

# TRANSVERSE EFFECTS DUE TO RANDOM DISPLACEMENT OF RESISTIVE WALL SEGMENTS AND FOCUSING ELEMENTS

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## Abstract

In this paper, we study the single bunch transverse beam dynamics in the presence of random displacements of resistive wall segments and focusing elements. Analytical formulas are obtained for long-range resistive wall wake, together with numerical results for short-range resistive wall wake. The results are applied to the LCLS project and some other proposed accelerators.

## EQUATION OF MOTION AND GENERAL SOLUTION

The lateral displacement of a beam under the influence of focusing and a transverse wakefield is given by [1-3]

$$\frac{\partial^2}{\partial s^2} x(s, \zeta) + \kappa^2 [x(s, \zeta) - d_f(s)] =$$

$$\varepsilon \int_0^\zeta d\zeta_1 w(\zeta - \zeta_1) F(\zeta_1) [x(s, \zeta_1) - d_w(s)],$$

where  $s$  is the distance from the entrance of the transport element;  $\zeta = t/t_0 - s/s_0$  measures the longitudinal position from the front of the bunch;  $\kappa$  is the transverse focusing strength;  $F(\zeta) = I(\zeta)/\bar{I}$ , the current form factor is the instantaneous current divided by the average current;  $d_f(s)$  and  $d_w(s)$  are the lateral displacements of the focusing element and the generator of the wakefield, respectively;  $w(\zeta)$  represents the time dependence of the wakefield while  $\varepsilon$  is its strength. When the wakefield is due to the resistive wall we have [4,5]

$$w(\zeta) = \sqrt{2\pi} e^{-\zeta} \left[ -\frac{2}{3} \cos(\sqrt{3}\zeta) + \frac{2}{\sqrt{3}} \sin(\sqrt{3}\zeta) \right] + \frac{16}{\sqrt{\pi}} \int_0^\infty dx \frac{\exp[-x^2\zeta]}{x^6 + 8},$$

with  $\varepsilon = \frac{4s_0^2}{\sqrt{2\pi}\gamma b^4} \frac{\bar{I}}{I_A}$ ,  $s_0 = \left( \frac{2b^2}{Z_0\sigma_c} \right)^{1/3} = ct_0$ ,  $b$  is the pipe radius, and  $I_A = 4\pi\epsilon_0 mc^3 / e \approx 17,045$  Amp.

And in particular, for  $\zeta \gg 1$  we have  $w(\zeta) = \zeta^{-1/2}$ , and

for  $\zeta \ll 1$  we have  $w(\zeta) = 2\sqrt{2\pi}\zeta$ .

In this paper we assume that the beam enters the transport system without lateral or angular displacement, but that the transport system (beam pipe and/or focusing elements) are randomly displaced with respect to the nominal beam line. Under those assumptions, the displacement of the beam as it propagates along the transport system is given by [3,6]

$$x(s, \zeta) = - \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) i_n(\kappa, s) * d_w(s) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) i_n(\kappa, s) * d_f(s),$$

$$x'(s, \zeta) = - \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) j_n(\kappa, s) * d_w(s) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) j_n(\kappa, s) * d_f(s).$$

The functions  $i_n(\kappa, s)$  and  $j_n(\kappa, s)$  are defined in terms of the Bessel functions of order integer plus one half

$$i_n(\kappa, s) = \frac{1}{n!} \left( \frac{s}{2\kappa} \right)^n \frac{1}{\kappa} \sqrt{\frac{\pi\kappa s}{2}} J_{n+1/2}(\kappa s),$$

$$j_n(\kappa, s) = \frac{1}{n!} \left( \frac{s}{2\kappa} \right)^n \sqrt{\frac{\pi\kappa s}{2}} J_{n-1/2}(\kappa s).$$

The functions  $f_n(\zeta)$  are defined by the recursions

$$f_0(\zeta) = 1,$$

$$f_{n+1}(\zeta) = \int_0^\zeta d\zeta_1 f_n(\zeta_1) F(\zeta_1) w(\zeta - \zeta_1),$$

and

$$i_n(\kappa, s) * d(s) = \int_0^s du i_n(\kappa, u) d(s-u)$$

is the convolution of  $i_n(\kappa, s)$  and  $d(s)$ .

## AUTOCORRELATION FUNCTION AND MEAN-SQUARE DISPLACEMENT

Ignoring for the moment the displacement of the focusing elements, and assuming that the transport system is made of segments of length  $l_w$  that are randomly displaced from the centerline with an rms displacement  $d_{w0}$  so that the autocorrelation function of the displacements of the wakefield elements can be modeled by

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$$R_{d_w}(u-v) = d_{w0}^2 l_w \delta(u-v), \quad (8)$$

the mean-square displacement of the beam is then [6]

$$\frac{\overline{x^2}(s, \zeta)}{d_{w0}^2 l_w} = \frac{\varepsilon^2}{\kappa^3} \sum_{N=0}^{\infty} \left( \frac{\varepsilon}{2\kappa^2} \right)^N \sum_{k=0}^N y_{k, N-k}(\kappa s) f_{k+1}(\zeta) f_{N+1-k}(\zeta), \quad (9)$$

where

$$y_{m,n}(t) = \frac{1}{m!n!} \int_0^t du u^{m+n} \left( \frac{\pi u}{2} \right) J_{m+1/2}(u) J_{n+1/2}(u). \quad (10)$$

In particular

$$y_{0,0}(t) = \frac{t}{2} - \frac{1}{2} \sin 2t, \quad (11)$$

$$y_{1,0}(t) = y_{0,1}(t) = \frac{t}{2} + \frac{t}{4} \cos 2t - \frac{3}{8} \sin 2t.$$

Similarly, if the focusing elements are randomly displaced with an rms displacement  $d_{f0}$  and auto-correlation function  $R_{d_f}(u-v) = d_{f0}^2 l_f \delta(u-v)$ , the mean-square displacement of the beam is [6]

$$\frac{\overline{x^2}(s, \zeta)}{d_{f0}^2 l_f} = \kappa \sum_{N=0}^{\infty} \left( \frac{\varepsilon}{2\kappa^2} \right)^N \sum_{k=0}^N y_{k, N-k}(\kappa s) f_k(\zeta) f_{N-k}(\zeta). \quad (12)$$

## LONG-RANGE WAKE AND UNIFORM CURRENT DISTRIBUTION

Equations (9) and (12) allow determination of the rms displacement of the beam as it propagates, once the functions  $f_n(\zeta)$  are known. For arbitrary wakefield and current distribution in the beam they would have to be calculated numerically from Eq. (6). However, for a uniform current distribution and in the case of the long-range resistive-wall wake field (bunch length  $\gg s_0$ ) where  $w(\zeta) = \zeta^{-1/2}$ , they can be easily obtained analytically and would provide a rough estimate of the displacement and distortion of the beam. Under these assumptions the functions  $f_n(\zeta)$  are given by [7]

$$f_n(\zeta) = \frac{[\pi \zeta]^{n/2}}{\Gamma(\frac{n}{2} + 1)}. \quad (13)$$

From Eq. (9) the rms beam displacement is then

$$\overline{x^2}(s, \zeta) = \frac{2d_{w0}^2 l_w \varepsilon^2 \zeta s}{\kappa^2} \left[ 1 + \frac{\pi}{2} \left( \frac{\varepsilon \zeta^{1/2}}{\kappa^2} \right) + \dots \right]. \quad (14)$$

Equation (14) assumes that the focusing is sufficiently strong that the total transport system is much longer than the focusing period. In the opposite case where focusing is absent, the rms beam displacement is given by [6]

$$\overline{x^2}(s, \zeta) = \frac{d_{w0}^2 l_w \varepsilon^2 \zeta s^3}{12} \left[ 1 + \frac{8\pi}{5} (\varepsilon \zeta^{1/2} s^2) + \dots \right]. \quad (15)$$

These results will be applied to two accelerators, LCLS and FLASH whose beam parameters are shown in Table 1. Since in both cases we have  $\zeta_B \gg 1$ , the use of the long-range wakefield  $w(\zeta) = \zeta^{-1/2}$  is justified. The rms

displacement of the tail of a bunch at the end of the transport system is obtained from Eq. (14) and is given by

$$\sqrt{\overline{x^2}(L, \zeta_B)} = \frac{d_{w0} \varepsilon}{\kappa} \sqrt{2l_w \zeta_B L} \quad (16)$$

Table 1: Summary of the parameters for the LCLS and FLASH projects

	LCLS	FLASH
Bunch length $t_B$ ( $\sqrt{2\pi}$ rms) (fs)	193	419
Pipe radius $b$ (mm)	3	4.75
Pipe length $L$ (m)	150	30
Conductivity $\sigma_c$ ( $10^7 \Omega^{-1} \text{m}^{-1}$ )	6	3.47
Focusing Strength $\kappa$ ( $\text{m}^{-1}$ )	1/18	1/4
Beam Energy (GeV)	14.35	1
Bunch Charge (nC)	1	1
$s_0$ ( $\mu\text{m}$ )	9.3	15.1
$t_0 = s_0 / c$ (fs)	31	50
Normalized bunch length $\zeta_b = t_B / t_0$	6.2	8.4
$\varepsilon$ ( $\text{m}^{-2}$ )	$1.8 \cdot 10^{-5}$	$5.1 \cdot 10^{-5}$

Assuming that the transport systems are made of sections of length  $l_w = 10$  m that are randomly displaced from each other by  $d_{w0} = 100$   $\mu\text{m}$ , then the rms displacement of the tail of a bunch at the end of the transport is, from Eq. (16), 4.4  $\mu\text{m}$  and 1.4  $\mu\text{m}$  for LCLS and FLASH, respectively.

If instead of the wakefield-generating elements, the focusing elements are displaced, then the mean-square displacement of the bunch is given by

$$\overline{x^2}(s, \zeta) = \frac{d_{f0}^2 l_f \kappa^2 s}{2} \left[ 1 + 2 \left( \frac{\varepsilon \zeta^{1/2}}{\kappa^2} \right) + \dots \right]. \quad (17)$$

The rms displacement of the tail of the bunch ( $\zeta = \zeta_B$ ) from the front ( $\zeta = 0$ ) is

$$\sqrt{\overline{x^2}(L, \zeta_B)} - \sqrt{\overline{x^2}(L, 0)} = \frac{d_{f0} \varepsilon}{\kappa} \sqrt{\frac{l_f L \zeta_B}{2}}. \quad (18)$$

With the same assumptions as above the rms displacement of the tail of a bunch with respect to the head would be, from Eq. (18), 2.2  $\mu\text{m}$  for LCLS and 0.72  $\mu\text{m}$  for FLASH.

## NON-UNIFORM CURRENT DISTRIBUTION

The results of the previous section can easily be extended to non-uniform current distributions while retaining the long-range wakefunction since the bunches for the accelerators under consideration satisfy  $t_B \gg t_0$ . If the wakefield is small enough, Eq. (9) reduces to

$$\overline{x^2}(s, \zeta) = \frac{d_{w0}^2 l_w \varepsilon^2 s}{2\kappa^2} f_1^2(\zeta), \quad (19)$$

with

$$f_1(\zeta) = \int_0^\zeta d\zeta_1 F(\zeta_1) \sqrt{\zeta - \zeta_1}. \quad (20)$$

These functions can be calculated for a variety of current distributions [7] and, in particular, for the profile shown in Fig. 1 and given by

$$F(\zeta) = 6 \left( \frac{\zeta}{\zeta_B} - \frac{1}{2} \right)^2 + \frac{1}{2}, \quad (21)$$

we have

$$f_1(\zeta) = \frac{4}{5} \zeta_B^{1/2} \left( \frac{\zeta}{\zeta_B} \right)^{1/2} \left[ 5 - 10 \frac{\zeta}{\zeta_B} + 8 \left( \frac{\zeta}{\zeta_B} \right)^2 \right]. \quad (22)$$

From Eq. (19) it is clear that the function  $f_1(\zeta)$  determines the rms distortion of the beam. The rms bunch distortion for a uniform current distribution and a “double-horn” current distribution, given by Eq. (21) and shown in Fig. 1, are shown in Fig. 2.

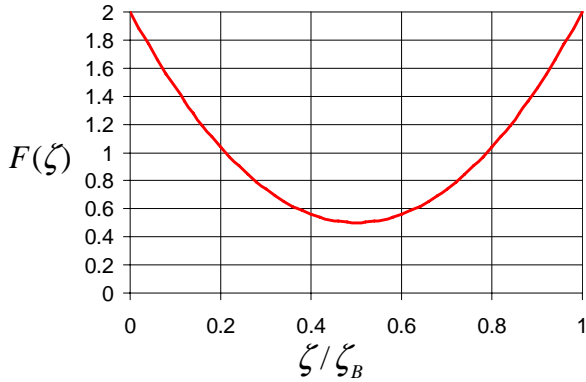


Figure 1: “Double-horn” current distribution given by Eq. (21).

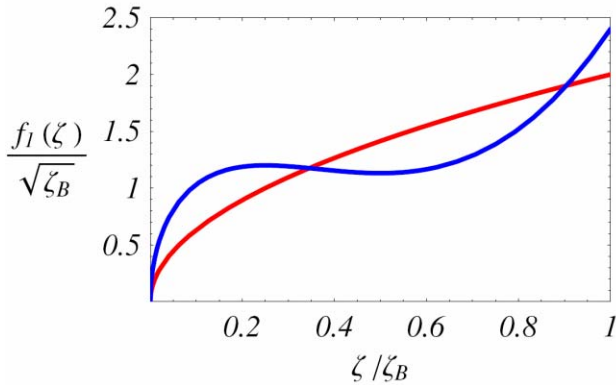


Figure 2: rms bunch distortion for a uniform (red) and “double-horn” current distribution (blue) for a long-range wakefield  $w(\zeta) = \zeta^{-1/2}$ .

## SHORT-RANGE WAKEFIELD

The rms distortion  $f_1(\zeta)$  of a bunch can be obtained for an arbitrary current profile and wakefunction from Eq. (6). In particular, it can be calculated for the full resistive-wall wakefunction given by Eq. (2). Figure 3 shows the bunch distortions for the uniform and “double-horn” current distribution shown in Fig. 1 for the wakefunction given by Eq. (2). In that plot it is assumed

that the normalized bunch length  $\zeta_B = t_B / t_0 = 6.2$  which is representative of LCLS. This plot can be compared with that of Fig. 2 which assumed the long-range wakefield  $w(\zeta) = \zeta^{-1/2}$ .

Figures 2 and 3 illustrate how the bunch distortion due to random displacement of the resistive wall segments is affected by the charge distribution in the bunch and the time dependence of the wakefield. Knowledge of the bunch distortion can then be used to estimate the emittance growth due to the resistive-wall wakefield.

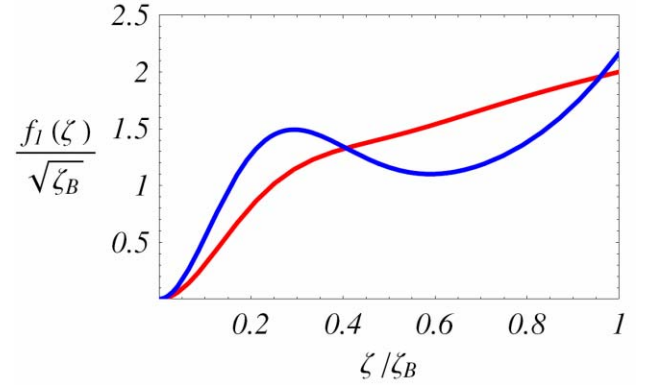


Figure 3: rms bunch distortion for a uniform (red) and “double-horn” current distribution (blue) for the wakefield given by Eq. (2) and a normalized bunch length  $\zeta_B = t_B / t_0 = 6.2$ .

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