May 2008

Preprint typeset in JHEP style - PAPER VERSION

YITP-SB-08-16
DFTT-06/2008
SLAC–PUB–13141

Universal structure of subleading infrared poles in gauge theory amplitudes

Lance J. Dixon
Stanford Linear Accelerator Center, Stanford University,
Stanford, CA 94309, USA
E-mail: lance@slac.stanford.edu

Lorenzo Magnea
Dipartimento di Fisica Teorica, Università di Torino, and
INFN, Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy
E-mail: magnea@to.infn.it

George Sterman
C. N. Yang Institute for Theoretical Physics, Stony Brook University,
SUNY, Stony Brook, NY 11794-3840, USA
E-mail: sterman@max2.physics.sunysb.edu

Abstract: We study the origin of subleading soft and collinear poles of form factors and amplitudes in dimensionally-regulated massless gauge theories. In the case of form factors of fundamental fields, these poles originate from a single function of the coupling, denoted $G(\alpha_s)$, depending on both the spin and gauge quantum numbers of the field. We relate $G(\alpha_s)$ to gauge-theory matrix elements involving the gluon field strength. We then show that $G(\alpha_s)$ is the sum of three terms: a universal eikonal anomalous dimension, a universal non-eikonal contribution, given by the coefficient $B_\delta(\alpha_s)$ of $\delta(1-z)$ in the collinear evolution kernel, and a process-dependent short-distance coefficient function, which does not contribute to infrared poles. Using general results on the factorization of soft and collinear singularities in fixed-angle massless gauge theory amplitudes, we conclude that all such singularities are captured by the eikonal approximation, supplemented only by the knowledge of $B_\delta(\alpha_s)$. We explore the consequences of our results for conformal gauge theories, where in particular we find a simple exact relation between the form factor and the cusp anomalous dimension.

Keywords: Factorization, Resummation, Wilson Lines, Gauge Theories.

Submitted to Physical Review D

Work supported in part by US Department of Energy contract DE-AC02-76SF00515

Contents

1. Introduction

The structure of soft and collinear singularities in perturbative QCD has been studied in depth for decades, uncovering a pattern of exponentiation dictated by gauge invariance and factorization. The prototype amplitude for these studies is the electromagnetic form factor of a colored particle, whose simple color structure and renormalization properties make it an ideal laboratory for isolating and evaluating long-distance contributions. In fact, the color-singlet QCD form factors of quarks and gluons can both be expressed in an elegant exponentiated form, each in terms of only two functions of the running coupling. In this paper, we will extract and interpret the universal functions that control subleading soft and collinear poles for the form factors. The same functions, as we will see, also control collinear poles for the full class of dimensionally-regulated fixed-angle scattering amplitudes.

Color-singlet parton form factors are the simplest amplitudes exhibiting the double logarithmic ('Sudakov') behavior characteristic of gauge theories in the massless limit. Following the early studies in the abelian theory [1], which were performed at leading logarithmic (LL) accuracy, the form factors of non-abelian gauge theory were shown to exponentiate to arbitrary logarithmic accuracy [2, 3, 4, 5]. Exponentiation occurs because the form factor obeys an evolution equation, which in turn is a consequence of factorization and gauge invariance [6]. Solving the evolution equation yields an especially transparent answer [7] if one employs dimensional regularization as an infrared regulator, as is routinely done in finite-order perturbative calculations.

Dimensional regularization, in this context, displays several remarkable features, going well beyond its properties of preserving gauge symmetry and simplifying calculations in massless theories. When performing a resummation, in fact, dimensional regularization expresses the solution to the appropriate evolution equation in terms of the d-dimensional running coupling, which vanishes in the infrared for $d > 4$ as a consequence of dimensional counting. On the one hand, this allows one to solve the equation in terms of a simple initial condition, since all perturbative contributions to the amplitude vanish as a power of the hard scale for $d > 4$; as a consequence, the resummed amplitude can be directly compared to finite-order Feynman diagram calculations. On the other hand, the d-dimensional running coupling in general displays a Landau pole with a nonvanishing imaginary part, which in turn allows an explicit evaluation of the resummed amplitude in terms of analytic functions of the coupling α_s and the dimension d [8].

Fixed-angle scattering amplitudes also have poles in dimensional regularization, and more generally double-logarithmic infrared enhancements at high energy. Evolution equations for on-shell high-energy scattering were developed first for theories without gauge bosons [9], while the infrared divergences associated with external colored particles remained an obstacle [10]. The recognition that soft and collinear (virtual) radiation can be factorized in a universal way from a hard QCD process (and specifically the result that soft radiation can be factorized from harder collinear radiation [11]) made it possible to generalize the exponentiation of the form factor to amplitudes with multiple colored legs [12, 13, 14].

Any gauge theory scattering amplitude can be treated as a vector in the space of available color configurations [15],

$$\begin{aligned} \mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= \sum_{L=1}^{N^{[\mathrm{f}]}} \mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}} \\ &= \left|\mathcal{M}^{[\mathrm{f}]}\right\rangle . \end{aligned} \tag{1.1}$$

Here the vector $\left|\mathcal{M}^{[\mathrm{f}]}\right\rangle$, associated with a scattering process with flavor structure labeled by [f], is represented by the coefficients $\mathcal{M}^{[\mathrm{f}]}_{L}$, in a basis defined by a set of $N^{[\mathrm{f}]}$ color tensors c_L; β_j are particle momenta p_j rescaled by a hard scale Q, for example as $p_j = (Q/\sqrt{2})\beta_j$. When the amplitude describes fixed-angle scattering, which we represent as

$$\mathrm{f} : p_1 + p_2 \to p_3 + \ldots + p_{n+2} \, , \tag{1.2}$$

each color component $\mathcal{M}^{[\mathrm{f}]}_{L}$ can be factorized into a product of 'jet' functions, describing the virtual color-singlet evolution of each external hard particle due to collinear radiation, times a 'soft' function organizing the effects of long-wavelength radiation [12, 13, 14],

$$\begin{aligned} \mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= \prod_{i=1}^{n+2} J^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) S^{[\mathrm{f}]}_{LI}\left(\beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon\right) \\ &\quad \times H^{[\mathrm{f}]}_{I}\left(\beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2)\right) , \end{aligned} \tag{1.3}$$

where Q' plays the role of a factorization scale separating soft and collinear momenta. The soft function, $S^{[f]}_{LI}$, is a matrix in the vector space spanned by the color tensors c_L. It acts on a vector of finite coefficient functions describing the effects of highly virtual particles. In this context, the form factor continues to play an important role: one can always, in fact, choose the factorization scheme so that the 'jet' functions for a generic amplitude may be identified with the square roots of the form factors of the corresponding hard partons,

$$J^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \left[\Gamma^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right]^{\frac{1}{2}}, \tag{1.4}$$

where $\Gamma^{[i]}(Q^2)$ represents the color-singlet form factor[1] for parton i.

The soft matrix $S^{[f]}_{LI}$ in Eq. (1.3) is responsible only for purely infrared single poles. For this reason, it can be computed using the eikonal approximation, in which hard partonic lines are replaced by Wilson lines. This fact leads to the important conclusion that the non-eikonal, hard collinear singularities in an arbitrary QCD amplitude, for fixed-angle scattering, can be completely organized in terms of the color-singlet form factors of quarks and gluons. Our discussion below will focus on the non-leading pole structure of the form factors.

Most of the results described above were originally derived with phenomenological applications in mind: long-distance singularities in the amplitudes, in fact, are the source of logarithmic enhancements in infrared- and collinear-safe cross sections near kinematic boundaries, which can have a sizable impact on perturbative predictions and often need to be resummed to all orders. Furthermore, the universal structure of long-distance singularities at fixed order [16] provides an important test for perturbative calculations, and is an essential ingredient in the construction of subtraction schemes, which are necessary to compute finite jet cross sections and event-shape distributions. Non-leading logarithmic enhancements were studied in deep-inelastic scattering and vector boson production in [17, 18, 19, 20]. These papers uncovered the same pattern for subleading enhancements that we will identify for poles in fixed-angle scattering. We will return to these studies in Sec. 5.2.

Beyond immediate phenomenological consequences, our analysis has a wide range of applicability, although derived within the context of QCD. It is based on universal properties of quantum field theories and of gauge theories in particular, and encodes general information about their long-distance behavior. In recent years, in fact, QCD results on the infrared structure of amplitudes have been applied to supersymmetric theories, and in particular to the maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory, which is of great theoretical interest because of its connections with string theory through the anti-de-Sitter-space-conformal-field-theory (AdS-CFT) correspondence [21].

This correspondence states that the strong-coupling, planar (large N_c) limit of $\mathcal{N} = 4$ super-Yang-Mills theory admits a simple description in terms of solutions of a classical gravitational theory, or of strings moving in a weakly curved background. On the other hand, the quantum conformal invariance of the theory, which has a vanishing β function to

[1] In Refs. [13, 14] the functions $\Gamma^{[i]}$ were denoted as $\mathcal{M}^{[i\to i]}$, as in the notation for amplitudes above.

all perturbative orders, implies a drastic simplification in the all-order perturbative resummation of infrared and collinear singularities. Together, these two observations suggest that perturbation theory for planar $\mathcal{N} = 4$ super-Yang-Mills amplitudes should have remarkable properties.

Indeed, motivated by a surprising iterative property of the two-loop four-gluon scattering amplitude [22, 23], an exponential form for the all-loop scattering amplitude in planar $\mathcal{N} = 4$ super-Yang-Mills theory was proposed [24]. This conjecture was constructed to be consistent with the all-order structure of soft and collinear divergences derived in Refs. [7] and [13]. At the level of finite terms, it was tested successfully for the three-loop four-gluon amplitude [24] and two-loop five-gluon amplitude [25, 26]. For four gluons, it received striking confirmation from the work of Alday and Maldacena [27], who computed the strong-coupling planar limit directly from the AdS-CFT correspondence, in dimensional regularization, finding the same exponential form in this limit. The conjecture is expected to hold for five gluons; recently, however, it has been found to break down for six gluons [28]. Such a breakdown was anticipated at strong coupling by an observed inconsistency in a particular kinematic limit with a very large number of gluons [29], and at two loops by an analysis of the high-energy limit of the six-gluon amplitude [30].

Irrespective of the form of the finite terms, one can also extract from the Alday-Maldacena solution the leading strong-coupling behavior of the same quantities that govern infrared evolution in perturbation theory [27, 31]. The leading (double) poles in the exponent for the amplitude, for example, are governed by the cusp anomalous dimension [32, 33, 34, 35]. The AdS-CFT correspondence can then be tested at the level of elementary fields, by looking for consistency between the perturbative series at weak coupling and the strong-coupling limit.

In the case of leading poles, the comparison can be made very precise because the cusp anomalous dimension has a well-defined non-perturbative definition in field theory, and a wealth of information is available about its properties. For planar $\mathcal{N} = 4$ super-Yang-Mills theory, it has been computed perturbatively up to four loops [24, 34, 36, 37, 38]. A strong-coupling expansion can be derived from string theory, in which the first three terms have now been computed [39, 40, 41]. Most remarkably, one can exploit the observed integrability properties of the theory to construct an integral equation [42], whose solution extends over all values of the coupling, and is in precise agreement with all four weak-coupling and all three strong-coupling [43] coefficients.

Subleading poles for scattering amplitudes are less well understood. In general, they cannot be expressed completely in terms of eikonal amplitudes, although we will see that eikonal amplitudes still play an important role in characterizing them. It is worth remarking that there has been considerable renewed interest of late in the perturbative study of eikonal amplitudes, or Wilson loop expectation values, inspired by their role in the strong-coupling approach of Alday and Maldacena. A close, but still not fully explained, relation between Wilson loop expectation values and the maximally-helicity-violating scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory has been uncovered at one loop [44] and verified at two loops [28, 45, 46].

It is the purpose of our paper to provide a more precise characterization of the subleading-pole singularities in a massless gauge theory such as QCD or $\mathcal{N} = 4$ super-Yang-Mills theory. We begin by reviewing briefly in Section 2 the known results about the Sudakov form factor, which we present in a simplified form and then apply to the case of a conformal theory. In Section 3, we revisit the standard factorization [47] in the context of dimensional regularization. This allows us to give explicit operator expressions for the functions that control the poles of the form factors, and thus the collinear poles of all other fixed-angle amplitudes, via Eq. (1.3) above. In Section 4, we derive evolution equations for these operators, and identify their anomalous dimensions. This leads to an explicit expression, in terms of these anomalous dimensions, for the function $G(\alpha_s)$, which determines the subleading poles of the form factor. In Section 5, we construct another expression for $G(\alpha_s)$, by relating collinear singularities of the form factor, for a given parton species, to the virtual contributions to the corresponding parton distribution. This leads us to identify explicitly the only non-eikonal long-distance contribution to $G(\alpha_s)$, which is given by the virtual term of the diagonal Altarelli-Parisi splitting function for the chosen parton, a fact that was pointed out at finite perturbative order in Refs. [48, 49]. Eikonal contributions to $G(\alpha_s)$, on the other hand, are related to the function responsible for soft single logarithms in threshold resummation for the Drell-Yan process [50]. As shown in Section 5 and in the Appendix, certain additional short-distance contributions to $G(\alpha_s)$ are given entirely by running-coupling effects. These contributions do not give rise to poles in scattering amplitudes, and are proportional to ϵ in a conformal gauge theory such as $\mathcal{N} = 4$ super-Yang-Mills theory.

We hope that our results will be helpful both in perturbative QCD studies, where a detailed knowledge of long-distance singularities to all orders is of direct phenomenological relevance, and in order to further our understanding of superymmetric gauge theories, where the striking discoveries of these recent years are beginning to map a precise connection between perturbation theory and strong coupling, possibly on the way to exact results.

2. Gauge Theory Form Factors

Let us begin by reviewing briefly the known results concerning the color-singlet form factors of massless colored particles. For a quark, one can define the form factor as a matrix element of the electromagnetic current. In the timelike case, for example, one can write

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = \overline{v}(p_2) \gamma_\mu u(p_1) \; \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \; . \tag{2.1}$$

Gluons do not couple directly to the electromagnetic current J_μ, but their form factor can be defined analogously as a matrix element of a gauge-invariant operator. A typical and useful example is the coupling of gluons to colorless scalar particles (such as the Higgs boson) through an effective vertex constructed by integrating out a heavy fermion loop (the top quark in the standard model). In this case the form factor is defined through the coupling to the operator $-\mathcal{C}_H \mathrm{Tr}\, [G_{\mu\nu} G^{\mu\nu}]/2$, where $G_{\mu\nu}$ is the Yang-Mills field strength and $\mathcal{C}_H$ is a matching coefficient containing the dependence on the mass of the heavy fermion.

Note that in this case the effective operator couples directly to soft gluons, but this does not change the structure of infrared singularities at leading power. The reasoning below therefore applies equally well to quarks and to gluons.

As was shown in Refs. [2, 3, 4, 5], and reviewed in Ref. [47], the momentum dependence of the form factor is determined by a simple evolution equation. The equation is a consequence of the factorization of soft and collinear modes from highly virtual exchanged particles and from each other [6], which in turn arises from the loss of quantum-mechanical coherence for processes occurring at widely separated scales. It can be proven by making use of Ward identities. In dimensional regularization, with $d = 4 - 2\epsilon$ and $\epsilon < 0$ in order to regulate mass divergences in the renormalized theory, the evolution equation takes the form

$$Q^2 \frac{\partial}{\partial Q^2} \log \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = \frac{1}{2} \left[K \left(\epsilon, \alpha_s(\mu^2) \right) + G \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] , \tag{2.2}$$

where the function $K(\epsilon, \alpha_s)$ is a pure counterterm, while the function $G(\xi^2, \alpha_s, \epsilon)$, which carries the momentum dependence, is finite as $\epsilon \to 0$. Furthermore, renormalization-group (RG) invariance of the form factor implies that

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \right) G \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) = - \left(\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \right) K \left(\epsilon, \alpha_s \right) \equiv \gamma_K(\alpha_s) \, , \tag{2.3}$$

which is how the cusp anomalous dimension $\gamma_K(\alpha_s)$ comes into play in this context.

In order to solve Eq. (2.2), one needs to introduce the d-dimensional running coupling, which solves the RG equation

$$\mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\epsilon, \alpha_s) = -2\epsilon \alpha_s + \hat{\beta}(\alpha_s) \, , \tag{2.4}$$

where $\hat{\beta}(\alpha_s)$ is the usual four-dimensional β function,

$$\hat{\beta}(\alpha_s) = - \frac{\alpha_s^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\alpha_s}{\pi} \right)^n \, , \tag{2.5}$$

with $b_0 = (11 C_A - 2 n_f)/3$ in our normalization. At the one-loop level, the solution to Eq. (2.4) is,

$$\overline{\alpha} \left(\frac{\mu^2}{\mu_0^2}, \alpha_s(\mu_0^2), \epsilon \right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} - \frac{1}{\epsilon} \left(1 - \left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} \right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} \, , \tag{2.6}$$

which clearly reduces to the usual four-dimensional result as $\epsilon \to 0$. Since the solution is RG invariant, we may use for the running coupling the simplified notation $\overline{\alpha}(\mu^2, \epsilon)$ whenever we do not need to adopt a specific boundary condition.

At tree level (and to all orders in a conformal theory) $\overline{\alpha}$ scales as a power of μ, $\overline{\alpha}(\mu^2, \epsilon) \sim \mu^{-2\epsilon} \alpha_s$. The running coupling thus vanishes in the infrared, as expected above the critical dimension $d = 4$. Given the RG invariance of the form factor, this has the important consequence of providing us with a simple initial condition for Eq. (2.2),

$$\Gamma \left(0, \alpha_s(\mu^2), \epsilon \right) = \Gamma \left(1, \overline{\alpha} \left(0, \epsilon \right), \epsilon \right) = 1 \, . \tag{2.7}$$

It is now straightforward to integrate Eq. (2.2), obtaining

$$\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[K\left(\epsilon,\alpha_s(\mu^2)\right)\right.\right.$$
$$\left.\left. + \, G\left(-1,\overline{\alpha}\left(\frac{\xi^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right),\epsilon\right)+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\frac{\lambda^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right)\right]\right\} , \tag{2.8}$$

where we integrated along the negative real axis to emphasize that the function G is real for negative Q^2.

The term proportional to $K(\epsilon,\alpha_s)$ in Eq. (2.8) has an apparent unregulated singularity due to the integration down to $\xi^2 = 0$: in fact, this term cancels exactly the ξ^2-independent terms arising from the integration of $\gamma_K(\overline{\alpha})$. This can be shown to all orders because the function $K(\epsilon,\alpha_s)$ is completely determined, through Eq. (2.3), by the coefficients of the perturbative expansions of γ_K and of the β function [8]. As a consequence, all poles in Eq. (2.8) arise from integrations over the scale of the running coupling in the infrared region. This cancellation can be made explicit by considering the RG equation for the counterterm function $K(\epsilon,\alpha_s)$, Eq. (2.3). Since K has no explicit scale dependence, one can write

$$\mu\frac{d}{d\mu}K(\epsilon,\alpha_s)=\beta(\epsilon,\alpha_s)\frac{\partial}{\partial\alpha_s}K(\epsilon,\alpha_s)=-\gamma_K(\alpha_s) \ . \tag{2.9}$$

Using, once again, the vanishing of the running coupling in the infrared, one has the boundary condition $K(\mu=0)=0$, so that Eq. (2.9) integrates to

$$K\left(\epsilon,\alpha_s(\mu^2)\right)=-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\bar{\alpha}(\lambda^2,\epsilon)\right) \ . \tag{2.10}$$

We now have two terms in Eq. (2.8) involving double scale integrals of the cusp anomalous dimension. They both diverge, but using Eq. (2.10), exchanging orders of integration, and choosing $\mu = Q$ they can be re-expressed as

$$\Gamma\left(Q^2,\epsilon\right)=\exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[G\Big(-1,\overline{\alpha}\left(\xi^2,\epsilon\right),\epsilon\Big)-\frac{1}{2}\gamma_K\Big(\overline{\alpha}\left(\xi^2,\epsilon\right)\Big)\log\left(\frac{-Q^2}{\xi^2}\right)\right]\right\} . \tag{2.11}$$

In QCD, Eq. (2.11) has phenomenological as well as theoretical interest. In fact, as described in Ref. [13], it is one of the building blocks for the analysis of mass singularities in general multiparton amplitudes at finite perturbative orders, which in turn is of relevance for the calculation of infrared-safe observables at high-energy colliders. We observe at this point that the function G in Eq. (2.11) not only generates next-to-leading poles in ϵ at each order in α_s, but it also serves as a complete infrared-safe coefficient function for the exponentiation of such poles, as well as finite parts, in the singlet form factor.

In the present context, it is interesting to notice that Eq. (2.8) drastically simplifies in a conformally-invariant theory such as $\mathcal{N}=4$ super-Yang-Mills theory. We emphasize again that the evolution equation, Eq. (2.2), is a consequence of gauge invariance and factorization, and hence holds for supersymmetric extensions of QCD, with, of course, different but related functions K and G. For a conformal theory, $\hat{\beta}(\alpha_s) = 0$; the coupling then runs according to its mass dimension in $d = 4 - 2\epsilon$, so that $\bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon} = \bar{\alpha}_s(\mu^2)\mu^{2\epsilon}$; as a consequence, all integrals in Eq. (2.8) can be performed trivially in this case [24]. Expanding the anomalous dimensions as

$$\gamma_K(\overline{\alpha}) = \sum_{n=1}^{\infty} \left(\frac{\overline{\alpha}}{\pi} \right)^n \gamma_K^{(n)} , \qquad G(-1, \overline{\alpha}, \epsilon) = \sum_{n=1}^{\infty} \left(\frac{\overline{\alpha}}{\pi} \right)^n G^{(n)}(\epsilon) , \tag{2.12}$$

one finds several remarkably simple results. First of all, the counterterm $K(\epsilon, \alpha_s)$ has only simple poles, and is easily expressed in terms of the perturbative coefficients of γ_K, as

$$K(\epsilon, \alpha_s) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \frac{\gamma_K^{(n)}}{2n\epsilon} . \tag{2.13}$$

Next, one observes that the logarithm of the form factor has only double and single poles in ϵ to any order in perturbation theory, in contrast to the situation in QCD, where the running of the coupling generates poles up to ϵ^{-p-1} at order α_s^p. In fact, one finds explicitly [24],

$$\begin{aligned} \log \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] &= -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^n \left(\frac{\mu^2}{-Q^2} \right)^{n\epsilon} \left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right] \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^n \mathrm{e}^{-\mathrm{i}\pi n\epsilon} \left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right] , \end{aligned} \tag{2.14}$$

displaying, as expected, exact RG invariance.

As a final remark, it is interesting to construct, in a conformal theory, the ratio of the time-like to the space-like form factor. This ratio was studied for QCD in Ref. [7]. In that case it is of phenomenological relevance, since it enters the resummed expression for the Drell-Yan cross section in the DIS factorization scheme [51, 52, 53]. In the conformal case the analytic continuation can be performed explicitly, and one finds

$$\log \left[\frac{\Gamma(Q^2)}{\Gamma(-Q^2)} \right] = \frac{\mathrm{i}}{2}\pi \left[K\left(\epsilon, \alpha_s(Q^2)\right) + G\left(-1, \alpha_s(Q^2), 0\right) \right] + \frac{\pi^2}{8}\gamma_K\left(\alpha_s(Q^2)\right) + \mathcal{O}(\epsilon) . \tag{2.15}$$

As observed in Ref. [7] in QCD, all poles in the ratio are given by an infinite phase, which in this case is simply related to γ_K via Eq. (2.13); the modulus squared of the ratio is thus finite in any gauge theory. For a conformally-invariant gauge theory in $d = 4$, one finds the very simple expression

$$\left| \frac{\Gamma(Q^2)}{\Gamma(-Q^2)} \right|^2 = \exp \left[\frac{\pi^2}{4} \gamma_K\left(\alpha_s(Q^2)\right) \right] . \tag{2.16}$$

Since all quantities in Eq. (2.16) have a precise nonperturbative definition, and since it provides a finite, unambiguous resummation of perturbation theory, Eq. (2.16) can be argued to be an exact result. It is easy to see that it agrees with the result for the four-loop form factor ratio for quarks in QCD [54], after setting all β function coefficients to zero.

3. Jet and Soft Functions in the Factorized Amplitude

The general arguments for the factorization of QCD amplitudes and cross sections, isolating the contributions responsible for long-distance singular behavior, have been known for some time [11]. In this section, we review the operator interpretation of this factorization, as described originally by Collins [47]. We then review the one-loop corrections to each of the relevant factors in dimensional regularization.

At all orders, three basic physical principles apply to hard-scattering amplitudes, including the form factors under consideration. First, soft gluons decouple from hard virtual partons at leading power in the hard scale, since each soft gluon insertion in a hard subdiagram adds to the diagram a new propagator far off the mass shell. Second, virtual hard partons collinear to an external hard parton effectively decouple from the remainder of the hard subdiagram, becoming insensitive to the energy and spin of fast partons moving in different directions. Finally, soft gluons decouple from jets, because their long wavelength does not allow them to discriminate features of a narrow jet other than its overall color and direction.

To express an amplitude in factorized form unambiguously, we identify operator expressions generating the leading contributions in each relevant region in momentum space, and make subtractions appropriate to avoid double counting. In the case of the form factor, the resulting factorization [47] is depicted in Figure 1 and can be expressed as

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = C\left(\frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \times \mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) \times \prod_{i=1}^{2} \left[\frac{J\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)} \right] . \tag{3.1}$$

For definiteness, in Eq. (3.1) we have in mind the timelike form factor for a massless quark, so that $p_1^2 = p_2^2 = 0$ and $(p_1 + p_2)^2 = Q^2$. We also define the quark and antiquark velocities β_i via $p_i^\mu = (Q/\sqrt{2})\beta_i^\mu$, so that $\beta_1 \cdot \beta_2 = 1$, while the vectors n_i^μ define the directions of auxiliary gauge links to be discussed below. The hard function C summarizes the short-distance contributions to the form factor, and is finite as $\epsilon \to 0$. In order to define the remaining functions appearing in Eq. (3.1), it is useful to introduce first a notation for the Wilson line which describes the eikonal couplings arising in the soft and collinear limits. These couplings are generated by the operator

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[{\rm i} g \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n) \right] , \tag{3.2}$$

describing a gauge link in direction n^μ.

The 'partonic jets' J appearing in Eq. (3.1) are matrix elements for the transition from partonic states to the vacuum, mediated by the corresponding partonic (in this case, quark) field. Their operator definition is

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) \; = \; \langle 0 \,| \Phi_n(\infty, 0) \, \psi(0) \,| p \rangle \, . \tag{3.3}$$

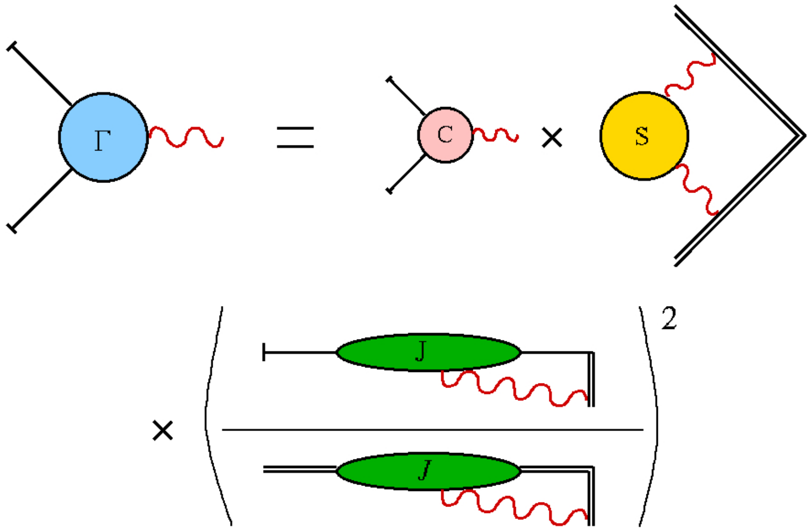

Figure 1: A graphical representation of the factorization of the form factor

The spinor $u(p)$ has been inserted to normalize the jet to unity at zeroth order; color and Dirac indices are implicit. The gauge link in the n^μ direction has a double role: on the one hand, it makes the matrix element gauge invariant; on the other hand, it mimics the coupling of gluons collinear to the incoming parton (say, a quark) to the opposite moving hard parton (say, the antiquark). Notice that, so long as $n^2 \neq 0$, the function J (and similarly $\mathcal{J}$, defined below) is invariant under rescalings of the vectors n_i. Factorization makes it convenient to consider the vectors n_i^μ in Eq. (3.1) to be spacelike [47]; a typical choice is $n_1 = \beta_1 - \beta_2 = -n_2$; the choice of vectors n_i^μ is otherwise free. This freedom can be used to derive the evolution equation (2.2) [6]. Clearly, J has infrared divergences, as well as collinear divergences associated with gluons collinear to p^μ.

Next, we introduce the soft function $\mathcal{S}$, as the vacuum expectation value of two lightlike Wilson lines in the directions β_1 and β_2,

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \Phi_{\beta_2}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, | 0 \rangle\ . \tag{3.4}$$

The soft function $\mathcal{S}$ is the eikonal limit of the full form factor, and thus contains double poles at every order, associated with gluons that are soft and collinear to either hard leg. It has several useful properties. First, it is a pure counterterm in any minimal regularization scheme, because all its Feynman diagrams have no mass scale. In addition, because it is defined purely in terms of Wilson lines, it exponentiates according to the general non-abelian exponentiation theorem [55, 56], and its logarithm can be expressed in terms of a specific subset of Feynman diagrams ('webs'), before the soft loop momentum is integrated over. An important feature of the defining matrix element (3.4) is that it is invariant under boosts along the $\beta_1\beta_2$ axis in any frame where these velocities are back to back. On the other hand, while one might expect $\mathcal{S}$, from its operator definition, to be invariant under rescalings of the velocities, for light-like β_i this invariance is broken (as we will further discuss below); thus, $\mathcal{S}$ can depend on the velocities, through the combination $\beta_1 \cdot \beta_2$ only.

There are a number of subtleties in the evaluation of purely eikonal functions, including $\mathcal{S}$. As a sum of scaleless integrals, $\mathcal{S}$ vanishes before renormalization on a diagram-by-diagram basis. This is perhaps why classic works on the renormalization of Wilson loops determine anomalous dimensions by introducing explicit mass scales [32, 33, 34], such as time-like lengths $\beta_i^2 > 0$ and/or cutoffs on the lengths of the Wilson lines Φ_{β_i}. This method was also used in Ref. [14]. Since here we are discussing specifically dimensionally-regularized amplitudes, we prefer to employ dimensional regularization throughout. To do so consistently, we must identify at each order the infrared-regularized coefficient of an ultraviolet pole. This is possible precisely because the logarithm of $\mathcal{S}$ has only a single ultraviolet divergence, which can be isolated systematically.

As a practical matter, to evaluate dimensionally-regulated integrals it is convenient to rescale the light-like velocities β_i to have units of mass. In this way, standard shifts of loop momenta can be carried out. This can be done without loss of generality in the Wilson lines Φ_{β_i} that define $\mathcal{S}$, by simple changes of variables. Below, we shall replace β_i by $(\mu/\sqrt{2})\beta_i$, with $\beta_1 \cdot \beta_2 = 1$. Then, for the choice $\mu = Q$, the velocities are identified with their corresponding momenta. For now, however, we keep the scales of the β_i arbitrary, and will continue to refer to them as 'velocities'.

We now turn to the exponentiation of poles in $\mathcal{S}$. Collinear contributions to the soft function $\mathcal{S}$ can be factorized from purely soft contributions in the same manner as for the full form factor, but with partonic jets replaced by eikonal jets, for which we will provide definitions shortly. As a consequence, $\mathcal{S}$ satisfies an evolution equation analogous to Eq. (2.2), and the resulting solution is of the same form as Eq. (2.8) for the partonic form factor, but with $-Q^2$ replaced by μ^2 everywhere, and with an ϵ-independent single-log function $G_{\rm eik}$ replacing G. The functions $\gamma_K(\alpha_s)$ and $K(\epsilon, \alpha_s)$, on the other hand, are the same as in Eq. (2.8), because the eikonal form factor matches exactly the partonic one in all infrared-singular regions, including the infrared-collinear ones. The arguments leading to Eq. (2.11) still hold, and we are led to

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2}\left[G_{\rm eik}\Big(\beta_1 \cdot \beta_2, \, \overline{\alpha}\left(\xi^2, \epsilon\right)\Big) - \frac{1}{2}\gamma_K\Big(\overline{\alpha}\left(\xi^2, \epsilon\right)\Big) \log\left(\frac{\mu^2}{\xi^2}\right)\right]\right\} \, . \qquad (3.5)$$

Notice that the lack of explicit ϵ dependence of $G_{\rm eik}$ ensures that $\mathcal{S}$ is a pure counterterm, consistent with its diagrammatic interpretation. We expect the real part of the function $G_{\rm eik}$, computed in back-to-back kinematics, to be related to the anomalous dimension $\Gamma_{\rm DY}$, defined and computed at two loops in Ref. [50][2].

[2] The "leading transcendentality" term in $\Gamma_{\rm DY}^{(2)}$, proportional to $\zeta(3)$, controls the single poles of the polygonal Wilson loop expectation values computed recently at two loops in $\mathcal{N} = 4$ super-Yang-Mills theory [45].

Including in the factorization both partonic jets and the full eikonal form factor clearly double counts the soft-collinear regions. This can be avoided if one divides by eikonal versions of the two jets, which are defined as

$$
\begin{aligned}
\mathcal{J}\left(\frac{(\beta_1 \cdot n_1)^2}{n_1^2}, \alpha_s(\mu^2), \epsilon\right) &= \langle 0 | \Phi_{n_1}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, | 0 \rangle \ , \\
\mathcal{J}\left(\frac{(\beta_2 \cdot n_2)^2}{n_2^2}, \alpha_s(\mu^2), \epsilon\right) &= \langle 0 | \Phi_{\beta_2}(\infty, 0)\, \Phi_{n_2}(0, -\infty)\, | 0 \rangle \ .
\end{aligned}
\tag{3.6}
$$

These jets are also pure counterterms in dimensional regularization, because they do not depend on any mass scale. They have soft-collinear enhancements from gluons moving in the β_i directions, matching those of both the soft function $\mathcal{S}$ or the partonic jets J. As a consequence the ratio $J/\mathcal{J}$ has only single collinear poles at every order, associated with hard collinear radiation, while the ratio

$$
\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{2} \mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}
\tag{3.7}
$$

has only single infrared poles associated with soft gluons emitted at wide angles from the hard partons. In Eq. (3.7) we have noted that the function $\overline{\mathcal{S}}$ can only depend on the homogeneous ratio

$$
\rho_{12} \equiv \frac{\left(-\beta_1 \cdot \beta_2\right)^2\, n_1^2\, n_2^2}{\left(-\beta_1 \cdot n_1\right)^2 \left(-\beta_2 \cdot n_2\right)^2} \, .
\tag{3.8}
$$

The reason is that invariance under separate rescalings of the velocities β_i, which was broken for $\mathcal{S}$, must be recovered in $\overline{\mathcal{S}}$, which contains the complete β dependence of the form factor; furthermore, homogeneity in n_i is built into the eikonal Feynman rules. The simplicity of these relations is a direct result of eikonal exponentiation. In the exponent, collinear and soft regions enter additively and universally [57].

The one-loop diagrams associated with the functions entering Eq. (3.1) are easily evaluated using eikonal Feynman rules where appropriate. We give the results below in the $\overline{\rm MS}$ scheme; thus all factors of $\log(4\pi)$ and γ_E are absent, having been absorbed into the definition of the renormalization scale μ.

The soft function receives a one-loop contribution only from the vertex correction diagram, since self-energies on eikonal light-like lines vanish like β_i^2. The full one-loop soft function is then given by the UV counterterm for the timelike vertex correction [58][3],

$$
\mathcal{S}^{(1)}\left(\beta_1 \cdot \beta_2, \epsilon\right) = -\frac{\alpha_s}{4\pi}\, C_F \left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log\left(-\beta_1 \cdot \beta_2\right)\right] \, .
\tag{3.9}
$$

[3]Note that the coefficient of $-\beta_1 \cdot \beta_2$ in the argument of the logarithm in Eq. (3.9) can be changed by rescaling the eikonal Feynman rules. Associating with each gluon emission, for example, a factor of $\kappa\beta_\mu/(\kappa\beta \cdot k)$, instead of the usual factor $\beta_\mu/(\beta \cdot k)$, rescales the argument of the logarithm by a factor κ^2. This ambiguity is associated with the broken invariance of the function $\mathcal{S}$ under rescalings of β_i, and corresponds to a choice of scheme in the renormalization of $\mathcal{S}$, which was discussed above. Once again, this ambiguity does not affect physical quantities: the dependence on κ cancels between the soft function $\mathcal{S}$ and the eikonal jets $\mathcal{J}$, as discussed below Eq. (3.8). Notice also that the invariance under rescalings of the vectors n_i^μ in the jet functions is not broken, since $n^2 \neq 0$, so that there are no collinear divergences associated with them.

Matching Eq. (3.5) and Eq. (3.9), we recover, as expected, that $\gamma_K^{(1)} = 2C_F$ at one loop, while[4].

$$G_{\rm eik}^{(1)}\left(\beta_1 \cdot \beta_2\right) = -\,\frac{\alpha_s}{\pi}\,C_F\,\log\left(-\beta_1 \cdot \beta_2\right)\,. \tag{3.10}$$

The eikonal jet $\mathcal{J}$ receives contributions at one loop from both the eikonal vertex correction and the self-energy diagram on the eikonal line along the n_i^μ direction. This eikonal self energy, $\mathcal{J}_{n^2}$ below, is a single pole pure counterterm at one loop, which is common to the partonic jet J and to the eikonal jet $\mathcal{J}$. It cancels in their ratio, but contributes to their respective evolution equations. The vertex correction to $\mathcal{J}$, which we denote by $\mathcal{J}_{\rm V}$, also contributes only through a counterterm. The complete one-loop eikonal jet is then

$$\begin{aligned}
\mathcal{J}^{(1)} &= \frac{1}{2}\,\mathcal{J}_{n^2}^{(1)} + \mathcal{J}_{\rm V}^{(1)}\,, \\
\mathcal{J}_{\rm V}^{(1)}\left(\frac{(\beta\cdot n)^2}{n^2},\,\epsilon\right) &= -\,\frac{\alpha_s}{4\pi}\,C_F\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\log\left(\frac{n^2}{2(-\beta\cdot n)^2}\right)\right]\,, \\
\mathcal{J}_{n^2}^{(1)} &= -\frac{\alpha_s}{2\pi}\,C_F\,\frac{1}{\epsilon}\,.
\end{aligned} \tag{3.11}$$

As expected, the soft-collinear double pole of the eikonal jet is one half of the corresponding pole in the eikonal form factor $\mathcal{S}$. Here and below, the one-loop self-energy counterterm (for the n eikonal) is multiplied by $1/2$, which reflects the removal of the square root of the residue of the relevant two-point function in a normalized S-matrix element.

Turning to the partonic jet J, we encounter in its vertex correction a one-loop diagram that is not simply a pure counterterm. The full one-loop result is the sum

$$J^{(1)} = \frac{1}{2}\,\mathcal{J}_{n^2} + J_{\rm V}^{(1)} + \frac{1}{2}\,J_{\rm P}^{(1)}\,, \tag{3.12}$$

with $\mathcal{J}_{n^2}^{(1)}$ given as above, while $J_{\rm V}^{(1)}$ is the quark-eikonal vertex correction, given by

$$\begin{aligned}
J_{\rm V}^{(1)}\left(\frac{(p\cdot n)^2}{n^2\mu^2},\,\epsilon\right) &= -\,\frac{\alpha_s}{4\pi}\,C_F\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(2 + \log\left(\frac{n^2\mu^2}{(-2p\cdot n)^2}\right)\right)\right. \\
&\quad \left. +\ \frac{1}{2}\log^2\left(\frac{n^2\mu^2}{(-2p\cdot n)^2}\right) + \log\left(\frac{n^2\mu^2}{(-2p\cdot n)^2}\right) + 2 + \frac{5}{12}\pi^2 + \mathcal{O}\left(\epsilon\right)\right]\,;
\end{aligned} \tag{3.13}$$

finally, $J_{\rm P}^{(1)}$ is the pure counterterm self-energy on the quark leg,

$$J_{\rm P}^{(1)}\left(\epsilon\right) = \frac{\alpha_s}{4\pi}\,C_F\,\frac{1}{\epsilon}\,. \tag{3.14}$$

Note that, as expected, the vertex correction $J_{\rm V}^{(1)}$ gives the same double pole as the eikonal jet, Eq. (3.11).

[4] Eq. (3.10) is consistent with Refs. [50, 59], which find that $\Gamma_{\rm eik}^{(1)} = \Gamma_{\rm DY}^{(1)} = 0$, provided one chooses the subtraction scheme for collinear poles corresponding to $\kappa = 1$ of the footnote above. In this scheme the argument of the logarithm is $-\beta_1 \cdot \beta_2$, as shown in (3.9), so that $G_{\rm eik}^{(1)}$ is purely imaginary for back-to-back time-like kinematics, giving a vanishing contribution to one-loop cross sections, while it vanishes for space-like kinematics. Note also that the "DIS" contour used in Ref. [59] differs from the space-like configuration considered here, so $G_{\rm eik}^{(1)}$ has no correspondence with the $\Gamma_{\rm DIS}^{(1)}$ defined there.

Collecting all the ingredients, and applying Eq. (3.1), we expect to reproduce all the infrared and collinear poles of the form factor at one loop. Indeed we find

$$\begin{aligned}\Gamma^{(1)}_{\rm pole}\left(\frac{Q^2}{\mu^2},\epsilon\right) &= \mathcal{S}^{(1)}\left(\beta_1\cdot\beta_2,\,\epsilon\right) + J^{(1)}_{\rm V,\,pole}\left(\frac{(p_1\cdot n_1)^2}{n_1^2\,\mu^2},\,\epsilon\right) + J^{(1)}_{\rm V,\,pole}\left(\frac{(p_2\cdot n_2)^2}{n_2^2\,\mu^2},\,\epsilon\right) \\ &\quad - \mathcal{J}^{(1)}_{\rm V}\left(\frac{(\beta_1\cdot n_1)^2}{n_1^2},\,\epsilon\right) - \mathcal{J}^{(1)}_{\rm V}\left(\frac{(\beta_2\cdot n_2)^2}{n_2^2},\,\epsilon\right) + J^{(1)}_{\rm P}\left(\epsilon\right) \\ &= \frac{\alpha_s}{4\pi}\,C_F\left[-\frac{2}{\epsilon^2}-\frac{2}{\epsilon}\log\left(\frac{\mu^2}{-Q^2}\right)-\frac{3}{\epsilon}\right]\,,\end{aligned} \tag{3.15}$$

which matches the pole structure of the one-loop form factor in dimensional regularization,

$$\Gamma^{(1)}\left(\frac{Q^2}{\mu^2},\epsilon\right) = -\,\frac{\alpha_s}{4\pi}\,C_F\left(\frac{\mu^2{\rm e}^{\gamma_E}}{-Q^2}\right)^{\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8+\mathcal{O}(\epsilon)\right)\,. \tag{3.16}$$

Eq. (3.16) also implies that

$$G^{(1)}\left(\frac{Q^2}{\mu^2},\epsilon\right) = \frac{\alpha_s}{\pi}\,C_F\left[\log\left(\frac{\mu^2}{-Q^2}\right)+\frac{3}{2}+\mathcal{O}(\epsilon)\right]\,, \tag{3.17}$$

in agreement with Ref. [7]. Our task is now to construct an all-order expression for the function G in terms of the anomalous dimensions of the various functions building up the form factor according to Eq. (3.1).

4. From Factorization to an Operator Intepretation for $G(\alpha_s)$

Let us begin our investigation of $G(\alpha_s)$ by considering the renormalization properties of the various functions entering Eq. (3.1). The partonic jet fuction J and the short-distance function C are multiplicatively renormalizable, with anomalous dimensions depending on the coupling α_s but not on the infrared regulator ϵ. Eikonal functions such as $\mathcal{S}$ and $\mathcal{J}$, on the other hand, require extra care: in general, their anomalous dimensions need infrared regularization in their own right, because of the overlap of collinear and ultraviolet divergences in the renormalization of any 'cusp' singularity involving light-like Wilson lines [34].

Consider first the soft eikonal function $\mathcal{S}$. The observation that it is a pure counterterm, of the general form of Eq. (3.5), leads to

$$\begin{aligned}\mu\frac{d}{d\mu}\log\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right) &= \beta(\epsilon,\alpha_s)\,\frac{\partial}{\partial\alpha_s}\log\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right) \\ &\equiv -\,\gamma_{\mathcal{S}}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right)\,,\end{aligned} \tag{4.1}$$

where the singular anomalous dimension $\gamma_{\mathcal{S}}$ is related to the function $G_{\rm eik}$ and to the cusp anomalous dimension γ_K by

$$\begin{aligned}\gamma_{\mathcal{S}}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right) &= -\,G_{\rm eik}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2)\right)+\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K\left(\overline{\alpha}(\xi^2,\epsilon)\right) \\ &= -\,G_{\rm eik}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2)\right)-K\left(\epsilon,\alpha_s(\mu^2)\right)\,.\end{aligned} \tag{4.2}$$

Here, in the second equality, we have used Eq. (2.10). We observe that $\gamma_{\mathcal{S}}$ has a single infrared pole determined by the cusp anomalous dimension γ_K. In order to work with infrared-finite anomalous dimensions, one can instead consider the function $\overline{\mathcal{S}}$, defined in Eq. (3.7), which carries only single infrared poles due to wide angle soft radiation. It obeys

$$\mu \frac{d}{d\mu} \log \overline{\mathcal{S}} \left(\rho_{12}, \alpha_s(\mu^2), \epsilon \right) = - \, \gamma_{\overline{\mathcal{S}}} \left(\rho_{12}, \alpha_s(\mu^2) \right) \, . \tag{4.3}$$

Finally, one may define

$$\begin{aligned} \mu \frac{d}{d\mu} \log J \left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \, &= - \, \gamma_J(\alpha_s) \, , \\ \mu \frac{d}{d\mu} \log C \left(\frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \, &= - \, \gamma_C(\rho_{12}, \alpha_s) \, , \end{aligned} \tag{4.4}$$

where the functional dependence of γ_C on μ is dictated by the requirement that the form factor as a whole not be renormalized, which implies, using Eqs. (3.1) and (3.7),

$$\gamma_{\overline{\mathcal{S}}} \left(\rho_{12}, \alpha_s \right) + \gamma_C \left(\rho_{12}, \alpha_s \right) + 2 \gamma_J \left(\alpha_s \right) = 0 \, . \tag{4.5}$$

At one loop, we can compute γ_J by combining the terms specified in Eq. (3.12); similarly, $\gamma_{\bar{\mathcal{S}}}$ is derived from the one-loop results for $\mathcal{S}$ and $\mathcal{J}$, combined as in Eq. (3.7). We find

$$\begin{aligned} \gamma_{\overline{\mathcal{S}}}^{(1)} \left(\rho_{12} \right) \, &= \, \frac{\alpha_s}{\pi} \, C_F \, \left[1 + \frac{1}{2} \log \left(\frac{\rho_{12}}{4} \right) \right] \, , \\ \gamma_J^{(1)} \, &= - \frac{\alpha_s}{\pi} \, \frac{3}{4} \, C_F \, , \end{aligned} \tag{4.6}$$

where we note that $\gamma_J^{(1)}$ equals the one-loop anomalous dimension of the quark field. We then derive for $\gamma_C^{(1)}$, using Eq. (4.5),

$$\gamma_C^{(1)} \left(\rho_{12} \right) \, = \, \frac{\alpha_s}{\pi} \, C_F \, \left[\frac{1}{2} - \frac{1}{2} \log \left(\frac{\rho_{12}}{4} \right) \right] \, . \tag{4.7}$$

Having exhibited these one-loop examples, we continue with the general discussion.

The next step is to consider the dependence on the vectors n_i^μ, which enter the form factor through the jet functions J and their eikonal counterparts $\mathcal{J}$. Following the reasoning of Refs. [4, 5] in axial gauge, generalized to arbitrary gauges in Ref. [6], we begin by observing that the form factor must be independent of n_i^μ. Defining $x_i \equiv \left(- \beta_i \cdot n_i \right)^2 / n_i^2$, we can write

$$x_i \, \frac{\partial}{\partial x_i} \log \Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = 0 \, . \tag{4.8}$$

When we apply this consistency condition to the factorized cross section in Eq. (3.1), derivatives with respect to n_i^μ of the partonic jet functions J_i separate into ultraviolet- and infrared-dominated terms, according to

$$\begin{aligned} x_i \, \frac{\partial}{\partial x_i} \log J_i \, &= \, - \, x_i \, \frac{\partial}{\partial x_i} \log C \, + \, x_i \, \frac{\partial}{\partial x_i} \log \mathcal{J}_i \\ &\equiv \frac{1}{2} \left[\mathcal{G}_i \Big(x_i, \alpha_s(\mu^2), \epsilon \Big) + \mathcal{K} \Big(\alpha_s(\mu^2), \epsilon \Big) \right] \, , \end{aligned} \tag{4.9}$$

where the second line defines the sum of the functions $\mathcal{G}_i$ and $\mathcal{K}$. As is clear from this definition, and in analogy with Eq. (2.2), the function $\mathcal{G}_i$ carries the scale dependence, but is finite as $\epsilon \to 0$, while $\mathcal{K}$ is a pure counterterm. At one loop we find, directly from Eq. (3.12),

$$\begin{aligned} \mathcal{G}_i^{(1)}(x_i, \epsilon) &= \frac{\alpha_s}{2\pi} C_F \left(\log \frac{n_i^2 \mu^2}{4(p_i \cdot n_i)^2} + 1 \right) + \mathcal{O}(\epsilon) \,, \\ \mathcal{K}^{(1)}(\epsilon) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\epsilon} \,. \end{aligned} \tag{4.10}$$

We can now relate the functions $\mathcal{G}_i$ and $\mathcal{K}$ to matrix elements of fields in the presence of Wilson lines. In fact, both partonic and eikonal jets depend on n^μ and on the velocity β^μ only through the combination $x = (-\beta \cdot n)^2/n^2$. One can thus simply relate their x dependence, given in Eq. (4.9), to their n^μ dependence, using

$$p \cdot n \frac{\partial J}{\partial p \cdot n} = - \frac{n^2}{p \cdot n} \, p^\nu \frac{\partial J}{\partial n^\nu} \,, \tag{4.11}$$

and similarly for $\mathcal{J}$, with p^μ replaced by β^μ. From the definitions of the jet functions, Eqs. (3.3) and (3.6), and from the behavior of an ordered exponential under variation with respect to the curve, we readily find expressions for the derivatives of the jet functions in Eq. (4.9), extending results found by Collins in QED [47].

For both partonic and eikonal jets, a derivative with respect to the vector n^μ replaces an ordered exponential in the n^μ direction by the integral of a field strength over the original path, sandwiched between Wilson lines. Written explicitly, we can use this result to determine the $p \cdot n$ dependence of the jet functions (3.3) in terms of matrix elements, as

$$\begin{aligned} & p \cdot n \frac{\partial}{\partial p \cdot n} \left[\log J \left(\frac{(p \cdot n)^2}{\mu^2 n^2}, \alpha_s(\mu^2), \epsilon \right) \right] \\ & \quad = - \frac{n^2}{p \cdot n} J^{-1} \frac{p^\nu}{2 p_0} u^\dagger(p) \langle 0 | \frac{\partial}{\partial n^\nu} \Phi_n(\infty, 0) \, \psi(0) \, | p \rangle \\ & \quad = - \frac{n^2}{p \cdot n} J^{-1} \frac{1}{2 p_0} u^\dagger(p) \int_0^\infty d\lambda \, \lambda \, \langle 0 | \, \Phi_n(\infty, \lambda) \; p^\mu \, n^\nu \; \left({\rm i} g \, F_{\mu\nu}(\lambda n) \right) \; \Phi_n(\lambda, 0) \, \psi(0) \, | p \rangle \\ & \quad \equiv \mathcal{G} \left(\frac{(p \cdot n)^2}{\mu^2 n^2}, \alpha_s(\mu^2), \epsilon \right) + \mathcal{K} \left(\alpha_s(\mu^2), \epsilon \right) \,, \end{aligned} \tag{4.12}$$

where the final equality is simply a restatement of Eq. (4.9). Similarly, as the first equality in (4.9) makes clear, $\mathcal{K}$ is found directly from $\mathcal{J}$ as

$$\begin{aligned} \beta \cdot n \frac{\partial}{\partial \beta \cdot n} \log \mathcal{J} \left(\frac{(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right) &= - \frac{n^2}{\beta \cdot n} \mathcal{J}^{-1} \beta^\nu \langle 0 | \frac{\partial}{\partial n^\nu} \Phi_n(\infty, 0) \, \Phi_\beta(0, -\infty) \, | 0 \rangle \\ & = - \frac{n^2}{\beta \cdot n} \mathcal{J}^{-1} \int_0^\infty d\lambda \, \lambda \, \langle 0 | \Phi_n(\infty, \lambda) \; \beta^\mu n^\nu \; \left({\rm i} g \, F_{\mu\nu}(\lambda n) \right) \; \Phi_n(\lambda, 0) \, \Phi_\beta(0, -\infty) \, | 0 \rangle \\ & \equiv \mathcal{K} \left(\alpha_s(\mu^2), \epsilon \right) \,. \end{aligned} \tag{4.13}$$

In the perturbative expansions of these matrix elements, the field strength operator prevents an unphysical gluon from coupling to the n^μ eikonal. At each order, the corresponding vertex cannot appear in any jet-like subdiagram that provides a collinear pole. It can, however, appear in subdiagrams that carry ultraviolet and infrared momenta [4], and, in the latter case, are associated with an infrared pole in dimensional regularization. The inverse jet and eikonal-jet factors multiplying the matrix elements cancel the residual collinear singularities, which factorize. The remaining terms give, order-by-order, the $\mathcal{G}$ and $\mathcal{K}$ fuctions above, from the short- and long-distance non-collinear regions to which the field strength vertex can contribute. The function $\mathcal{G}$ is seen to be the difference of two gauge-invariant matrix elements, both involving the field strength and Wilson lines, derived from, and normalized by, the partonic and eikonal jet functions.

It is worth noting that the $n^2 \to 0$ limits of Eqs. (4.12) and (4.13) are singular. In this limit, the matrix elements become boost-invariant, and at the same time develop collinear singularities for gluons in the n^μ direction. As for the soft function $\mathcal{S}$, the jet functions, which are sums of pole contributions only, are no longer scale invariant in the light-like vectors β and n.

We can now turn to the determination of the Q dependence of the full form factor, Eq. (2.2). Γ depends on Q directly through the short-distance function C, and indirectly through the partonic jets J_i, which depend on external momenta through $p_i \cdot n_i$. From the factorized expression, Eq. (3.1), using Eq. (4.5), one easily derives

$$Q \frac{\partial}{\partial Q} \log \Gamma = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log C - \gamma_{\bar{\mathcal{S}}} - 2\, \gamma_J + \sum_{i=1}^{2} \left(\mathcal{G}_i + \mathcal{K} \right) \, . \tag{4.14}$$

Because the pole terms $\mathcal{K}(\epsilon, \alpha_s)$ are independent of the kinematic variables, they are equal, and we have $K = 2\,\mathcal{K}$. Comparing Eq. (4.14) with the original evolution equation, Eq. (2.2), we finally find an expression for G, in terms of the anomalous dimensions of the soft and eikonal jet functions, and in terms of the functions $\mathcal{G}_i$, defined by the matrix elements of Eqs. (4.12) and (4.13). Explicitly, we have

$$G(\alpha_s) = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log C - \gamma_{\bar{\mathcal{S}}} - 2\gamma_J + \sum_{i=1}^{2} \mathcal{G}_i \, , \tag{4.15}$$

which is easily verified to hold at one loop, making use of the results of Section 3.

5. Eikonal and Collinear Contributions to Subleading Poles

The single poles of form factors are generated by functions $G(\alpha_s)$, one for each type of parton, which have been related to gauge theory matrix elements via Eq. (4.15). In this section, we will link $G(\alpha_s)$ to two anomalous dimensions, one stemming from the collinear evolution of parton distributions, the other from the eikonal form factor, $\mathcal{S}$. A relation following from this structure was verified empirically to three loops in Ref. [49], based on an earlier observation of Ref. [48]. A similar connection has been established at finite order between single-logarithmic contributions to the Drell-Yan cross section and collinear evolution kernels in Ref. [17]. These empirical observations are established here to all orders in perturbation theory, exploiting a connection between the form factor and parton-in-parton distributions, which follows from factorization and which was noted already in

Ref. [60]. To derive our result, we first have a look at the analog of the form factor in parton evolution.

5.1 Factorization for Virtual Contributions to Parton Distributions

Let us begin by considering the standard definition of the light-cone distribution for a parton of flavor i, carrying momentum fraction x, in a parent parton of the same flavor. For a quark, for example, one writes

$$\phi_{q/q}\left(x,\epsilon\right) = \frac{1}{4N_c}\int\frac{d\lambda}{2\pi}\mathrm{e}^{-\mathrm{i}\lambda x p\cdot\beta}\langle p|\overline{\psi}_q(\lambda\beta)\,\gamma\cdot\beta\,\Phi_\beta(\lambda,0)\,\psi_q(0)|p\rangle\,, \tag{5.1}$$

where p is the momentum of the parent quark, which we can take, say, along the $(+)$ direction; β is then an auxiliary light-cone vector along the $(-)$ direction, and the Dirac projector $\gamma\cdot\beta$ selects the relevant components of the quark field, while the Wilson line Φ_β ensures gauge invariance. An analogous definition applies for the gluon-in-gluon distribution, with the Wilson line in the adjoint representation. Note that there is no explicit $p\cdot\beta$ dependence in the parton distributions when they are defined in this boost-invariant fashion in the $\overline{\mathrm{MS}}$ prescription.

In order to single out the virtual contributions to the parton distribution, we proceed as follows. The gauge link can be split by extending it to light-like infinity along the β direction, according to

$$\Phi_\beta(\lambda,0) = \Phi_\beta(\lambda,\infty)\Phi_\beta(\infty,0)\,; \tag{5.2}$$

one can now insert a complete set of states between the two Wilson lines, and then isolate the contribution of the vacuum. This gives the virtual contribution to the parton distribution at the amplitude level as the correlator

$$\overline{\Gamma}_{q/q}\left(\frac{p\cdot\beta}{\mu},\alpha_s(\mu^2),\epsilon\right) \equiv \langle 0|\Phi_\beta(\infty,0)\,\psi_q(0)\,|p\rangle\,, \tag{5.3}$$

coupling a single-particle state to the vacuum through the action of the partonic field ψ_q and of a gauge link in the same color representation. We shall define this matrix element as a sum of pure pole terms, consistent with its interpretation as part of an $\overline{\mathrm{MS}}$ parton distribution function.

Clearly, the amplitude $\overline{\Gamma}_{q/q}$ in Eq. (5.3) is closely related to the partonic jet, J, in Eq. (3.3). In fact, the only difference is that the gauge link is now in a light-like direction opposite to the parton momentum. As a consequence, $\overline{\Gamma}_{q/q}$ can be factorized in the same manner as the full partonic amplitude, Eq. (3.1), into short-distance, jet, and soft functions. Now, however, we need a separate partonic jet only for the incoming line, since the collinear singularities of the outgoing gauge link match the collinear singularities of the soft function $\mathcal{S}$ in Eq. (3.1). We can then write

$$\begin{aligned}\overline{\Gamma}_{q/q}\left(\frac{p\cdot\beta}{\mu},\alpha_s(\mu^2),\epsilon\right) &= \mathcal{S}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right)\\ &\quad\times C_J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)\left[\frac{J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)}{\mathcal{J}\left(\frac{(\beta_p\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)}\right]\,,\end{aligned} \tag{5.4}$$

where we have introduced the velocity four-vector β_p associated with the momentum p. The function C_J is a short-distance coefficient chosen to cancel all terms that are finite for $\epsilon \to 0$ in J, because $\overline{\Gamma}_{q/q}$ is defined as a sum of pole terms only. We are assured that the function C_J exists, because of the exponentiation of all pole terms. Using the fact that $\mathcal{S}$ and $\mathcal{J}$ are also pure pole terms, we may write,

$$C_J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)\, J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right) = \left[J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)\right]_{\rm pole}. \tag{5.5}$$

Because $\overline{\Gamma}_{q/q}$ is n-independent, all n-dependence in poles on the right-hand side of Eq. (5.4) is guaranteed to cancel in the ratio of jet functions, leaving only finite n-dependence, which is cancelled by C_J. This cancellation is possible simply because collinear singularities are independent of n^μ [4], leaving only soft contributions, whose (exponentiating) poles match between J and $\mathcal{J}$, and short-distance contributions, which can be cancelled by C_J. We observe that the ratio of partonic to eikonal jets in Eq. (5.4) is the same as in the basic factorized form, Eq. (3.1).

Inserting Eq. (5.5) into Eq. (5.4), $\overline{\Gamma}_{q/q}$ can be represented as,

$$\overline{\Gamma}_{q/q}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right) = \mathcal{S}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right)\,\frac{\left[J\left(\frac{(\beta_p\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)\right]_{\rm pole}}{\mathcal{J}\left(\frac{(\beta_p\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)}\,, \tag{5.6}$$

where we have chosen the magnitude of the four-velocity β_p so that $p = (\mu/\sqrt{2})\beta_p$, and the scalar product $p\cdot\beta = (\mu/\sqrt{2})\,\beta_p\cdot\beta$. Thus μ is the only remaining scale, which appears only as the argument of the coupling. As usual, Eq. (5.6) involves exponentiating double poles from $\mathcal{S}$, which cancel when combined with the real emission contributions to the parton distribution, leaving behind only single, collinear poles that define the splitting functions.

Now, following Ref. [60], we introduce the eikonal counterpart of the correlator (5.3), which will represent the virtual contribution to an eikonal parton distribution. This is naturally defined as the soft function $\mathcal{S}$ itself,

$$\overline{\Gamma}^{\rm eik}_{q/q}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right) \equiv \mathcal{S}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right)\,, \tag{5.7}$$

which, once again, is an exponential consisting entirely of pole terms. We can now exploit the fact that $\overline{\rm MS}$ parton distributions can be defined, in moment space, simply as exponentials of the integrated collinear anomalous dimension, as was done in Ref. [60]. Furthermore, the eikonal approximation is accurate for real final-state radiation, up to inverse powers of the Mellin variable N. The ratio of the virtual contribution for the parton distribution to its eikonal counterpart must thus be given entirely by the virtual term of the corresponding diagonal splitting function, $B_\delta^{[i]}(\alpha_s)$, whose normalization is defined by

$$P_{ii}(x) = \frac{\gamma_K^{[i]}(\alpha_s)}{2}\left[\frac{1}{1-x}\right]_+ + B_\delta^{[i]}(\alpha_s)\,\delta(1-x) + \mathcal{O}\Big((1-x)^0\Big)\,. \tag{5.8}$$

Taking the ratio of Eq. (5.7) and Eq. (5.6) we then find,

$$\frac{\left[J\left(\frac{(\beta_p\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)\right]_{\rm pole}}{\mathcal{J}\left(\frac{(\beta_p\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)} = \frac{\overline{\Gamma}_{q/q}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right)}{\overline{\Gamma}^{\rm eik}_{q/q}\left(\beta_p\cdot\beta,\alpha_s(\mu^2),\epsilon\right)}$$
$$= \exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\,B_\delta^{[q]}\left(\overline{\alpha}(\xi^2,\epsilon)\right)\right]\,, \qquad (5.9)$$

where the second relation[5] mirrors the results obtained in Ref. [60] for the complete parton distribution. At one loop, $B_\delta^{[q]}(\alpha_s) = (3/4)\,C_F\,(\alpha_s/\pi)$ for quarks, and $B_\delta^{[g]}(\alpha_s) = (b_0/4)\,(\alpha_s/\pi)$ for gluons. From Eq. (5.7), the eikonal vertex $\overline{\Gamma}^{\rm eik}$ and $\mathcal{S}$ are to be computed in the same fashion in perturbation theory, identifying their eikonal velocities with momenta $(\mu/\sqrt{2})\beta_p$ and $(\mu/\sqrt{2})\beta$, for incoming and outgoing lines, respectively. Consistency in the factorization formula Eq. (3.1) then requires the same treatment of the velocity β_p in the function $\mathcal{J}$ on the left-hand side of the first equality in Eq. (5.9).

The result in Eq. (5.9) expresses the purely collinear single poles of the ratio between the partonic jet function and its eikonal counterpart in terms of the virtual contribution $B_\delta(\alpha_s)$ to the splitting kernel of the appropriate parton flavor. We now use this result in conjunction with our basic factorization formula, Eq. (3.1), to get another simple expression for the function G.

5.2 Relating Form Factors to Collinear Evolution Kernels

We proceed by using Eq. (5.5), followed by Eq. (5.9), in Eq. (3.1) for each jet, and exploiting renormalization-group invariance of the full form factor to set $\mu^2 = Q^2$. This expresses $\Gamma(Q^2,\epsilon)$ in terms of the eikonal soft function $\mathcal{S}$, the virtual evolution kernel B_δ, and the *finite* factors in the partonic jet functions, as

$$\begin{aligned}\Gamma\left(1,\alpha_s(Q^2),\epsilon\right) &= C\left(1,\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(Q^2),\epsilon\right)\left[\prod_{i=1}^{2} C_J\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right)\right]^{-1}\\ &\quad\times\exp\left[\int_0^{Q^2}\frac{d\xi^2}{\xi^2}\,B_\delta\left(\overline{\alpha}(\xi^2,\epsilon)\right)\right]\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(Q^2),\epsilon\right)\,,\\ &\equiv \overline{C}\left(\alpha_s(Q^2),\epsilon\right)\\ &\quad\times\exp\left[\int_0^{Q^2}\frac{d\xi^2}{\xi^2}\,B_\delta\left(\overline{\alpha}(\xi^2,\epsilon)\right)\right]\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(Q^2),\epsilon\right)\,,\end{aligned} \qquad (5.10)$$

where in the second relation we define the function $\overline{C}(\alpha_s,\epsilon)$ to include all factors that are finite at vanishing ϵ. In this function, all dependence on jet directions cancels.

In Eq. (3.5), we have an exponentiated form for the soft function $\mathcal{S}$, but to make direct contact with the standard form factor notation, Eq. (2.11), in terms of G and γ_K, we

[5] There is a factor of 1/2 in the exponential of Eq. (5.9) relative to the corresponding relation in Ref. [60], because we are computing an amplitude here rather than a parton distribution.

need an expression for $\overline{C}$ as well. This expression can be simply obtained, using again the integrability of the coupling α_s for $\epsilon < 0$. We write

$$\overline{C}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(Q^2), \epsilon\right) = \exp\left[\frac{1}{2}\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left\{2\,\xi^2 \frac{d}{d\xi^2}\,\log \overline{C}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)\right\}\right]$$

$$\equiv \exp\left[\frac{1}{2}\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\, G_{\overline{C}}\left(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)\right]. \tag{5.11}$$

In the definition of $G_{\overline{C}}$ we have used the independence of the form factor on the choice of the eikonal vectors n_i, and have inserted a factor of $1/2$ to conform with the normalization of Eq. (2.8). We note that although $G_{\overline{C}}$ is finite at $\epsilon = 0$, the integral in Eq. (5.11) produces no poles, since the logarithmic derivative with respect to the scale generates a positive power of ϵ. Inserting Eq. (5.11) and Eq. (3.5) in Eq. (5.10), we find our final exponentiated result,

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left[G_{\rm eik}\left(1, \overline{\alpha}(\xi^2, \epsilon)\right) + 2\, B_\delta\left(\overline{\alpha}(\xi^2, \epsilon)\right)\right.\right.$$

$$\left.\left. + \, G_{\overline{C}}\left(\overline{\alpha}(\xi^2, \epsilon), \epsilon\right) - \frac{1}{2}\,\gamma_K\left(\overline{\alpha}(\xi^2, \epsilon)\right)\,\log\left(\frac{Q^2}{\xi^2}\right)\right]\right\}, \tag{5.12}$$

where we have set $\beta_1 \cdot \beta_2 = 1$ in $G_{\rm eik}$. Comparing this result with Eq. (2.11), we conclude that

$$G(1, \alpha_s, \epsilon) = 2B_\delta\left(\alpha_s\right) + G_{\rm eik}\left(1, \alpha_s\right) + G_{\overline{C}}\left(\alpha_s, \epsilon\right)\ , \tag{5.13}$$

The function G is thus the sum of three terms: twice the coefficient of $\delta(1-x)$ in the relevant parton splitting function, the single-logarithmic anomalous dimension of the eikonal form factor, and finally a term associated with the running of the coupling in the infrared-finite hard-scattering function. The latter term is proportional to the d-dimensional beta function, Eq. (2.4) (see Eq. (5.11) and the Appendix); hence it vanishes as $\epsilon \to 0$ in a scale-invariant theory.

Finally, comparing Eqs. (4.15) and (5.13) for $G(\alpha_s)$, and referring to Eq. (4.12) and Eq. (4.13), which relate $\mathcal{G}$ to nonlocal matrix elements involving the field strength, we find that the moment-independent term in the evolution kernel is determined by the same matrix elements of the field strength, and by a combination of anomalous dimensions of eikonal and local operators, including the logarithmic derivative of the functions C_J introduced in Eq. (5.4). Explicitly,

$$B_\delta\left(\alpha_s\right) = \frac{1}{2}\sum_{i=1}^{2}\left[\mathcal{G}_i + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\log C_{J_i}\right] - \frac{1}{2}G_{\rm eik} - \frac{1}{2}\gamma_{\bar{\mathcal{S}}} - \gamma_J\,. \tag{5.14}$$

Although individual terms in this new relation for the function $B_\delta(\alpha_s)$ depend on n_i and β_i, this dependence cancels in the sum.

Once again, Eq. (5.13) can easily be tested at one loop using the results of Section 3. It can be tested further, up to the three-loop level, by comparing with the results of Ref. [49], where the relation between the function G and the virtual splitting kernel was emphasized. Indeed, Eq. (20) of Ref. [49] shows that up to three loops the perturbative coefficients $G^{(k)}$ of the function G are given by the sum of 'maximally non-abelian' terms $f^{(k)}$, plus twice the virtual splitting kernel contributions $B_\delta^{(k)}$, as in Eq. (5.13), plus remainders proportional either to ϵ or to the β function coefficients. In the Appendix we show that such terms are precisely the ones that arise in the expansion of a function of the coupling that is defined as a total derivative with respect to the scale, as is the case for $G_{\overline{C}}$. It is natural then to identify $f^{(k)}$ with the corresponding perturbative coefficient of $G_{\rm eik}$. Indeed, $f^{(1)} = 0$, consistently with Eq. (3.10) in the $\overline{\rm MS}$ scheme and for space-like kinematics. Furthermore, one easily verifies that, when brought to the same normalization, $f^{(2)}$ is one half of $\Gamma^{(2)}_{\rm eik} \equiv \Gamma^{(2)}_{\rm DY}$, as computed in Ref. [50]. The factor of 1/2 is expected, since $f^{(2)}$ contributes to an amplitude while $\Gamma^{(2)}_{\rm DY}$ contributes to a cross section.

Equations similar to Eq. (5.13) have appeared in the description of the anomalous dimensions of effective currents (or their matching coefficients) in soft collinear effective theory. In Ref. [18] it was noticed that the empirical relations found through three loops [17, 48, 49] imply that the subleading-logarithmic part of the effective-current anomalous dimension, denoted there $B_1(\alpha_s)$, is given just in terms of $B_\delta(\alpha_s)$ and the function f of Ref. [49]; the $G_{\overline{C}}$ terms drop out of the anomalous dimension. In Ref. [20] a similar relation was found, and f (called γ^W in Ref. [20]) was identified with the anomalous dimension for a momentum-space Wilson loop associated with the Drell-Yan process. An analogous relation for deep-inelastic scattering was obtained in Ref. [19].

The identification of the coefficients $f^{(k)}$ with the eikonal (Wilson-line) quantities $G^{(k)}_{\rm eik}$ neatly explains a couple of their properties found empirically through three loops [48, 49]: the relation $f_g^{(k)}/C_A = f_q^{(k)}/C_F$, and the maximally non-abelian color structure of these quantities. The non-abelian exponentiation theorem for eikonal graphs [55, 56] implies that only "color-connected" graphs composed of single gluon webs (along with fermion-loop insertions) contribute to $G_{\rm eik}$. Through $k = 3$ loops, all such graphs have color factors of the "maximally non-abelian" form $C_i C_A^{k-l-1} n_f^l$, where C_i is the Casimir factor for the eikonal line, C_F for quarks and C_A for gluons. This form breaks down at four loops, due to the existence of color factors that cannot be expressed in terms of quadratic Casimir operators, as in the case of the four-loop beta function in QCD [61].

As mentioned above, in a conformal theory there is no contribution to Eq. (5.13) from $G_{\overline{C}}$ as $\epsilon \to 0$, so we have

$$G(1, \alpha_s, 0) = G_{\rm eik}\left(1, \alpha_s\right) + 2 B_\delta\left(\alpha_s\right) . \tag{5.15}$$

The eikonal quantity $G_{\rm eik}$ carries no information about the spin of the parton, only its color (representation under the gauge group). Thus the spin-dependence of $G(1, \alpha_s, 0)$ is all carried by the virtual part of the splitting kernel, $B_\delta(\alpha_s)$. Many conformal theories are supersymmetric: in this case, if two partonic states belong to the same supersymmetry multiplet, then they are in the same gauge-group representation, and $G_{\rm eik}$ is the same for both. The leading-twist operators whose anomalous dimensions yield $B_\delta(\alpha_s)$ will also be related by supersymmetry. By Eq. (5.15), the values of $G(1, \alpha_s, 0)$ for these states should be the same too. This result can also be seen via supersymmetry Ward identities which relate the appropriate S-matrix elements [62], and thereby imply that the corresponding single $1/\epsilon$ poles have to be identical. It would be interesting to see if the simple compound representation of $G(1, \alpha_s, 0)$ in Eq. (5.15) can help in the study of its properties in $\mathcal{N} = 4$ super-Yang-Mills theory.

6. Concluding Remarks

We have reviewed the resummation of poles in dimensionally-regularized singlet form factors for QCD and related theories, using the factorization properties of their amplitudes. Revisiting the basic evolution equation and its solution, we observed the strikingly simple connection between the analytic continuation of the form factor in a conformal theory and the cusp anomalous dimension, given by Eq. (2.16).

In view of of the universal nature of collinear poles in dimensionally-regularized amplitudes, Eq. (1.1), we have investigated the origin of subleading poles in the form factor. Our analysis extends the familiar relationship between the leading poles and the cusp anomalous dimension. We have determined the origin of the non-singular function $G(\alpha_s)$ in the basic evolution equation, Eq. (2.2). Relying on the operator content of the factorized jet and soft functions, we found Eq. (4.15), which relates $G(\alpha_s)$ to the short-distance function C, and to $\gamma_{\bar{\mathcal{S}}}$ and γ_J, the anomalous dimensions of soft and jet functions respectively, as well as to matrix elements involving the field strength and Wilson lines. The structure of this result is made more transparent in Eq. (5.13), which expresses $G(\alpha_s)$ in terms of two universal quantities (the single-pole anomalous dimension of the eikonal form factor, and the coefficient of $\delta(1-z)$ in the diagonal partonic evolution kernel), plus a process-dependent short-distance contribution, which generates no singularities and is proportional to ϵ in the conformal limit. The same pattern has been noted in deep-inelastic scattering and the Drell-Yan process [17, 18, 19, 20], and, through the relation between form factors and amplitudes, it will appear in subleading logarithmic corrections to any threshold-resummed jet cross section [63]. Equating our two expressions for $G(\alpha_s)$ provides an interesting new relation between $B_\delta(\alpha_s)$, other anomalous dimensions, and matrix elements of the field strength.

In summary, unlike leading poles, nonleading poles in form factors and fixed-angle scattering amplitudes have a compound structure, even in conformal theories. Nevertheless, all contributions to $G(\alpha_s)$ that generate infrared poles have a well-defined and universal origin, in terms of matrix elements and anomalous dimensions in the massless gauge field theory. In the context of planar $\mathcal{N} = 4$ super-Yang-Mills theory, the explicit operator interpretations for $G(\alpha_s)$ may aid efforts to use integrability to determine its exact coupling-constant dependence.

Acknowledgments

We thank T. Becher, Z. Bern, J. Maldacena, S. Moch and M. Staudacher for stimulating conversations, and E. Gardi and G. Grunberg for a timely exchange relevant to integrals over the running coupling. L.M. thanks the C.N. Yang Institute for Theoretical Physics at SUNY Stony Brook and the CERN Theory Division for hospitality during the completion of this work. This work was supported in part by the US Department of Energy under contract DE–AC02–76SF00515, by the National Science Foundation, grants PHY-0354776, PHY-0354822 and PHY-0653342, by MIUR under contract 2006020509_004, and by the European Community's Marie-Curie Research Training Network 'Tools and Precision Calculations for Physics Discoveries at Colliders' ('HEPTOOLS'), under contract MRTN-CT-2006-035505.

A. Comparison to Fixed-Order

The form factors for quarks and gluons in QCD have been evaluated recently at three loops in Refs. [49, 54], allowing for a stringent test of our result, Eq. (5.13). Ref. [49] (MVV below), in particular, gives an explicit expression for the single-logarithmic function $G(\alpha_s, \epsilon)$ up to three loops, as a sum essentially in the form given in Eq. (5.13) above. In this sum, the function that we denote by $G_{\overline{C}}(\alpha_s, \epsilon)$ corresponds to a set of terms containing the coefficients $\widetilde{G}_i^p$ in Eq. (20) of MVV. Similarly, $G_{\rm eik}$ can be identified with terms labeled f in MVV, whose universality is noted, without an explicit discussion of their origin.

The key feature of Eq. (20) in MVV is that the terms in the function G that are not accounted for by the virtual splitting kernel B_δ or by the eikonal function f are proportional either to ϵ, or to the coefficients of the β function. We want to verify that all these terms are precisely of the form that follows by requiring that they are coefficients in the expansion of a total derivative with respect to the scale of the running coupling, as is the case for our function $G_{\overline{C}}(\alpha_s, \epsilon)$, Eq. (5.11).

Let us begin by working out the consequences of our definition of $G_{\overline{C}}(\alpha_s, \epsilon)$, which is of the form

$$G_{\overline{C}}\left(\alpha_s(\mu^2), \epsilon\right) = 2\,\mu^2 \frac{d}{d\mu^2}\, E\left(\alpha_s(\mu^2), \epsilon\right)\,, \tag{A.1}$$

for some function $E(\alpha_s(\mu^2), \epsilon)$ that is an expansion in both $\alpha_s(\mu^2)$ and ϵ, and which is finite at $\epsilon = 0$. In terms of our factorization analysis, $E(\alpha_s, \epsilon)$ is simply the logarithm of the finite coefficient function $\overline{C}$ in the factorized form factor, in the scheme in which the coefficient function is defined to absorb all finite terms in the expansion of the partonic jet functions. Expanding $G_{\overline{C}}$ and E in powers of α_s and ϵ, we write

$$G_{\overline{C}}\left(\alpha_s, \epsilon\right) = \sum_{n=1}^{\infty} G_{\overline{C}}^{(n)}(\epsilon) \left(\frac{\alpha_s}{\pi}\right)^n = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} G_{\overline{C},\,m}^{(n)}\, \epsilon^m\, \left(\frac{\alpha_s}{\pi}\right)^n\,, \tag{A.2}$$

and similarly for E. Next, we use the fact that E depends on the scale only through the coupling $\alpha_s(\mu^2)$. As a consequence, we can easily find an expression for the perturbative coefficients of $G_{\overline{C}}$ in terms of those of E. Using our normalizations for the β function, Eqs. (2.4) and (2.5), we find

$$G_{\overline{C}}^{(n)}(\epsilon) = -2\, n\, \epsilon\, E^{(n)}(\epsilon) - \frac{1}{2} \sum_{k=1}^{n-1} k\, b_{n-k-1}\, E^{(k)}(\epsilon)\,. \tag{A.3}$$

Since all $E^{(k)}(\epsilon)$ are finite as $\epsilon \to 0$, this implies

$$G^{(n)}_{\overline{C}}(0) = -\frac{1}{2} \sum_{k=1}^{n-1} k\, b_{n-k-1}\, E^{(k)}(0)\,. \tag{A.4}$$

We observe from Eq. (A.3) that the perturbative coefficients of $G_{\overline{C}}$ are sums of terms that are proportional either to ϵ or to the coefficients of the β function, as expected. Furthermore, combining Eq. (A.4) with Eq. (A.3), it is clear that one can determine $G^{(n)}_{\overline{C}}(0)$ recursively in terms of $G^{(k)}_{\overline{C}}(\epsilon)$, with $k < n$. To match the notation of MVV, we proceed by defining

$$\widetilde{g}(\epsilon) \equiv \frac{1}{\epsilon}\Big[g(\epsilon) - g(0)\Big]\,, \tag{A.5}$$

for any function $g(\epsilon)$ with a finite limit as $\epsilon \to 0$. The recursion starts with $G^{(1)}_{\overline{C}}(0) = 0$, so that Eq. (5.13) gives

$$G^{(1)} = 2B^{(1)}_{\delta} + G^{(1)}_{\rm eik} + \epsilon\widetilde{G}^{(1)}_{\overline{C}}\,, \tag{A.6}$$

matching the one-loop Eq. (20) of MVV.

Proceeding recursively, it is easy to see that at two loops one can write

$$G^{(2)}_{\overline{C}}(0) = -\frac{b_0}{2}E^{(1)}(0) = \frac{b_0}{4}\widetilde{G}^{(1)}_{\overline{C}}(0)\,. \tag{A.7}$$

Using $G^{(2)}_{\overline{C}}(\epsilon) = G^{(2)}_{\overline{C}}(0) + \epsilon\widetilde{G}^{(2)}_{\overline{C}}$, and taking into account the different normalizations, Eq. (A.7) matches the two-loop result in Eq. (20) of MVV,

$$G^{(2)} = 2B^{(2)}_{\delta} + G^{(2)}_{\rm eik} + \frac{b_0}{4}\widetilde{G}^{(1)}_{\overline{C}}(0) + \epsilon\widetilde{G}^{(2)}_{\overline{C}}\,. \tag{A.8}$$

A short calculation yields also the three-loop expression

$$\begin{aligned} G^{(3)}_{\overline{C}}(0) &= -b_0\, E^{(2)}(0) - \frac{b_1}{2}\, E^{(1)}(0) \\ &= \frac{b_0}{4}\,\widetilde{G}^{(2)}_{\overline{C}}(0) - \frac{b_0^2}{16}\,\widetilde{\widetilde{G}}{}^{(1)}_{\overline{C}}(0) + \frac{b_1}{4}\,\widetilde{G}^{(1)}_{\overline{C}}(0)\,, \end{aligned} \tag{A.9}$$

which again matches the β-function terms in the three-loop result in Eq. (20) of MVV, provided the normalizations are taken into account.

An alternative way of expressing the solution of this recursion problem, without making use of the subtraction in Eq. (A.5), is to expand explicitly the coefficients $G^{(n)}_{\overline{C}}(\epsilon)$ in powers of ϵ, as done in Eq. (A.2). It is straightforward to express the solution, at any order, in terms of the coefficients $G^{(n)}_{\overline{C},m}$. One finds

$$G^{(n+1)}_{\overline{C},0} = \frac{b_0}{4}G^{(n)}_{\overline{C},1} - \frac{b_0^2}{16}G^{(n-1)}_{\overline{C},2} + \frac{b_1}{4}G^{(n-1)}_{\overline{C},1} + \frac{b_2}{4}G^{(n-2)}_{\overline{C},1} - \frac{b_0 b_1}{8}G^{(n-2)}_{\overline{C},2} + \ldots\,. \tag{A.10}$$

Eq. (A.10) was derived in Ref. [53], as a solution to the problem of finding a function $\overline{G}(\alpha_s(\mu^2))$, independent of ϵ, but capable, upon integration over the scale, of matching the simple poles of the form factor, generated by the function $G(\alpha_s(\mu^2), \epsilon)$. The result of Ref. [53] can now be rephrased by stating that the function $\overline{G}(\alpha_s(\mu^2))$ must be given by $G(\alpha_s(\mu^2), \epsilon = 0)$ plus the total derivative with respect to the scale of a finite function of ϵ.

References

[1] V. V. Sudakov, Sov. Phys. JETP **3** (1956) 65; P. M. Fishbane and J. D. Sullivan, Phys. Rev. **D 4** (1971) 458; T. Appelquist and J. R. Primack, Phys. Rev. **D 4** (1971) 2454.

[2] A. H. Mueller, Phys. Rev. **D 20** (1979) 2037.

[3] J. C. Collins, Phys. Rev. **D 22** (1980) 1478.

[4] J. C. Collins and D. E. Soper, Nucl. Phys. **B 193** (1981) 381 [Erratum-ibid. **B 213** (1983) 545].

[5] A. Sen, Phys. Rev. **D 24** (1981) 3281.

[6] H. Contopanagos, E. Laenen and G. Sterman, Nucl. Phys. **B 484** (1997) 303, `hep-ph/9604313`.

[7] L. Magnea and G. Sterman, Phys. Rev. **D 42** (1990) 4222.

[8] L. Magnea, Nucl. Phys. **B 593** (2001) 269, `hep-ph/0006255`.

[9] G. C. Marques, Phys. Rev. **D 9** (1974) 386;
M. Creutz and L. L. Wang, Phys. Rev. **D 10**(1974) 3749;
H. B. Thacker and W. I. Weisberger, Phys. Rev. **D 14** (1976) 2658.

[10] C. G. Callan and D. J. Gross, Phys. Rev. **D 11** (1975) 2905.

[11] J. C. Collins, D. E. Soper and G. Sterman, Adv. Ser. Direct. High Energy Phys. **5** (1988) 1, `hep-ph/0409313`.

[12] A. Sen, Phys. Rev. **D 28** (1983) 860.

[13] G. Sterman and M. E. Tejeda–Yeomans, Phys. Lett. **B 552** (2003) 48, `hep-ph/0210130`.

[14] S. Mert Aybat, L. J. Dixon and G. Sterman, Phys. Rev. **D 74** (2006) 074004, `hep-ph/0607309`.

[15] S. Catani and M. H. Seymour, Phys. Lett. **B 378** (1996) 287, `hep-ph/9602277`; Nucl. Phys. **B 485** (1997) 291 [Erratum-ibid. **B 510** (1998) 503], `hep-ph/9605323`.

[16] S. Catani, Phys. Lett. **B 427** (1998) 161, `hep-ph/9802439`.

[17] E. Laenen and L. Magnea, Phys. Lett. **B 632** (2006) 270, `hep-ph/0508284`.

[18] A. Idilbi, X. d. Ji and F. Yuan, Nucl. Phys. **B 753** (2006) 42, `hep-ph/0605068`.

[19] T. Becher, M. Neubert and B. D. Pecjak, JHEP **0701** (2007) 076, `hep-ph/0607228`.

[20] T. Becher, M. Neubert and G. Xu, `arXiv:0710.0680 [hep-ph]`.

[21] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113], `hep-th/9711200`;
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B 428** (1998) 105 `hep-th/9802109`;
E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253, `hep-th/9802150`.

[22] Z. Bern, J. S. Rozowsky and B. Yan, Phys. Lett. **B 401** (1997) 273, `hep-ph/9702424`.

[23] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **91** (2003) 251602, `hep-th/0309040`.

[24] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. **D 72** (2005) 085001, `hep-th/0505205`.

[25] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. **D 74** (2006) 045020, hep-th/0602228.

[26] Z. Bern, M. Czakon, D. A. Kosower, R. Roiban and V. A. Smirnov, Phys. Rev. Lett. **97** (2006) 181601, hep-th/0604074.

[27] L. F. Alday and J. M. Maldacena, JHEP **0706** (2007) 064, arXiv:0705.0303 [hep-th].

[28] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, arXiv:0803.1465 [hep-th].

[29] L. F. Alday and J. M. Maldacena, JHEP **0711** (2007) 068, arXiv:0710.1060 [hep-th].

[30] J. Bartels, L. N. Lipatov and A. Sabio Vera, arXiv:0802.2065 [hep-th].

[31] L. F. Alday and J. M. Maldacena, JHEP **0711** (2007) 019, arXiv:0708.0672 [hep-th].

[32] G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. **B 171** (1986) 459.

[33] S. V. Ivanov, G. P. Korchemsky and A. V. Radyushkin, Yad. Fiz. **44** (1986) 230 [Sov. J. Nucl. Phys. **44** (1986) 145].

[34] G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. **B 283** (1987) 342.

[35] G. P. Korchemsky, Phys. Lett. **B 220** (1989) 629.

[36] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, Phys. Rev. **D 75** (2007) 085010, hep-th/0610248.

[37] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. **D 75** (2007) 105011, hep-th/0612309.

[38] G. P. Korchemsky and G. Marchesini, Nucl. Phys. **B 406** (1993) 225, hep-ph/9210281.

[39] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. **B 636** (2002) 99, hep-th/0204051.

[40] S. Frolov and A. A. Tseytlin, JHEP **0206** (2002) 007, hep-th/0204226.

[41] R. Roiban, A. Tirziu and A. A. Tseytlin, JHEP **0707** (2007) 056, arXiv:0704.3638 [hep-th];
R. Roiban and A. A. Tseytlin, JHEP **0711** (2007) 016, arXiv:0709.0681 [hep-th].

[42] N. Beisert, B. Eden and M. Staudacher, J. Stat. Mech. **0701** (2007) P02, hep-th/0610251.

[43] M. K. Benna, S. Benvenuti, I. R. Klebanov and A. Scardicchio, Phys. Rev. Lett. **98** (2007) 131603, hep-th/0611135;
B. Basso, G. P. Korchemsky and J. Kotański, Phys. Rev. Lett. **100** (2008) 091601, arXiv:0708.3933 [hep-th].

[44] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. **B 795** (2008) 385, arXiv:0707.0243 [hep-th];
A. Brandhuber, P. Heslop and G. Travaglini, Nucl. Phys. **B 794** (2008) 231, arXiv:0707.1153 [hep-th].

[45] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. **B 795** (2008) 52, arXiv:0709.2368 [hep-th]; arXiv:0712.1223 [hep-th]; arXiv:0712.4138 [hep-th].

[46] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, arXiv:0803.1466 [hep-th].

[47] J. C. Collins, Adv. Ser. Direct. High Energy Phys. **5** (1989) 573, hep-ph/0312336.

[48] V. Ravindran, J. Smith and W. L. van Neerven, Nucl. Phys. **B 704** (2005) 332, hep-ph/0408315.

[49] S. Moch, J. A. M. Vermaseren and A. Vogt, Phys. Lett. **B 625** (2005) 245, hep-ph/0508055.

[50] A. V. Belitsky, Phys. Lett. **B 442** (1998) 307, hep-ph/9808389.

[51] G. Parisi, Phys. Lett. **B 90** (1980) 295.

[52] G. Sterman, Nucl. Phys. **B 281** (1987) 310.

[53] T. O. Eynck, E. Laenen and L. Magnea, JHEP **0306** (2003) 057, hep-ph/0305179.

[54] S. Moch, J. A. M. Vermaseren and A. Vogt, JHEP **0508** (2005) 049, hep-ph/0507039.

[55] J. G. M. Gatheral, Phys. Lett. **B 133** (1983) 90.

[56] J. Frenkel and J. C. Taylor, Nucl. Phys. **B 246** (1984) 231.

[57] C. F. Berger, Phys. Rev. **D 66** (2002) 116002, hep-ph/0209107.

[58] J. Botts and G. Sterman, Nucl. Phys. **B 325** (1989) 62.

[59] G. P. Korchemsky and G. Marchesini, Phys. Lett. **B 313** (1993) 433.

[60] E. Laenen, G. Sterman and W. Vogelsang, Phys. Rev. **D 63** (2001) 114018, hep-ph/0010080.

[61] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. **B 400** (1997) 379, hep-ph/9701390.

[62] M. T. Grisaru, H. N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. **D 15** (1977) 996; M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B 124** (1977) 81.

[63] N. Kidonakis, G. Oderda and G. Sterman, Nucl. Phys. **B 525** (1998) 299, hep-ph/9801268.