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TRANSITION AMPLITUDES IN THE N-BODY FADDEEV-TYPE THEORY*

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ABSTRACT

Within the framework of the Faddeev-type N-body scattering formalism, we find the transition operators for all N-body processes that start from two-cluster initial states, and construct the N-body Faddeev-type equations they satisfy. In particular, we show that our previous N-body generalizations of the three-body operator $K_{\beta\alpha}$ correspond to the N-body scattering amplitudes for complete break-up, and that the Alt, Grassberger and Sandhas N-body generalizations of the three-body transition operator $U_{\beta\alpha}$ correspond to N-body elastic and rearrangement amplitudes. As an introduction, we give a simple derivation of the N-body Faddeev-type equations for the wavefunction components.

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I. INTRODUCTION

The different treatments of the N-body Faddeev-type theory that can be found in the literature are given in terms of N-body operators whose relation to the physically interesting N-body transition amplitudes has not been investigated in detail. In particular, the Faddeev-Yakubovskii[1] (FY) treatment is made in terms of N-body generalizations of the Faddeev three-body operator $M_{\beta\alpha}$[2]. As in the three-body case, these N-body operators are not directly related to transition operators; rather, the transition amplitudes can be obtained as residues at appropriate singularities of plane wave matrix elements of these operators.

The Alt, Grassberger and Sandhas (AGS) matrix formalism[3] involves N-body operators that are generalizations of the AGS three-body transition operator $U_{\beta\alpha}$[4]. It has been shown[5] that the four-body versions of this operator are also transition operators (in a special case). The formalism suggests that the N-body AGS operators are in fact the transition operators for N-body elastic and rearrangement processes; it is shown below that this is indeed the case.

A third version of the N-body theory is obtained[6] by considering the N-body generalizations of the three-body operator $K_{\beta\alpha}$[7]. The essential feature of these operators is that they are the kernel operators for all N-body Faddeev-type equations. However, since in the three-body case $K_{\beta\alpha}$ is also the transition operator for break-up processes, its N-body generalization will be related to such transition operators as well.

In the following sections we show how to construct a transition operator within the N-body Faddeev-type theory that yields the transition amplitudes

for all N-body processes when appropriate matrix elements between Faddeev-type components of the initial and final states are taken (only two-cluster initial states will be considered). This N-body matrix of transition operators $\mathcal{B}$ reduces to the matrix of AGS N-body U-operators for the case of elastic and rearrangement scattering, and to the matrix of N-body K-operators for the case of complete break-up. For partial break-up processes, we find operators not considered before. However, in all cases the transition operators satisfy N-body Faddeev-type equations similar to the equations for U and K.

As an introduction we present a rederivation of the N-body FY equations for the Faddeev-type components of the full N-body wavefunction. Besides illustrating in a particularly simple case the FY procedure for constructing more connected kernels by successive splittings of the total wavefunction, this derivation is useful in demonstrating that for all scattering processes appropriate matrix elements of $\mathcal{B}$ reduce to the conventional wavefunction expression for the transition amplitude.

II. N-BODY FADDEEV-TYPE WAVEFUNCTION EQUATIONS

Prior to the discussion of the N-body transition operators, we present in this section a rederivation of the N-body Faddeev-type wavefunction equations. We do this because some intermediate results from this derivation will be useful in later sections, and as an illustration (in the most simple case) of the general procedure for deriving such N-body equations.

We first proceed to describe our notation for the labeling of subsystems. Following Yakubovskii[1], we use the concept of a partition a_i to describe a splitting of the set of N particles into i groups with the understanding that only

particles belonging to the same group can interact. For instance, a_2, b_2, ... denote different possible partitions of the N particles into two groups that do not interact with each other; a_{N-2}, b_{N-2}, ... denote configurations of the type 1+...+1+1+3 or 1+...+1+2+2 - for which we shall sometimes use the notation τ, σ, ... - and a_{N-1}, b_{N-1}, ... denote interacting pairs, sometimes labeled by α, β, ... Partitions can be performed one after another to form a sequence. In such a case they will be denoted by the same letter, and symbolized by $b_i \subset b_k$, $i>k$ [8]. Only sequences whose last partition is of the type b_{N-1} will be considered; they will be denoted by a greek letter, and their subscript will be that of the first partition in the sequence. In this way, $\beta_2 = (b_2, \ldots, b_{N-1})$ determines one way in which the N-body system is ultimately separated into N-1 groups that do not interact with each other. At times, we will make part of a sequence explicit, as in $\beta_2 = (b_2, \beta_3)$. Finally, each possible partition b_i defines a particular disconnected N-body scattering problem, in which specific interactions have been set equal to zero.

The operators of the N-body scattering theory will be labeled by pairs of full sequences β_2, α_2, as in $A_{\beta_2\alpha_2}$. Sometimes it will be convenient to consider these sequences (or parts of them) as matrix indices, and we will write

$$A = \left\{ A^{b_2 a_2} \right\} = \left\{ A_{\beta_2 \alpha_2} \right\} \quad . \tag{2.1}$$

In (2.1), the first bracket contains the submatrices of A obtained when the first partition label is made explicit, and the second the simple elements of A. Matrices of operators related to disconnected (partitioned) N-body subsystems labeled by b_k, say, will be denoted

$$A^{b_k} = \left\{ A^{b_k}_{b_{k+1}, a_{k+1}} \right\} = \left\{ A^{b_k}_{\beta_{k+1}, \alpha_{k+1}} \right\} , \tag{2.2}$$

where the partition index b_i appears as a single upper index, and it is understood that $\beta_{k+1}, \alpha_{k+1} \subset b_k$.

In order to obtain Faddeev-type equations for the N-body wavefunction components, we start with the Schrödinger equation for the full N-body wavefunction,

$$(H_0 + V - E)\,\Psi = 0 \quad , \tag{2.3}$$

where $V = \sum_\gamma V_\gamma$ is the sum of all potentials between pairs of particles. We are interested in outgoing wave scattering solutions, corresponding to two-cluster initial states labeled, say, by a partition a_2. Eq.(2.3) is equivalent to

$$\Psi = C\phi - G_0(E+iO)V\Psi \quad , \tag{2.4}$$

where ϕ is a solution of $(H_0 - E)\phi = 0$, and C is a constant. Since we have chosen to consider only two-cluster configurations as initial states, $C = 0$. As indicated in (2.4), all operators of this section are to be taken at an energy corresponding to to that of the initial state.

Faddeev components of the N-body wavefunctions are defined through

$$\Psi_\beta = - G_0 V_\beta \Psi \tag{2.5}$$

so that

$$\Psi = \sum_\beta \Psi_\beta \quad , \tag{2.6}$$

where $\beta = b_{N-1}$ is the partition label corresponding to a splitting of N particles into N-1 groups.

We now apply the Faddeev procedure of removing two-body disconnected pieces from the kernel of (2.5), obtaining

$$(1 + G_0 V_\beta)\Psi_\beta = - G_0 V_\beta \sum_\gamma \bar{\delta}_{\beta\gamma} \Psi_\gamma \tag{2.7}$$

where $\bar{\delta}_{\beta\gamma} = 1 - \delta_{\beta\gamma}$. Let t_β be defined by the equation $t_\beta = V_\beta - V_\beta G_0 t_\beta$. Operating on (2.7) with $(1 - G_0 t_\beta)$, we get

$$\Psi_\beta = C\,\phi_\beta - G_0 t_\beta \sum_\gamma \bar{\delta}_{\beta\gamma} \Psi_\gamma \quad , \tag{2.8}$$

where ϕ_β satisfies $(1 + G_0 V_\beta)\,\phi_\beta = 0$. Again, due to our choice of initial state, $C = 0$.

The kernel of (2.8) must be further modified since it still contains disconnected pieces corresponding to 2, 3, ..., N-2 non-interacting clusters. We now proceed to remove those pieces that correspond to N-2 non-interacting clusters, i.e. to clusters of the type 1+...+1+1+3 and 1+...+1+2+2, labeled by the partition index $b_{N-2} = \sigma$. From (2.8) we define the components

$$\Psi_\beta^\sigma = - G_0 t_\beta \sum_{\gamma\subset\sigma} \bar{\delta}_{\beta\gamma} \Psi_\gamma \quad , \tag{2.9}$$

where $\sigma\supset\beta$, and such that

$$\Psi_\beta = \sum_{\sigma\supset\beta} \Psi_\beta^\sigma \quad . \tag{2.10}$$

As before, we take the diagonal piece in (2.9) to the left, and get

$$\sum_{\gamma\subset\sigma} (\delta_{\beta\gamma} + G_0 t_\beta \bar{\delta}_{\beta\gamma})\Psi_\gamma^\sigma = - G_0 t_\beta \sum_{\gamma\subset\sigma} \bar{\delta}_{\beta\gamma} \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho} \Psi_\gamma^\rho \quad . \tag{2.11}$$

Consider now an operator $K_{\beta\alpha}^\sigma$ [7] defined in the subsystem σ through the equation

$$K_{\beta\alpha}^\sigma = \bar{\delta}_{\beta\gamma} t_\beta - \sum_{\gamma'\subset\sigma} K_{\beta\gamma'}^\sigma G_0 t_{\gamma'} \bar{\delta}_{\gamma'\gamma} \quad . \tag{2.12}$$

Operating on (2.11) with the expression $(\delta_{\beta\beta'} - G_0 K_{\beta\beta'}^\sigma)$, $\beta, \beta'\subset\sigma$, we get

$$\Psi_\beta^\sigma = C\phi_\beta^{(\sigma)} - \sum_{\gamma\subset\sigma} G_0 K^\sigma_{\beta\gamma} \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho} \Psi_\gamma^\rho \quad , \quad \beta\subset\sigma \tag{2.13}$$

where $\sum_{\gamma\subset\sigma}(\delta_{\beta\gamma} + G_0 t_\beta \bar{\delta}_{\beta\gamma})\phi_\gamma^{(\sigma)} = 0$, i.e. $\phi_\beta^{(\sigma)}$ is the Faddeev component of the wavefunction corresponding to a bound state in the subsystem σ. With our choice of initial state, again $C = 0$ (unless $N = 4$, in which case $C = \delta^{\sigma\tau}$ and we obtain the four-body FY equations[9]).

The kernel of (2.13) is now more connected than that of (2.8), in the sense that its third (or higher) power does not contain disconnected pieces corresponding to N-2 non-interacting clusters.

It is clear that this procedure of splitting the wavefunction into components that satisfy progressively more connected equations can be continued. We will now prove by induction that after (N-k+1) such splittings, the resulting components $\Psi_{\beta_{k-1}} = \Psi_{\beta_k}^{b_{k-1}}$ of the full wavefunction satisfy the equations

$$\Psi_{\beta_k}^{b_{k-1}} = - \sum_{\delta_k\subset b_{k-1}} G_0 K^{b_{k-1}}_{\beta_k\delta_k} \sum_{d_{k-1}} \bar{\delta}^{b_{k-1},d_{k-1}} \Psi_{\delta_k}^{d_{k-1}} \quad , \tag{2.14}$$

where

$$\sum_{b_{k-1}\supset b_k} \Psi_{\beta_k}^{b_{k-1}} = \Psi_{\beta_k} \quad , \quad \sum_{\beta_k} \Psi_{\beta_k} = \Psi \quad , \tag{2.15}$$

and the operators $K^{b_{k-1}}_{\beta_k\delta_k}$ satisfy the equations[6]

$$K^{b_{k-1}}_{\beta_k\delta_k} = \bar{\delta}^{b_k d_k} K^{b_k}_{\beta_{k+1}\delta_{k+1}} - \sum_{\substack{c_k\subset b_{k-1} \\ c_k\supset d_{k+1}}} \sum_{\gamma_{k+1}} K^{b_{k-1}}_{\beta_k\gamma_k} G_0 K^{c_k}_{\gamma_{k+1}\delta_{k+1}} \bar{\delta}^{c_k d_k} \quad , \tag{2.16}$$

in terms of subsystem operators $K^{b_k}_{\beta_{k+1},\delta_{k+1}}$. By β_k, γ_k and δ_k we understand

the sequences of partitions $\beta_k = (b_k, \ldots, b_{N-1})$, $\gamma_k = (c_k, \ldots, c_{N-1})$ and $\delta_k = (d_k, \ldots, d_{N-1})$, and $\sum_{\beta_k}$ stands for $\sum_{b_k} \sum_{b_{k+1}} \cdots \sum_{b_{N-1}}$. (As always, for sequences of partitions it is understood that $b_k \supset b_{k+1} \supset \cdots \supset b_{N-1}$.)

Here it is advantageous to use the matrix notation of Eqs. (2.1) and (2.2) and of Ref. 3, since in that case the indices β_k, δ_k of $K^{b_{k-1}}_{\beta_k \delta_k}$ and of $\Psi^{b_{k-1}}_{\beta_k}$ in (2.14) and (2.16) can be suppressed,

$$N^{b_{k-1}} = \left\{ N^{b_{k-1}}_{b_k d_k} \right\} = \left\{ G_0 K^{b_{k-1}}_{\beta_k \delta_k} \right\} \quad , \tag{2.17}$$

$$\Psi^{b_{k-1}} = \left\{ \Psi^{b_{k-1}}_{b_{k-2}} \right\} = \left\{ \Psi_{\beta_{k-1}} \right\} \quad . \tag{2.18}$$

As always, b_{k-1} in (2.17) indicates which partitioned subsystem is described by $N^{b_{k-1}}$ (recall that in a partitioned system the interactions between particles belonging to different clusters are set to be zero). In (2.18), however, it should be stressed that the elements of $\Psi^{b_{k-1}}$ are the components of the full N-body wavefunction, and b_{k-1} indicates that (N-k+1) splittings have been carried out. On the other hand, when we refer to wavefunctions describing partitioned systems, we will use upper indices in parenthesis, as in $\Phi^{(b_k)}$.

Using (2.17) and (2.18), Eqs. (2.14) and (2.16) can now be written in compact form,

$$\Psi^{b_{k-1}} = - N^{b_{k-1}} \sum_{d_{k-1}} \bar{\delta}^{b_{k-1}, d_{k-1}} \Psi^{d_{k-1}} \quad , \tag{2.19}$$

$$N^{b_{k-1}}_{b_k d_k} = \bar{\delta}^{b_k d_k} N^{b_k} - \sum_{c_k \subset b_{k-1}} N^{b_{k-1}}_{b_k c_k} N^{c_k} \bar{\delta}^{c_k d_k} \quad . \tag{2.20}$$

The proof now proceeds as follows: we split the wavefunction once again,

$$\Psi^{b_{k-2}}_{b_{k-1}} = - N^{b_{k-1}} \sum_{d_{k-1} \subset b_{k-2}} \bar{\delta}^{b_{k-1}, d_{k-1}} \quad \Psi^{d_{k-1}} \quad , \quad b_{k-2} \supset b_{k-1} \tag{2.21}$$

with

$$\Psi^{b_{k-1}} = \sum_{b_{k-2} \supset b_{k-1}} \Psi^{b_{k-2}}_{b_{k-1}} , \quad \sum_{\beta_{k-1}} \Psi_{\beta_{k-1}} = \Psi \quad . \tag{2.22}$$

Shifting the diagonal term as before, we get

$$\sum_{d_{k-1} \subset b_{k-2}} (1^{b_{k-1}, d_{k-1}} + N^{b_{k-1}} \bar{\delta}^{b_{k-1}, d_{k-1}}) \quad \Psi^{b_{k-2}}_{d_{k-1}} =$$

$$= - N^{b_{k-1}} \sum_{d_{k-1} \subset b_{k-2}} \bar{\delta}^{b_{k-1}, d_{k-1}} \sum_{d_{k-2} \supset d_{k-1}} \bar{\delta}^{b_{k-2}, d_{k-2}} \quad \Psi^{d_{k-2}}_{d_{k-1}} \quad . \tag{2.23}$$

Defining an operator $N^{b_{k-2}}_{b_{k-1}, d_{k-1}}$, $b_{k-1}, d_{k-1} \subset b_{k-2}$, through

$$N^{b_{k-2}}_{b_{k-1}, d_{k-1}} = \bar{\delta}^{b_{k-1}, d_{k-1}} \quad N^{b_{k-1}} - \sum_{c_{k-1} \subset b_{k-2}} N^{b_{k-2}}_{b_{k-1}, c_{k-1}} N^{c_{k-1}} \bar{\delta}^{c_{k-1}, d_{k-1}} , \tag{2.24}$$

we can operate on (2.23) with the expression $(1^{b_{k-1}, b'_{k-1}} - N^{b_{k-2}}_{b_{k-2}, b'_{k-2}})$, and finally obtain

$$\Psi^{b_{k-2}} = C \; \Phi^{(b_{k-2})} - N^{b_{k-2}} \sum_{d_{k-2}} \bar{\delta}^{b_{k-2}, d_{k-2}} \quad \Psi^{d_{k-2}} \quad . \tag{2.25}$$

Here,

$$\sum_{c_{k-1} \subset b_{k-2}} (1^{b_{k-1}, c_{k-1}} + N^{b_{k-1}} \bar{\delta}^{b_{k-1}, c_{k-1}}) \quad \Phi^{(b_{k-2})}_{c_{k-1}} = 0 \quad , \tag{2.26}$$

and again C=0 due to our choice of initial state.

Now, it is clear that Eqs. (2.25) and (2.24) are the same as (2.19) and (2.20), but with $k \to k-1$; in addition, (2.19) and (2.20) reduce to Eqs. (2.13) and (2.12) for $k = N-1$, $(K^{b_{N-1}}_{b_N, c_N} = t_{b_{N-1}})$.

With this observation, we conclude our proof by induction, and establish that Eqs. (2.19) and (2.20) are indeed obtained by performing the stepwise splitting of the full N-body wavefunction (N-k+1) times.

The final Faddeev-type equations for the wavefunction components are obtained by taking k=4 in (2.25). In this last step, however, the solution $\Phi^{(b_2)} = \left\{ \Phi^{(b_2)}_{c_3} \right\} = \left\{ \phi^{(b_2)}_{\gamma_3} \right\}$ of the homogeneous equation

$$\sum_{c_3 \subset b_2} \left(\mathbb{1}^{b_3 c_3} + \mathbb{N}^{b_3} \bar{\delta}^{b_3 c_3} \right) \Phi^{(b_2)}_{c_3} = 0 \tag{2.27}$$

does correspond to our choice of initial state, and we get

$$\Psi^{b_2[a_2]} = \delta^{b_2 a_2} \Phi^{(a_2)} - \mathbb{N}^{b_2} \sum_{d_2} \bar{\delta}^{b_2 d_2} \Psi^{d_2[a_2]} , \tag{2.28}$$

where $[a_2]$ labels the initial two-cluster subsystem. Eq. (2.28) is identical to the FY equation for the N-body wavefunction components, as presented in Ref. 6. In particular, the operators $\mathbb{N}$ are precisely the kernel operators for the Faddeev-type scattering equations, and they are discussed at length in this reference.

We conclude this section by deriving an expression for $\Psi^{[a_2]} = \left\{ \Psi^{b_2[a_2]} \right\}$ in terms of $\Phi^{[a_2]} = \left\{ \delta^{b_2 a_2} \Phi^{(b_2)} \right\}$ that will turn out to be useful in a later section. Acting on (2.28) with the operator $(\mathbb{1} - \mathbb{N})$, where $\mathbb{N} = \left\{ \mathbb{N}_{b_2 a_2} \right\}$ satisfies (2.24) with k=3, we directly obtain

$$\Psi^{[a_2]} = (1 - N)\,\Phi^{[a_2]} \tag{2.29}$$

or, more explicitly,

$$\Psi^{[a_2]}_{\beta_2} = \delta^{b_2 a_2}\,\phi^{(a_2)}_{\beta_3} - G_0 \sum_{\delta_3} K^{b_2 a_2}_{\beta_3 \delta_3}\,\phi^{(a_2)}_{\delta_3} \,. \tag{2.30}$$

III. THE AGS MATRIX FORMALISM

In this section we briefly consider some aspects of the matrix formulation of the Faddeev-type N-body theory[3,6] which we have not discussed so far. In addition to the matrix N considered before, other matrices of operators can be defined, such as V, G_0 and T. These matrices can be obtained inductively in terms of subsystem matrices as generalizations of the two-body operators V, G_0 and t, and their equations form a hierarchy of Lippman-Schwinger equations, accordingly generalized (to matrix form). In particular, T forms an N-body generalization of the two-body transition operator, and obeys a two-body like equation

$$T = V - VG_0T \,. \tag{3.1}$$

More precisely, the elements of the matrices V and G_0 are defined as follows: for N=3,

$$V = \left\{ -\delta_{\beta\alpha} G_0^{-1} \right\} \quad , \qquad G_0 = \left\{ -\delta_{\beta\alpha} G_0 t_\beta G_0 \right\} \quad , \tag{3.2}$$

and for N=4, analogously,

$$\begin{aligned} V &= \left\{ -\delta^{\sigma\tau} (G_0)^{-1} \right\} = \left\{ \bar{\delta}^{\sigma\tau} \delta_{\beta\alpha} (G_0 t_\beta G_0)^{-1} \right\} , \\ G_0 &= \left\{ -\delta^{\sigma\tau} (G_0 T^{\sigma} G_0) \right\} = \left\{ -\delta^{\sigma\tau} G_0 t_\beta G_0 U^{\sigma}_{\beta\alpha} G_0 t_\alpha G_0 \right\} , \end{aligned} \tag{3.3}$$

where $\beta \subset \sigma$, $\alpha \subset \tau$. The matrix indices of the operators inside the parentheses are understood to be within σ and τ, respectively.

With these definitions, (3.1) becomes for N=3 the Faddeev equation for $U_{\beta\alpha}$, and for N=4 the AGS Faddeev-type equation for the four-body operator $T = \left\{ U^{\sigma\tau}_{\beta\alpha} \right\}$.

For the partitioned N-body scattering problem labeled by b_{k-1}, we have likewise

$$V = \left\{ -\bar{\delta}^{b_k a_k} (G_0^{-1}) \right\} \quad , \quad G_0 = \left\{ -\delta^{b_k a_k} (G_0 T^{a_k} G_0) \right\} \tag{3.4}$$

where b_k, $a_k \subset b_{k-1}$, and again the matrix indices for the operators inside the parentheses are within b_k and a_k, respectively.

The matrix of operators N of section II is related to the operators of this section through[6]

$$N = G_0 T \quad , \quad T = V(1 - N) \quad , \tag{3.5}$$

so that in particular

$$(G_0 V)^{b_{k-1}} = \left\{ \bar{\delta}^{b_k a_k} N^{b_k} \right\} \quad , \tag{3.6}$$

where b_k, $a_k \subset b_{k-1}$.

As compared to the Yakubovski formulation, the AGS matrix formulation of the N-body Faddeev-type theory has the advantage of explicitly providing the N-body matrix of operators T. As we shall see in the next section, the elastic and rearrangement scattering amplitudes for the N-body problem are simply matrix elements of this operator.

IV. THE SCATTERING AMPLITUDES

In order to obtain the explicit form for the scattering amplitudes in the N-body Faddeev-type theory, it is convenient to again make use of the AGS matrix notation. We first consider the case N=3. For elastic and rearrangement scattering, the transition amplitude from an initial state $|\phi_{(\alpha)}>$ representing a bound pair α and a free particle, to a similar final state $|\phi_{(\beta)}>$ is given by

$$<\phi_{(\beta)}|U_{\beta\alpha}|\phi_{(\alpha)}> = <\Phi_{(\beta)}|\,T\,|\Phi_{(\alpha)}> \tag{4.1}$$

where $|\Phi_{(\alpha)}>$ is the column vector $\delta_{\gamma\alpha}|\phi_{(\alpha)}>$, $<\Phi_{(\beta)}|$ the row vector $\delta_{\beta\gamma}<\phi_{(\beta)}|$, and $T = \left\{ U_{\gamma'\gamma} \right\}$.

For break-up scattering, on the other hand, it has been shown[7] that the transition amplitude is given by

$$<\phi_0|\sum_\beta K_{\beta\alpha}|\phi_{(\alpha)}> = <\phi_{(0)}|G_0^{-1}\sum_\beta N_{\beta\alpha}|\phi_{(\alpha)}> \quad . \tag{4.2}$$

Here, $<\phi_{(0)}|$ represents the final state of three free particles, $K_{\beta\alpha}$ is the K-operator of Ref. 7 satisfying Eq. (2.12) for N=3, and $<\phi_{(0)}|K_{\beta\alpha}|\phi_{(\alpha)}> = \mathscr{K}_{\beta\alpha}$ is the component of the break-up amplitude defined by Faddeev[2].

In this section we shall give a general expression for all transition amplitudes from an initial two-cluster state to a final state of k clusters (for arbitrary N), in terms of the operator $B^{(k,2)} = \left\{ B^{(k,2)}_{\beta_2\alpha_2} \right\}$,

$$B^{(k,2)} = \left\{ B^{(k,2)}_{b_2\ldots b_{k-1};a_2\ldots a_{k-1}} \right\} = \left\{ (G_0^{-1})^{b_{k-1}} \; N_{b_2\ldots b_{k-1};a_2\ldots a_{k-1}} \right\}. \tag{4.3}$$

In (4.3), the partitions $b_2,\ldots,b_{k-1}$ of the sequence β_2 and the partitions

$a_2, \ldots, a_{k-1}$ of α_2 have been made explicit.

More precisely, it will be shown that all transition amplitudes are given by

$$< \Phi^{[b_k]} | \sum_{b_2} \cdots \sum_{b_{k-1}} B^{(k,2)} | \Phi^{[a_2]} > , \quad B^{(k,2)} = \left\{ B^{(k,2)}_{\beta_2 \alpha_2} \right\} , \quad (4.4)$$

where

$$| \Phi^{[b_k]} > = \left\{ \delta^{b_k c_k} | \phi^{(b_k)}_{\gamma_{k+1}} > \right\} , \qquad \gamma_k = (c_k, \ldots, c_{N-1}) \qquad (4.5)$$

is the vector of Faddeev-type components representing the k-cluster bound state wavefunction.

In the special case of elastic and/or rearrangement scattering, the amplitude is obtained by taking k=2 in (4.4). Recalling (3.5), we see that in this case $B^{(2,2)}$ in (4.3) reduces to the AGS matrix T , and (4.4) becomes

$$< \Phi^{[b_2]} | T | \Phi^{[a_2]} > \qquad (4.6)$$

which is a natural generalization of (4.1) for the N-body system.

The complete break-up amplitude is obtained by taking k=N in (4.4), in which case $< \Phi^{[b_N]} | \equiv < \Phi_{(0)} |$, the final state of N free particles. $B^{(N,2)}$ in (4.3) reduces here to $\left\{ K_{\beta_2 \alpha_2} \right\}$, and (4.4) becomes

$$<\phi_{(0)} | \sum_{b_2} \sum_{\beta_3} \sum_{\alpha_3} K^{b_2 a_2}_{\beta_3 \alpha_3} | \phi^{(a_2)}_{\alpha_3} > \quad , \qquad (4.7)$$

which is a natural generalization of (4.2).

We now proceed to show that Eq. (4.4) is indeed the transition amplitude for all physical processes. Using Eq. (2.30) and the definition of $B^{(k,2)}$ in (4.3), (4.4) can be written as

$$< \Phi^{[b_k]} | (G_0^{-1})^{b_{k-1}} \sum_{b_2 \ldots b_{k-1} \supset b_k} \cdots \sum (\delta^{b_2 a_2} | \Phi^{(a_2)}_{b_3 \ldots b_{k-1}} > - | \Psi^{[a_2]}_{b_2 \ldots b_{k-1}} >) \tag{4.8}$$

where the indices $b_2, \ldots, b_{k-1}$ have been made explicit. The first term in (4.8) can be shown to vanish, as follows: we combine (2.26) and (3.6) into

$$| \Phi^{(b_k)} > = - (G_0 V)^{b_k} | \Phi^{(b_k)} > \quad , \tag{4.9}$$

or using (3.1),

$$| \Phi^{(b_k)} > = - (G_0 T [1 + G_0 V])^{b_k} | \Phi^{(b_k)} > \quad . \tag{4.10}$$

From (3.4), however, $(G_0^{-1})^{b_{k-1}} = - \delta^{b_k c_k} (G_0 T^{b_k} G_0)^{-1}$, so that

$$(G_0^{-1})^{b_{k-1}} | \Phi^{(b_k)} > = (G_0^{-1})^{b_k} (1 + G_0 V)^{b_k} | \Phi^{(b_k)} > = 0 \quad . \tag{4.11}$$

The vanishing of (4.11) follows from (4.9), and the same is true for the first term in (4.8). Such an argument does not apply to (4.10), of course, due to the presence of singularities in the operator T . Similarly, a singularity in $| \Psi^{[a_2]}_{b_2 \ldots b_{k-1}} >$ prevents the second term in (4.8) from vanishing, and we have

$$- < \Phi^{[b_k]} | (G_0^{-1})^{b_{k-1}} \sum_{b_2} \cdots \sum_{b_{k-1}} | \Psi^{[a_2]}_{b_2 \ldots b_{k-1}} > =$$

$$= - \sum_{d_k} < \Phi^{(b_k)} | (G_0^{-1})_{b_k d_k} | \Psi^{d_k [a_2]} > \quad , \tag{4.12}$$

where, following the conventions of section II (cf. (2.22) and (4.5)), we have used the relations

$$\sum_{b_2} \cdots \sum_{b_{k-1}} | \Psi^{[a_2]}_{b_2 \ldots b_{k-1}} > = | \Psi^{b_k [a_2]} >$$

$$| \Phi^{[b_k]} > = \left\{ \delta^{b_k c_k} | \Phi^{(b_k)} > \right\} . \tag{4.13}$$

In (4.12), the sum over d_k is unrestricted. With the aid of (2.19) and (3.6), (4.12) can be written

$$\sum_{d_k} < \Phi^{(b_k)} | V_{b_k d_k} | \Psi^{d_k [a_2]} > \quad . \tag{4.14}$$

Using (3.4) we obtain from (4.14)

$$- \sum_{d_k} < \Phi^{(b_k)} | \bar{\delta}^{b_k d_k} (G_0^{-1}) | \Psi^{d_k [a_2]} > =$$

$$= \sum_{d_k} \sum_{b_{k+1} \subset d_k} \sum_{d_{k+1}} < \Phi^{(b_k)}_{b_{k+1}} | \bar{\delta}^{b_k d_k} V^{d_k}_{b_{k+1} d_{k+1}} | \Psi^{d_{k+1} [a_2]} > \quad , \tag{4.15}$$

where (2.21) and (3.6) have been used to obtain the right hand side (note that by using (2.21) we undo the corresponding splitting of the wavefunction).

Applying the same relations repeatedly, we obtain for (4.15)

$$\sum_{d_k \neq b_k} \sum_{b_{k+1} \subset d_k} \sum_{d_{k+1} \neq b_{k+1}} \sum_{b_{k+2} \subset d_{k+1}} \sum_{d_{k+2} \neq b_{k+2}} \cdots \sum_{b_{N-1} \subset d_{N-2}} \sum_{d_{N-1} \neq b_{N-1}}$$

$$< \phi^{(b_k)}_{\beta_{k+1}} | V_{d_{N-1}} | \Psi^{[a_2]} > \quad , \tag{4.16}$$

or, after interchange of the order of summation,

$$\sum_{b_{k+1}} \cdots \sum_{b_{N-1}} < \phi^{(b_k)}_{\beta_{k+1}} | \sum_{\substack{d_k \neq b_k \\ d_k \supset b_{k+1}}} \sum_{\substack{d_{k+1} \neq b_{k+1} \\ d_{k+1} \supset b_{k+2}}} \cdots \sum_{d_{N-1} \neq b_{N-1}} V_{d_{N-1}} | \Psi^{[a_2]} > . \tag{4.17}$$

However, it is shown in the appendix that

$$\sum_{\substack{d_k \neq b_k \\ d_k \supset b_{k+1}}} \cdots \sum_{d_{N-1} \neq b_{N-1}} V_{d_{N-1}} = \sum_{d_{N-1} \not\subset b_k} V_{d_{N-1}} \quad . \tag{4.18}$$

Therefore, the sums over β_{k+1} can be carried out, with the result

$$\sum_{\beta_{k+1}} \phi^{(b_k)}_{\beta_{k+1}} = \phi^{(b_k)} \tag{4.19}$$

where $\phi^{(b_k)}$ is the total wave function for the final, k-cluster system.

Collecting results, we finally conclude that (4.4) reduces to

$$< \Phi^{[b_k]} | \sum_{b_2} \cdots \sum_{b_{k-1}} B^{(k,2)} | \Phi^{[a_2]} > = < \phi^{(b_k)} | \sum_{d_{N-1} \not\subset b_k} V_{d_{N-1}} | \Psi^{[a_2]} > \quad , \tag{4.20}$$

where the right hand side can be recognized as the standard wavefunction expression for the elastic, rearrangement or breakup amplitude.

We have thus completed the demonstration that the left hand side of (4.20) is the N-body Faddeev-type theory expression for the scattering amplitude for any process that starts from a two-cluster initial state.

V. FADDEEV-TYPE EQUATIONS FOR THE TRANSITION OPERATORS

Returning to the definition (4.3) of $B^{(k,2)}$, we realize that the Faddeev-type equations for the transition operators can easily be obtained from the corresponding equations for N , (c.f. (4.3)). The result is

$$B^{(k,2)}_{b_2 a_2} = \bar{\delta}^{b_2 a_2} A^{b_2} - \sum_{d_2} B^{(k,2)}_{b_2 d_2} N^{d_2} \bar{\delta}^{d_2 a_2} \tag{5.1}$$

where

$$A^{b_2} = \left\{ (G_0^{-1})^{b_{k-1}} \quad N^{b_2}_{b_3 \ldots b_{k-1}; a_3 \ldots a_{k-1}} \right\} .$$

When summed over $b_2 \ldots b_{k-1}$, Eqs. (5.1) are therefore the Faddeev-type equations for the N-body transition operators for all physical processes. Since N also satisfies Faddeev-type equations with the kernel to the left[6], so does $B^{(k,2)}$:

$$B^{(k,2)}_{b_2 a_2} = \bar{\delta}^{b_2 a_2} \; A^{b_2} - C^{b_2} \sum_{d_2} \bar{\delta}^{b_2 d_2} \; B^{(k,2)}_{d_2 a_2} \tag{5.2}$$

where

$$C^{b_2} = \left\{ (G_0^{-1})^{b_{k-1}} \quad N^{b_2}_{b_3 \ldots b_{k-1}; d_3 \ldots d_{k-1}} \; (G_0)^{d_{k-1}} \right\} .$$

For elastic and rearrangement processes, Eqs. (5.1) and (5.2) reduce of course to the AGS equations for the N-body U-operators, while for complete breakup processes, (5.1) and (5.2) reduce to the N-body K-operator equations of Ref. 6.

VI. CONCLUSIONS

In this work we have first shown how to construct the equations for the Faddeev-type components of the N-body full wavefunction in a straightforward manner, by repeatedly applying the Faddeev procedure of removing disconnected pieces from the kernel of the equations. Each further application of this procedure separates the wavefunction into components that satisfy more connected equations, until finally a fully connected set of equations for the fully-split

wavefunction components is obtained.

We have further shown that, within the Faddeev scattering formalism, it is possible to construct a matrix of N-body transition operators that gives the scattering amplitudes for all physical processes that start from a two-cluster initial state, when appropriate matrix elements between Faddeev-type components of the initial and final states are taken. This was demonstrated by showing that these matrix elements reduce in all cases to the conventional expressions for the scattering amplitudes in terms of potentials and full wavefunctions.

The N-body transition operators reduce to the N-body generalization of the AGS U-operators for the case of elastic and rearrangement scattering, and to the N-body Faddeev-type kernel operators for the case of complete breakup. For other processes the transition operators form a set that has not been encountered previously; however, since they are simply related to the N-body kernel operators, it is easy to obtain the equations they satisfy. We have therefore concluded by giving the Faddeev-type equations for the N-body transition operators for all processes discussed above.

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APPENDIX

Here, we present the details of the proof of relation (4.18). In the special case N=4 and k=2 this is easily done, since in this case it reduces to

$$\sum_{\substack{\rho \neq \sigma \\ \rho \supset \beta}} \sum_{\substack{\gamma \neq \beta \\ \gamma \subset \rho}} V_\gamma = \sum_{\gamma \not\subset \sigma} V_\gamma \quad .$$

The equality simply follows from the fact that since $\gamma \neq \beta$, we can write

$$\sum_{\substack{\rho \neq \sigma \\ \rho \supset \beta}} \sum_{\substack{\gamma \neq \beta \\ \gamma \subset \rho}} V_\gamma = \sum_{\gamma \neq \beta} \sum_{\substack{\rho \supset \beta, \gamma \\ \rho \neq \sigma}} V_\gamma = \sum_{\gamma} \delta(\gamma \not\subset \sigma)\, V_\gamma \quad .$$

For arbitrary N and k, we first note that if $d_{N-1} \not\subset b_k$, the sequence $d_k \ldots d_{N-1}$ is uniquely determined by d_{N-1} and the conditions $d_{N-1} \neq b_{N-1}$, $d_{N-2} \supset b_{N-1}$, etc., and that this sequence is different from $b_k, \ldots, b_{N-1}$. Hence, each sum contributes only one term, except for that over d_{N-1}. However, if $d_{N-1} \subset b_k$, there is an i such that $b_{i-1} \supset d_{N-1}$, and $b_i \not\supset d_{N-1}$. Since d_i contains both b_{i+1} and d_{i+1}, and d_{i+1} contains both b_{i+2} and d_{i+2}, etc., it follows that d_i contains $b_{i+1}, d_{i+1}, \ldots, b_{N-1}, d_{N-1}$. But then it also follows that $d_i \subset b_{i-1}$. We now have $d_i \neq b_i$, and both contained in d_{i-1} and b_{i-1}. Hence $d_{i-1} = b_{i-1}$ and there is no contribution to the sum. The only contribution to the left hand side of (4.18) is therefore given by the right hand side of (4.18), and we thus obtain the proof.

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