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BEAM LOADING IN HIGH-ENERGY STORAGE RINGS*

P. B. Wilson

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Summary

The analysis of beam loading in the RF systems of high-energy storage rings (for example, the PEP e^-e^+ ring[1]) is complicated by the fact that the time, T_b, between the passage of successive bunches is comparable to the cavity filling time, T_f. In this paper, beam loading expressions are first summarized for the usual case in which $T_b \ll T_f$. The theory of phase oscillations in the heavily-beam-loaded case is considered, and the dependence of the synchrotron frequency and damping constant for the oscillations on beam current and cavity tuning is calculated. Expressions for beam loading are then derived which are valid for any value of the ratio T_b/T_f. It is shown that, for the proposed PEP e^-e^+ ring parameters, the klystron power required is increased by about 3% over that calculated using the standard beam loading expressions. Finally, the analysis is extended to take into account the additional losses associated with the excitation of higher-order cavity modes. A rough numerical estimate is made of the loss enhancement to be expected for the PEP RF system. It is concluded that this loss enhancement might be substantial unless appropriate measures are taken in the design and tuning of the accelerating structure.

Summary of Beam Loading Expressions for $T_b \ll T_f$

We consider first the case for which the time between bunch passages is short compared to the filling time for the fundamental and all higher-order cavity modes up to a cut-off frequency determined by the bunch length and the dimensions of apertures coupling the cavity to the outside world. We assume also that these higher-order mode frequencies are sufficiently removed from being integral multiples of $1/T_b = N_b f_0$ (where N_b is the number of bunches and f_0 is the revolution time) so that no higher-order mode is resonately excited. Beam loading can then be characterized by a continuous RF current at the fundamental mode frequency having a peak value (for short bunches) equal to twice the average circulating current. Beam loading in circular machines in this limit has been analyzed previously.[2] We will use here a somewhat different approach and notation. In this notation, a tilde is used to denote a complex (phasor) quantity, while a quantity without a tilde denotes absolute value. For convenience, $-\tilde{i}_0$ is taken as the reference direction (real axis), where $\tilde{i}_0$ is the beam current; the accelerating component of a phasor voltage is then obtained by taking the real part. All phasors are assumed to vary as $e^{j\omega t}$. In Fig. 1, $\tilde{i}_g$ gives the phase of the incident wave from the external generator. Angle θ is the phase of the external generator with respect to $-\tilde{i}_0$, and ϕ is the phase between $-\tilde{i}_0$ and the net cavity voltage, $\tilde{V}_c$. Using superposition, $\tilde{V}_c$ is the sum of a generator-produced voltage, $\tilde{V}_g$, and a beam-induced voltage, $\tilde{V}_b$. The vector addition $\tilde{V}_g + \tilde{V}_b = \tilde{V}_c$ is illustrated in Fig. 1.

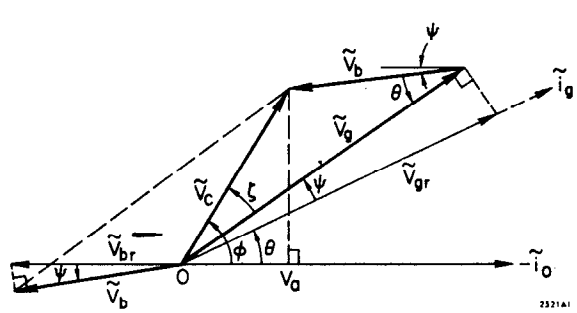

FIG. 1--Diagram showing vector addition of voltages in an RF cavity.

The voltage $\tilde{V}$ produced across a parallel resonant circuit with shunt resistance R and driving current $\tilde{i}$ is $\tilde{V}/R = \tilde{i}/(1+j\xi) = \tilde{i}\cos\psi\, e^{j\psi}$. In this expression ψ is the tuning angle shown in Fig. 1 and defined by $\tan\psi = -\xi = -2Q_L(\omega-\omega_0)/\omega_0$, where Q_L is the loaded Q, ω is the driving frequency and ω_0 is the cavity resonant frequency. We define now a coupling coefficient β, such that β/R is the internal shunt conductance of the external generator. Physically, β is the ratio of the power emitted from the coupling aperture to the power, P_c, dissipated in the cavity walls with the external generator off and the cavity internally excited (by beam loading, for example). The following expressions for the magnitudes of $\tilde{V}_g$ and $\tilde{V}_b$ can now be derived:

$$V_g = V_{gr}\cos\psi; \qquad V_{gr} = \sqrt{RP_g}\,\frac{2\sqrt{\beta}}{1+\beta} \tag{1a}$$

$$V_b = V_{br}\cos\psi; \qquad V_{br} = \frac{i_0 R}{1+\beta} \tag{1b}$$

Here P_g is the incident (available) power from the generator, i_0 is the average circulating current,* and V_{gr} and V_{br} are the magnitudes of $\tilde{V}_g$ and $\tilde{V}_b$ at resonance.

From the vector diagram in Fig. 1, the basic expression for the accelerating voltage is

$$V_a = V_c\cos\phi = V_{gr}\cos\psi\cos(\theta+\psi) - V_{br}\cos^2\psi \tag{2}$$

In designing an RF system for a storage ring, we are normally given the synchronous energy gain, eV_s, determined by the magnet lattice; the synchronous phase angle, ϕ, required to maintain an adequate quantum lifetime (alternatively, the peak cavity voltage $V_c = V_s/\cos\phi$ can be specified); and the circulating current i_0. We want to know the required generator power as a function of tuning angle and coupling coefficient. From Fig. 1 we have

$$V_c\sin\phi = V_{gr}\cos\psi\sin(\theta+\psi) - V_{br}\cos\psi\sin\psi. \tag{3}$$

Using this expression in Eq. (2) to eliminate $(\theta+\psi)$, and Eq. (1a) to express the result in terms of the available generator voltage $\sqrt{RP_g}$, we obtain

$$\sqrt{RP_g} = \frac{V_c}{\cos\psi}\left[\frac{1+\beta}{2\sqrt{\beta}}\right]\left\{\left[\cos\phi + \frac{i_0 R}{V_c(1+\beta)}\cos^2\psi\right]^2 + \left[\sin\phi + \frac{i_0 R}{V_c(1+\beta)}\cos\psi\sin\psi\right]^2\right\}^{1/2}. \tag{4}$$

Suppose now that a feedback system is available which can keep the reflected voltage wave in the input line to the cavity real by adjusting the cavity tuning; that is, to cancel out the reactive component of the induced beam-loading voltage. The reflected voltage wave can be real only if $\tilde{V}_c$ and $\tilde{i}_g$ as shown in Fig. 1 are co-linear. This condition in turn implies that $\zeta+\psi=0$. Since $\zeta+\psi+\theta=\phi$, then we must have also that $\theta=\phi$. Using the laws of sines on the vector triangle in Fig. 1 together with these relationships between

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*The expression for V_{br} is valid in the limit of short bunches. For long bunches, i_0 must be multiplied by a bunch form factor, b_0, given in Appendix A. Note also that the shunt impedance is defined such that power and voltage are related by $P = V^2/R$.

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angles, $V_{br}\cos\psi/V_c = \sin\zeta/\sin\theta = -\sin\psi/\sin\phi$. The tuning angle for real reflected power becomes

$$\tan\psi = -\frac{V_{br}}{V_c}\sin\phi = -\frac{i_0 R}{V_c(1+\beta)}\sin\phi \ . \qquad (5)$$

For this case, Eq. (4) reduces to

$$\sqrt{RP_g} = \frac{1+\beta}{2\sqrt{\beta}}\left[V_c + V_{br}\cos\phi\right] \ . \qquad (6)$$

If we now differentiate Eq. (6) with respect to β to find the value of coupling coefficient, β_m, which minimizes the required generator power, we obtain

$$\beta_m = 1 + \frac{i_0 R\cos\phi}{V_c} = 1 + \frac{i_0 R\, V_s}{V_c^2} \ . \qquad (7a)$$

The generator power at β_m is calculated, using Eq. (7a) in Eq. (6), to be $\sqrt{RP_g} = V_c\sqrt{\beta_m}$. The efficiency for the transfer RF power to the beam is $\eta = P_b/P_g = i_0 V_s/P_g = (\beta_m - 1)/\beta_m$. The power dissipated in the cavity walls is $P_c = V_c^2/R = P_g/\beta_m = P_b/(\beta_m - 1)$. By conservation of power, the reflected power is in general $P_r = P_g - P_b - P_c$. It is seen that the reflected power reduces to zero for $\beta = \beta_m$. The tuning angle at $\beta = \beta_m$ is obtained, using Eq. (7a) in Eq. (5), as

$$\tan\psi_m = -\left[(\beta_m - 1)/(\beta_m + 1)\right]\tan\phi \ . \qquad (7b)$$

Phase Oscillations and Phase Stability

If the phase oscillations of the bunch about the synchronous phase angle are to be stable, then the above transition $dV_a/d(t-t_s)$ must be negative, where $(t-t_s)$ is the arrival time of the bunch at the cavity gap measured with respect to the arrival time of a synchronous particle. The arrival time of the bunch relative to a synchronous particle is related to their difference in phase by $\omega(t-t_s) = \theta - \theta_s$. Therefore, another way of saying the same thing is that $dV_a/d\theta$ must be negative. From Eq. (2) we then obtain the simple condition that $\sin(\theta+\psi) > 0$, or

$$0 < (\theta + \psi) < \pi \ . \qquad (8a)$$

It is easy to show that this condition is one of the stability conditions first derived by Robinson.[2] Using Eq. (3), this condition can be written

$$V_c \sin\phi + V_{br}\cos\psi\sin\psi > 0 \ . \qquad (8b)$$

If a feedback circuit adjusts the cavity tuning to keep the reflected voltage wave real in accordance with Eq. (5), then Eq. (8b) reduces to $V_{br}\cos\phi < V_c$. At optimum coupling, using Eq. (7a), this becomes $[(i_0 R\cos\phi)/(i_0 R\cos\phi + 2V_c)] < 1$. At optimum coupling the system is always stable.

The physical meaning of the stability condition of Eq. (8a) is clear when it is realized that $\theta+\psi$ is the phase angle between the bunch and the crest of the generator-produced wave. Since the beam-induced wave changes in phase with the bunch, only the generator wave is effective in producing a net restoring force on the bunch. Instability arises when $\theta+\psi = 0$ and the bunch lies at the crest of the generator-produced wave. Since the effective restoring force depends on $\theta+\psi$, and since this angle is a function of beam current for fixed V_s and V_c, then the synchrotron frequency will depend on beam current. From Eqs. (2) and (3),

$$dV_a/d\theta = V_{gr}\cos\psi\sin(\theta+\psi) = V_c\sin\phi + V_{br}\cos\psi\sin\psi \ .$$

If cavity tuning is adjusted according to Eq. (5), and using also the fact that the synchrotron frequency is proportional to $(dV_a/d\theta)^{1/2}$, the change in synchrotron frequency with current can be written

$$\frac{\omega_s}{\omega_{s0}} = \frac{\omega_s(i_0)}{\omega_s(0)} = \left[\frac{1 - (V_{br}/V_c)^2\cos^2\phi}{1 + (V_{br}/V_c)^2\sin^2\phi}\right]^{1/2} \qquad (9)$$

The high-current stability limit, $V_{br}\cos\phi = V_c$, is also the condition for $\omega_s = 0$.

The preceding analysis is strictly valid only in the limit in which the synchrotron oscillation frequency is small compared to the cavity bandwidth. If ω_s is large compared to the bandwidth, the induced beam voltage cannot respond to changes in phase of the driving beam current. The accelerating voltage is then simply $V_c\cos\phi$, and the stability condition is $V_c\sin\phi > 0$, independent of current. For synchrotron frequencies which are comparable to the cavity bandwidth, the dynamic response of the cavity field to phase changes in the driving current must be taken into account. This calculation is carried out in Appendix B. It is shown that for $\omega_s T_f \ll 1$, ω_s/ω_{s0} is given by Eq. (9), and that for $\omega_s T_f \gg 1$, $\omega_s = \omega_{s0}$. It is also shown that for $\xi > 0$ (or $\omega > \omega_0$) the oscillations are damped, while the opposite is true for $\omega < \omega_0$. This is the dynamic stability condition for a system without feedback, considered by Robinson.[2] Lee[3] has considered the case of a system with feedback.

It is worth point out that there are stability conditions similar to the ones just discussed in any high-frequency systems in which the stored energy can vary parametrically at a low frequency rate. Ceperley[4] has made a clear and concise analysis of a closely related problem, that of electromechanical (ponderomotive) oscillations which result from the modulation of the resonant frequency of a cavity by mechanical vibrations. He concludes that for these oscillations the system is antidamped (unstable) for $\omega > \omega_0$, and that on the opposite side of the resonance curve a "static" instability occurs in the limit of zero modulation frequency, corresponding to Eq. (8b) for the case of phase oscillations.

Beam Loading for $T_b \sim T_f$

The energy per turn extracted by a bunch passing through a cavity is $\Delta W = qV_s = i_0 T_b V_s = P_b T_b$, where q is the charge in the bunch. Using $Q_L = \omega T_f/2 = \omega W/[P_c(1+\beta)]$, together with $P_b/P_c = \beta_m - 1$ at optimum coupling, we have $\Delta W/W = (2T_b/T_f)[(\beta_m - 1)/(\beta_m + 1)]$. The ratio $\tau \equiv T_b/T_f$ is therefore an approximate measure of the fraction of the stored energy removed from the cavity by the passage of each bunch. This expression breaks down for $\tau \sim 1$, since ΔW cannot be greater than W. It is clear that the expressions for beam loading derived so far must be modified for $\tau \sim 1$.

Suppose charge dq crosses the cavity gap, producing an increment dV_b in the induced beam loading voltage. Using $R/Q = (V_c^2/P_c)/(\omega W/P_c) = V_c^2/\omega W$, the change in stored energy is $dW = 2V_c dV_c/[\omega(R/Q)]$. By conservation of energy this must be equal to $-V_c\cos\phi\, dq$. Since dV_b is in the direction of i_0, the change in cavity voltage will be $dV_c = -\cos\phi\, dV_b$. Therefore $dV_b = (\omega/2)(R/Q)\, dq$. Assuming that all of the charge crosses the gap in a time short compared to the RF period, then

$$\Delta V_b = \frac{\omega}{2}\left(\frac{R}{Q}\right) q = \frac{i_0 R\tau}{1+\beta} \ . \qquad (10)$$

For long bunches, the above expression must be multiplied by the bunch form factor given by Eq. (A. 1). The primary role of R/Q in determining the beam-loading characteristics for the passage of a single bunch is also clearly seen.

The steady-state excitation of the cavity by a periodic train of bunches is now readily calculated by taking the vector sum of the fields induced on successive passages.

The total induced beam voltage, calculated by taking the sum immediately after the passage of each bunch and denoted by $\tilde{V}_b^-$, is

$$\tilde{V}_b^- = \Delta V_b \left[1 + e^{-\tau + j\delta} + e^{2(-\tau + j\delta)} \ldots\right] = \frac{\Delta V_b}{1 - e^{-\tau} e^{j\delta}} \quad , \qquad (11)$$

where $\delta = (\omega_0 - \omega)T_b = \tau \tan\psi$. In other words, between one bunch passage and the next, the residual field from the previous passage decays by a factor $e^{-\tau}$ and shifts phase by an angle δ. The average induced beam voltage is given by $\tilde{V}_b = \tilde{V}_b^- - \Delta V_b/2$. The relationship between these induced beam voltage vectors, and the transient behavior of the cavity field between bunch passages, is shown in Fig. 2.

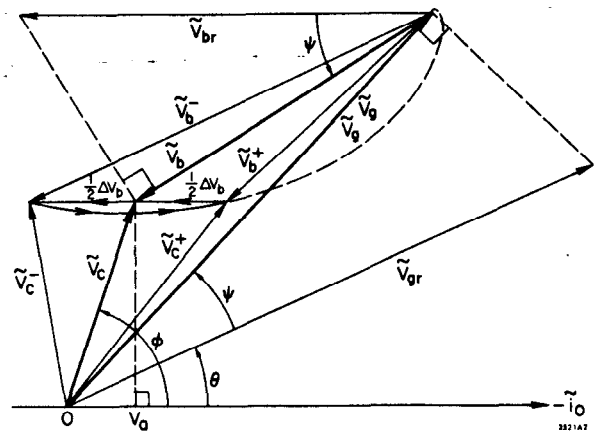

FIG. 2--Vector diagram showing transient behavior of cavity fields during beam loading for $\tau \sim 1$.

Using Eqs. (10) and (11) in this expression for V_b,

$$\tilde{V}_b = \left(\frac{i_0 R}{1+\beta}\right)\left[F_1(\tau) + jF_2(\tau)\right] \quad , \qquad (12)$$

where

$$F_1(\tau) = \frac{\tau(1 - e^{-2\tau})}{2\left[1 - 2e^{-\tau}\cos(\tau\tan\psi) + e^{-2\tau}\right]} \quad , \qquad (13a)$$

$$F_2(\tau) = \frac{\tau e^{-\tau}\sin(\tau\tan\psi)}{1 - 2e^{-\tau}\cos(\tau\tan\psi) + e^{-2\tau}} \quad . \qquad (13b)$$

In the limit $\tau \ll 1$, these relations reduce to $F_1 = \cos^2\psi$ and $F_2 = \sin\psi\cos\psi$, as expected. The functions $F_1(\tau)/\cos^2\psi$ and $F_2(\tau)/\cos\psi\sin\psi$ are plotted in Fig. 3 as a function of τ.

All of the expressions listed previously for the case $\tau \ll 1$ can be corrected so that they are valid for any τ by using Eq. (12) wherever $\tilde{V}_b$ appears. In particular, Eq. (4) for the available generator voltage becomes

$$\sqrt{RP_g} = \frac{V_c}{\cos\psi}\left(\frac{1+\beta}{2\sqrt{\beta}}\right)\left\{\left[\cos\phi + \frac{V_{br}}{V_c}F_1(\tau)\right]^2 + \left[\sin\phi + \frac{V_{br}}{V_c}F_2(\tau)\right]^2\right\}^{\frac{1}{2}} . \qquad (14)$$

This result can be applied to the proposed PEP RF system, for which $i_0 = 200$ mA, $R = 950$ MΩ, $V_c = 44$ MV and $V_s = 26$ MV. We also define $\tau_0 = \tau/(1+\beta) = T_b/T_{f0}$, where $T_{f0} = 2Q_0/\omega$ is the unloaded filling time. For PEP, $T_{f0} = 25$ μsec, $T_b = 2.4$ μsec and $\tau_0 = .096$. Using these parameters, the preceding expression can be minimized as a function of β and ψ. The result is $\sqrt{RP_g} = 84.1$ MV for $\beta_m = 3.41$ and $\psi_m = -37.3^\circ$. These values can be compared

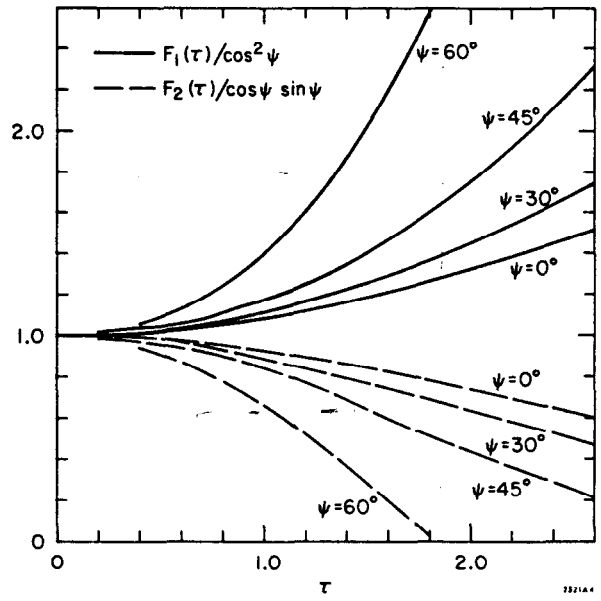

FIG. 3--The functions $F_1(\tau)$ and $F_2(\tau)$ for various values of ψ.

to $\sqrt{RP_g} = 82.9$ MV, $\beta_m = 3.55$ and $\psi_m = -37.4^\circ$ for $\tau = 0$. Thus P_g is increased by 3% over that computed assuming $\tau = 0$.

Beam-Loading Enhancement Due to Excitation of Higher Modes

From the analysis in the preceding section it is seen that the accelerating voltage can be written in the following two alternative forms:

$$V_a = V_{ga} - i_0\left(\frac{R}{1+\beta}\right)F_1(\tau) \quad ; \qquad (15a)$$

$$\Delta U = qV_a = qV_{ga} - q^2\left[\frac{\omega}{4}\left(\frac{R}{Q}\right)\right]\left[\frac{2F_1(\tau)}{\tau}\right] \quad , \qquad (15b)$$

where V_{ga} is the accelerating component of the generator voltage and ΔU is the energy gain per turn. For $\tau \ll 1$, $F_1(\tau)$ reduces to $\cos^2\psi$, while in the limit of large τ it is seen from Eq. (13a) that $F_1(\tau)$ approaches $\tau/2$. Thus $(\omega/4)$ (R/Q) represents the energy lost to the fundamental mode per unit charge for a single passage of the bunch, while the factor $2F_1/\tau$ takes account of the cumulative effect of a charge passing through the cavity on successive revolutions.

It is now simple to generalize the preceding expressions to take into account the additional loss due to the excitation of higher-order modes. This loss can be represented as an additional beam-loading voltage with real component

$$V_b(n>0) = \sum_{n>0} \frac{(w_n/q)\left(1 - e^{-2\tau_n}\right)}{1 - 2e^{-\tau_n}\cos\delta_n + e^{-2\tau_n}} \quad . \qquad (16)$$

Here $\tau_n = T_b/T_{fn}$, $\delta_n = (\omega_n - \omega)T_b = \omega_n T_b$, $w_n = q^2 b_n^2(\omega_n/4)(R/Q)_n$ is the energy a single bunch would deposit in the nth mode in an "empty" cavity, and b_n is the bunch length correction factor of Eq. (A.1). For small τ_n, the ratio of the total beam-loading voltage, $V_{bt} = V_b(n>0) + V_{b0}$ to V_{b0} for the

fundamental mode at resonance is

$$f(\tau) = \frac{V_{bt}}{V_{b0}} = 1+\tau^2 \sum_{n>0} \frac{(w_n/w_0)(\tau_n/\tau)}{2(1-\cos\delta_n)}$$

For large τ, Eq. (16) shows that $f(\tau)$ approaches a limiting value of $f(\infty) = 1+\Sigma_{n>0}(w_n/w_0)$. Explicit expressions for the w_n's are given by Morton and Neil[5] for cylindrical cavities. As a function of τ, $f(\tau)$ should look like the curve in Fig. 4, with an inflection point at roughly $\tau \approx 1$. Shown also are resonances at particular values of τ for which higher-mode frequencies are exact multiples of $1/T_b$.

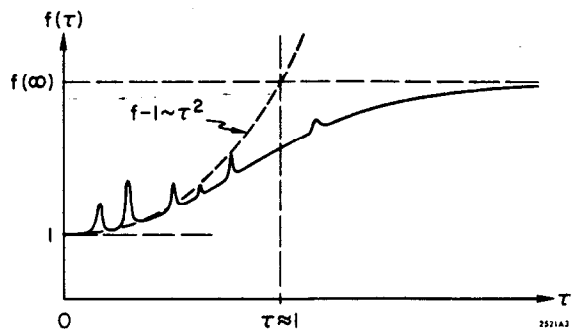

FIG. 4--The beam loading loss enhancement factor $f(\tau)$.

A measurement of V_{bt} in the limit $\tau >> 1$ has been made for a typical accelerating structure. An energy loss of 38 MeV has been measured[6] for a bunch of 10^9 electrons passing through the SLAC structure. The loss if only the fundamental mode were excited would be $V_{b0} = q(\omega_0/4)(rL/Q)$. Substituting in $q = 1.6\times10^{-10}$, $\omega = 2\pi\times2856$ MHz, $L = 2880$ m and $r/Q = 4100\ \Omega/m$, we calculate that $V_{b0} = 8.5$ MV. Thus the single-bunch loss in the SLAC structure is enhanced by a factor of 4.5 due to the excitation of higher modes. This corresponds to the limit $f(\infty)$ in Fig. 4. For PEP, $\tau \approx 0.4$. If the PEP RF structure were to behave similarly to the SLAC structure, and if the enhancement factor for the SLAC structure were to behave as shown in Fig. 4, for PEP f would be about 1.5. Because the bunch length is relatively longer in a storage ring than in a linac, this is an overestimate of the enhancement factor. However, even an enhancement of $f = 1.25$ would make it necessary to increase the klystron power by 3.4 MW, or about 50% over the present design value of 7.2 MW. (See Appendix C for details.)

In principle, the enhancement factor f can be calculated for any value of τ using Eq. (16) if the frequencies, decay times and R/Q's of all modes with wavelengths longer than the bunch length are known. However, the measurement of these quantities with the necessary degree of completeness and precision is indeed a difficult problem. It may be necessary to measure the loss enhancement experimentally by sending a train of bunches at an energy of least several MeV, spaced apart in time by T_b, through a prototype structure. By adjusting the shapes of the cells within the structure and by trimming the higher-order mode frequencies with tuners, it should be possible to minimize the loss enhancement factor for a given value of T_b.

Acknowledgements

I would like to thank M. Allen, P. Morton, M. Lee, G. Loew and R. McConnell for helpful discussions on various aspects of the problem of beam loading in storage rings. Discussions with M. Sands and J. Rees were instrumental in the development of the analysis given in Appendix C.

Appendix A. The Bunch Form Factor

In Ref. 7 it is shown that the amplitude of the field induced in the nth mode of a cavity by the passage of a bunch having a current i(t) is reduced below the field induced by a short bunch by the factor

$$b_n = \left(I_{ns}^2 + I_{na}^2\right)^{1/2} , \qquad (A.1)$$

where

$$I_{ns} = \frac{1}{q\omega_n}\int i(t')\cos(\omega_n t')\, d(\omega_n t') \ ;$$

$$I_{na} = \frac{1}{q\omega_n}\int i(t')\sin(\omega_n t')\, d(\omega_n t') \ .$$

The phase of the induced field with respect to the real axis of a rotating coordinate system $e^{j\omega_n t}$ is given by $\tan\theta_n = (I_{na}/I_{ns})$.

Appendix B

Assume a driving current i having a phase modulation of amplitude A, assumed small, such that $\tilde{i} = \tilde{i}_0(1+jA\cos\omega_s t)$. The response of a resonant circuit to this driving current is

$$\tilde{V} = R\, i_0 \left\{ \frac{1}{1+j\xi} + \frac{jA}{2}\left[\frac{e^{j\omega_s t}}{1+j(\xi+\eta)} + \frac{e^{-j\omega_s t}}{1+j(\xi-\eta)}\right]\right\} ,$$

where $\xi = (\omega-\omega_0)T_f$ and $\eta = \omega_s T_f$. The terms in $e^{j\omega_s t}$ and $e^{-j\omega_s t}$ represent two counter-rotating vectors with origins at the tip of the vector $R\,\tilde{i}_0\cos\psi\, e^{j\psi}$, where $\tan\psi = -\xi$. It is readily shown that the resultant of the two rotating vectors traces out an ellipse with semi-major axis

$$a = (A/2)\left\{\left[1+(\xi+\eta)^2\right]^{-1/2} + \left[1+(\xi-\eta)^2\right]^{-1/2}\right\}$$

and semi-minor axis

$$b = (A/2)\left\{\left[1+(\xi+\eta)^2\right]^{-1/2} - \left[1+(\xi-\eta)^2\right]^{-1/2}\right\} .$$

The ellipse is rotated through an angle $\gamma = \pi/2 + (\psi_+ + \psi_-)/2$ with respect to the real axis, where $\tan\psi_+ = -(\xi+\eta)$ and $\tan\psi_- = -(\xi-\eta)$. This result is illustrated in Fig. 5.

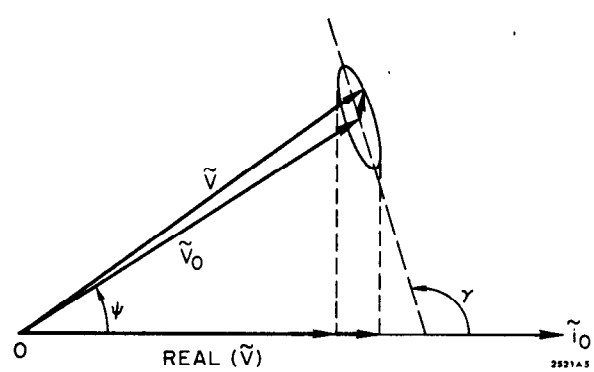

FIG. 5--Response of a resonant circuit to a phase-modulated driving current. Here $\tilde{V}_0$ is the unmodulated portion of the response, and REAL $(\tilde{V})$ is the projection of $\tilde{V}$ on $\tilde{i}_0$.

If $\tilde{i}_0$ is a beam-loading current, the projection of the ellipse on $\tilde{i}_0$ provides a component in the effective restoring force for phase oscillations. For ξ or $\eta = 0$ the ellipse collapses to a straight line, and the restoring force is in phase with the oscillations. For $|\xi| > 0$, the restoring force

is shifted in phase with respect to the oscillations, if η is also > 0, leading to a growth or decay of the oscillations.

Assume that the phase oscillations have the form $\Delta = \Delta_0 e^{(j\omega_s + \alpha)t}$. The differential equation for Δ can be written as

$$\frac{d^2\Delta}{dt^2} = -\omega_{s0}^2 \widetilde{T}\Delta = (j\omega_s + \alpha)^2 \Delta \qquad \text{(B. 1)}$$

where ω_{s0} is the usual relation for the synchrotron frequency. The function $\widetilde{T}$ includes the effect of the cavity response on restoring force as mentioned above, and is

$$\widetilde{T} = \left\{1 - \frac{V_{br}}{V_c \sin\phi}\left[F_a(\xi,\eta) + jF_b(\xi,\eta)\right]\right\}$$

where

$$F_a = \frac{\xi(1+\xi^2-\eta^2)}{[1+(\xi+\eta)^2][1+(\xi-\eta)^2]} \qquad \text{(B. 2a)}$$

$$F_b = \frac{-2\xi\eta}{[1+(\xi+\eta)^2][1+(\xi-\eta)^2]} \qquad \text{(B. 2b)}$$

Substituting for $\widetilde{T}$ in Eq. (B. 1), and equating real and imaginary parts, we obtain

$$\omega_s^2 - \alpha^2 = \omega_{s0}^2\left[1 - \left(\frac{V_{br}}{V_c \sin\phi}\right) F_a\right] \qquad \text{(B. 3a)}$$

$$2\omega_s\alpha = \omega_{s0}^2\left(\frac{V_{br}}{V_c \sin\phi}\right) F_b \qquad \text{(B. 3b)}$$

It is seen that, for $\xi \sim \omega - \omega_0 > 0$, the damping constant is negative and the phase oscillations are damped. In the limit of low oscillation frequency, $F_b = 0$ and $F_a = \xi/(1+\xi^2)$. Using also the real reflected power condition of Eq. (5), $\xi = (V_{br}/V_c)\sin\phi$, Eq. (B. 3a) reduces to the result obtained previously in Eq. (9). For $\eta >> 1$, $F_a = 0$ and $\omega_s = \omega_{s0}$. For $\eta \sim 1$, Eqs. (B. 3a) and (B. 3b) can be combined to give a fifth-order polynomial in ω_s. After solving this polynomial for ω_s, the damping constant is readily obtained from Eq. (B. 3b).

If $\Delta\omega_s = \omega_{s0} - \omega_s$ is small compared to ω_{s0}, we can write $\alpha/\Delta\omega_s \approx F_b/F_a = -2\eta/(1+\xi^2-\eta^2)$. Since α is also the half-width of the line for damped oscillations, this result shows that for $\eta \approx 1$ the line width is of the same order as the frequency shift.

Appendix C. Calculation of Power Loss due to Higher-Order Mode Excitation

A revised energy gain expression, which includes the additional energy loss due to higher-order cavity mode excitation, can be written as follows:

$$V_s + V_{hm} = V_{ga} - V_{b_0} F_1(\tau) = V_c \cos\phi \quad .$$

The additional energy loss to higher modes, $qV_{hm} = q(f-1)V_{b0}$, is equivalent to an enhancement in the synchrotron radiation loss per turn in its effect on the synchronous phase angle and over-voltage ratio for the fundamental mode. The power transferred to the beam is increased by an amount $i_0 V_{hm}$. In addition, a higher peak cavity voltage is required to contain quantum fluctuations. It is shown by Sands[9] that, in the absence of higher-mode cavity losses, the height of the potential barrier which must be overcome by energy fluctuations is proportional to $V_s(\tan\phi - \phi)$. Thus, if the same potential barrier is to be maintained in the presence of higher-mode losses, a new synchronous phase angle ϕ' is required, where ϕ' is obtained from

$$\tan\phi' - \phi' = \frac{V_s}{V_s + V_{hm}}(\tan\phi - \phi) \quad . \qquad \text{(C. 1)}$$

The revised peak cavity voltage required to give the same quantum lifetime is then

$$V_c' = \frac{V_s + V_{hm}}{\cos\phi'} \quad , \qquad \text{(C. 2)}$$

and the revised rf power requirement is

$$P_g' = \frac{V_c'^2}{R} + i_0(V_s + V_{hm}) \quad . \qquad \text{(C. 3)}$$

For PEP, $V_{b0} = i_0 R/(1+\beta_m) = 42$ MV, using $i_0 = 0.2$A, $R = 950$ MΩ and $\beta_m = 3.55$. For $f = 1.25$, $V_{hm} = (f-1)V_{b0} = 10.5$ MV. Using this value in Eq. (C. 1), together with $V_s = 26$ MV and $\phi = \cos^{-1}(26\text{ MV}/44\text{ MV}) = 53.8^\circ$, we calculate $\phi' = 49.4^\circ$. From Eq. (C. 2) we obtain $V_c' = 56$ MV, which can be compared to $V_c = 44$ MV with $V_{hm} = 0$. From Eq. (C. 3) we find that $P_g' = 10.6$ MW, an increase by 3.4 MW over the power requirement for $V_{hm} = 0$. The cavity wall losses in the fundamental mode are increased from 2.0 to 3.3 MW as the result of the higher peak voltage requirement, and the power transferred to the beam is increased from 5.2 to 7.3 MW. However, this additional 2.1 MW also ends up as power dissipated in the cavity walls.

It must be emphasized that the preceding calculation is based on a very crude estimate of the enhancement factor f. The PEP structure, which consists of a chain of inductively-coupled shaped cells with re-entrant nose cones, may have a behavior which is quite different from that of the disk-and-cylinder structure measured at SLAC[6] and investigated theoretically by Keil.[8] Further experimental and theoretical work is clearly required to resolve this important question in detail.

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DISCUSSION

James Leiss (NBS): Have you investigated what happens when bunches don't have the same population?

Wilson (SLAC): The answer is no. It seems to be messy enough for the case where the bunches have the same population, but it certainly should be looked into. The answer is, I have not.

Mark Barton (BNL): Have you investigated a case where the energy extracted by the beam in one pass is a sizeable fraction of the energy stored in the cavity?

Wilson: The expressions I showed are valid also in that case. In fact, they include the case where the stored energy goes through zero, and the beam induces a field of the opposite polarity.

Gordon Danby (BNL): If you have that case, where the loading is very heavy and the cavity fields haven't fully recovered by the time the next bunch arrives, then when you start, you have a different condition. How do you cope with that case?

Wilson: It is an equilibrium situation. In equilibrium the cavity recovers exactly back to the point where it was before the preceding bunch came by. If that does not happen by the time the next bunch comes by, you are not in equilibrium. Part of the calculation was to insure that that was exactly the case, that the cavity fields have recovered back exactly to the same point each time before the following bunch comes by.