## A TWO-PARTICLE COLLISION IN AESTHETIC FIELD THEORY*

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#### Abstract

We have found a new computer solution to the aesthetic field equations. This solution describes a two particle system with more structure than previously found. The contour lines show an arm structure. We have observed four arms around the maximum center. The location of the maximum (minimum) center is not along a straight line. This is the first time that such an effect has been observed for any kind of nonlinear partial differential equation so far as we know. A further discussion of the aesthetic principles leading to the field equations is given.


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[^0]I. A New Solution of $\Gamma_{\mathrm{jk} ; \ell}^{\mathrm{i}}=0$

We have found a set of data for the equations 1,2

$$
\begin{equation*}
\frac{\partial \Gamma_{j k}^{i}}{\partial x^{l}}+\Gamma_{j k}^{m} \Gamma_{m \ell}^{i}-\Gamma_{m k}^{i} \Gamma_{j l}^{m}-\Gamma_{j m}^{i} \Gamma_{k \ell}^{m}=0 \tag{1}
\end{equation*}
$$

which give a more complex two-particle system than we have obtained up to now. Planar maps for $z=0,1$, and 2 for a representative field component, chosen here to be $\Gamma_{11}^{1}$, are given in Figs. 1, 2, and 3. $\mathrm{A} z=0$ plot at $\mathrm{t}=4$ is given in Fig. 4. We have also studied the motion of the maximum (minimum) center in time. We find that the location of the maximum (minimum) center is not on a straight line. This is the first time that such an effect has been observed for any kind of nonlinear partial differential equation, so far as we know.

We write

$$
\begin{equation*}
\Gamma_{j \mathrm{k}}^{\mathrm{i}}=\mathrm{e}_{\alpha}^{\mathrm{i}} \mathrm{e}_{\mathrm{j}}^{\beta} \mathrm{e}_{\mathrm{k}}^{\gamma} \Gamma_{\beta \gamma}^{\alpha} \tag{2}
\end{equation*}
$$

The values chosen for $\Gamma_{\beta \gamma}^{\alpha}$ are:

$$
\begin{array}{llll}
\Gamma_{11}^{1}=.1 & \Gamma_{12}^{1}=.1 & \Gamma_{13}^{1}=0 & \Gamma_{10}^{1}=.1 \\
\Gamma_{21}^{1}=0 & \Gamma_{22}^{1}=0 & \Gamma_{23}^{1}=.1 & \Gamma_{20}^{1}=-.1 \\
\Gamma_{31}^{1}=.1 & \Gamma_{32}^{1}=-.1 & \Gamma_{33}^{1}=0 & \Gamma_{30}^{1}=.1 \\
\Gamma_{01}^{1}=.1 & \Gamma_{02}^{1}=0 & \Gamma_{03}^{1}=0 & \Gamma_{00}^{1}=0 \\
\Gamma_{11}^{2}=0 & \Gamma_{12}^{2}=.1 & \Gamma_{13}^{2}=-.1 & \Gamma_{10}^{2}=.1 \\
\Gamma_{21}^{2}=0 & \Gamma_{22}^{2}=.1 & \Gamma_{23}^{2}=.1 & \Gamma_{20}^{2}=.1 \\
\Gamma_{31}^{2}=.1 & \Gamma_{32}^{2}=0 & \Gamma_{33}^{2}=0 & \Gamma_{30}^{2}=-.1 \\
\Gamma_{01}^{2}=0 & \Gamma_{02}^{2}=.1 & \Gamma_{03}^{2}=0 & \Gamma_{00}^{2}=0
\end{array}
$$

$$
\begin{array}{llll}
\Gamma_{11}^{3}=0 & \Gamma_{12}^{3}=.1 & \Gamma_{13}^{3}=0 & \Gamma_{10}^{3}=-.1 \\
\Gamma_{21}^{3}=-.1 & \Gamma_{22}^{3}=0 & \Gamma_{23}^{3}=.1 & \Gamma_{20}^{3}=.1 \\
\Gamma_{31}^{3}=.1 & \Gamma_{32}^{3}=0 & \Gamma_{33}^{3}=.1 & \Gamma_{30}^{3}=.1 \\
\Gamma_{01}^{3}=0 & \Gamma_{02}^{3}=0 & \Gamma_{03}^{3}=.1 & \Gamma_{00}^{3}=0 \\
\Gamma_{11}^{0}=-.1 & \Gamma_{12}^{0}=-.1 & \Gamma_{13}^{0}=.1 & \Gamma_{10}^{0}=0 \\
\Gamma_{21}^{0}=.1 & \Gamma_{22}^{0}=-.1 & \Gamma_{23}^{0}=-.1 & \Gamma_{20}^{0}=0 \\
\Gamma_{31}^{0}=-.1 & \Gamma_{32}^{0}=.1 & \Gamma_{33}^{0}=-.1 & \Gamma_{30}^{0}=0 \\
\Gamma_{01}^{0}=0 & \Gamma_{02}^{0}=0 & \Gamma_{03}^{0}=0 & \Gamma_{00}^{0}=.1
\end{array}
$$

The values chosen for $\mathrm{e}_{\mathrm{i}}^{\alpha}$ are:
$\mathrm{e}_{1}^{1}=.88$
$\mathrm{e}_{2}^{1}=-.42$
$\mathrm{e}_{3}^{1}=-.32$
$\mathrm{e}_{0}^{1}=.22$
$\mathrm{e}^{2}{ }_{1}=.5$
$\mathrm{e}_{2}^{2}=.9$
$\mathrm{e}_{3}^{2}=-.425$
$\mathrm{e}_{0}^{2}=.3$
$e^{3}=.2$
$\mathrm{e}_{2}^{3}=-.55$
$\mathrm{c}_{3}^{3}=.89$
$\mathrm{e}_{0}^{3}=.6$
$\mathrm{e}_{1}^{0}=.44$
$e_{2}^{0}=-.16$
$\mathrm{e}_{3}^{0}=.39$
$e_{0}^{0}=1.01$

The data above satisfies $R_{j k \ell}^{i} \neq 0$ integrability equations. It is not completely general as we have $\Gamma_{\mathrm{tk}}^{\mathrm{t}}=\Gamma_{\mathrm{kt}}^{\mathrm{t}}$.

## II. Description of the Solution

The $\mathrm{z}=$ constant maps show as much as four planar maxima and minima centers. This is more than we have obtained previously. However, we have only been able to find one three-dimensional maximum and one three-dimensional minimum. The maximum is at $x=6.161, y=2.541, z=3.451$. Here the field component takes on the value 4.25 (rounded off). The minimum is at $x=8.146$,
$\mathrm{y}=.410, \mathrm{z}=-1.450$ with $\Gamma_{11}^{1}=-1.94$ (rounded off). The particle is seen to have an arm structure. We have found four arms, or tubes, coming out of the maximum. This is the first time particles of such complexity have been uncovered from a nonlinear field equation.

There is no sign of singularities anywhere and as one goes far outside the particle system, the field components all tend to zero. Thus, we have a similar situation here as in our previous particle solutions. ${ }^{2,3,4}$ As we go away from the origin, we have not found any more particles. Also, long runs along the coordinate axes show no additional interesting structure.

We next studied the way the system changes as a function of time. The contour lines undergo changes in time (see Fig. 4). The location of the maximum and minimum center as a function of time is given in Tables 1 and 2.

When the particles are away from each other, we see that the data is consistent with a straight line motion. From the results at $t=0$ and $t=10$, we obtain the following straight lines. For the maximum we have

$$
\begin{align*}
& \mathrm{x}=-.9592 \mathrm{t}+6.161 \\
& \mathrm{y}=-.4314 \mathrm{t}+2.541  \tag{3}\\
& \mathrm{z}=-.9100 \mathrm{t}+3.951
\end{align*}
$$

For the minimum we have

$$
\begin{align*}
& x=-1.0566 t+8.146 \\
& y=-.3268 t+.410  \tag{4}\\
& z=-.6449 t-1.450
\end{align*}
$$

All our results for $\mathrm{x}, \mathrm{y}, \mathrm{z}$ fall within .001 of this (except in one instance when it is . 002). The grid used in mapping the regions around the maximum and minimum centers was .001. Thus, these results accurately describe a straight line motion when the particles are apart.

At $t=20.37$ we can see from the above equations that the $x, y, z$ positions of the maximum is within .001 of the corresponding $x, y, z$ locations of the minimum. Thus, the results are consistent with an impact in the vicinity of $t=20.37$. This is a rather remarkable and unexpected occurrence.

As the maximum and minimum centers approach one another, the fields become greater in magnitude. Furthermore the field changes markedly within small dimensions. This necessitates smaller and smaller grids in order to follow the collision. Here, limitations in computer time becomes a factor. Thus, the decision was made to follow the particle at times somewhat after the impact. The maximum (minimum) center no longer lies on a straight line trajectory. A schematic picture of what occurs is given in Fig. 5. The results are best explained by the maximum and minimum centers "bouncing" off one another. A further examination shows that the particles reflect off one another (see Fig. 5).* This is a simple form of scattering.

Another possible interpretation of Fig. 5 is that the maximum (minimum) evolves into a minimum (maximum) much like a sine wave maximum can evolve into a minimum as time goes on. However, we have not found any evidence, from our computer results, for a maximum going into a minimum in a continuous manner.

At $t=100$, we verified with a very course grid that along $C O B$ in Fig. 5, $\Gamma_{11}^{1}$ gets smaller in magnitude. Its value here was found to be -1.08 .

[^1]
## III. Aesthetic Principles

In the course of our work on aesthetic field theory, we have introduced several modifications from time to time. We would like here to give a rederivation of the basic equations in order to clarify several points.

As most field theories allow for a vector field, we shall start off by assuming the existence of a vector field.

We write for the change of the vector between two points in cartesian space

$$
\begin{equation*}
d A_{i}=\Gamma_{i k}^{j} A_{j} d x^{k} \tag{5}
\end{equation*}
$$

Using continuity, we have

$$
\begin{equation*}
\frac{\partial A_{i}}{\partial x^{k}}=\Gamma_{i k}^{j} A_{j} \tag{6}
\end{equation*}
$$

$\mathrm{dA}_{\mathrm{i}}$ should depend on the displacement between the two points. We drop possible contributions of order (dx) ${ }^{2}$ and higher. $d A_{i}$ should also depend on $A_{i}$. To make things more general, we could allow $\Gamma_{j k}^{i}$ to be a function of $A_{i}$, among other things. $\quad \Gamma_{j k}^{i}$ is called the change function as it determines the change of $d A_{i}$.

In standard tensor analysis, there is no difference between covariant and contravariant indices in an orthogonal coordinate system. However, our introduction of upper and lower indices has a different purpose. In our approach, the coordinate system is just an arena for the dynamical fields. If we consider a field $g_{i j}$, we can introduce a dual or inverse field $g^{j k}$ such that $g_{i j} g^{j k}=\delta_{i}^{k}$. Thus, $g^{i j}$ is just the cofactor of $g_{i j}$ divided by the determinant of $g_{i j}$. In other words, $\mathrm{g}^{\mathrm{jk}}$ is just an abbreviation for certain combinations of the field components $g_{i j}$. Thus, the upper indices carry dynamical information and does not represent parallel projection of a vector on the coordinate axes as distinguished from perpendicular projection.

Thus, we have argued that it is legitimate to introduce upper indices in a cartesian coordinate system. Fields with upper indices carry dynamical information.

This is all well and good if $\mathrm{g}^{\mathrm{ij}}$ exists at all points. In our initial work on aesthetic field theory, we assumed that $g_{i j} \rightarrow(-1,-1,-1,1)$ at infinity, and it was supposed that $g^{i j}$ could then be defined at all points. However, in such a theory it was necessary for an infinite number of restrictions to be imposed on $\Gamma_{j k}^{i}$ at the origin point. ${ }^{5}$ All invariants formed from $\Gamma_{j k}^{i}$ and $g_{i j}$ had to be zero at the origin. Such a set of initial data is not easy to come by. Nevertheless, we did find some examples of such data. However, in none of these examples did we find a bounded solution.

In the case that $g^{i j}$ exists at all points, we may introduce $A^{i}$ by $A_{i}=g_{i j} A^{j}$. Then we get

$$
\begin{equation*}
d A^{i}=-\Gamma_{j k}^{i} A^{j} d x x^{k} \tag{7}
\end{equation*}
$$

We have found better computer results with data that appears to satisfy $g_{i j} \rightarrow 0$ at infinity. Then since $g=0$ at infinity, we conclude that $g^{i j}$ is not defined at all points. Thus, we have to be careful and not introduce inverse fields when they are not defined.

If we introduce a set of basis vector fields $e_{i}^{\alpha}$, the dual field would be defined by $\mathrm{e}_{\mathrm{i}}^{\alpha} \mathrm{e}_{\alpha}^{\mathrm{j}}=\delta_{\mathrm{i}}^{\mathrm{j}}$. Then we could use $\mathrm{e}^{\alpha}{ }_{\mathrm{i}}$ to introduce a dual field, if $\mathrm{e}^{\alpha}{ }_{\mathrm{i}} \rightarrow \delta_{\mathrm{i}}^{\alpha}$ at infinity. However, we have never given any proof that this boundary condition can indeed be satisfied.

On the other hand, once we have established the point that the upper indices have dynamical character, we may realize that it is not necessary to discuss the introduction of upper indices in terms of inverse fields for $g_{i j}$ and $e_{i}^{\alpha}$.

The difference between two vectors in a cartesian space is a vector. The role of the upper indices in (6) is to denote scalar products. This is a dynamical way for introducing a scalar product. We make the requirement that $\Gamma_{\text {tk }}^{t}$ act like a vector. This fixes the change of upper index fields, to have the same structure as (7) (we make use of the field equation (9) which is obtained below).

Thus, in order for $\Gamma_{\mathrm{tk}}^{\mathrm{t}}$ to behave like a vector so far as its change is concerned, the change of upper indices is fixed according to (7).

For the change of a 2 nd vector field, we write

$$
\begin{equation*}
\mathrm{dB}_{\mathrm{i}}=\Gamma_{\mathrm{ik}}^{\mathrm{j}} \mathrm{~B}_{\mathrm{j}} \mathrm{dx} \tag{8}
\end{equation*}
$$

We are assuming the existence of a 2 nd vector field $B_{i}$. (This is no problem since from $\Gamma_{j k}^{i}$ we can construct $\Gamma_{\mathrm{tk}}^{\mathrm{t}}$ and $\Gamma_{\mathrm{kt}}^{\mathrm{t}}$.) We are assuming, also, that $\Gamma_{j k}^{\mathbf{i}}$ is a universal change function that determines the change of all vector fields in a uniform way. It is our philosophy that a vector being an array of numbers, should behave like any other vector field so far as its change is concerned.

Going one step further, the change function should determine the change of all tensor fields with a tensor behaving like a product of vectors so far as its change is concerned. This leads to the field equations for the change function

$$
\begin{equation*}
\frac{\partial \Gamma_{\mathrm{jk}}^{\mathrm{i}}}{\partial \mathrm{x}}+\Gamma_{\mathrm{jk}}^{\mathrm{m}} \Gamma_{\mathrm{ml}}^{\mathrm{i}}-\Gamma_{\mathrm{mk}}^{\mathrm{i}} \Gamma_{\mathrm{jl}}^{\mathrm{m}}-\Gamma_{\mathrm{jm}}^{\mathrm{i}} \Gamma_{\mathrm{kl}}^{\mathrm{m}}=0 \tag{9}
\end{equation*}
$$

Another way to justify the field equations is to argue that (9) constitutes a field equation with the following desirable aesthetic properties:
a) All products of the field, and contractions of the field, are treated in a uniform way as far as their change is concerned.
b) All higher derivatives of the field are treated in a uniform so far as their change is concerned.
c) Computer studies suggest that the field is analytic.
d) Computer studies suggest that $\Gamma_{j \mathrm{k}}^{\mathrm{i}} \rightarrow 0$ at infinity may be satisifed.
e) No arbitrary functions appear in the theory.

Thus (9) can be considered a "fait accompli" so far as these aesthetic ideas are concerned.

Note, in this approach, it is not necessary to introduce $g_{i j}$ as an independent field (by independent we mean that arbitrary parameters are assigned to $g_{i j}$ at the origin point).

From $\Gamma_{j k}^{i}$ we can form objects like $\bar{g}_{i j} \equiv \Gamma_{t i}^{t} \Gamma_{s j}^{s}$. However, since $\Gamma_{j k}^{i} \rightarrow 0$ at infinity, $\overline{\mathrm{g}}_{\mathrm{ij}}$ would also go to zero at infinity. Then, $\overline{\mathrm{g}}^{\mathrm{ij}}$ would not be defined at all points. This tells us that we can not indiscriminately introduce inverse fields once we impose the condition $\Gamma_{j k}^{i} \rightarrow 0$ at infinity.

## IV. Discussion

By now we have obtained many solutions to the integrability equations. These solutions consitute the initial data for the partial differential equations. The resulting maps are very dependent on the choice of initial data. For example, if we choose all $\Gamma_{j k}^{i}$ equal, then it can be shown that singularities develop. Also, we found ${ }^{5}$ another set of data for which singularitics can be shown to exist. We have, as yet, been unable to produce data that can be considered general. For example, the data in this paper obeys $\Gamma_{t k}^{t}=\Gamma_{k t}^{\mathrm{t}}$. This restriction is maintained at all points as a consequence of the field equations. Thus, a hypothesis we can make is that the most aesthetic type solution of the integrability equations should be free from unjustifiable constraints that are preserved at all points.

In our minds, we can think of all sorts of functions that allow for many particle particle behavior and obey natural boundary conditions and have no singularities associated with them. Even though we can conceive of such functions in our minds, it would be a more difficult matter to write down a mathematical form for them. Our viewpoint is - of all these possible functions that can be constructed, the correct one would conform to mathematically aesthetic ideas. The many solutions we have found over the years can be looked at within this context. Solutions without unnatural constraints would be considered more aesthetic.

We have pointed out that our data is not entirely general. Thus, there is no reason to believe that our present solutions are the most complex that the aesthetic field equations are capable of. We can expect that there is still a wealth of information within aesthetic field theory yet to be uncovered.

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Table I: Location of Maximum Center as a Function of Time

$$
\begin{aligned}
& \text { Time Location of Maximum Center Value of } \Gamma_{11}^{1} \text { (rounded off) } \\
& \mathrm{x}=9.997 \\
& t=-4 \\
& \mathrm{y}=4.266 \\
& 3.55 \\
& z=7.591 \\
& \mathrm{x}=6.161 \\
& \mathrm{t}=0 \\
& \mathrm{y}=2.541 \\
& \text { 4. } 25 \\
& \mathrm{z}=3.951 \\
& \mathrm{x}=2.324 \\
& t=4 \\
& y=.815 \\
& 5.29 \\
& \mathrm{z}=.311 \\
& x=-3.431 \\
& t=10 \\
& y=-1.773 \\
& 8.24 \\
& z=-5.149 \\
& \mathrm{x}=-13.022 \\
& t=20 \\
& y=-6.086 \\
& 231.67 \\
& z=-14.249 \\
& x=-13.831 \\
& t=20.8 \\
& y=-6.387 \\
& z=-14.864 \\
& \mathrm{x}=-14.042 \\
& t=21 \\
& \mathrm{y}=-6.452 \\
& 63.23 \\
& z=-14.993 \\
& \mathrm{x}=-18.268 \\
& \mathrm{t}=25 \\
& \mathrm{y}=-7.758 \\
& 8.56 \\
& z=-17.572 \\
& x=-23.551 \\
& t=30 \\
& \mathrm{y}=-9.393 \\
& 4.11 \\
& \mathrm{z}=-20.797
\end{aligned}
$$

- Table II: Location of Minimum Center as a Function of Time

| Time | Location of Minimum Center | Value of $\Gamma_{11}^{1}$ (rounded off) |
| :---: | :---: | :---: |
|  | $x=12.372$ |  |
| $t=-4$ | $y=1.717$ | -1.62 |
|  | $\mathrm{z}=1.130$ |  |
|  | $x=8.146$ |  |
| $\mathrm{t}=0$ | $y=.410$ | -1.95 |
|  | $\mathrm{z}=-1.450$ |  |
|  | $\mathrm{x}=3.92$ |  |
| $\mathrm{t}=4$ | $\mathrm{y}=-.897$ | -2.42 |
|  | $\mathrm{z}=-4.030$ |  |
|  | $\mathrm{x}=-2.420$ |  |
| $\mathrm{t}=10$ | $y=-2.858$ | $-3.82$ |
|  | $z=-7.899$ |  |
|  | $\mathrm{x}=-17.817$ |  |
| $\mathrm{t}=25$ | $\mathrm{y}=-8.243$ | -18.71 |
|  | $\mathrm{z}=-18.799$ |  |
|  | $\mathrm{x}=-89.750$ |  |
| $\mathrm{t}=100^{*}$ | $y=-40.592$ | -1.08 |
|  | $z=-87.048$ |  |

[^2]
## Figure Captions

1. Map of $\Gamma_{11}^{1}$ at $z=0, t=0$. Approximate ranges: $x=-.2$ to 11.8; $y=-9$ to 9.6 . Contour numbers are truncated. That is .49 is read as .4 and not .5. However, the zero is roughly the true zero.

Contour lines around the maximum and minimum are described by an additional decimal place.
2. Map of $\Gamma_{11}^{1}$ at $z=1, t=0$. Approximate ranges: $x=-.2$ to 11.8 ; $\mathrm{y}=-9$ to 9.2 .
3. Map of $\Gamma_{11}^{1}$ at $\mathrm{z}=2, \mathrm{t}=0$. Approximate ranges: $\mathrm{x}=-.2$ to 11.8; $y=-9$ to 9.2 .
4. Map of $\Gamma_{11}^{1}$ at $z=0, t=4$. Approximate ranges: $x=-4.2$ to 12.2; $y=-10.6$ to 11.4 .
5. Schematic drawing of collision process. The location of the maximum (minimum) center is not on a straight line.


FIGUPE NO. 1


FIGURE NO. 2


FIGUPE NO. 3


Fig. 4


Fig. 5


[^0]:    *Work supported in part by the U. S. Atomic Energy Commission.

[^1]:    *Zabusky and Kruskal ${ }^{6}$ worked with a one-dimensional nonlinear equation and found "pulse" like solutions that travel through one another. No evidence for a "bouncing" off effect was found. The "bouncing" off effect has not been observed previously, even in one dimension.

[^2]:    *Grid here is too course for an accurate determination of location.

