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SOME INVARIANT PROPERTIES OF THE REAL HADAMARD MATRIX*

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ABSTRACT

Applications of well-known matrix theory reveal some interesting and possibly useful invariant properties of the real Hadamard matrix and transform (including the Walsh matrix and transform).

Subject to certain conditions that can be fulfilled for many orders of the matrix, the space it defines can be decomposed into two invariant subspaces defined by two real, singular, mutually orthogonal (although not self-orthogonal) matrices, which differ from the Hadamard matrix only on the principal diagonal. They are their own Hadamard transforms, within a scalar multiplier, so their columns (or rows) are the eigenvectors of the Hadamard matrix.

This relationship enables us to determine the eigenvalues of the Hadamard matrix, and to construct its Jordan normal form. The transformation matrix for converting the one to the other is equal to the sum of the Hadamard matrix and the Jordan form matrix. Unfortunately, therefore, it appears to be more complicated structurally than the Hadamard matrix itself, and does not lead to a simple method of generating the Hadamard matrix directly from its easily constructed Jordan form.

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I. INTRODUCTION

Applications of certain well-known matrix theory reveal some very interesting and possibly useful invariant properties of the real Hadamard matrix H, and of the Hadamard transform

$$F = H f \tag{1}$$

of a discrete-variable function f, where H is a square orthogonal matrix of order 4m (where m is a positive integer) each of whose elements is either 1 or -1, and F and f are column matrices of 4m elements.

In this paper the Hadamard matrices include the Walsh matrices, which are of order 2^n (where n is a positive integer), and the results can be modified to apply more explicitly to the Walsh matrix and Walsh transform by making the simple substitution

$$4m = 2^n \; ; \; n > 1 \tag{2}$$

and by writing W in place of H.

Subject to certain conditions specified in Section II, which can be fulfilled for H of most orders 200 or less (and of indefinitely many higher orders), the space defined by H can be decomposed into two invariant subspaces defined by two real matrices S_1 and S_2, whose elements differ from those of H only on the principal diagonal. Although not self-orthogonal, S_1 and S_2 are mutually orthogonal, so the scalar product of their transforms F_1 and F_2 of f is zero.

The sum and difference of S_1 and S_2 are respectively H and, within a scalar multiplier, the identity matrix I (i.e., a scalar matrix), while the sum and difference of F_1 and F_2 are respectively F and, within a scalar multiplier, f.

Both S_1 and S_2 are singular, so they cannot be inverted. But each is its own Hadamard transform, within a scalar multiplier, so their columns (or rows) are the eigenvectors of H. From this fact and applicable matrix theory we can show that H has 4m real eigenvalues, all of the same modulus, but half of them positive and half negative.

From these eigenvalues we can construct the Jordan normal form J of H. It was hoped that the transformation matrix T (and its inverse) for converting H to J, or J to H, might be of some simple structure that would lead to a new method of generating H from J. Unfortunately T is equal to the sum of H and J, and therefore appears even more complicated structurally than H itself.

II. CONDITIONS

It is customary to arrange the Walsh functions (the rows of H of order 2^n) in some standard order, so that H is symmetric and possesses certain other desirable structural properties [1]. Although the more general Hadamard matrices (of order $4m \neq 2^n$) do not possess some of these properties, the author has shown that any such matrix constructed by the methods of R.E.A.C. Paley, and thus of any order 200 or less except at most six that have been constructed by other methods (92, 116, 156, 172, 184, 188) can be converted by one or two elementary matrix operations to a form that, like the standard forms of Walsh matrices, is symmetric and has zero trace [2].

Throughout this paper, therefore, it is assumed that H is of that form, with the reservation that the results may not apply to the six excepted orders. The invariant properties derived, since they are not affected by elementary matrix operations, will remain true for any other form of H, either asymmetric or having nonzero trace or both, obtained from the canonical form by any combination of permuting rows and/or columns and multiplying rows and/or columns by -1. However, the derivations herein would be extremely difficult if not impossible without the stipulated assumption.

III. THE SUBSPACES OF H

In matrix theory a vector subspace V' of a vector space V is defined as a collection of vectors v' with the property that any linear combination of them belongs only to V', and it is shown that V can be decomposed into two subspaces V' and V'' if every vector $v \in V$ can be represented as a sum

$$v = v' + v'' \; ; \; v' \in V' \; , \; v'' \in V'' \tag{3}$$

and if V' and V'' have only the null vector in common [3].

Thus the space defined by H (of order 4m) can be decomposed into two subspaces defined by the square (not necessarily self-orothogonal) matrices S_1 and S_2 (of order 4m) if

$$S_1 + S_2 = H \tag{4}$$

$$S_1 S_2 = N \tag{5}$$

where N is the null matrix. If

$$S_1 = \tfrac{1}{2} H + m^{\frac{1}{2}} I \; ; \; m^{\frac{1}{2}} > 0 \tag{6}$$

$$S_2 = \tfrac{1}{2} H - m^{\frac{1}{2}} I \; ; \; m^{\frac{1}{2}} > 0 \tag{7}$$

where I is the identity matrix, then clearly (4) is satisfied. Since

$$H^2 = 4mI \tag{8}$$

as is now well known, then by (6) - (8)

$$S_1 S_2 = \tfrac{1}{4} H^2 - mI = N \tag{9}$$

so (5) is satisfied too. The subspaces defined by S_1 and S_2 are said to be orthogonal complements of one another [4].

It is also of interest that by (6) and (7)

$$S_1 - S_2 = 2m^{\frac{1}{2}} I = (4m)^{\frac{1}{2}} I \tag{10}$$

IV. INVARIANCE OF THE SUBSPACES

Again, in matrix theory a subspace $V' \subset V$ is said to be invariant with respect to an operator O if the operator carries any vector of the subspace into the same subspace [5], and consequently if

$$O\, V' \subset V' \tag{11}$$

We can show easily that H transforms S_1 into S_1 and S_2 into S_2, within a scalar multiplier in either case, and thus that the subspaces defined by S_1 and S_2 are essentially invariant with respect to H. By (6) and (8)

$$\begin{aligned} HS_1 &= H(\tfrac{1}{2} H + m^{\frac{1}{2}} I) \\ &= \tfrac{1}{2} H^2 + m^{\frac{1}{2}} H \\ &= 2m\, I + m^{\frac{1}{2}} H \\ &= 2m^{\frac{1}{2}} (\tfrac{1}{2} H + m^{\frac{1}{2}} I) \\ &= (4m)^{\frac{1}{2}} S_1 \end{aligned} \tag{12}$$

and similarly, by (7) and (8)

$$HS_2 = -\,(4m)^{\frac{1}{2}} S_2 \tag{13}$$

V. INVARIANCE OF THE TRANSFORMS F_1 AND F_2

What is true of the subspaces is true of the transforms, of course, since the latter are simply linear combinations of the vectors (i.e., of the rows of the matrices S_1 and S_2) defining the subspaces. If, corresponding to (1),

$$F_1 = S_1 f \tag{14}$$

$$F_2 = S_2 f \tag{15}$$

where the transforms F_1 and F_2 are column matrices of 4m elements, then by (14), (15), (4), and (1)

$$F_1 + F_2 = (S_1 + S_2) f = Hf = F \tag{16}$$

By (14) and (15)

$$F_1^t F_2 = (S_1 f)^t \; S_2 f \tag{17}$$

where the superscript t signifies the transpose of the matrix.

Inasmuch as H is stipulated to be symmetric, and inasmuch as I is symmetric, then since a sum of symmetric matrices is also symmetric, it follows from (6) and (7) that S_1 and S_2 are both symmetric, and thus that

$$S_1^t = S_1 \tag{18}$$

$$S_2^t = S_2 \tag{19}$$

It is well known that the transpose of a product of matrices is equal to the product of the transposes of the factor matrices in the reverse order [6], so by (18) and (9) we can rewrite (17) as

$$F_1^t F_2 = f^t S_1^t S_2 f = f^t S_1 S_2 f$$

$$= f^t N f = 0 \tag{20}$$

As vectors, therefore, F_1 and F_2 are mutually perpendicular.

It is also of interest that by (14), (15), and (10)

$$F_1 - F_2 = (S_1 - S_2) f = (4m)^{\frac{1}{2}} f \tag{21}$$

Finally, we can show easily that H transforms F_1 into F_1 and F_2 into F_2, within a scalar multiplier in either case, and thus that the transforms F_1 and F_2 are invariant with respect to H. By (14) and (12)

$$HF_1 = HS_1 f = (4m)^{\frac{1}{2}} S_1 f = (4m)^{\frac{1}{2}} F_1 \tag{22}$$

Similarly, by (15) and (13)

$$HF_2 = HS_2 f = -(4m)^{\frac{1}{2}} S_2 f = -(4m)^{\frac{1}{2}} F_2 \tag{23}$$

VI. THE SQUARES OF S_1 AND S_2

Unlike the square of H, as defined by (4), the squares of S_1 and S_2 are not scalar matrices. Instead, by (6) and (8)

$$S_1^2 = \left(\tfrac{1}{2} H + m^{\frac{1}{2}} I\right)^2$$

$$= \tfrac{1}{4} H^2 + m^{\frac{1}{2}} H + m I$$

$$= m^{\frac{1}{2}} H + 2m I$$

$$= 2m^{\frac{1}{2}} \left(\tfrac{1}{2} H + m^{\frac{1}{2}} I\right)$$

$$= (4m)^{\frac{1}{2}} S_1 \tag{24}$$

Similarly, by (7) and (8)

$$S_2^2 = -(4m)^{\frac{1}{2}} S_2 \tag{25}$$

Thus S_1 and S_2 are their own transforms, within a scalar multiplier in either case. Although they are mutually orthogonal, as shown by (9), clearly they are not self-orthogonal.

Comparing the right sides of (12) and (24), and of (13) and (25), we can observe the interesting and perhaps rather startling result that even though

$$H \neq S_1 \neq (4m)^{\frac{1}{2}} I \neq H \tag{26}$$

$$H \neq S_2 \neq -(4m)^{\frac{1}{2}} I \neq H \tag{27}$$

nevertheless

$$HS_1 = S_1^2 = (4m)^{\frac{1}{2}} S_1 \tag{28}$$

$$HS_2 = S_2^2 = -(4m)^{\frac{1}{2}} S_2 \tag{29}$$

VII. NORMALIZATION OF S_1 AND S_2

For some purposes it may be desired to normalize S_1 and S_2. If we define the normalized forms as

$$\mathcal{S}_1 = (4m)^{-\frac{1}{2}} S_1 \tag{30}$$

$$\mathcal{S}_2 = -(4m)^{-\frac{1}{2}} S_2 \tag{31}$$

then by (30) and (24)

$$\mathcal{S}_1^{\,2} = (4m)^{-1} S_1^2 = (4m)^{-\frac{1}{2}} S_1 = \mathcal{S}_1 \tag{32}$$

and by (31) and (25)

$$\mathcal{S}_2^{\,2} = (4m)^{-1} S_2^2 = -(4m)^{-\frac{1}{2}} S_2 = \mathcal{S}_2 \tag{33}$$

In the terminology of matrix theory $\mathscr{S}_1$ and $\mathscr{S}_2$ are idempotent [7]. By (32), (33), and (10)

$$\mathscr{S}_1^{\,2} + \mathscr{S}_2^{\,2} = \mathscr{S}_1 + \mathscr{S}_2 = (4m)^{-\frac{1}{2}} (S_1 - S_2) = I \tag{34}$$

It is also of interest that by (32), (33), and (4)

$$\mathscr{S}_1^{\,2} - \mathscr{S}_2^{\,2} = \mathscr{S}_1 - \mathscr{S}_2 = (4m)^{-\frac{1}{2}} (S_1 + S_2) = (4m)^{-\frac{1}{2}} H \tag{35}$$

VIII. THE SINGULARITY OF S_1 AND S_2

The transforms defined by (14) and (15) are in essence, and in contrast to that defined by (1), "one-way streets." Given F_1 and F_2, we can compute f by the relation

$$f = H^{-1} (F_1 + F_2) \tag{36}$$

obtained from (16), or by the relation

$$f = (4m)^{-\frac{1}{2}} (F_1 - F_2) \tag{37}$$

obtained from (21).

But given F_1 only or F_2 only, we cannot determine f. It is futile to multiply both sides of (14) by S_1^{-1} or (15) by S_2^{-1}, because we can show that S_1 and S_2 are singular. We might endeavor to treat (36) and (37) as a pair of simultaneous matrix equations, combine them in a way to eliminate whichever of F_1 and F_2 is not given, and solve for f in terms of the other. Unfortunately that process leads to an expression involving the inverse of the right side of (6) or (7), so it too is futile.

Although the task is rather laborious, the determinants of S_1 and S_2 can be computed conveniently by a well-known method involving the traces of the powers of S_1 and S_2 and what are called the Bôcher formulas [8], [9].

First, we compute the first 4m powers of S_1 and S_2. To simplify the notation slightly, in (24) and (25) let

$$(4m)^{\frac{1}{2}} = p \; ; \; p > 0 \tag{38}$$

where p is a constant. Then by (24)

$$S_1^{\ 2} = p\ S_1 \tag{39}$$

$$S_1^{\ 3} = S_1^{\ 2} S_1 = p\ S_1^{\ 2} = p^2 S_1 \tag{40}$$

$$S_1^{\ 4} = S_1^{\ 3} S_1 = p^2 S_1^{\ 2} = p^3 S_1 \tag{41}$$

$$\vdots$$

$$S_1^{\ a} = S_1^{\ a-1} S_1 = p^{a-2} S_1^{\ 2} = p^{a-1} S_1 \; ; \; a > 1 \tag{42}$$

where a is a positive integer. Similarly, by (25)

$$S_2^{\ 2} = -p\ S_2 \tag{43}$$

$$S_2^{\ 3} = S_2^{\ 2} S_2 = -p\ S_2^{\ 2} = p^2 S_2 \tag{44}$$

$$S_2^{\ 4} = S_2^{\ 3} S_2 = -p^2 S_2^{\ 2} = -p^3 S_2 \tag{45}$$

$$\vdots$$

$$S_2^{\ a} = S_2^{\ a-1} S_2 = (-p)^{a-2} S_2^{\ 2} = (-p)^{a-1} S_2 \; ; \; a > 1 \tag{46}$$

Next, we compute the traces of these powers of S_1 and S_2. As stipulated in Section II,

$$\text{tr } H = 0 \tag{47}$$

for any order 4m of H. We can readily see that for any order 4m of I

$$\text{tr } I = 4m \tag{48}$$

Since the trace of a sum of matrices is equal to the sum of the traces of the term matrices [10], [11], then by (6), (47), (48), and (38)

$$\mathrm{tr}\ S_1 = \tfrac{1}{2}\mathrm{tr}\ H + m^{\frac{1}{2}}\mathrm{tr}\ I = 4m^{\frac{3}{2}} = \tfrac{1}{2}p^3 \tag{49}$$

and by (7), (47), (48), and (38)

$$\mathrm{tr}\ S_2 = \tfrac{1}{2}\mathrm{tr}\ H - m^{\frac{1}{2}}\mathrm{tr}\ I = -4m^{\frac{3}{2}} = -\tfrac{1}{2}p^3 \tag{50}$$

For convenience let

$$\tau_{1a} = \mathrm{tr}\ S_1{}^a \ ; \ a \geqq 1 \tag{51}$$

$$\tau_{2a} = \mathrm{tr}\ S_2{}^a \ ; \ a \geqq 1 \tag{52}$$

Then by (39) - (42), (49), and (51)

$$\tau_{11} = \mathrm{tr}\ S_1 = \tfrac{1}{2}p^3 \tag{53}$$

$$\tau_{12} = \mathrm{tr}\ S_1{}^2 = p\ \mathrm{tr}\ S_1 = \tfrac{1}{2}p^4 \tag{54}$$

$$\tau_{13} = \mathrm{tr}\ S_1{}^3 = p^2\ \mathrm{tr}\ S_1 = \tfrac{1}{2}p^5 \tag{55}$$

$$\vdots$$

$$\tau_{1a} = \mathrm{tr}\ S_1{}^a = p^{a-1}\mathrm{tr}\ S_1 = \tfrac{1}{2}p^{a+2} \tag{56}$$

Similarly, by (43) - (46), (50), and (52)

$$\tau_{21} = \mathrm{tr}\ S_2 = -\tfrac{1}{2}p^3 \tag{57}$$

$$\tau_{22} = \mathrm{tr}\ S_2{}^2 = -p\ \mathrm{tr}\ S_2 = \tfrac{1}{2}p^4 \tag{58}$$

$$\tau_{23} = \mathrm{tr}\ S_2{}^3 = p^2\ \mathrm{tr}\ S_3 = -\tfrac{1}{2}p^5 \tag{59}$$

$$\vdots$$

$$\tau_{2a} = \mathrm{tr}\ S_2{}^a = (-p)^{a-1}\mathrm{tr}\ S_2 = \tfrac{1}{2}(-p)^{a+2} \tag{60}$$

Next, we compute the coefficients γ_{1a} of the characteristic polynomial of S_1, and the coefficients γ_{2a} of that of S_2, by means of the Bôcher formulas [8], [9]. By (53) - (56) and these formulas

$$\gamma_{11} = -\tau_{11} = -\frac{1}{2}p^3 = -\frac{p}{(1!)(2)}\left(p^2\right) \tag{61}$$

$$\gamma_{12} = -\frac{1}{2}\left(\gamma_{11}\tau_{11} + \tau_{12}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{4}\right)p^4\ (-p^2 + 2)$$

$$= -\frac{p^2}{(2!)(2^2)}\left(p^2\right)\left(2-p^2\right) \tag{62}$$

$$\gamma_{13} = -\frac{1}{3}\left(\gamma_{12}\tau_{11} + \gamma_{11}\tau_{12} + \tau_{13}\right)$$

$$= -\frac{1}{3}\left(\frac{1}{16}\right)p^5\left[-p^2\left(2-p^2\right) - 4p^2 + 8\right]$$

$$= -\frac{1}{6}\left(\frac{1}{8}\right)p^5\left[-p^2\left(2-p^2\right) + 4\left(2-p^2\right)\right]$$

$$= -\frac{p^3}{(3!)(2^3)}\left(p^2\right)\left(2-p^2\right)\left(4-p^2\right) \tag{63}$$

$$\vdots$$

$$\gamma_{1a} = -\frac{p^a}{(a!)(2^a)}\prod_{i=0}^{a-1}\left(2i-p^2\right) \tag{64}$$

Similarly, by (57) - (60) and these formulas

$$\gamma_{21} = -\tau_{21} = -\frac{(-p)}{(1!)(2)}\left(p^2\right) \tag{65}$$

$$\gamma_{22} = -\frac{(-p)^2}{(2!)(2^2)}\left(p^2\right)\left(2-p^2\right) \tag{66}$$

$$\gamma_{23} = -\frac{(-p)^3}{(3!)(2^3)}\left(p^2\right)\left(2-p^2\right)\left(4-p^2\right) \tag{67}$$

$$\vdots$$

$$\gamma_{2a} = -\frac{(-p)^a}{(a!)(2^a)}\prod_{i=0}^{a-1}\left(2i-p^2\right) \tag{68}$$

Finally, making use of (38), and of (64) and (68) with

$$a = 4m \tag{69}$$

we obtain the desired results

$$\det S_1 = (-1)^{4m}\gamma_{1,4m}$$

$$= -\frac{(4m)^{2m}}{(4m)!\,(2^{4m})}\prod_{i=0}^{4m-1}(2i-4m) \tag{70}$$

$$\det S_2 = (-1)^{4m}\gamma_{2,4m}$$

$$= -\frac{(-4m)^{2m}}{(4m)!\,(2^{4m})}\prod_{i=0}^{4m-1}(2i-4m) \tag{71}$$

Now, in both (70) and (71) there is one factor

$$(2i-4m)\Big|_{i=2m} = 0 \tag{72}$$

Thus for any order 4m of H

$$\det S_1 = \det S_2 = 0 \tag{73}$$

and consequently S_1 and S_2 are singular.

IX. THE EIGENVALUES AND EIGENVECTORS OF H

In matrix theory real nonzero numbers λ_i and real non-null column matrices s_j that satisfy the relation

$$H\, s_j = \lambda_i s_j \; ; \; i,j = 0,1,\ldots 4m-1 \tag{74}$$

are called respectively the eigenvalues and eigenvectors of H [12]. It is shown that if M is a non-null square matrix satisfying the relation

$$HM = \lambda_i M \; ; \; i = 0,1,\ldots\; 4m-1 \tag{75}$$

then every non-null column of M is an eigenvector of H corresponding to one of the eigenvalues [13].

Comparing (28) and (29) with (75), we can conclude that among the eigenvalues of H are the values

$$\lambda_i = \pm (4m)^{\frac{1}{2}} ; \; i = 0,1,\ldots \tag{76}$$

and that every non-null column of S_1 and S_2 is an eigenvector of H (or every row of S_1 and S_2, since S_1 and S_2 are symmetric if H is symmetric, as discussed in Sec. V).

But is is shown in matrix theory that the eigenvalues of a real symmetric matrix are all real [14], [15], and that all eigenvalues of a real orthogonal matrix have the same modulus [16]. Therefore, although we have not followed the usual and much more laborious textbook method of finding the eigenvalues of H, we can conclude at once that (76) gives the only eigenvalues of H.

If H is of some other form, either asymmetric or having nonzero trace or both, obtained by any combination of permuting rows and/or columns and multiplying rows and/or columns by -1, its orthogonality guarantees that all its eigenvalues will still be of the same modulus

$$|\lambda_i| = (4m)^{\frac{1}{2}} ; \; i = 0,1,\ldots 4m-1 \tag{77}$$

but some or all of them will be complex instead of real numbers.

It is shown in matrix theory that the trace of a matrix is equal to the sum of its eigenvalues [17] - [19]. Since H is stipulated to have zero trace, it follows that

$$\mathrm{tr}\, H = \sum_{i=0}^{4m-1} \lambda_i = 0 \tag{78}$$

which implies that since all of the eigenvalues are real and of equal modulus, exactly half of them must be positive and half of them negative.

X. THE DETERMINANT OF H

It is well known that the determinant of a product of square matrices is equal to the product of the determinants of the factor matrices [20]. Since H is stipulated to be symmetric, and since 4mI is a scalar matrix, all of whose 4m nonzero elements are 4m, then by (8)

$$(\det H)^2 = \det H^2 = \det (4mI) = (4m)^{4m} \tag{79}$$

It is shown in matrix theory that the determinant of a matrix is equal to the product of its eigenvalues [17] - [19]. Thus, taking the square root of both sides of (79), we can deduce that

$$|\det H| = (4m)^{2m} = \left|\prod_{i=0}^{4m-1} \lambda_i\right| \tag{80}$$

Since there are 4m eigenvalues, all of the same modulus (as shown in Sec. IX), it follows from (80) that

$$|\lambda_i| = \left[(4m)^{2m}\right]^{\frac{1}{4m}} = (4m)^{\frac{1}{2}} \tag{81}$$

which agrees with (77).

As established in Sec. IX, exactly half of the eigenvalues are negative. Since half of 4m is 2m, an even number, it also follows from (80) that

$$\det H = \prod_{i=0}^{4m-1} \lambda_i = (4m)^{2m} \tag{82}$$

If H is of some other form, either asymmetric or having nonzero trace or both, obtained by any combination of permuting rows and/or columns and multiplying rows and/or columns by -1, then since the effect of each such elementary matrix operation is merely to change the sign of the determinant, (80) will remain true, but det H may be either positive or negative, according as the number of such elementary matrix operations is even or odd.

XI. THE JORDAN NORMAL FORM OF H

It is shown in matrix theory that an arbitrary matrix, H in our case, is similar to a diagonal or quasi-diagonal matrix J, called the Jordan normal form of H, which means that

$$H = TJT^{-1} \tag{83}$$

where T is a nonsingular transformation matrix [21].

It is also shown that the elementary divisors of H [22] are of the form

$$(\lambda-\lambda_i)^{q_i} \; ; \; i = 0,1,\dots z \tag{84}$$

in which some of the λ_i may be equal, but subject to the constraint

$$q_0 + q_1 + \dots + q_z = 4m \tag{85}$$

where the value of z depends upon the values of the q_i [23].

We have shown in Sections IX and X that, under the conditions stipulated in Sec. II, H has 4m real eigenvalues of equal modulus, of which 2m are positive and 2m are negative. Therefore, we can conclude that in (84)

$$q_i \equiv 1 \; ; \; i = 0,1,\dots 4m-1 \tag{86}$$

and that in (85)

$$z = 4m - 1 \tag{87}$$

It is shown further in matrix theory that if all of the elementary divisors of H are of the first degree, as indicated by (86), then J is strictly diagonal [24], so we can construct it as the diagonal matrix [25]

$$J = \left\{\lambda_0, \lambda_1, \dots \lambda_{4m-1}\right\} \tag{88}$$

where

$$\lambda_i = \begin{cases} (4m)^{\frac{1}{2}} \; ; \; i = 0,1,\dots 2m-1 \\ -(4m)^{\frac{1}{2}} \; ; \; i = 2m, 2m+1,\dots 4m-1 \end{cases} \tag{89}$$

Now, if T in (83) possessed some distinctive, perhaps recursive, properties, possibly simpler than those of H, which would enable us to construct it for any value (or at least many values) of 4m, then we would have a new method of generating H for all such values. Disappointingly, it appears that the structure of T is more complicated than that of H instead of less so, for we can show easily that

$$T = H + J \tag{90}$$

Assume that (90) is valid. We can write (83) in the form

$$HT = TJ \tag{91}$$

Because H and J are both self-orthogonal, then by (8), (88), and (89)

$$H^2 = J^2 = 4mI \tag{92}$$

By (91), (90), and (92)

$$HT = H(H + J) = H^2 + HJ = 4mI + HJ \tag{93}$$

$$TJ = (H + J)J = HJ + J^2 = HJ + 4mI \tag{94}$$

Except in the order of terms, the last members of (93) and (94) are identical, as we should expect from (91), thus verifying (90).

If we could determine the nature of T and construct it easily, then by (90) an alternative and simpler method of generating H (not requiring the inversion of T) would be

$$H = T - J \tag{95}$$

XII. CONCLUSIONS

It was hoped that the research described herein might cast further light on the problem of the existence or nonexistence of unknown orders of H. In this respect it appears not to have been very fruitful.

Nevertheless, the approach has yielded many valuable new insights to the properties of H of most known orders 200 or less, and it is hoped that the results will serve as firm groundwork for further investigation.

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