49)$$

This complex (CPT-noninvariant) action describes a plasma which is composed solely of monopoles with no antimonopoles. (Due to the existence of the affine root, one can have a neutral plasma composed solely of monopoles!) Interestingly, the soliton spectrum of the affine Toda theory is exactly computable. When reduced to one dimension, this theory is an integrable system as shown by Hollowood [36], using techniques due to Hirota [37, 38].

As discussed above, the k -string tension T_k is equal to the action of the kink solution with topological charge k . Borrowing the exact soliton spectrum from Ref. [36], one finds that the k -string tensions in the affine Toda theory are given by

$$T_k^{\text{affine Toda}} = TN \sin \frac{\pi k}{N}, \quad k = 1, N-1, \quad (3.50)$$

with T given above in Eq. (3.46).

The long distance effective theory (3.25) for our deformed Yang-Mills theory (when $L N \Lambda \ll 1$) is a deformation of the affine Toda system by complex conjugation. Unfortunately, unlike the integrable affine Toda system, when $N > 2$ the resulting CPT invariant system is no longer exactly integrable according to Hirota's criteria.¹⁶ Consequently, we do not expect k -string tensions in the deformed Yang-Mills theory to have the sine-law form (3.50).

¹⁵An analog of this double scaled limit was previously discussed by Douglas and Shenker in mass (m) deformed $SU(N)$ $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on \mathbb{R}^4 [26] down to $\mathcal{N} = 1$. This theory, just like our deformed Yang-Mills theory, possess a regime in which the long distance gauge structure reduce to abelian subgroup $U(1)^{N-1}$. Ref. [26] shows that in the $N \rightarrow \infty$ limit of the mass deformed theory, the abelian long distance regime is preserved only if the mass deformation is taken to zero $m/\Lambda \sim \frac{1}{N^4}$ in a correlated fashion, i.e., in a vanishingly small window of mass. In particular, at any finite m , if one takes $N \rightarrow \infty$ first, there is no regime of the supersymmetric gauge theory in which the long distance dynamics remains abelian. Although this phenomena only appeared previously in the context of supersymmetric gauge theories, it is generic in deformed Yang-Mills and other deformed QCD-like theories.

¹⁶In the absence of the complex conjugate term in the potential, there is a change of variables which converts the soliton equation of motion into ‘‘Hirota bilinear type,’’ which is synonymous with solvability [36]. The presence of the complex conjugate term spoils the bi-linearity.

Recently, there have been attempts [39], to model the *strongly* coupled confined regime of Yang-Mills theory assuming the Wilson line has the center-symmetric form (3.1). (See also earlier related work in Refs. [40, 41, 42, 35].) A few remarks concerning the connection with Ref. [39] may be in order. First, our results for the k -string tensions do not support the claim of Ref. [39], which asserts that k -string tensions will have the sine law form (3.50). As just noted, sine-law string tensions are a property of the affine Toda subsystem, whereas the center-stabilized Yang-Mills in a weak coupling regime is dual to a real deformation of the affine Toda theory. We see no reason to believe that the k -dependence of the string tensions will be unaffected by the deformation. Secondly, it should be emphasized that the deformation (2.5) stabilizes the center symmetric vacuum in the *weakly* coupled regime, and thereby provides a window in which a semiclassical analysis is reliable. Many earlier discussions of center symmetric backgrounds do not clearly distinguish the weakly coupled “Higgs” regime, in which fluctuations of the Wilson line eigenvalues are small, from the strong coupling regime in which the eigenvalues have large fluctuations and are essentially randomized over the unit circle. In our deformed Yang-Mills theory, both regimes exist. As the compactification size L increases, the theory moves from the weakly coupled regime to the strongly coupled regime. These two regimes are expected to be smoothly connected — no physical order parameter sharply distinguishes the two regimes. Nevertheless, the long distance physics of the weak coupling Higgs regime is effectively Abelian, while in the strong coupling regime there is no length scale beyond which the dynamics can be described accurately in terms of Abelian degrees of freedom.

Acknowledgments

This work was supported in part by the U.S. Department of Energy under Grant No. DE-AC02-76SF00515 and DE-FG02-96ER40956.

A. Monopole measure

The appropriate one-loop measure for integrating over configurations of a single monopole (of any type) may be expressed as¹⁷

$$d\mu_{\text{monopole}} = \mu^4 e^{-(S_{\text{YM}} + \Delta S)} \frac{d^3 \mathbf{a}}{(2\pi)^{3/2}} J_{\mathbf{a}} \frac{d\phi}{(2\pi)^{1/2}} J_{\phi} [\det'(-D_{\text{adj}}^2)]^{-1}, \quad (\text{A.1})$$

where $\mathbf{a} \in \mathbb{R}^3$ is the monopole position, $\phi \in [-\pi, \pi]$ is the internal $U(1)$ angle of the monopole, and μ is the (Pauli-Villars) renormalization scale. Global $U(1)$ gauge transformations (in the $U(1)$ subgroup associated with the given type of monopole) shift the angle ϕ . Fluctuations in the position and $U(1)$ angle of the monopole represent the four zero modes in the monopole

¹⁷The following summary is an adaptation of the appendix of Ref. [35], which treats the monopole measure in supersymmetric Yang-Mills theory.

small fluctuation operator;¹⁸ the factor of μ^4 can be viewed as the contributions of the Pauli-Villars regulator fields associated with these bosonic zero modes. The exponential factor is, of course, the exponential of minus the classical action of the monopole. The collective coordinate Jacobians are given by [35]

$$J_a = S_{\text{YM}}^{3/2}, \quad J_\Omega = \frac{2\pi S_{\text{YM}}^{1/2}}{\alpha_j \cdot \lambda} = NL S_{\text{YM}}^{1/2}, \quad (\text{A.2})$$

where $\lambda = \{0, 2\pi, 4\pi, \dots, 2\pi(N-1)\}/(NL)$ are the eigenvalues of $-i \ln \Omega$. The primed determinant represents the result of Gaussian integrals over all fluctuations other than zero modes; the prime on the determinant denotes omission of the zero modes. The contributions from gauge bosons and ghosts,

$$\underbrace{[\det'(-D^2\delta_{\mu\nu} - 2F_{\mu\nu})_{\text{adj}}]^{-1/2}}_{\text{gauge bosons}} \times \underbrace{\det(D^2)_{\text{adj}}}_{\text{ghosts}}, \quad (\text{A.3})$$

combine to give this simple form because

$$[\det'(-D^2\delta_{\mu\nu} - 2F_{\mu\nu})_{\text{adj}}]^{-1/2} = [\det(-D^2)_{\text{adj}}]^{-2} \quad (\text{A.4})$$

in any self-dual background. These functional determinants may be regularized using the Pauli-Villars scheme.

The fields of fundamental monopoles reside entirely within an $SU(2)$ subgroup of $SU(N)$, and the characteristic size of these monopoles is given by the inverse of the lightest W -boson mass, $m_W^{-1} \sim NL$. (This is the only scale which appears in the classical equations for the monopole.) The regularized scalar determinant depends on the cube root of the renormalization scale, $\det(-D^2) \sim \mu^{1/3}$. Since the determinant is dimensionless, it must have the form

$$[\det(-D^2)]^{-1} = 2\pi C (\mu NL)^{-1/3}, \quad (\text{A.5})$$

where C is a pure number (N -independent). Consequently, the one-loop monopole measure equals

$$d\mu_{\text{monopole}} = C\mu^{11/3} (NL)^{2/3} (S_{\text{YM}})^2 e^{-S_{\text{YM}} + \Delta S} d^3\mathbf{a} d\Omega. \quad (\text{A.6})$$

Performing the trivial integral over the angle Ω , the result is the $\zeta d^3\mathbf{a}$, with ζ the monopole fugacity. Choosing to use m_W as the value of the renormalization point yields the expression (3.23) for the fugacity.

¹⁸For comparison, recall that an instanton in $SU(N)$ Yang-Mills theory has $4N$ zero-modes. (See, for example, Ref. [43].) For $SU(2)$ gauge theory, these are the four zero modes corresponding to changes in the instanton position, one for its size, and three global gauge rotations. For $SU(N)$ with $N > 2$, there are in addition $4N-8$ zero modes associated with changes in the embedding of $SU(2)$ within $SU(N)$. When compactified in one dimension, with non-trivial holonomy Ω , one may regard an instanton as being composed of N independent monopole constituents [41, 40], each of which carries four zero-modes.

References

- [1] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.
- [2] L. G. Yaffe, *Large N limits as classical mechanics*, *Rev. Mod. Phys.* **54** (1982) 407.
- [3] G. Bhanot, U. M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett. B* **113** (1982) 47.
- [4] R. Narayanan and H. Neuberger, *Large N reduction in continuum*, *Phys. Rev. Lett.* **91** (2003) 081601, [hep-lat/0303023](#).
- [5] J. Kiskis, R. Narayanan and H. Neuberger, *Does the crossover from perturbative to nonperturbative physics in QCD become a phase transition at infinite N ?*, *Phys. Lett. B* **574** (2003) 65, [hep-lat/0308033](#).
- [6] B. Lucini, M. Teper and U. Wenger, *Properties of the deconfining phase transition in $SU(N)$ gauge theories*, *J. High Energy Phys.* **0502** (2005) 033, [hep-lat/0502003](#).
- [7] B. Bringoltz and M. Teper, *In search of a Hagedorn transition in $SU(N)$ lattice gauge theories at large- N* , *Phys. Rev. D* **73** (2006) 014517, [hep-lat/0508021](#).
- [8] T. D. Cohen, *Large N_c continuum reduction and the thermodynamics of QCD*, *Phys. Rev. Lett.* **93** (2004) 201601, [hep-ph/0407306](#).
- [9] Y. Makeenko, *Methods of contemporary gauge theory*, Cambridge, 2002.
- [10] A. Gonzalez-Arroyo and M. Okawa, *The twisted Eguchi-Kawai model: A reduced model for large N lattice gauge theory*, *Phys. Rev. D* **27** (1983) 2397.
- [11] A. Gonzalez-Arroyo and M. Okawa, *A twisted model for large N lattice gauge theory*, *Phys. Lett. B* **120** (1983) 174.
- [12] M. Teper and H. Vairinhos, *Symmetry breaking In twisted Eguchi-Kawai models*, *Phys. Lett. B* **652** (2007) 359, [hep-th/0612097](#).
- [13] T. Azeyanagi, M. Hanada, T. Hirata and T. Ishikawa, *Phase structure of twisted Eguchi-Kawai model* [arXiv:0711.1925](#) [hep-lat].
- [14] P. Kovtun, M. Unsal, and L. G. Yaffe, *Volume independence in large N_c QCD-like gauge theories*, [hep-th/0702021](#).
- [15] M. Unsal and L. G. Yaffe, *(In)validity of large N orientifold equivalence*, *Phys. Rev. D* **74** (2006) 105019, [hep-th/0608180](#).
- [16] A. Armoni, M. Shifman and G. Veneziano, *SUSY relics in one-flavor QCD from a new $1/N$ expansion*, *Phys. Rev. Lett.* **91** (2003) 191601, [hep-th/0307097](#).
- [17] A. Armoni, M. Shifman and G. Veneziano, *Refining the proof of planar equivalence*, *Phys. Rev. D* **71** (2005) 045015, [hep-th/0412203](#).
- [18] M. Schaden, *A center-symmetric $1/N$ expansion*, *Phys. Rev. D* **71** (2005) 105012, [hep-th/0410254](#).
- [19] R. D. Pisarski, *Effective theory of Wilson lines and deconfinement*, *Phys. Rev. D* **74** (2006) 121703, [hep-ph/0608242](#).

- [20] J. C. Myers and M. C. Ogilvie, *New phases of $SU(3)$ and $SU(4)$ at finite temperature*, [arXiv:0707.1869](#) [hep-lat].
- [21] M. Shifman and M. Unsal, *QCD-like theories on $R_3 \times S_1$: a smooth journey from small to large $r(S_1)$ with double-trace deformations*, [arXiv:0802.1232](#) [hep-th].
- [22] A. M. Polyakov, *Quark confinement and topology of gauge groups*, *Nucl. Phys. B* **120** (1977) 429–458.
- [23] A. R. Zhitnitsky, *Confinement-deconfinement phase transition and fractional instanton quarks in dense matter*, [hep-ph/0601057](#).
- [24] D. Toublan and A. R. Zhitnitsky, *Confinement-deconfinement phase transition at nonzero chemical potential*, *Phys. Rev. D* **73**, 034009 (2006), [hep-ph/0503256](#).
- [25] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)], [hep-th/9407087](#).
- [26] M. R. Douglas and S. H. Shenker, *Dynamics of $SU(N)$ supersymmetric gauge theory*, *Nucl. Phys. B* **447** (1995) 271, [hep-th/9503163](#).
- [27] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, eds., *Quantum fields and strings: A course for mathematicians. vol. 1, 2*, Providence, USA: AMS (1999) 1–1501.
- [28] P. Kovtun, M. Ünsal and L. G. Yaffe, *Non-perturbative equivalences among large N gauge theories with adjoint and bifundamental matter fields*, *J. High Energy Phys.* **0312** (2003) 034, [hep-th/0311098](#).
- [29] P. Kovtun, M. Ünsal and L. G. Yaffe, *Necessary and sufficient conditions for non-perturbative equivalences of large N orbifold gauge theories*, *J. High Energy Phys.* **0507** (2005) 008, [hep-th/0411177](#).
- [30] F. R. Brown and L. G. Yaffe, *The Coherent State Variational Algorithm: A numerical method for solving large N gauge theories*, *Nucl. Phys. B* **271** (1986) 267.
- [31] T. A. Dickens, U. J. Lindqwister, W. R. Somsy and L. G. Yaffe, *The Coherent State Variational Algorithm. 2. Implementation and testing*, *Nucl. Phys. B* **309** (1988) 1.
- [32] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43.
- [33] Y. M. Makeenko and A. A. Migdal, *Exact equation for the loop average in multicolor QCD*, *Phys. Lett. B* **88** (1979) 135 [Erratum-ibid. B **89** (1980) 437].
- [34] M. Unsal, in preparation, 2008.
- [35] N. M. Davies, T. J. Hollowood, and V. V. Khoze, *Monopoles, affine algebras and the gluino condensate*, *J. Math. Phys.* **44** (2003) 3640–3656, [hep-th/0006011](#).
- [36] T. J. Hollowood, *Solitons in affine Toda field theories*, *Nucl. Phys. B* **384**, 523 (1992).
- [37] R. Hirota and J. Satsuma, *A variety of nonlinear network equations generated from the Bäcklund transformation for the Toda lattice*, *Prog. Theor. Physics Supp.* **59** (1976) 64–100 :
- [38] M. Toda, *Nonlinear waves and solitons*, Springer, 1989.

- [39] D. Diakonov and V. Petrov, *Confining ensemble of dyons*, *Phys. Rev. D* **76** (2007) 056001, [arXiv:0704.3181](#) [hep-th].
- [40] K. M. Lee and P. Yi, *Monopoles and instantons on partially compactified D-branes*, *Phys. Rev. D* **56** (1997) 3711, [hep-th/9702107](#).
- [41] T. C. Kraan and P. van Baal, *Periodic instantons with non-trivial holonomy*, *Nucl. Phys. B* **533** (1998) 627, [hep-th/9805168](#).
- [42] F. Bruckmann, D. Negradi and P. van Baal, *Higher charge calorons with non-trivial holonomy*, *Nucl. Phys. B* **698** (2004) 233, [hep-th/0404210](#).
- [43] E. J. Weinberg and P. Yi, *Magnetic monopole dynamics, supersymmetry, and duality*, *Phys. Rept.* **438**, 65 (2007), [hep-th/0609055](#).