COHERENT SYNCHROTRON RADIATION AND SPACE CHARGE FOR A 1-D SOURCE ON AN ARBITRARY PLANAR ORBIT * †

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Abstract

Realistic modeling of coherent synchrotron radiation (CSR) and the space charge force in single-pass systems and rings usually requires at least a two-dimensional (2-D) description of the charge/current density of the bunch. Since that leads to costly computations, one often resorts to a 1-D model of the bunch for first explorations. This paper provides several improvements to previous 1-D theories, eliminating unnecessary approximations and physical restrictions.

INTRODUCTION

Numerical modeling of CSR and the space charge force is an important task in the design and study of various advanced accelerators, for instance in bunch compressors for X-ray free electron lasers where these effects can degrade beam quality and prevent lasing. Coherent motion of particles can be treated self-consistently by the macroparticle method or the Vlasov equation, either of which provides a description of the charge/current density $(\rho, \mathbf{J})$ at any time step. Given present and past values of the source $(\rho, \mathbf{J})$, we must solve Maxwell's equations to update the fields $(\mathbf{E}, \mathbf{B})$ for the next time step. This is an expensive step if one aspires to a realistic 3-D or 2-D model of the source. A way to economize is to project the realistic source from macroparticles or Vlasov onto a 1-D manifold, then solve the Maxwell equations with the resulting 1-D source. It is plausible that this gives a reasonable picture of gross qualities of CSR and the space charge force, if not the fine details.

Such a projection method is implemented in the useful code Elegant [1], but the code does not make a full solution of the Maxwell equations even with the 1-D source. Rather, for CSR it uses a formula of Saldin *et al.* [2] which approximates the effect of a line charge traversing a single bend, entering and leaving the bend on a straight orbit. The profile of charge density $\lambda(z)$ is invariant in time. Among more general codes working in higher dimensions, some such as CSRtrack [4] have a 1-D source option which may be less restrictive but still involves assumptions that might be avoided.

The object of the present work is to develop a 1-D model with a minimum of restrictions on the physics. The resulting theory has the following features:

(1) The beam has zero transverse extent, but vertical extent with an arbitrary but fixed vertical charge distribution. The longitudinal distribution is arbitrary, and may be time dependent.

(2) The bunch moves on an arbitrary planar orbit, so that the finite separation of bends is properly accounted for.

(3) Longitudinal and transverse forces are finite and are given in a neighborhood of the bunch by singularity-free 1-D integrals over retarded times.

(4) Since the full fields are computed with arbitrary γ, the space charge force is included.

(5) The theory is easily extended to account for shielding of radiation by the vacuum chamber, represented by infinite parallel plates [3].

After the bunch description is restricted as in item (1), the only additional assumptions are (A) *The maximum bunch dimension σ and the distance of the observation point from the bunch centroid must be small compared to the minimum bending radius*, and (B) *The phase space distribution in the beam frame must undergo little change during a time in which the bunch moves a distance $\mathcal{O}(\sigma)$.*

The charge and current densities satisfy the continuity equation to a certain approximation. Ideas to achieve more exact continuity are under study.

DESCRIPTION OF THE MODEL

Position in the laboratory frame is denoted by the vector $(Z, X, Y) = (\mathbf{R}, Y)$, where the (Z, X) plane is called horizontal, the Y direction vertical. An arbitrary reference trajectory in the horizontal plane is parametrized by arc length s and written as $\mathbf{R}_o(s) = \big(Z_o(s), X_o(s)\big)$. Unit vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$, tangent and normal to $\mathbf{R}_o$ respectively, satisfy the relations

$$\mathbf{t}(s) = \begin{pmatrix} Z_o'(s) \\ X_o'(s) \end{pmatrix}, \ \mathbf{n}(s) = \begin{pmatrix} -X_o'(s) \\ Z_o'(s) \end{pmatrix} \quad (1)$$

$$\mathbf{t}'(s) = -\kappa(s)\mathbf{n}(s) \,, \quad \mathbf{n}'(s) = \kappa(s)\mathbf{t}(s) \quad (2)$$

$$\kappa(s) = \pm 1/\rho(s) \,, \quad 1/\rho(s) = |\mathbf{R}_o''(s)| \quad (3)$$

The signed curvature κ is the reciprocal of the radius of curvature ρ, modulo a sign. The sign of κ is determined by equations (2).

An arbitrary point in the horizontal plane can be represented in Frenet-Serret coordinates (s, x) as $\mathbf{R} = \mathbf{R}_o(s) + x\mathbf{n}(s)$. The reference particle representing unperturbed motion has speed βc, and has location $\mathbf{R}_o(\beta ct)$. To locate points close to the reference particle it is convenient to replace s by the distance (in arc length) to the reference

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particle, namely $z = s - \beta u$, $u = ct$. Now when z and x are small compared to the radius of curvature, as assured by assumption (A), we can expand the representation of a point $\mathbf{R}$ as follows using (2,3):

$$\begin{aligned}\mathbf{R} &= \mathbf{R}_o(z+\beta u) + x\mathbf{n}(z+\beta u)\\ &= \mathbf{R}_o(\beta u) + \mathbf{t}(\beta u)z + \mathbf{n}(\beta u)x + \mathcal{O}((z^2, xz)/\rho)\end{aligned} \tag{4}$$

We refer to (z, x) as *beam frame coordinates*, and write $\mathbf{r} = (z, x)$. Keeping just the linear terms, we have a good approximation for the lab-to-beam frame transformation,

$$z = \mathbf{t}(\beta u)\cdot(\mathbf{R}-\mathbf{R}_o(\beta u)) , \quad x = \mathbf{n}(\beta u)\cdot(\mathbf{R}-\mathbf{R}_o(\beta u)) . \tag{5}$$

The solution of Maxwell's equations is carried out in the laboratory frame, in which the charge and current densities are denoted by $\rho_L(\mathbf{R}_L, u),\ \mathbf{J}_L(\mathbf{R}_L, u),\ \mathbf{R}_L = (\mathbf{R}, Y)$. A causal solution of the wave equation $\triangle F - \partial^2 F/\partial u^2 = \mathcal{S}$ with source $\mathcal{S}(\mathbf{R}_L, u)$ is

$$F(\mathbf{R}_L, u) = \frac{1}{4\pi}\int d\mathbf{R}_L' \frac{\mathcal{S}(\mathbf{R}_L', u - |\mathbf{R}_L' - \mathbf{R}_L|)}{|\mathbf{R}_L' - \mathbf{R}_L|} . \tag{6}$$

For scalar and vector potentials $\phi,\ \mathbf{A}$ in the Lorenz gauge the sources are $(\rho_L/\epsilon_o,\ mu_o\mathbf{J}_L) = Z_0(c\rho_L,\ \mathbf{J}_L/c)$, where Z_0 is the impedance of free space.

Details of the vertical distribution of charge are probably not very important for CSR, since bending of our planar orbits is independent of Y. Consequently we simplify by taking a factored form for the densities, $\rho_L(\mathbf{R}_L, u) = H(Y)\tilde{\rho}_L(\mathbf{R}, u)$, and similarly for $\mathbf{J}_L$, where $H(Y)$ is an arbitrary time-independent function with $\int H(Y)dY = 1$. According to the analysis in [5], under assumptions (A),(B) the lab frame densities are related to beam frame densities by approximate equations as follows:

$$\begin{aligned}&\tilde{\rho}_L(\mathbf{R}, u) = \rho(\mathbf{r}, \beta u) \ , \ \tilde{\mathbf{J}}_L(\mathbf{R}, u) = \mathbf{j}(\mathbf{r}, \beta u) =\\ &\beta c[\rho(\mathbf{r}, \beta u)\mathbf{t}(\beta u) + \tau(\mathbf{r}, \beta u)\mathbf{n}(\beta u)],\\ &\rho(\mathbf{r}, s) = Q\int f(\mathbf{r}, \mathbf{p}, s)d\mathbf{p} \ ,\\ &\tau(\mathbf{r}, s) = Q\int p_x f(\mathbf{r}, \mathbf{p}, s)d\mathbf{p} \ ,\end{aligned} \tag{7}$$

where $f(\mathbf{r}, \mathbf{p}, s)$ is the beam frame phase space density, normalized to unit integral, Q is the total charge, $p_x = dx/ds$ and $p_z = (\gamma - \gamma_o)/\gamma_o$. Since $\rho(\mathbf{r}, \beta u)$ and $\tau(\mathbf{r}, \beta u)$ are concentrated in $\mathbf{r}$ near $\mathbf{R}_o(\beta u)$, it is valid to use in (7) the linear relation of $\mathbf{r}$ to $\mathbf{R}$ as given in (5). The transverse current $\beta c\tau\mathbf{n}$ is usually negligible, but we retain it for generality.

The 1-D model is defined by concentrating the beam frame densities at the mean value of x,

$$\begin{aligned}\rho(z, x, \beta u) &\rightarrow \delta(x - \bar{x}(u))\int \rho(z, x, \beta u)\\ &\doteq \delta(x - \bar{x}(u))\rho(z, \beta u) \ ,\end{aligned} \tag{8}$$

and similarly for $\mathbf{j}$. Should this step appear as too radical, one could divide space into bins of p_z (or even 2-D bins of p_z and p_x) and use a reduction such as (8) for the contribution of each bin B_i. Then the resulting lab frame density is

$$\tilde{\rho}_L(\mathbf{R}, u) = \sum_i H_i(Y)\rho_i(z, \beta u)\delta(x - \bar{x}_i(u)) \ . \tag{9}$$

We proceed to find the fields for the i-th bin, dropping subscript i, and putting $\bar{x} = 0$ to reduce clutter. It will be obvious how to restore $\bar{x}$ if needed. Fields will be expressed in terms of the longitudinal densities

$$\rho(z, s), \ \ \mathbf{j}(z, s) = \beta c\big[\rho(z, s)\mathbf{t}(s) + \tau(z, s)\mathbf{n}(s)\big]. \tag{10}$$

Derivatives will be notated by $\mathbf{D}_1 = \partial/\partial z,\ \mathbf{D}_2 = \partial/\partial s$. We suppose that $\mathbf{D}_1^2\rho$ and $\mathbf{D}_1^2\tau$ are continuous.

FIELD COMPUTATION

Again downplaying the importance of Y-dependence, we shall average the fields in Y with respect to the distribution H. This can be done at the level of potentials rather than fields. Applying (6) we change integration variable from Y' to $\xi = Y' - Y$ to obtain

$$\begin{aligned}\langle\phi\rangle(\mathbf{R}, u) &= \frac{1}{4\pi\epsilon_o}\int \chi(\xi)d\xi\\ &\cdot\int d\mathbf{R}'\frac{\tilde{\rho}_L(\mathbf{R}', u - [(\mathbf{R}'-\mathbf{R})^2 + \xi^2]^{1/2})}{[(\mathbf{R}'-\mathbf{R})^2 + \xi^2]^{1/2}} ,\end{aligned} \tag{11}$$

and a similar formula for $\langle\mathbf{A}\rangle$, where $\chi(\xi) = \int H(Y)H(Y+\xi)dY$. Henceforth we write $\phi, \mathbf{A}$ for $\langle\phi\ \rangle, \langle\mathbf{A}\rangle$.

To evaluate the $\mathbf{R}'$-integral in (11) it is convenient to use polar coordinates centered at $\mathbf{R}$ and then use the retarded time v in place of the radial coordinate [3]:

$$\mathbf{R}' = \mathbf{R} + \zeta\mathbf{e}(\theta) \ , \ v = u - (\zeta^2 + \xi^2)^{1/2} \ , \tag{12}$$

where $\mathbf{e}(\theta) = (\cos\theta, \sin\theta)$. This gets rid of the vanishing denominator and also avoids the problem of computing retarded times. The transformed integral is

$$\phi(\mathbf{R}, u) = \frac{1}{4\pi\epsilon_o}\int \chi(\xi)d\xi \ \cdot \tag{13}$$

$$\int_{-\infty}^{u-|\xi|} dv\int_0^{2\pi}\tilde{\rho}_L(\mathbf{R} + \zeta\mathbf{e}(\theta), v)d\theta \ , \tag{14}$$

with $\zeta = [(u-v)^2 - \xi^2]^{1/2}$. Substituting the 1-D distribution we have

$$\begin{aligned}\phi(\mathbf{R}, u) &= \frac{1}{4\pi\epsilon_o}\int \chi(\xi)d\xi\int_{-\infty}^{u-|\xi|} dv\int_0^{2\pi} d\theta \ \cdot\\ &\delta(x_c + \zeta\mathbf{e}(\theta)\cdot\mathbf{n}(\beta v))\ \rho(z_c + \zeta\mathbf{e}(\theta)\cdot\mathbf{t}(\beta v), \beta v) \ ,\end{aligned} \tag{15}$$

where x_c and z_c are components of the chord from the retarded point to the observation point,

$$x_c(\mathbf{R},\beta v)=\mathbf{n}(\beta v)\cdot(\mathbf{R}-\mathbf{R}_o(\beta v))\ ,$$
$$z_c(\mathbf{R},\beta v)=\mathbf{t}(\beta v)\cdot(\mathbf{R}-\mathbf{R}_o(\beta v))\ . \quad (16)$$

Taking $\zeta\mathbf{e}\cdot\mathbf{n}$ as variable of integration in place of θ, and noting $(\mathbf{e}\cdot\mathbf{n})^2+(\mathbf{e}\cdot\mathbf{n})^2=1$, we can carry out the θ-integral. Taking care to get the right signs of $\mathbf{e}\cdot\mathbf{n}$ and $\mathbf{e}\cdot\mathbf{t}$ in the four quadrants of θ we find

$$\frac{1}{4\pi\epsilon_o}\int\chi(\xi)d\xi\int_{-\infty}^{u-\epsilon(x,\xi)}\frac{\bar\rho(\mathbf{R},u,v,\xi)}{\sqrt{b^2-\xi^2}}dv\ , \quad (17)$$

where

$$b=\sqrt{(u-v)^2-x_c^2},\quad x=x_c(\mathbf{R},\beta u), \quad (18)$$
$$\epsilon(x,\xi)=\sqrt{x^2+\xi^2}\ , \quad (19)$$
$$\bar\rho(\mathbf{R},u,v,\xi)=\rho(z_c-\sqrt{b^2-\xi^2},\beta v)+\rho(z_c+\sqrt{b^2-\xi^2},\beta v)\ . \quad (20)$$

The upper limit of v is less than it was in (15), because the δ function does not hit for all $v<u-|\xi|$. The upper limit in (17) is a good approximation to the precise upper limit, justified by assumption (A). Notice that x is just the transverse component of the displacement of the observation point from the reference particle.

The next step is to reverse the order of v and ξ integrals. Suppose that $H(Y)=0,\ |Y|>h/2$, implying that $\chi(\xi)=0,\ |\xi|>h$. Then (17) takes the form

$$\frac{1}{4\pi\epsilon_o}\Big[\int_{-\infty}^{u-\epsilon(x,h)}dv\int_{-h}^{h}d\xi+\int_{u-\epsilon(x,h)}^{u-\epsilon(x,0)}dv\int_{-b}^{b}d\xi\Big]\cdot\frac{\chi(\xi)}{\sqrt{b^2-\xi^2}}\bar\rho(\mathbf{R},u,v,\xi))\ . \quad (21)$$

By an approximation based on Assumption A, derived from Taylor expansions, we can replace the small argument ξ of $\bar\rho$ by h in the first integral and by 0 in the second. At a consistent level of approximation, the argument x_c of b can be replaced by x when $v\geq u-\epsilon(x,h)$. This means that the first integral has $b>h$ and the second $0\leq b\leq h$. Now the ξ-integral is the known function

$$\Phi(b,h)=\int_{-\bar\xi}^{\bar\xi}\frac{\chi(\xi)d\xi}{\sqrt{b^2-\xi^2}}\ ,\quad \bar\xi=\begin{cases}h\ , & b>h\\ b\ . & 0\leq b\leq h\end{cases} \quad (22)$$

In the square step model with $H(Y)=\mathrm{const.},\ |Y|<h/2$, we have

$$b>h:$$
$$\Phi(b,h)=\frac{2}{h}\left[\sin^{-1}\frac{h}{b}+\frac{1}{h}([b^2-h^2]^{1/2}-b)\right]\ ,$$
$$0\leq b\leq h:$$
$$\Phi(b,h)=\frac{1}{h}\left[\pi-\frac{2b}{h}\right]\ . \quad (23)$$

Here Φ and $\partial\Phi/\partial b$ are continuous at $b=h$, as is true in more general models of H.

After analogous reductions for $\mathbf{A}$ we are prepared to compute $\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/\partial t$ and $B_Y=(\nabla\times\mathbf{A})_Y$. It is useful to change the variable of integration from v to $w=u-v$ in the integral for $\mathbf{A}$, before taking $\partial/\partial t$. Results are stated in terms of the following definitions, where f can be either ρ or $\mathbf{j}$. For $v<u-\epsilon(x,h)\ \ (b>h)$,

$$f_\pm(\mathbf{R},u,v)=f(z_c-\sqrt{b^2-h^2},\beta v)\pm f(z_c+\sqrt{b^2-h^2},\beta v). \quad (24)$$

For $u-\epsilon(x,h)\leq v\leq u-|x|\ \ (0\leq b\leq h)$,

$$f_+(\mathbf{R},u,v)=2f(z_c,\beta v)\ ,\quad f_-(\mathbf{R},u,v)=0\ . \quad (25)$$

In summary the fields near the reference particle are

$$\mathbf{E}(\mathbf{R},u)=\frac{Z_o}{4\pi}\int_{-\infty}^{u-|x|}dv\Big[\frac{x_c}{b}\frac{\partial\Phi}{\partial b}(c\rho_+\mathbf{n}+x_c'\mathbf{j}_+)$$
$$-\Phi\mathbf{D}_1\big[c\rho_+\mathbf{t}+z_c'\mathbf{j}_++\frac{x_c}{\sqrt{b^2-h^2}}(c\rho_-\mathbf{n}+x_c'\mathbf{j}_-)\big]$$
$$-\Phi\beta\mathbf{D}_2\mathbf{j}_+\Big]+\frac{Z_oc}{2\pi}\frac{x}{|x|}\mathbf{n}(\beta u)\Phi(0,h)\rho(z_c,\beta v)\big|_{u-|x|}, \quad (26)$$

$$B_Y(\mathbf{R},u)=\frac{Z_o}{4\pi c}\int_{-\infty}^{u-|x|}dv\Big[-\frac{x_c}{b}\frac{\partial\Phi}{\partial b}\mathbf{n}\times\mathbf{j}_+$$
$$+\Phi\mathbf{D}_1\big[\mathbf{t}\times\mathbf{j}_++\frac{x_c}{\sqrt{b^2-h^2}}\mathbf{n}\times\mathbf{j}_-\big]\Big]_Y$$
$$-\frac{Z_o}{2\pi c}\frac{x}{|x|}\Phi(0,h)(\mathbf{n}(\beta u)\times\mathbf{j}(z_c,\beta v))_Y\big|_{u-|x|}\ . \quad (27)$$

Under the integrals $\mathbf{t}$ and $\mathbf{n}$ are evaluated at βv, and prime $(')$ means $\partial/\partial v$, from which it follows that

$$x_c'=\beta\kappa(\beta v)z_c\ ,\quad z_c'=-\beta(1+\kappa(\beta v)x_c)\ . \quad (28)$$

The other components of the integrand are defined in (10,24,25,16,18,22). Transverse fields blow up as $1/h$ in the limit of small h, whereas the longitudinal electric field diverges as $\ln(h)$, a typical behavior for the space charge effect. In the transverse Lorentz force the final terms in (26) and (27) combine to give a term $\mathcal{O}(1/(h\gamma^2))$. Also, there is a cancelation producing $1/\gamma^2$ in the tangential part of $\mathbf{D}_1(c\rho_+\mathbf{t}+z_c'\mathbf{j}_+)$, which identifies a space charge effect. For non-zero h the integrals are free of singularities and can be evaluated numerically, but care is required in handling a sharp peak of the integrand near the upper limit.

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