# SINGLE PARTICLE CONTRIBUTIONS TO CERTAIN CLASSES 

of ALGEBRAIC SUM RULES*

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ABSTRACT

Phenomenological and theoretical aspects of single particle contributions to sum rules derived from commutation relations are considered. A derivation of sum rules arising from an equal time axial-charge algebra evaluated between arbitrary single-particle states is given. A phenomenological analysis of these sum rules is carried out. An analogous derivation of sum rules associated with the sigma operator is shown to be invalid. An amended form for the sum rules is derived. Finally, we comment on relations obtained by taking vacuum-vacuum or vacuum-single particle matrix elements of certain commutators.
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[^0]
## I. INTRODUCTION

We present in this paper a study of phenomenological and theoretical aspects of single particle contributions to sum rules arising from matrix elements of various commutator algebras. In all cases, we restrict the invariant momentum transfer, $q^{2}$, across these matrix elements to be either zero or small $\left(\left|q^{2}\right| \lesssim 0.5 \mathrm{GeV}^{2}\right)$.

The motivation for undertaking this study arose from a calculation by Golowich and Holstein ${ }^{1}$ in which a model for vector and axial-vector current excitation of the pion into arbitrary-spin single particle states was formulated and solved. This model parameterized the momentum transfer dependence, for small $q^{2}$, of matrix elements of the operators $V_{a}^{\mu}, A_{a}^{\mu}, \partial_{\mu} A_{a}^{\mu}(a=1,2,3)$ by means of $\rho, \mathrm{A}_{1}, \pi$ poles, as well as allowing for higher mass contributions by means of constants. Current algebra, the "partially conserved axial-vector current" hypothesis (PCAC), and the Bjorken-Johnson-Low (BJL) theorem were used to constrain the parameters of the model. A rigorous consequence of the conditions just enumerated is that excitation of the pion to single-particle states with spin $\mathrm{J} \geq 4$ is forbidden. This result led to a study of sum rules associated with commutators of time components of currents at zero or small $q^{2}$ taken between single pion states at infinite momentum. In particular, a Fourier transform of the charge density algebra

$$
\begin{equation*}
\left[\mathrm{A}_{+}^{0}(0, \overrightarrow{\mathrm{x}}), \mathrm{A}_{-}^{0}(0)\right]=2 \delta^{3} \overrightarrow{(\mathrm{x})} \mathrm{V}_{3}^{0}(0) \tag{1}
\end{equation*}
$$

(in this paper, $J_{ \pm} \equiv J_{1} \pm i J_{2}$ for any isospin carrying operator $J_{a}$ ), its first derivative with respect to $q^{2}$, and the first derivative with respect to $q^{2}$ of the Fourier transform of

$$
\begin{equation*}
\left[\mathrm{V}_{+}^{0}(0, \overrightarrow{\mathrm{x}}), \mathrm{V}_{-}^{0}(0)\right]=2 \delta^{3}(\overrightarrow{\mathrm{x}}) \mathrm{V}_{3}^{0}(0) \tag{2}
\end{equation*}
$$

were all evaluated at $q^{2}=0$. The sum rules were evaluated in resonance approximation. With experimentally determined decay widths as input, it was found that resonances of spin $\mathrm{J}<4$ were successful in nearly saturating the sum rules. ${ }^{1}$ On this basis, it was conjectured in Ref. 1 that damping of small $q^{2}$ current-induced transitions for which the difference in spin exceeds some moderate value (perhaps $\Delta J \sim 4$ ) might be a general hadronic phenomenon.

Further thought has tempered, to some extent, our enthusiasm for this outlook. It is possible that the underlying reason for the success of the saturated sum rules lies in the subtractive nature of the commutators in Eqs. (1) and (2). That is, the terms which arise from the two different orderings of the operators in these particular commutators contribute to the sum rule with opposite relative signs. Thus, terms associated with large mass contributions may have little effect on the commutator due to cancellation, although each could be individually large.

Without deeper theoretical understanding, it is not easy to judge the relative importance of these two mechanisms. Conceivably, either can be true to a greater or lesser degree. At any rate, we have been stimulated to examine a phenomenological aspect of this subject, a numerical evaluation of resonance saturated sum rules associated with the equal time charge algebra

$$
\begin{equation*}
\left[F_{+}^{5}(0), F_{-}^{5}(0)\right]=2 \mathrm{~F}_{3}(0) \tag{3}
\end{equation*}
$$

taken between arbitrary, diagonal single-particle states. This analysis is given in Section II. Our primary aim is simply to ascertain how the numbers come out in light of existing experimental data. As a matter of principle, it is important to keep subjecting relations like Eq. (3) to new experimental tests, even though previous studies ${ }^{2}$ lead us to accept its validity. The cleanest
signal of something wrong with Eq. (3) would be oversaturation, in which contributions to the left-hand side exceed the bound given by the right-hand side. We also wish to exhibit the problems one encounters in practice while attempting to evaluate the charge algebra sum rules. What are the phenomenological limitations to these sum rules? Finally, we hope to stimulate experimental work in the difficult subject of higher meson and baryon resonances. The sum rules can provide, in individual cases, a quantitative measure of the extent to which further couplings to a given hadron are to be anticipated in order that the sum rule be saturated.

A natural extension of the work (based on Eq. (3)) just described is to apply the same methods to the equal time commutator

$$
\begin{equation*}
\mathrm{i}\left[\mathrm{~F}_{+}^{5}(0), \partial_{\mu} \mathrm{A}_{-}^{\mu}(0)\right]=2 \sigma(0) \tag{4}
\end{equation*}
$$

where $\sigma$ (the "sigma operator") is assumed for simplicity to carry zero isospin. As will be described in Section III, this turns out to be impossible. The sum rules which result are not valid because the mathematical procedures used in deriving them are not legitimate. A method suggested by R. Jaffe and coworkers ${ }^{3}$ for eliminating this difficulty is discussed, and an amended class of sum rules is written down. The emphasis in this Section is almost entirely theoretical.

Thus far, we have discussed the contribution of single particle intermediate states to certain commutation relations evaluated between single particle states. In the interest of thoroughness, we devote Section IV to a brief survey of the status of "simpler" commutator matrix elements, in which one or both of the external states is the vacuum state.

The paper concludes in Section V with a summary of our results and a discussion of their significance.

## II. AXIAL CHARGE COMMUTATOR

Our goal, to perform a phenomenological analysis involving the commutation relation of Eq. (3) taken between arbitrary diagonal single-particle states, actually dictates that the single-particle intermediate states play a central role. Otherwise we would end up with formulae generally having no realistically obtainable experimental content.

We shall begin with derivations containing enough detail to establish our notation as well as to make the paper self-contained for the reader. ${ }^{4}$ Suppose the commutation relation in Eq. (3) is sandwiched between initial and final states $|\alpha(\vec{p}, r)\rangle,\left\langle\alpha\left(\overrightarrow{p^{\prime}}, r\right)\right|$ respectively, where $r$ is a helicity label. In the numerical work to be discussed later, we shall consistently choose $\alpha$ to be the state of highest weight in its isospin multiplet. ${ }^{5}$ For definiteness, we shall assume that it carries charge +1 in the following derivation. Let us insert an intermediate state consisting of some particle $\gamma$, not belonging to the same isotopic multiplet as $\alpha$, and also sum over the helicity r of particle $\alpha$. We find

$$
\begin{align*}
\sum_{r, r^{\prime}} \int \frac{d^{3} q}{(2 \pi)^{3} N_{\gamma}} & \left\{\left\langle\alpha \overrightarrow{p^{\prime}}, r\right)\left|F_{+}^{5}(0)\right| \gamma^{0}\left(\vec{q}, r^{\prime}\right)\right\rangle\left\langle\gamma^{0}\left(\vec{q}, r^{\prime}\right)\right| F_{-}^{5}(0)|\alpha(\vec{p}, r)\rangle \\
& \left.\left.-\left\langle\alpha \overrightarrow{\left(p^{\prime}\right.}, r\right)\left|F_{-}^{5}(0)\right| \gamma^{++}\left(\vec{q}, r^{\prime}\right)\right\rangle\left\langle\gamma^{++}\left(\vec{q}, r^{\prime}\right)\right| F_{+}^{5}(0)|\alpha(\vec{p}, r)\rangle\right\} \\
+\ldots & =2(2 \pi)^{3} \mathrm{~T}_{3}(\alpha)\left(2 J_{\alpha}+1\right) \mathrm{N}_{\alpha} \delta^{3}\left(\overrightarrow{p^{\prime}}-\vec{p}\right) \tag{5}
\end{align*}
$$

where the normalization of single particle states is given by

$$
\begin{equation*}
\left.\left\langle\alpha\left(\overrightarrow{p^{\prime}}, \mathrm{r}^{\prime}\right) \mid \alpha(\overrightarrow{\mathrm{p}}, \mathrm{r})\right\rangle=(2 \pi)^{3} \mathrm{~N}_{\alpha} \delta_{\mathrm{rr}^{\prime}} \delta^{3} \overrightarrow{\left(\overrightarrow{p^{\prime}}\right.}-\overrightarrow{\mathrm{p}}\right) . \tag{6}
\end{equation*}
$$

The normalization factor $N_{\alpha}$ need not be specified any further in this Section because it will cancel out of our equations. Next, express each axial-charge in
terms of its charge density and use translation invariance to carry out the spatial integrals. As a result, all states $\alpha$ and $\gamma$ have the same momentum $\overrightarrow{\mathrm{p}}$. Thus, if we employ
$\langle\alpha(\vec{p}, r)| \partial_{\mu} A_{+}^{\mu}(0)\left|\gamma^{0}\left(\vec{p}, r^{\prime}\right)\right\rangle=i\left(p_{\alpha}^{0}-p_{\gamma}^{0}\right)\langle\alpha(\vec{p} ; r)| A_{+}^{0}(0)\left|\gamma^{0}\left(\vec{p}, r^{\prime}\right)\right\rangle$,
along with the PCAC relation
$\langle\alpha(\overrightarrow{\mathrm{p}}, \mathrm{r})| \partial_{\mu} \mathrm{A}_{+}^{\mu}(0)\left|\gamma^{0}\left(\overrightarrow{\mathrm{p}}, \mathrm{r}^{\prime}\right)\right\rangle=\frac{\left.\mathrm{m}_{\pi}^{2} \mathrm{~F}_{\pi}\langle\alpha \overrightarrow{\mathrm{p}}, \mathrm{r})\left|J_{+}^{\pi}(0)\right| \gamma^{0}\left(\overrightarrow{\mathrm{p}}, \mathrm{r}^{\prime}\right)\right\rangle}{\mathrm{m}_{\pi}^{2}-\left(\mathrm{p}_{\alpha}-\mathrm{p}_{\gamma^{\prime}}\right)^{2}}$
where $J_{+}^{\pi}$ is the pion current ${ }^{6}$ and $F_{\pi} \cong 94 \mathrm{MeV}$, we obtain

$$
\begin{gather*}
\sum_{r, r^{\prime}} \frac{\left(m_{\pi}^{2} \mathrm{~F}_{\pi}\right)^{2}}{\left(\mathrm{~m}_{\pi}^{2}-\left(\mathrm{p}_{\alpha}-\mathrm{p}_{\gamma}\right)^{2}\right)^{2}} \frac{\left\{1<\alpha(\overrightarrow{\mathrm{p}}, \mathrm{r})\left|J_{+}^{\pi}(0)\right| \gamma^{0}\left(\overrightarrow{\mathrm{p}}, \mathrm{r}^{\prime}\right)>\left.\right|^{2}-|<\alpha(\overrightarrow{\mathrm{p}}, \mathrm{r})| J_{-}^{\pi}(0)\left|\gamma^{++}\left(\overrightarrow{\mathrm{p}}, \mathrm{r}^{\prime}\right)>\right|^{2}\right\}}{2 \mathrm{~N}_{\alpha} \mathrm{N}_{\gamma} \mathrm{T}_{3}(\alpha)\left(2 \mathrm{~J}_{\alpha}+1\right)\left(\mathrm{p}_{\alpha}^{0}-\mathrm{p}_{\gamma}^{0}\right)^{2}} \\
+\ldots=1 \tag{9}
\end{gather*}
$$

Upon taking the limit $|\vec{p}| \rightarrow \infty$, we may relate the above squared matrix elements to decay widths if $\left|m_{\gamma}-m_{\alpha}\right|>m_{\pi}$. For definiteness, we temporarily take $m_{\gamma}>m_{\alpha}+m_{\pi^{*}}$. The association of the above matrix elements with physical decay widths is not exact. The latter are proportional to squared matrix elements having momentum transfer $q^{2}=m_{\pi}^{2}$, whereas the former have $q^{2}=0$ in the limit $|\overrightarrow{\mathrm{p}}| \rightarrow \infty$. We shall assume that the physical decay widths can be used without appreciable error. ${ }^{7}$ This is the main point at which we employ the PCAC hypothesis. We can then express Eq. (9) in the form

$$
\begin{equation*}
\frac{8 \pi \eta}{\mathrm{~T}_{3}(\alpha)} \frac{2 \mathrm{~J}_{\gamma}+1}{2 \mathrm{~J}_{\alpha}^{+1}} \frac{\mathrm{~F}_{\pi}^{2} \mathrm{~m}_{\gamma}^{2}}{\mathrm{k}\left(\mathrm{~m}_{\gamma}^{2}-\mathrm{m}_{\alpha}^{2}\right)^{2}} \quad\left\{\Gamma\left(\gamma^{0} \rightarrow \alpha \pi^{-}\right)-\Gamma\left(\gamma^{++} \rightarrow \alpha \pi^{+}\right)\right\}+\ldots=1 \tag{10}
\end{equation*}
$$

where $\eta=2$ if the particle $\alpha$ is a pion, $\eta=1$ otherwise, and k is the decay momentum evaluated in the parent rest frame. There is a question as to whether $k$ should be evaluated with the pion mass taken as zero or physical. We have chosen to use the former, thus implying

$$
\begin{equation*}
\mathrm{k}=\frac{\left(\mathrm{m}_{\gamma}^{2}-\mathrm{m}_{\alpha}^{2}\right)}{2 \mathrm{~m}_{\gamma}} \tag{11}
\end{equation*}
$$

A relation analogous to Eq. (10) can be written down for the case $\mathrm{m}_{\alpha}>\mathrm{m}_{\gamma}+\mathrm{m}_{\pi}$ where now, $\eta=2$ if $\gamma$ is a pion. Summing over all contributions of single particle states $\gamma$, we finally obtain

$$
\begin{align*}
16 \pi \mathrm{~F}_{\pi}^{2}\{ & \sum_{\gamma} \frac{2 J_{\gamma}^{+1}}{2 J_{\alpha}^{+1}} \frac{\eta}{\mathrm{~T}_{3}(\alpha)} \frac{\Gamma\left(\gamma^{0} \rightarrow \alpha \pi^{-}\right)-\Gamma\left(\gamma^{++} \rightarrow \alpha \pi^{+}\right)}{\mathrm{m}_{\gamma}^{3}\left(1-\frac{\mathrm{m}_{\alpha}^{2}}{\mathrm{~m}_{\gamma}^{2}}\right)^{3}} \\
& \left.+\sum_{\gamma}^{(2)} \frac{\eta}{\mathrm{T}_{3}^{(\alpha)}} \frac{\Gamma\left(\alpha \rightarrow \gamma^{0} \pi^{+}\right)-\Gamma\left(\alpha \rightarrow \gamma^{++} \pi^{-}\right)}{\mathrm{m}_{\alpha}^{3}\left(1-\frac{\mathrm{m}_{\gamma}^{2}}{\mathrm{~m}_{\alpha}^{2}}\right)^{3}}\right\}+\ldots=1 . \tag{12}
\end{align*}
$$

The superscripts on the summation symbols refer to (1) $m_{\gamma}>m_{\alpha}+m_{\pi}$ and (2) $m_{\alpha}>m_{\gamma}+m_{\pi}$. Contributions not explicitly included in Eq. (12) must have mass within a band $m_{\alpha^{\prime}} \pm m_{\pi^{*}}$. There are only a finite number of these.

Equation (12) will form the basis of our phenomenological analysis. Its content can be clarified somewhat by writing it in terms of spin-averaged total cross sections $\bar{\sigma}_{ \pm}$pertaining to the center-of-mass scattering of charged ( $\pm$) pions off the particle $\alpha$. Starting from a relation like Eq. (9) and taking account of the initial state flux factor and final state phase space, it is not difficult to
express the sum rule as

$$
\begin{equation*}
-\frac{2 \mathrm{~F}_{\pi}^{2}}{\pi} \int_{\mathrm{s}_{0}}^{\infty} \frac{\mathrm{ds}}{\mathrm{~s}-\mathrm{m}_{\alpha}^{2}}\left(\bar{\sigma}_{-}(\mathrm{s})-\bar{\sigma}_{+}(\mathrm{s})\right)+\underset{\text { terms }}{\text { discrete }}=2 \mathrm{~T}_{3}(\alpha) \tag{13}
\end{equation*}
$$

where $s$ is the invariant energy and $s_{0}=\left(m_{\alpha}+m_{\pi}\right)^{2}$. The "discrete terms" in Eq. (13) correspond to contributions with mass less than $m_{\alpha}+m_{\pi}$. We can recover Eq. (12) from Eq. (13) by using the narrow resonance Breit-Wigner formula

$$
\begin{equation*}
\bar{\sigma}_{ \pm}=\frac{4 \pi}{\mathrm{k}^{2}} \frac{2 \mathrm{~J}_{\gamma}+1}{2 \mathrm{~J} \alpha^{+1}} \frac{\pi \Gamma\left(\pi^{ \pm} \alpha\right)}{2} \quad \delta\left(\mathrm{~m}_{\gamma}-\mathrm{W}\right) \tag{14}
\end{equation*}
$$

where $\mathrm{W}=\mathrm{s}^{1 / 2}$ and k is given in Eq. (11).
Before commencing our numerical study of Eq. (12), we wish to point out two features of Eqs. (12) and (13). First is the convergent nature of the sum rules for large mass contributions, as evidenced especially in Eq. (13) via the Pomeranchuk theorem. This is one of the mechanisms mentioned in the Introduction which tend to made this class of sum rules approximable in terms of single particle contributions. Its origin lies in the antisymmetric behavior of Eq. (3) under ( $+\leftrightarrow-$ ), a property not universally shared by all commutators as we shall see in the next Section. A second noteworthy feature of the sum rules, more easily apparent in Eq. (12), is the existence of contributions not expressible in terms of decay widths or cross sections. These terms are known only in special cases - more often, we lack even a reasonable theoretical estimate of them. It is this, along with the fact that these terms become more numerous (albeit finite in number) as the mass of the state chosen for $\alpha$ is increased, which constitutes the major limitation in confronting the sum
rules (12) with experimental data. As candidates for the external states $\alpha$, we shall consider first the baryons, then the mesons. Unless otherwise specified, the data is taken from Ref. 8. ${ }^{9}$ In order of their appearance, the baryons to be surveyed here are $\mathrm{N}^{+}(938), \Delta^{++}(1233), \mathrm{N}^{+}{ }^{+}(1470), \Sigma^{+}(1189), \mathrm{Y}_{1}^{+}(1384)$, and $z^{\circ}(1315):$
$\mathrm{N}^{+}(938)$ : This is naturally the case for which, of all the hadrons, the most data is available. Numerics are exhibited in Table Ia. A summary of contributions is given by

$$
\begin{equation*}
\underbrace{\mathrm{g}_{\mathrm{A}}^{2}}_{\text {nucleon }}+\underbrace{0.544}_{\mathrm{T}=1 / 2}-\underbrace{0.975}_{\Delta(1233)}-\underbrace{0.175}_{\mathrm{T}=3 / 2}+\ldots=1 \tag{15}
\end{equation*}
$$

where the $T=1 / 2,3 / 2$ contributions group together all resonances of a given isospin. There are two ways in which Eq. (15) might naturally be interpreted:
(i) Simply insert the existing experimental value for $g_{A}$, thereby testing how well Eq. (15) is saturated. With $\mathrm{g}_{\mathrm{A}}=1.25$, we obtain $0.96+\ldots=1$. (ii) Use Eq. (15) to compute $g_{A}$. This is a traditional way of using algebras, often with poor success because the number of intermediate states taken into account is truncated too severely. From Eq. (15), we find $\mathrm{g}_{\mathrm{A}}^{2}=1.606$ or $\mathrm{g}_{\mathrm{A}} \cong 1.27$. In this case, nature has supplied us with enough data to give a reasonably good estimate of $\mathrm{g}_{\mathrm{A}}$.
$\Delta^{++}$(1233): The only intermediate state contribution not estimable in terms of a decay width is that of $\Delta(1233)$ itself. The parameter which characterizes this contribution is a form factor $F_{A}\left(q^{2}\right)$,
$\left\langle\Delta^{++}\left(\overrightarrow{p^{r}}, r^{\prime}\right)\right| A_{+}^{\mu}(0)\left|\Delta^{+}(\vec{p}, r)\right\rangle=i \bar{u}_{\sigma}\left(\overrightarrow{p^{\prime}}, r^{\prime}\right) \gamma_{5}\left\{\gamma_{g}^{\mu} \sigma_{\rho} F_{A}\left(q^{2}\right)+\ldots\right\} u_{\rho}(\vec{p}, r)$
evaluated at $q^{2}=0$. Analogous to $g_{A}$ of the nucleon axial-vector matrix element, let us define $\mathrm{f}_{\mathrm{A}} \equiv \mathrm{F}_{\mathrm{A}}(0)$ for the $\Delta(1233)$ axial-vector matrix element. Referring to the values given in Table Ib , we have

$$
\begin{equation*}
\underbrace{\frac{5}{27} \mathrm{f}_{\mathrm{A}}^{2}}_{\Delta}+\underbrace{0.244}_{\text {nucleon }}+\underbrace{0.149}_{\mathrm{T}=1 / 2}+\underbrace{0.03}_{\mathrm{T}=3 / 2}+\ldots=1 \tag{17}
\end{equation*}
$$

Unfortunately, it is not realistic to expect an experimental determination of $f_{A}$. Rather than anticipate Eq. (17) will provide a good value for $\mathrm{f}_{\mathrm{A}}$ provided that we assume the above numbers already saturate the sum rule, we prefer instead to adopt the more conservative stance of testing the degree of saturation in (17) by obtaining some estimate of $f_{A}$. This is done by first expressing $f_{A}$ in terms of the coupling constant $g_{\pi \Delta \Delta}$ via a Goldberger-Treiman relation and then using $\mathrm{SU}(6)_{W}$ to relate $\mathrm{g}_{\pi \Delta \Delta}$ to the known quantity $\mathrm{g}_{\pi \mathrm{NN}^{*}}$. We find $\mathrm{f}_{\mathrm{A}}^{2} \cong 2.1$, whereupon the sum rule (17) reads $0.82+\ldots=1$.
$\mathrm{N}^{+}(1470)$ : This state is the lowest in mass of the essentially continuous spectrum of highly excited $\pi \mathrm{N}$ resonances. As such, it represents the first case where our ability to test the sum rule (12) becomes seriously hindered. There is a band of width $2 m_{\pi}$ surrounding $N^{*}(1470)$ for which contributions to the sum rule cannot be estimated experimentally. This band contains the states $\mathrm{N}^{*}(1520)$ and $\mathrm{N}^{*}(1535)$ with spin-parity $\mathrm{J}^{\mathrm{P}}=\frac{3^{-}}{2}, \frac{1^{-}}{2}$ respectively. Of course, there is also the contribution of $N^{*}(1470)$, expressed in terms of a parameter $\mathrm{g}_{\mathrm{A}}^{*}$ entirely analogous to $\mathrm{g}_{\mathrm{A}}$. The sum rule reads

$$
\begin{equation*}
\underbrace{\mathrm{g}_{\mathrm{A}}^{2}}_{\mathrm{N}^{*}(1470)}+\underbrace{0.126}_{\text {nucleon }}-\underbrace{0.207}_{\Delta(1233)}+\mathrm{N}^{*}(1520)+\mathrm{N}^{*}(1535)+\ldots=1 \tag{18}
\end{equation*}
$$

Naturally, for the more massive states $\alpha$, the amount of data pertaining to resonances which decay into $\alpha$ plus a pion gets scarcer. This explains the paucity of numerical information in Eq. (18) relative to that in Tables Ia and Ib. However, this does not constitute a fundamental difficulty like the discrete contributions just discussed. If experiments in hadron spectroscopy continue, we can hope that transitions from one higher resonance to another can ultimately be unravelled. 'Ihis is not easy but at least it is possible. At any rate all we can infer from the numbers in Eq. (18) is that the sum of the discrete contributions equals 1.1 , given the nucleon and $\Delta(1233)$ contributions.
$\underline{\Sigma}^{+}(1189)$ : This state is of interest because it has the lowest mass for which the sum rule (12) is testable in a channel with non-zero strangeness and baryon number one. There are two discrete contributions, so that the sum rule reads

$$
\begin{equation*}
\frac{1}{2}\left(\mathrm{~g}_{\mathrm{A}}^{\left(\Sigma^{+} \Sigma^{\mathrm{o}}\right)}\right)^{2}+\frac{1}{2}\left(\mathrm{~g}_{\mathrm{A}}^{\left(\Sigma^{+} \Lambda\right)}\right)^{2}+\text { resonances }=1 \tag{19}
\end{equation*}
$$

We use $\operatorname{SU}(3)$, with an $F / D$ parameter $\alpha \cong 2 / 3$, to estimate the quantities $\mathrm{g}_{\mathrm{A}}{\left(\Sigma^{+} \Sigma^{0}\right)}$ and $\left.\mathrm{g}_{\mathrm{A}}{\left(\Sigma^{+}\right.}^{+}\right)$. The resonance contributions are exhibited in Table Ic. Altogether, the numbers are

$$
\begin{equation*}
\underbrace{0.174}_{\Sigma}+\underbrace{0.231}_{\Lambda}+\underbrace{0.142}_{\mathrm{T}=0}+\underbrace{0.055}_{\mathrm{T}=1}+\ldots=1 \tag{20}
\end{equation*}
$$

or in total, $0.60+\ldots=1$. There remains a fairly substantial contribution to be made from as yet unobserved decays of hyperchange zero baryon resonances into $\Sigma \pi$. Note that the low mass $\frac{1}{2}^{+}$and $\frac{3}{2}^{+}$contributions to the proton sum rule add to 0.588 whereas for the $\Sigma^{+}$sum rule they give 0.442 . However, the higher resonances for the intensively studied $\pi \mathrm{N}$ system sum to 0.369 whereas the more
complicated system of hypercharge-zero higher resonances contributes only 0.16.
$\vec{Y}_{1}^{+}(1385)$ : The problem of unobservable contributions to the $Y_{1}^{+}(1385)$ sum rule is not serious. There is the contribution of $Y_{1}^{0}(1385)$ which we can estimate from $\operatorname{SU}(6)_{W}$ just as we did for the $\Delta^{+}(1233)$ contribution to the $\Delta^{++}(1233)$ sum rule. In addition, the unitary singlet state $\mathrm{Y}_{0}^{*}(1405)$ can contribute in principle. However, this transition proceeds only through $\operatorname{SU}(3)$ breaking effects. Thus, we can hope that its effect on the sum rule is minor. Our numerical analysis gives

or $0.62+\ldots=1$, which is a reasonably high amount of saturation considering the large mass of the external state.
$z^{0}(1315):$ Despite the experimental effort which has been put into the hypercharge - 1 baryon channel, distressingly little is known about the particle spectrum at this time. Our $\Xi^{0}$ sum rule reads

$$
\begin{equation*}
\underbrace{0.173}_{\Xi^{-}}+\underbrace{0.196}_{\Xi^{*}(1530)}+\ldots= \tag{22}
\end{equation*}
$$

where we have used $\operatorname{SU}(3)$ to estimate the $\Xi^{-}$(discrete) contributions. Equation (22) sums to $0.37+\ldots=1$, so the dominant contributions to the $\Xi^{0}$ sum rule remain to be detected. We can only await a correct interpretation of the resonant behavior around energy 1820 MeV , which has for so long resisted efforts at classification. This concludes our survey of the baryon sum rules.

The main difference between the structure of the baryon and meson sum rules is that those intermediate states $\gamma$ which lie in the same isotopic spin multiplet as the external state $\alpha$ are forbidden to contribute to the latter by G-parity. We shall consider, in turn, sum rules for the following meson states, $\pi^{+}, \rho^{+}(765), \mathrm{A}_{2}^{+}(1310), \mathrm{K}^{+}(494), \mathrm{K}_{\mathrm{V}}^{+}(892), \mathrm{K}_{\mathrm{T}}^{+}(1421)$, where the subscripts $V, A, T$ denote vector $\left(1^{-}\right)$, axial $\left(1^{+}\right)$, and tensor $\left(2^{+}\right)$respectively. $\underline{\pi}^{+}$: This sum rule, also studied in Ref. 1, gives

$$
\begin{equation*}
\underbrace{0.292}_{\epsilon(700)}+\underbrace{0.047}_{\epsilon^{\prime}(1000)}+\underbrace{0.097}_{\mathrm{f}(1260)}+\underbrace{0.47}_{\rho(765)}+\underbrace{0.043}_{\mathrm{g}(1680)}+\ldots=1 \tag{23}
\end{equation*}
$$

where $\pi \pi$ total widths $\Gamma(\epsilon \pi \pi)=300 . \mathrm{MeV}, \Gamma\left(\epsilon^{\prime} \pi \pi\right)=50 . \mathrm{MeV}$ have been employed. ${ }^{10}$ The above contributions come close to saturating the sum rule, yielding $0.95+\ldots=1$. Thus the lowest mass baryon and meson states have their resonant-dominated sum rules saturated to within five percent.
$\rho^{+}(765)$ : The only contribution to the $\rho^{+}$sum rule which is not directly measurable comes from the $\omega$ intermediate state. However, we may use Eq. (9) in conjunction with the Gell Mann-Sharp-Wagner model ${ }^{11}$ of the $\omega \rightarrow \pi \gamma$ transition to estimate it. We define a coupling constant $\mathrm{g}_{\omega \rho \pi}$ for the process $\omega\left(\overrightarrow{\mathrm{k}}, \mathrm{r}^{\prime}\right) \rightarrow \rho(\overrightarrow{\mathrm{q}}, \mathrm{r})+\pi(\overrightarrow{\mathrm{p}})$ with a momentum space interaction amplitude

$$
\begin{equation*}
\mathbf{g}_{\omega_{\rho} \pi} \epsilon_{\alpha \beta \mu \nu} \mathrm{k}^{\alpha} \epsilon^{\beta}\left(\mathrm{k}, \mathrm{r}^{\prime}\right) \mathrm{q}^{\mu} \epsilon^{\dagger \nu}(\mathrm{q}, \mathrm{r}) . \tag{24}
\end{equation*}
$$

The $\rho^{+}$sum rule can then be written as

$$
\begin{equation*}
\frac{1}{6}\left(\mathrm{~F}_{\pi} \mathrm{g}_{\omega \rho \pi}\right)^{2}+\underbrace{0.078}_{\pi}+\underbrace{0.189}_{\mathrm{A}_{1}(1100)}+\underbrace{0.042}_{\mathrm{A}_{2}(1310)}+\ldots=1 . \tag{25}
\end{equation*}
$$

A recent determination ${ }^{12}$ of $\mathrm{g}_{\omega \rho \pi}$ is quoted as $\mathrm{g}_{\omega \rho \pi}=14.4 \mathrm{GeV}^{-1}$, so the degree of saturation of the $\rho^{+}$sum rule is $0.62+\ldots=1$.
$\hat{\mathrm{A}}_{2}^{+} \underline{(1310)}$ : This is the largest mass non-strange meson whose sum rule we shall analyze numerically. The extent to which contributions from states lying within energy $m_{\pi}$ above or below the $A_{2}$ mass affect the sum rule is not clear because of our relative ignorance of the particle spectrum at these energies. An amusing example of the difficulty in estimating one of these contributions is provided by the axial-vector meson $\mathrm{B}(1237)$. In principle, $\mathrm{A}_{2}(1310)$ can decay into $\mathrm{B}(1237) \pi$ because the finite widths of these resonances provides a certain amount of phase space. ${ }^{13}$ The decay $\mathrm{A}_{2} \rightarrow \omega \pi \pi$ has been observed, ${ }^{14}$ and noting that $B \rightarrow \omega \pi$ is essentially the only decay mode of $B(1237)$, we can obtain an upper bound on the rate for $\mathrm{A}_{2} \rightarrow \mathrm{~B} \pi$. This upper bound would imply a whopping contribution from $\mathrm{B}(1237)$ of 0.78 to the $\mathrm{A}_{2}$ sum rule. However, it would also imply a dimensionless p-wave coupling constant $g^{2}\left(\mathrm{~A}_{2} \mathrm{~B} \pi\right) / 4 \pi \cong 70$, which in our opinion is too large to be believed. Therefore, we summarize the present situation as

$$
\begin{equation*}
\underbrace{0.024}_{\rho(765)}+\underbrace{0.01}_{\eta(549)}+\mathrm{B}(1237) \text { term }+\ldots=1 \tag{26}
\end{equation*}
$$

for the $\mathrm{A}_{2}^{+}$sum rule, with the $" \mathrm{~B}(1237)$ term" $\lesssim 0.78$.
K(494): There is data at present for us to take the contribution of just two resonances into account,

$$
\begin{equation*}
\underbrace{0.375}_{\mathrm{K}_{\mathrm{V}}(892)}+\underbrace{0.084}_{\mathrm{K}_{\mathrm{T}}(1421)}+\ldots=1 \tag{27}
\end{equation*}
$$

or $0.46+\ldots=1$. The analogous two states contribute 0.57 to the pion sum rule, so at this level the difference in convergence between the $\pi$ and K sum rules is not large. However, the predominantly isoscalar meson $\epsilon(700)$ contributes significantly to the pion sum rule but not at all to that of the kaon. The kaon sum rule must make up the difference with the more massive states, ${ }^{15}$ a situation which suggests a deep relation between the chiral algebra and the spectrum of hadron states.
$\mathrm{K}_{\mathrm{V}}(892):$ The status of the $\mathrm{K}_{\mathrm{V}}(892)$ sum rule would be clarified if more information on the axial-vector kaons in the mass range $1200-1400 \mathrm{MeV}$ were available. For our calculation, we have assumed that the axial-vector kaon with mass 1242 decays into $\mathrm{K}_{\mathrm{V}}(892) \pi$ with a width of $127 \mathrm{MeV} .{ }^{8}$ We find

$$
\begin{equation*}
\underbrace{0.0168}_{\mathrm{K}(494)}+\underbrace{0.0272}_{\mathrm{K}_{\mathrm{T}}(1421)}+\underbrace{0.33}_{\mathrm{K}_{\mathrm{A}}(1242)}+\ldots=1 \tag{28}
\end{equation*}
$$

or $0.48+\ldots=1$, a fair degree of saturation.
$\mathrm{K}_{\mathrm{T}} \xrightarrow{(1421):}$ Like its SU(3) partner $\mathrm{A}_{2}^{+}(1310)$, the $\mathrm{K}_{\mathrm{T}}^{+}(1421)$ has but two well determined contributions to its sum rule,

$$
\begin{equation*}
\underbrace{0.017}_{K(494)}+\underbrace{0.027}_{K_{V}(892)}+\ldots=1 \tag{29}
\end{equation*}
$$

or $0.044+\ldots=1$. The $0^{-}$and $1^{-}$states are thus seen to contribute almost negligibly to the sum rules of the tensor mesons $\mathrm{A}_{2}^{+}(1310)$ and $\mathrm{K}_{\mathrm{T}}^{+}(1421)$.

Further comments on the analysis just presented are reserved for the Conclusion. In the next Section, we consider an algebra, which at first sight, appears amenable to a similar treatment.

## II. SIGMA OPERATOR

The sigma operator has been defined in terms of an equal time commutation relation in Eq. (4). Not much empirical knowledge exists regarding matrix elements of this operator. The nucleon matrix element is thought to be given, to a good approximation, by the combination of isospin-even pion-nucleon amplitudes $\mathrm{A}^{(+)}+\nu \mathrm{B}^{(+)}$evaluated at $\mathrm{s}=\mathrm{m}_{\mathrm{N}}^{2}, \mathrm{t}=2 \mathrm{~m}_{\pi}^{2}$. A variety of recent phenomenological efforts ${ }^{16}$ points to a value $40 \lesssim\langle N| \sigma|N\rangle \lesssim 70 \mathrm{MeV}$ although a definitive evaluation has yet to be performed. The pion matrix element can be estimated in terms of a low energy theorem to be $\langle\pi| \sigma|\pi\rangle \cong \mathrm{m}_{\pi}^{2}$. This evaluation is suspect because it involves extrapolation over a distance $m_{\pi}^{2}$ of a quantity itself of order $\mathrm{m}_{\pi}^{2}$. However, it does provide an order of magnitude estimate.

We may use the methods of the previous Section to derive a class of sum rules for the sigma operator matrix element taken between single particle states. Upon doing so we find ${ }^{17}$

$$
\begin{gather*}
\phi_{\alpha}<\alpha(\overrightarrow{\mathrm{p}}, \lambda)|\sigma(0)| \alpha(\overrightarrow{\mathrm{p}}, \lambda)=-16 \pi \mathrm{~F}_{\pi}^{2}\left\{\sum_{\mathrm{m}_{\gamma}\left(1-\frac{\mathrm{m}_{\alpha}^{2}}{\mathrm{~m}_{\gamma}^{2}}\right)^{3}}^{\frac{2 \mathrm{~J}^{+1}}{2 \mathrm{\gamma}_{\alpha}+1}} \frac{\Gamma\left(\gamma^{\circ} \rightarrow \alpha \pi^{-}\right)+\Gamma\left(\gamma^{++} \rightarrow \alpha \pi^{+}\right)}{\mathrm{m}^{\mathrm{o}}\left(1-\frac{\mathrm{m}_{\alpha}^{2}}{\mathrm{~m}_{\gamma}^{2}}\right)^{3}}\right.
\end{gather*}
$$

where the superscripts on the summation symbols have the same meaning as in Eq. (12). The constant

$$
\phi_{\alpha}=\left\{\begin{align*}
1 & \text { mesons }  \tag{31}\\
2 \mathrm{~m}_{\alpha} & \text { baryons }
\end{align*}\right.
$$

occurs because meson and baryon matrix elements of the sigma operator have different units. A formula relating the sigma operator matrix elements to cross sections, analogous to Eq. (13) can also be derived,
$\left.\phi_{\alpha}<\alpha(\vec{p}, \lambda)|\sigma(0)| \alpha(\overrightarrow{\mathrm{p}}, \lambda)\right\rangle=-\frac{2 \mathrm{~F}_{\pi}^{2}}{\pi} \int \mathrm{ds}\left(\bar{\sigma}_{-}(\mathrm{s})+\bar{\sigma}_{+}(\mathrm{s})\right)+\begin{gathered}\text { discrete } \\ \text { terms }\end{gathered}$.
There are several features of EqS. (30) and (32) that warrant immediate discussion. The widths and cross sections are seen to add whereas in Eqs. (12) and (13) they contribute with opposite relative sign. This is not a mistake, but rather reflects the behavior of the commutation relation (4) under the interchange ( $+\leftrightarrow-$ ). Of greater significance is that, given existing estimates (e.g., Regge) of the asymptotic behavior of hadronic cross sections, our formulae for the sigma-operator matrix elements are seen to diverge. Even if these asymptotic estimates turn out to be wrong and the integrals in (32) actually converge, the situation is still bleak because the dominant contributions to (30) and (32) are of the wrong sign. ${ }^{18}$ As an example, the resonances exhibited in Table Ia give $\langle\mathrm{N}| \sigma(0)|\mathrm{N}\rangle \cong-0.92 \mathrm{~m}_{\mathrm{N}}$ according to Eq. (30). In the same manner, we calculate $\langle\pi| \sigma(0)|\pi\rangle \cong-36 \mathrm{~m}_{\pi}^{2}$ upon using the $\pi \pi$ resonances listed in the previous Section. Neither of these values can coexist with the estimates discussed at the beginning of this Section.

It turns out that one can employ a rather different derivation for the $q^{2}=0$ sum rule (32), the result of which exhibits remnants of its original form, while at the same time patching up its weak spots. Much of the work is already done in Ref. 3 so in the following, we shall just outline the necessary steps.

First, we rederive Eq. (32) as a $q^{2}=0$ sum rule involving structure functions for neutrino scattering. ${ }^{19}$ Consider the spin-averaged quantity

$$
\begin{equation*}
\left.\left.\left.\mathrm{W}^{\mu \nu}(\mathrm{q}, \mathrm{p})=\frac{1}{4 \pi} \int \mathrm{~d}^{4} \mathrm{x} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}\langle\alpha(\overrightarrow{\mathrm{p}})|\left[\mathrm{J}_{+}^{\mu}(\mathrm{x}), \mathrm{J}_{-}^{\nu}(0)\right] \right\rvert\, \alpha \overrightarrow{\mathrm{p}}\right)\right\rangle \tag{33}
\end{equation*}
$$

where $J_{+}^{\mu}$ is the $\Delta Y=0$ weak current which raises the hadronic charge. The sigma-operator matrix element is obtained by contracting Eq. (33) with $q_{\nu}$ and setting $\vec{q}=0$,

$$
\begin{equation*}
\phi_{\alpha}\langle\alpha(\vec{p})| \sigma(0)|\alpha(\vec{p})\rangle=-\int_{-\infty}^{\infty} \mathrm{dq}^{0} q^{0} W^{00}\left(q^{0}, p\right) . \tag{34}
\end{equation*}
$$

If we define a set of kinematic singularity free structure functions ${ }^{20}$

$$
\begin{align*}
\mathrm{W}^{\mu \nu}=-\mathrm{g}^{\mu \nu} \mathrm{W}_{1} & +\frac{\mathrm{p}^{\mu} \mathrm{p}^{\nu}}{\mathrm{m}_{\alpha}^{2}} \mathrm{~W}_{2}-\mathrm{i} \frac{\epsilon^{\mu \nu \alpha \beta} \mathrm{p}_{\alpha} \mathrm{q}_{\beta}}{2 \mathrm{~m}_{\alpha}^{2}} \mathrm{~W}_{3} \\
& +\frac{\mathrm{q}^{\mu} \mathrm{q}^{\nu}}{\mathrm{m}_{\alpha}^{2}} \mathrm{~W}_{4}+\frac{\left.\hat{\mathrm{p}}^{\mu} \mathrm{q}^{\nu}+\mathrm{p}^{\nu} \mathrm{q}^{\mu}\right)}{2 \mathrm{~m}_{\alpha}^{2}} \mathrm{~W}_{5} \tag{35}
\end{align*}
$$

where $\mathrm{W}_{\mathrm{i}}=\mathrm{W}_{\mathrm{i}}\left(\mathrm{q}^{2}, \nu\right), \nu=\mathrm{q} \cdot \mathrm{p}$, and use the crossing property

$$
\begin{equation*}
\mathrm{W}_{2}^{\nu}\left(\mathrm{q}^{2}, \nu\right)=-\mathrm{W}_{2}^{\bar{\nu}}\left(\mathrm{q}^{2},-\nu\right), \tag{36}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\phi_{\alpha}\langle\alpha(\overrightarrow{\mathrm{p}})| \sigma(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle=-\frac{1}{\mathrm{~m}_{\alpha}^{2}} \int_{0}^{\infty} \mathrm{dxx}\left(\mathrm{~W}_{2}^{\nu}(0, \mathrm{x})+\mathrm{W}_{2}^{\bar{\nu}}(0, \mathrm{x})\right) . \tag{37}
\end{equation*}
$$

But if the lepton mass is neglected, we can use the $q^{2}=0$ relation

$$
\begin{equation*}
\mathrm{W}_{2}^{\bar{\nu}}, \nu(0, \nu)=\frac{2 \mathrm{~m}_{\alpha}^{2} \mathrm{~F}_{\pi}^{2}}{\pi \mathrm{~kW}} \quad \sigma_{ \pm}(\mathrm{W}) \tag{38}
\end{equation*}
$$

(where $W=s^{1 / 2}, k=\left(s-m_{\alpha}^{2}\right) / 2 s$ ) in conjunction with Eq. (37) to regain Eq. (32). In other words, the steps leading to Eq. (37) are not valid.

The method suggested in Ref. 3 to deal with these difficulties is to consider the spin-averaged amplitude

$$
\begin{align*}
\mathrm{T}^{\mu \nu}\left(\mathrm{q}^{2}, \nu\right)=\mathrm{i} \int \mathrm{~d}^{4} \mathrm{xe} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}} \theta\left(\mathrm{x}^{0}\right) & <\alpha(\mathrm{p})\left|\left[\mathrm{J}^{\mu}(\mathrm{x}), \mathrm{J}^{\nu}(0)\right]\right| \alpha(\mathrm{p})> \\
& + \text { seagull terms } \tag{39}
\end{align*}
$$

where

$$
\begin{equation*}
\operatorname{Im} \mathrm{T}^{\mu \nu}\left(\mathrm{q}^{2}, \nu\right)=2 \pi \mathrm{~W}^{\mu \nu}\left(\mathrm{q}^{2}, \nu\right) \tag{40}
\end{equation*}
$$

The seagull terms are polynomials in $q$ which might be needed in order that $\mathrm{T}^{\mu \nu}$ be a legitimate second-rank tensor. The decomposition used in Ref. 3 for $\mathrm{W}^{\mu \nu}$ is the same as our Eq. (35) except for the presence of

$$
\begin{align*}
& \mathrm{W}_{4}^{\mathrm{t}}=\mathrm{W}_{4}-\frac{\mathrm{m}_{\alpha}^{2}}{\mathrm{q}^{2}} \mathrm{~W}_{1}-\frac{\nu^{2}}{\mathrm{q}^{4}} \mathrm{~W}_{2} \\
& \mathrm{~W}_{5}^{\mathrm{v}}=\mathrm{W}_{5}+\frac{2 \nu}{\mathrm{q}^{2}} \mathrm{~W}_{2} \tag{41}
\end{align*}
$$

in place of our $W_{4,5}$. The above decomposition, although not kinematic singularity free for $\mathrm{q}^{2} \rightarrow 0$, has the advantage that only $\mathrm{W}_{4,5}^{\prime}$ contribute to $\mathrm{q}_{\mu} \mathrm{W}^{\mu \nu}$. The amplitude $\mathrm{T}^{\mu \nu}$ of Eq. (39) has a similar decomposition in terms of amplitudes $T_{i}^{\prime}, i=1, \ldots, 5$. The key point is that appropriately subtracted dispersion relations can be written for each of the $\mathrm{T}_{\mathrm{i}}^{\prime}\left(\mathrm{q}^{2}, \nu\right)$. Thus, the high energy $(\nu \rightarrow \infty)$ behavior is properly accounted for. In particular, the dispersion relation for $\mathrm{T}_{4}^{\prime}\left(\mathrm{q}^{2}, \nu\right)$ is seen to contain a subtraction constant $\mathrm{T}_{4}^{\prime}\left(\mathrm{q}^{2}, 0\right)$. Information regarding the sigma operator is obtained by taking the BJL limit of
$\mathrm{q}_{\nu} \mathrm{T}^{\mu \nu}\left(\mathrm{q}^{2}, \nu\right)$. It is found that

$$
\begin{equation*}
\left.-\phi_{\alpha}<\alpha(p)|\sigma(0)| \alpha(p)\right\rangle=\frac{1}{2 \mathrm{~m}_{\alpha}^{2}} q^{2} \lim ^{\lim } q^{4} \overline{\mathrm{~T}}_{4}^{1}\left(q^{2}, 0\right) \tag{42}
\end{equation*}
$$

where the quantity $\widetilde{T}_{4}^{\prime}\left(q^{2}, 0\right)$ is that part of $T_{4}^{\prime}\left(q^{2}, 0\right)$ which varies as $q^{-4}$ in the limit $q^{2} \rightarrow \infty$. At this point, the situation for expressing $\langle\alpha| \sigma|\alpha\rangle$ in terms of measurable quantities admittedly looks hopeless.

One way out of this impasse is to conjecture the existence of $\mathrm{J}=0$ fixed poles ${ }^{21}$ in the $T_{4,5}^{\prime}$ amplitudes. The fixed poles can be extracted from $T_{4,5}^{\prime}$ by subtracting off from these amplitudes all Regge contributions in the range $1 \geq \alpha(0)>0$. One finds for their residues in the limit $q^{2} \rightarrow-\infty$,

$$
\begin{align*}
& \mathrm{C}_{4}\left(\mathrm{q}^{2}\right) \rightarrow \mathrm{T}_{4}^{\prime}\left(\mathrm{q}^{2}, 0\right)-\frac{8 \mathrm{~m}_{\alpha}^{4}}{\mathrm{q}^{4}} \int_{0}^{\infty}\left(\widetilde{\mathrm{F}}_{4}^{\nu}+\widetilde{\mathrm{F}}_{4}^{\bar{\nu}}\right) \frac{\mathrm{dx}}{3} \\
& \mathrm{C}_{5}\left(\mathrm{q}^{2}\right) \rightarrow \frac{4 \mathrm{~m}_{\alpha}^{2}}{\mathrm{q}^{2}} \int_{0}^{\infty}\left(\widetilde{\mathrm{F}}_{5}^{\nu}+\widetilde{\mathrm{F}}_{5}^{\bar{\nu}}\right) \frac{\mathrm{dx}}{\mathrm{x}^{2}} \tag{43}
\end{align*}
$$

where $\mathrm{F}_{4,5}$ are the scaling limits of $\nu^{2} \mathrm{~W}_{4,5} / \mathrm{m}_{\alpha}^{4}$, and $\widetilde{\mathrm{F}}_{4,5}=\mathrm{F}_{4,5}-\mathrm{F}_{4,5}^{(\mathrm{R})}$ where $F_{4,5}^{(R)}$ are the Regge fits which include all singularities with $1 \geq \alpha(0)>0$. Notice that Eq. (43) gives information on the subtraction constant $T_{4}^{\prime}\left(q^{2}, 0\right)$. The $C_{4,5}\left(q^{2}\right)$ can be related to fixed pole residues $P_{i}\left(q^{2}\right)(i=1, \ldots, 5)$ of the kinematic singularity free amplitudes by an equation identical in form to (41). With the assumptions that the $P_{i}\left(q^{2}\right)$ are polynomials in $q^{2}$ and that $\nu^{2} T_{4,5}^{\prime} / m_{\alpha}^{4}$ scale, it follows from Eqs. (42), (43) that

$$
\begin{equation*}
\lim _{q^{2} \rightarrow-\infty} q^{4} \stackrel{\rightharpoonup}{\mathrm{~T}}_{4}^{\prime}\left(q^{2}, 0\right)=-m_{\alpha}^{4} P_{2}(0)+8 m_{\alpha}^{4} \int_{0}^{\infty}\left(\widetilde{\mathrm{F}}_{4}^{\nu}+\widetilde{\mathrm{F}}_{4}^{\bar{\nu}}\right) \frac{d x}{\mathrm{x}^{3}} \tag{44}
\end{equation*}
$$

But by its very definition,

$$
\begin{equation*}
P_{2}(0)=\frac{2}{\mathrm{~m}_{\alpha}^{4}} \int_{0}^{\infty} \mathrm{dxx}\left(\tilde{W}_{2}^{v}(0, \mathrm{x})+\tilde{W}_{2}^{\bar{\nu}}(0, \mathrm{x})\right) \tag{45}
\end{equation*}
$$

Finally, we have

$$
\begin{align*}
\phi_{\alpha}\langle\alpha(p)| \sigma(0)|\alpha(\mathrm{p})\rangle & =4 \mathrm{~m}_{\alpha}^{2} \int_{0}^{\infty} \mathrm{dxx}\left(\widetilde{\mathrm{~F}}_{4}^{\nu}+\widetilde{\mathrm{F}}_{4}^{\bar{\nu}}\right) \\
& -\frac{1}{\mathrm{~m}_{\alpha}^{2}} \int_{0}^{\infty} \mathrm{dxx}\left(\widetilde{\mathrm{~W}}_{2}^{\nu}(0, \mathrm{x})+\widetilde{\mathrm{w}}_{2}^{\bar{\nu}}(0, \mathrm{x})\right) . \tag{46}
\end{align*}
$$

Upon comparison with the original formula Eq. (37) for the sigma operator matrix element, Eq. (46) is seen to solve the divergence problem as well as include the necessary positive contribution. However, the triumph is rather hollow because even for the nucleon, the structure function $\mathrm{F}_{4}$ will be extremely difficult to measure. That is, the formula (46), while sound in principle, is not likely to be of any use in practice.

## IV. VACUUM MATRIX ELEMENTS

Thus far, we have studied diagonal single-particle matrix elements of the axial-charge and sigma operator commutation relations. The sum rules thereby generated are expressible in terms of pion cross sections and structure functions pertaining to neutrino-induced processes. We have shown that at least two of the axial-charge sum rules are almost entirely saturated by single particle intermediate states and that several others give promise of behaving accordingly as more data becomes available. In this Section, we allow one or both of the external states to be the vacuum. ${ }^{22}$ Again, we focus on the contributions of the single particle intermediate states and also clarify the physical content of the algebra sum rules.

First, we treat the vacuum-vacuum matrix elements. Suppose we have an algebra in which some charge operator $Q$ is commuted at equal times with local operator $\mathrm{A}(0)$ to produce local operator $\mathrm{B}(0)$,

$$
\begin{equation*}
\mathrm{i}[Q(0), \mathrm{A}(0)]=\mathrm{B}(0) . \tag{47}
\end{equation*}
$$

Sandwiching this commutator between vacuum states and inserting singleparticle intermediate states yields

$$
\begin{equation*}
\langle 0| \mathrm{B}(0)|0\rangle=-\sum_{\gamma}\langle 0| \partial_{\mu} \mathrm{J}^{\mu}(0)|\gamma\rangle\langle\gamma| \mathrm{A}(0)|0\rangle \tag{48}
\end{equation*}
$$

where $\partial_{\mu} J^{\mu}$ is the divergence of the current associated with charge Q. The sum rule (48) can also be derived as a low energy theorem associated with the propagator

$$
\begin{equation*}
\Delta\left(q^{2}\right)=i \int d^{4} x e^{i q \cdot x}<0\left|T \partial_{\mu} J^{\mu}(x) A(0)\right| 0> \tag{49}
\end{equation*}
$$

Therefore, this type of sum rule relates the vacuum expectation value of a local operator to the zero-energy value of a related propagator.

Let us briefly explore the consequences of single particle dominance in a model where the chiral non-symmetric part of the energy density is

$$
\begin{equation*}
\theta_{00}^{\prime}-u_{0}+c u_{8} \tag{50}
\end{equation*}
$$

and the trace of the energy momentum tensor is

$$
\begin{equation*}
\theta=(4-d)\left(u_{0}+c u_{8}\right) \tag{51}
\end{equation*}
$$

where $u_{0,8}$ transform as $0^{+}$isoscalar members of $\left(3,3^{*}\right)+\left(3^{*}, 3\right)$ with dimension d. Letting operators $\partial_{\mu} J^{\mu}$ and A of Eq. (49) become $\partial_{\mu} A_{a}^{\mu}, \partial_{\mu} A_{b}^{\mu}$, first with $\mathrm{a}, \mathrm{b}=1,2,3$, then with $\mathrm{a}, \mathrm{b}=4,5,6,7$, we find

$$
\begin{equation*}
\left.\mathrm{m}_{\pi}^{2} \mathrm{~F}_{\pi}^{2}=-\frac{\sqrt{2}+\mathrm{c}}{3}\left(v \dot{2}<0\left|\mathrm{u}_{0}\right| 0\right\rangle+\langle 0| \mathrm{u}_{8}|0\rangle\right) \tag{52}
\end{equation*}
$$

and

$$
\begin{equation*}
m_{K}^{2} F_{K}^{2}=-\frac{2 \sqrt{2}-c}{2 \sqrt{3}}\left(\sqrt{\frac{2}{3}}\langle 0| u_{0}|0\rangle-\frac{1}{2 \sqrt{3}}\langle 0| u_{8}|0\rangle\right) . \tag{53}
\end{equation*}
$$

If the vacuum is taken to be approximately $\mathrm{SU}(3)$ invariant, $\langle 0| \mathrm{u}_{8}|0\rangle \cong 0$, then the approximate numerical relation

$$
\begin{equation*}
c=-2 \sqrt{2}\left[\frac{13\left(\mathrm{~F}_{\mathrm{K}} / \mathrm{F}_{\pi}\right)^{2}-1}{26\left(\mathrm{~F}_{\mathrm{K}} / \mathrm{F}_{\pi}\right)^{2}+1}\right] \tag{54}
\end{equation*}
$$

is obtained. ${ }^{23}$ Next replace $\partial_{\mu} J^{\mu}$ of Eq. (49) by the operator $\theta$ and for A substitute first $\theta$, then $\sigma$. We therefore find

$$
\begin{gather*}
\left(\frac{\mathrm{F}_{\epsilon}}{\mathrm{F}_{\pi}}\right)^{2}=\frac{3}{\sqrt{2}} \frac{\mathrm{~d}(4-\mathrm{d})}{\sqrt{2}+\mathrm{c}}\left(\frac{\mathrm{~m}_{\pi}}{\mathrm{m}_{\epsilon}}\right)^{2}  \tag{55}\\
-23-
\end{gather*}
$$

and then

$$
\begin{equation*}
\frac{\mathrm{F}_{\pi}}{\mathrm{F}_{\epsilon}}=\mathrm{g}_{\epsilon} \tag{56}
\end{equation*}
$$

where $F_{\epsilon}, g_{\epsilon}$ are defined by

$$
\begin{equation*}
\langle 0| \theta(0)|\epsilon(p)\rangle=m_{\epsilon}^{2} F_{\epsilon} \tag{57}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle 0| \sigma(0)|\epsilon(\mathrm{p})\rangle=\mathrm{m}_{\pi}^{2} \mathrm{~F}_{\pi} \mathrm{g}_{\epsilon} . \tag{58}
\end{equation*}
$$

We have obtained Eqs. (52)-(58) by approximating zero energy two point functions in terms of $0^{-}(\pi, \mathrm{K})$ and then $0^{+}(\epsilon)$ intermediate states. The overall picture given by Eqs. (54) and (55)-(58) is that $c$ is near $-\sqrt{2}$ and that $F_{\pi}$ and $F_{\epsilon}$ are of the same order of magnitude.

There is nothing outlandish about the results just derived: They are in qualitative accord with estimates of $c$ and $F_{\pi} / F_{\epsilon}$ arising from, at least, nominally different approaches. ${ }^{24}$ In fact, given the structure of the vacuumvacuum matrix elements, dominance of the $\pi, K, \epsilon$ states in the relations (52)(56) can be given an aura of respectibility by appealing to the "near-by singularity" argument of analytic function theory. However, in our opinion, justification for these single-particle truncations is not so clear. Unfortunately, because of the difficulty in detecting low spin hadrons with high mass and then revealing their properties, it is not likely that more than a few intermediate states can be explicitly taken into account in the vacuum-vacuum sum rules (48). Thus, calculable corrections to these relations are not expected to be forthcoming. This is in marked contrast to the sum rules of Section II. Moreover, the "near-by singularity" justification mentioned above is probably specious.

In a recent paper, ${ }^{25}$ Baluni and Broadhurst have used rigorous theoretical bounds on $\mathrm{K}_{\ell 3}$ form factors along with reliable experimental data to show that the dimension of the $\left(3,3^{*}\right)+\left(3^{*}, 3\right)$ operators more than likely exceeds the value two. Since the high $q^{2}$ behavior of two-point functions goes as $\left(q^{2}\right)^{d-2}$, this means that spectral representations of the $\left(3,3^{*}\right)+\left(3^{*}, 3\right)$ propagators must be at least once subtracted. Thus, the singular high energy behavior is capable of upsetting zero energy estimates by introducing an unknown subtraction constant into the calculation. The only means of evasion from this dilemma is to view propagator pole and cut contributions perturbatively in the context of chiral and scale symmetry breaking. It is then argued that any effect arising from the cut is of second order in symmetry breaking and hence negligible relative to the pole. However, this argument is certainly not compelling for the kaon channel and even less so for the $\epsilon$ channel.

It is instructive to consider commutation relations of various of the $\left(3,3^{*}\right)+\left(3^{*}, 3\right)$ operators taken between a vacuum and a single particle state. The problems associated with truncating the number of contributing intermediate states remain but without the "successes" of the vacuum-vacuum matrix elements. It will suffice to give some examples. Employing the notation of Eq. (47), the general form of the "vacuum-single particle" sum rule becomes

$$
\begin{align*}
-\langle 0| \mathrm{B}(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle & =\sum_{\gamma} \frac{\langle 0| \mathrm{A}(0)|\gamma(\overrightarrow{\mathrm{p}})\rangle\langle\gamma(\overrightarrow{\mathrm{p}})| \partial_{\mu} \mathrm{J}^{\mu}(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle}{2 \mathrm{p}_{\gamma}^{0}\left(\mathrm{p}_{\gamma}^{0}-\mathrm{p}_{\alpha}^{0}\right)} \\
& +\sum_{\beta} \frac{\langle 0| \partial_{\mu} J^{\mu}(0)|\beta(0)\rangle\langle\beta(0)| \mathrm{A}(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle}{2 \mathrm{~m}_{\beta}^{2}} \tag{59}
\end{align*}
$$

In the following we shall choose the charge operator to be $\mathrm{F}_{\mathrm{a}}^{5}, \mathrm{a}=1,2,3$, and we shall take the limit $|\vec{p}| \rightarrow \infty$. If hadronic form factors of local operators vanish for infinite momentum transfer $q^{2} \rightarrow \infty$, then a truncated form of the second term in Eq. (59) will not contribute. The physical content of the sum rule at this level of approximation is seen to involve relations between $q^{2}=0$ form factors and various constants associated with vacuum-single particle matrix elements of local operators,
$-\langle 0| \mathrm{B}(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle=\sum_{\gamma} \frac{\langle 0| \mathrm{A}(0)|\gamma(\overrightarrow{\mathrm{p}})\rangle\langle\gamma(\overrightarrow{\mathrm{p}})| \partial{ }_{\mu} J^{\mu}(0)|\alpha(\overrightarrow{\mathrm{p}})\rangle}{\mathrm{m}_{\gamma}^{2}-\mathrm{m}_{\alpha}^{2}}$.

In our examples, $\partial_{\mu} J^{\mu}$ will be the axial-vector divergence, so we can estimate the $q^{2}=0$ form factors by means of Goldberg-Treiman formulae. Substituting $\partial_{\mu} \mathrm{A}_{\mathrm{b}}^{\mu}(\mathrm{b}=1,2,3)$ for $\mathrm{A}, \sigma$ for B , and $\epsilon$ for $\alpha$ in Eq. (60), we obtain

$$
\begin{equation*}
\mathrm{g}_{\epsilon}=\frac{\mathrm{F}_{\mathrm{\pi}} \mathrm{~g}_{\epsilon \pi \pi}}{\mathrm{m}_{\epsilon}^{2}-\mathrm{m}_{\pi}^{2}}+\ldots \tag{61a}
\end{equation*}
$$

whereas the replacements $\sigma$ for $\mathrm{A},-\partial_{\mu} \mathrm{A}_{\mathrm{a}}^{\mu}$ for B , and $\pi$ for $\alpha$ yield

$$
\begin{equation*}
1=\frac{\mathrm{g}_{\epsilon} \mathrm{F}_{\pi} \mathrm{g}_{\epsilon \pi \pi}}{\mathrm{m}_{\epsilon}^{2}-\mathrm{m}_{\pi}^{2}}+\ldots \tag{61b}
\end{equation*}
$$

From the estimate $\Gamma(\epsilon \pi \pi) \cong 300 \mathrm{MeV}$, we obtain $\mathrm{F}_{\pi} \mathrm{g}_{\epsilon \pi \pi} / \mathrm{m}_{\epsilon}^{2} \cong 0.5$. Thus Eq. (61a) implies $g_{\epsilon} \cong 0.5$ whereas (61b) gives $g_{\epsilon} \cong 2$. 0 . This disagreement can probably be blamed on the deficiency of the truncation approximation not enough of the intermediate states have been taken into account. A more striking failure of the truncation approximation emerges upon making the replacements $\theta$ for $\mathrm{A},-(4-\mathrm{d}) \partial_{\mu} \mathrm{A}_{\mathrm{a}}^{\mu}$ for B , and $\epsilon$ for $\alpha$. Then we find from

Eq. (60) that

$$
\begin{equation*}
4-\mathrm{d}=\left(\frac{\mathrm{m}_{\epsilon}}{\mathrm{m}_{\pi}}\right)^{2} \frac{\mathrm{~F}_{\epsilon} \mathrm{g}_{\epsilon \pi \pi}}{\mathrm{m}_{\epsilon}^{2}-\mathrm{m}_{\pi}^{2}}+\ldots \tag{61c}
\end{equation*}
$$

The left hand side is an order of magnitude or so smaller than the right hand side of this "equation". A result like this increases one's appreciation for the success of hard-meson off-shell methods. ${ }^{26}$

We have derived Eqs. (59)-(61c) by considering scalar operators and states. Analogous relations can be obtained for operators (like $\mathrm{V}^{\mu}, \mathrm{A}^{\mu}, \theta^{\mu \nu}$ ) and states (like $\rho, \mathrm{A}_{1}$, f) which carry spin. ${ }^{27}$ Similar negative results are found.

## V. CONCLUSION

The underlying theme of our study has been to survey the contributions of single-particle intermediate states to sum rules generated by various commutation relations.

For the class of axial-charge sum rules catalogued in Section II, considered as a whole, single-particle contributions afford the only realistic phenomenological test. Aside from using theoretical estimates for certain nonmeasurable terms, our analysis was phenomenologically oriented. In particular, we made no effort to classify intermediate states according to algebraic representations. We found almost complete saturation for two sets of external states ( $\pi, N$ ), and varying degrees of saturation of all the rest: $82 \%$ for $\Delta(1233)$, roughly $60 \%$ for $\Sigma, \mathrm{Y}_{1}(1385), \rho$, somewhat under $50 \%$ for $\mathrm{K}_{\mathrm{V}}, \mathrm{K}_{\mathrm{V}}(892)$, and under $40 \%$ for $\Xi(1315)$, and not much information for the remaining cases examined. Notice from Eq. (13) that if the asymptotic equality of particle and antiparticle cross sections were not valid, we might expect to see some effect of non-convergence in our sum rules. However, in none of the cases was oversaturation detected. We are optimistic that, with further experimental effort in hadron spectroscopy, enough information can be gathered to allow almost complete saturation of the $\Delta(1233), \Sigma, \mathrm{Y}_{1}(1385), \rho, \mathrm{K}$, and $\mathrm{K}_{\mathrm{V}}(892)$ sum rules, and substantially more information regarding the $\Xi(1315)$ sum rule. In principle, there is nothing to prevent this. However, we do not envisage there being substantial phenomenological applicability of the sum rules associated with external states, $\alpha$, of higher mass. Since contributions for which the mass of the external state exceeds that of the intermediate state do not appear to be large, decay widths where $\alpha$ appears in a final state will be needed. These are hard to measure.

Moreover, the number of non-measurable contributions will increase. Whether our theories of hadrons will improve enough to allow calculation of these is a matter of conjecture.

The work of Section III essentially speaks for itself and warrants little discussion here. While it is commendable to see that the $q^{2}=0$ sigma-operator sum rule can be written in such a way that its original diseases are cured, the resulting phenomenological disfigurement is such that the sum rule loses almost all its attractiveness. In particular, the low energy pion-nucleon system will remain the best area in which to attempt determination of the nucleon matrix element of the sigma operator.

Despite the rather extensive employment by previous workers of the vacuum-vacuum and vacuum-single particle sum rules of the general type in Section IV, it is our conclusion that these systems are far from being under theoretical control. At the very least, the probable need for subtraction constants ${ }^{25}$ in the $\left(3,3^{*}\right)+\left(3^{*}, 3\right)$ propagators is an ominous signal that the usual truncation procedures adopted might be inadequate. In addition, there is the recent ee annihilation data, which, for example, shows that the $\rho$ contribution to Weinberg's first sum rule is only $1 / 30$ that of the higher mass continuum ${ }^{27}$ in the vector current propagator probed so far by the experiments. It remains to be seen whether other calculations based on single particle dominance of the vacuum-vacuum and vacuum-single particle matrix elements will fail so resoundingly.

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4. The overall approach we follow in this Section is standard and is described in Ref. 2. The only stipulation we wish to add is that unstable states are treated in the narrow-resonance approximation throughout.
5. However, $\alpha$ cannot have isospin zero as this reduces the content of Eq. (3) to a triviality. The right hand side vanishes because $\mathrm{T}_{3}(\alpha)=0$ and the left hand side is zero term-by-term due to cancellation between $F_{+}^{5} F_{-}^{5}$ and $F_{-}^{5} F_{+}^{5}$ with isospin one intermediate states.
6. Beware that $J_{+}^{\pi}=J_{1}^{\pi}+\mathrm{i} J_{2}^{\pi}$, so that an inverse factor of $\sqrt{2}$ must ultimately be appended to $J_{+}^{\pi}$ for comparison with physical processes.
7. An additional assumption in this approach is that the limit $|\overrightarrow{\mathrm{p}}| \rightarrow \infty$ can be taken term-by-term.
8. Particle Data Group, Rev. Mod. Phys. Suppl. 45, S1 (1973).
9. In those cases where ranges for masses and/or widths are given, we use the average value.
10. We take the $\epsilon^{\prime}(1000)$ width from D. M. Binnie et al., Phys. Rev. Letters 31, 1534 (1973) and S. D. Protopopescu et al., Phys. Rev. D7, 1279 (1973).
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13. The problem of calculating a transition rate given this kinematical situation is discussed by E. Golowich, "Finite Width Effects in Resonance Decay Near Threshold", Stanford Linear Accelerator Center Report No. SLAC-PUB-1426 (May, 1974).
14. For example, see U. Karshon et al., Phys. Rev. Letters 32, 852 (1974), J. Diaz et al., Phys. Rev. Letters, 32, 260 (1974).
15. We do not anticipate a very large contribution from the lowest mass $0^{+}$ kaon, whose effect can be estimated from SU(3) in terms of the $\epsilon^{\prime}(1000)$ contribution to the pion sum rule. Admittedly, mixing effects involving the $\epsilon^{\prime}$ could strain this argument somewhat.
16. For example, see Y. -C. Liu and J.A. M. Vermaseren, Phys. Rev. D8, 1602 (1973) and references cited therein.
17. At this point we specify our normalization of states by choosing $N_{\alpha}=2 p_{\alpha}^{0}$ (mesons) and $\mathrm{N}_{\alpha}=\mathrm{p}_{\alpha}^{0} / \mathrm{m}_{\alpha}$ (baryons).
18. A further criticism of Eqs. (30) and (32) is that their derivation involves an extrapolation over a range $\mathrm{m}_{\pi}^{2}$ and so cannot be used with absolute confidence to provide an accurate estimate of $\langle\alpha| \sigma(0)|\alpha\rangle$. However, our interest in these relations is not so much phenomenological as theoretical. The real issue is to determine what modifications are necessary to obtain meaningful sum rules.
19. Hereafter, we concentrate on the continuum contribution to $\langle\alpha| \sigma|\alpha\rangle$, completely ignoring the discrete terms arising from contributions below the $\pi \alpha$ physical threshold.
20. We assume time reversal invariance.
21. A $\mathrm{J}=0$ fixed pole shows up as a contribution to $\operatorname{Re}_{\mathrm{T}}^{\mathrm{t}}$ in the limit $\nu \rightarrow \infty$ with energy dependence the same as that of an $\alpha=0$ Regge pole.
22. Unfortunately the charge algebra reduced to a triviality in this case.
23. We realize that this formula is already familiar to many readers. We are simply using it as an example in the course of our discussion.
24. For example, see M. Gell Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968) and P.A. Carruthers, Phys. Rev. D3, 959 (1971) respectively.
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26. For example, see J. Ellis, P.H. Weisz, and B. Zumino, Phys. Letters 34B, 91 (1971).
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TABLE Ia
Resonance contributions to axial-charge sum rule with proton external state. The first four columns list properties of each intermediate state and the final two columns give individual and cumulative contributions to the sum rule Eq. (12).

| Mass | Spin | Isospin | Partial <br> Width | Individual | Cumulative |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1470 | 0.5 | 0.5 | 140 | 0.125 | 0.125 |
| 1525 | 1.5 | 0.5 | 64 | 0.089 | 0.215 |
| 1550 | 0.5 | 0.5 | 37 | 0.023 | 0.238 |
| 1678 | 2.5 | 0.5 | 58 | 0.067 | 0.305 |
| 1685 | 2.5 | 0.5 | 86 | 0.097 | 0.402 |
| 1715 | 0.5 | 0.5 | 120 | 0.041 | 0.443 |
| 1755 | 0.5 | 0.5 | 40 | 0.012 | 0.455 |
| 1815 | 1.5 | 0.5 | 64 | 0.032 | 0.488 |
| 2130 | 3.5 | 0.5 | 74 | 0.035 | 0.522 |
| 2223 | 4.5 | 0.5 | 44 | 0.0214 | 0.544 |
| 1233 | 1.5 | 1.5 | 115 | -0.975 | $-0.431$ |
| 1655 | 0.5 | 1.5 | 46 | -0.019 | -0.451 |
| 1695 | 1.5 | 1.5 | 36 | -0.026 | $-0.477$ |
| 1880 | 2.5 | 1.5 | 48 | -0.030 | $-0.507$ |
| 1858 | 0.5 | 1.5 | 68 | -0.015 | $-0.522$ |
| 1955 | 3.5 | 1.5 | 99 | $-0.069$ | -0.591 |
| 2385 | 5.5 | 1.5 | 34 | -0.015 | $-0.606$ |

TABLE Ib
Resonance contributions to axial-charge sum rule with
$-\Delta^{+}(1233)$ external state.

| Mass | Spin | Isospin | Partial <br> Width | Individual | Contribution |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 939 | 0.5 | 0.5 | 115 | 0.244 | 0.244 |
| 1470 | 0.5 | 0.5 | 58 | 0.052 | 0.296 |
| 1525 | 1.5 | 0.5 | 35 | 0.035 | 0.331 |
| 1678 | 2.5 | 0.5 | 80 | 0.039 | 0.369 |
| 1685 | 2.5 | 0.5 | 38 | 0.018 | 0.387 |
| 1755 | 0.5 | 0.5 | 60 | 0.006 | 0.393 |
| 1655 | 0.5 | 1.5 | 57 | 0.008 | 0.402 |
| 1695 | 1.5 | 1.5 | 62 | 0.014 | 0.416 |
| 1858 | 0.5 | 1.5 | 27 | 0.001 | 0.418 |
| 1953 | 3.5 | 1.5 | 36 | 0.005 | 0.423 |
|  |  |  |  |  |  |

TABLE Ic
Resonance contributions to axial-charge sum rule with
$\Sigma^{+}(1189)$ external state

| Mass | Spin | Isospin | Partial <br> Width | Individual | Cumulative |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1405 | 0.5 | 0.0 | 40 | 0.094 | 0.094 |
| 1520 | 1.5 | 0.0 | 7 | 0.010 | 0.104 |
| 1670 | 0.5 | 0.0 | 11 | 0.003 | 0.107 |
| 1690 | 1.5 | 0.0 | 31 | 0.015 | 0.122 |
| 1815 | 2.5 | 0.0 | 9 | 0.004 | 0.125 |
| 1830 | 2.5 | 0.0 | 42 | 0.016 | 0.141 |
| 2100 | 3.5 | 0.0 | 5 | 0.001 | 0.142 |
| 1385 | 1.5 | 1.0 | 4 | 0.037 | 0.179 |
| 1670 | 1.5 | 1.0 | 20 | 0.016 | 0.195 |
| 1765 | 2.5 | 1.0 | 1 | 0.007 | 0.196 |
| 2030 | 3.5 | 1.0 | 5 | 0.002 | 0.197 |


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