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WAS GALILEO CORRECT ABOUT FALLING BODIES?*

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Abstract

The question is asked whether, as is often claimed, the Galilean conclusion is rigorously correct that bodies of different weight falling through the same height reach the ground in the same time. For purposes of this inquiry, a comparison is made of the Aristotelian and Galilean notions about falling bodies in both real and idealized cases with special emphasis on a nonrelativistic analysis of the Galilean idealization of a body falling in vacuum. This analysis shows that Galileo was not rigorously correct in the general case.

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I. CONTEXT OF THE PROBLEM

As gravitational phenomena are the most dramatic of any manifestation of the four basic interactions, they have been of great interest to scientists of all ages. In popular discussions regarding falling bodies, Aristotle is often made out to be the casual expounder who fails to make experimental observations, and Galileo as the innovative intellectual hero, who, by experiment, demonstrates that bodies of different weight falling through the same height reach the ground at the same time.

If we examine Aristotle's statement[1], made in the fourth century B.C., it is important to note that his theory is qualitatively correct in many real physical cases:

"If a certain weight moves (falls) a certain distance in a given time, a greater will move the same distance in a less time, and the proportion which the weights bear to one another, the times, too, will bear to one another, e.g., if one weight is twice another, if the half weight cover the distance in x, the whole weight will cover it in $x/2$." If we compare bodies of the same shape and size falling in a medium such as air or water, then they do reach terminal velocities nearly proportional to their weights as in the quotation. Of course, the terminal velocity is reached much sooner in the denser medium.

If Aristotle did conduct experiments, he more likely did it in a liquid such as water to slow down the falling body to an easily observable speed than in air.

From a historical point of view it is not true that Galileo was the first to challenge Aristotle and in so doing to introduce the experimental method. As early as the 5th centry A.D., and he may very well have had predecessors, Ioannes Philoponus challenged Aristotle's foregoing statement:

"But this is completely erroneous, and our view may be corroborated by actual observation more effectively than by any sort of verbal argument. For if you let fall from the same height two weights of which one is many times as heavy as the other, you will see that the ratio of the times required for the motion does not depend on the ratio of the weights, but that the difference in time is a very small one."[2]

In the early 17th century, Galileo[3] made essentially the same observation as Philoponus: "But I, Simplicio, who have made the test, can assure you that a cannon ball weighing one or two hundred pounds, or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits."

He argued that the slight difference in time could be ascribed to the resistance offered by the medium to the motion of the falling body. In air, feathers do fall more slowly than rocks. Galileo then made the idealization that in a medium totally devoid of resistance (a vacuum), all bodies will fall at the same speed. This idealization neglected the complexity of the fall of objects in media accessible to Galileo and his predecessors, and was indeed a significant advance toward a deeper understanding of the motion of bodies. Yet, as we shall see, even with this idealization, Galileo's conclusion is only approximately correct -- contrary to the statements made in widely used physics and philosophy of science texts.

II. ANALYSIS OF THE MOTION OF TWO GRAVITATIONALLY ATTRACTED BODIES

Consider two masses m_a and m_b interacting with one another gravitationally in vacuum. The force of mutual attraction is

$$F = -Gm_a m_b/r^2 \ , \tag{1}$$

where G is the universal gravitational constant and $\vec{r} = \vec{r}_b - \vec{r}_a$ is the distance between the centers of mass of m_a and m_b. Since this is a central force, the motion of the two bodies about their center of mass may be formally reduced to an equivalent one-body problem with a body of reduced mass

$$\mu = m_a m_b/(m_a + m_b) \ . \tag{2}$$

If the bodies are released from rest with inital separation r_1 and final separation r_2, by the principle of conservation of energy we have

$$\tfrac{1}{2}\mu v^2 + \frac{-Gm_a m_b}{r_2} = \frac{-Gm_a m_b}{r_1} \ , \tag{3}$$

where $\vec{v} = \vec{v}_b - \vec{v}_a$ is the relative velocity of the two bodies acquired as they fall toward one another. Solving equation (3), we find for the relative velocity

$$v = \left[2G(m_a + m_b)(r_2^{-1} - r_1^{-1})\right]^{\frac{1}{2}} \ . \tag{4}$$

Thus we see that the velocity of approach of the two masses (which is what an observer making measurements from the surface of the earth would measure) depends on the sum of the two masses. That is, it depends on both the mass of the earth and on the mass of the falling body. However, since the mass of the earth (6×10^{24} kg) is so much greater than the mass of any test bodies that are likely to fall, to a good approximation we may neglect the mass of the test body.

It is only in this sense that Galileo may be considered to be approximately correct, although he is not rigorously correct in the general case. The earth is not rigorously an inertial frame of reference for such a measurement since it, too, accelerates towards the falling body. The acceleration,

$$a = \frac{dv}{dt} = \frac{G(m_a + m_b)}{{r_2}^2} \tag{5}$$

is proportional to the sum of the masses. This is only a virtual violation of Einstein's General Theory of Relativity, since the measurement is from a noninertial frame. The velocities of the two bodies relative to their center of mass (which is an inertial frame) are:

$$\vec{v}_a = - [m_b/(m_a + m_b)]\vec{v} , \tag{6}$$

and

$$v_b = [m_a/(m_a + m_b)]\vec{v} . \tag{7}$$

Substituting equation (4) into equations (6) and (7), we have

$$v_a = -m_b\left[\frac{2G}{(m_a + m_b)}(r_2^{-1} - r_1^{-1})\right]^{\frac{1}{2}} , \qquad (8)$$

$$v_b = m_a\left[\frac{2G}{(m_a + m_b)}(r_2^{-1} - r_1^{-1})\right]^{\frac{1}{2}} . \qquad (9)$$

Thus we see that the velocities of the two bodies relative to the center of mass inertial frame as a function of position depend on the sum of the two masses. However, the accelerations a_a and a_b relative to the common center of mass or to any inertial reference frame are independent of the respective masses. These are:

$$a_a = \frac{dv_a}{dt} = -Gm_b/r_2^{\,2} , \qquad (10)$$

$$a_b = \frac{dv_b}{dt} = Gm_a/r_2^{\,2} \qquad (11)$$

in accord with General Relativity.

The solution for the three-body problem in which two test bodies are dropped simultaneously is not readily obtainable. However, the results should be similar to those we have obtained when the test bodies are not close together. It is interesting to note that for sequential earthbound experiments, where the sum of the test mass, m_a, and the mass of the rest of the earth, m_b, is a constant, Galileo can be correct, since the relative velocity depends on the sum $m_a + m_b$. However, it is clear from his writings that Galileo did not consider such a subtle point. If Simplicio had asked, "Will a meteor weighing ten thousand pounds fall faster to the earth than one weighing only ten pounds?", it is most likely that Galileo would have answered that, neglecting air resistance, both would fall with the same speed provided they fell from the same height with the same initial velocity.

Let us convince ourselves that the results obtained are consistent with the standard elementary approach to the problem. Let m_a be the mass of the falling test body dropped from a height h from the surface of the earth, and m_b be the mass of the earth; by Newton's Second Law:

$$m_a g = m_a a \ , \tag{12}$$

thus

$$v = (2ah)^{\frac{1}{2}} = (2gh)^{\frac{1}{2}} \ , \tag{13}$$

$$m_a g = G m_a m_b / r^2 \ . \tag{14}$$

Substituting equation (14) in equation (13), the velocity of free fall is:

$$v = \left[2Gm_b h/r^2\right]^{\frac{1}{2}} \ . \tag{15}$$

In the limit $m_b >> m_a$, we should get the same result from equation (4) which becomes:

$$v \doteq \left[2Gm_b\left(\frac{r_1 - r_2}{r_1 r_2}\right)\right]^{\frac{1}{2}} \quad . \tag{16}$$

Now $r_1 - r_2 = h$ and $r_1 \doteq r_2 = r$, so equation (16) becomes

$$v \doteq \left[2Gm_b h/r^2\right]^{\frac{1}{2}} \quad , \tag{17}$$

which is the same as equation (15). So the two approaches are equivalent in the limit $m_b >> m_a$.

III. DISCUSSION

In the light of our nonrelativistic theory, let us consider an argument made by Galileo[3] to show that Aristotle's hypothesis is logically inconsistent and to promote his own view. Tie m, a light stone, together with M, a heavy one, to form a double stone. Then in falling, m should retard M, since it falls more slowly than M. Hence the combination should fall at some speed between that of m and M. However, according to Aristotle, the double body $(m + M)$, being heavier than M, should fall faster than M. Galileo would have us believe that this physics question has been resolved by his _reductio ad absurdum_ argument and that not only is Aristotle absurdly wrong, but since the body $(m + M)$ cannot fall both more slowly and more quickly than the body M, it must, therefore, fall at the same speed as M, as he expounds. However, we know from our analysis that Aristotle was at least qualitatively correct (at least if the bodies are brought in from outside the earth) and that the double body $(m + M)$ does indeed fall faster than the body M, as measured from the earth.

Galileo's logic is _non sequitur_. Combining the mass m with the mass M does not necessarily slow down the double body $(m + M)$. This is the fallacy in Galileo's logic and is indicative of the pitfalls in reaching broad physical conclusions by means of rhetorical logic. To make the fallacy clear, let us consider a simple example. Let us drop two hollow bodies of the same size, but of widely different densities, in a liquid. The heavier one M will indeed fall at a greater speed than the light one m. Now let us compact the lighter body m so that it will fit inside the hollow heavier body M. Now dropping the double body $(m + M)$ in the liquid, it will clearly fall at an even greater speed than the body M. This illustrates not only the flaw in Galileo's logic but the need for experiment to decide physical questions. Galileo did perform experiments, which is indeed to his credit.

What is regrettable, is not Galileo's failure to discover the rigorously correct general law of falling bodies as measured from the earth, but rather the promulgation of his approximately correct result as if it were an exact law. This is commonly done in both physics and in philosophy of science texts. For example, in one excellent physics textbook[4] which is currently widely used we find the following statement which is typical of that found in other books:

"The most common example of motion with (nearly) constant acceleration is that of a body falling toward the earth. In the absence of air resistance it is found that all bodies, regardless of their size, weight, or composition, fall with the same acceleration at the same point of the earth's surface, and if the distance covered is not too great, the acceleration remains constant throughout the fall. This ideal motion, in which air resistance and the small change in acceleration with altitude are neglected, is called 'free fall'."

This entire paragraph was quoted so that it would be clear that the statement has not been taken out of context. Qualifications are given in it regarding air resistance and small change in acceleration with altitude. But unfortunately no qualification is made regarding the slight change in acceleration as a function of the mass of the falling body (considering the rest of the mass of the earth to be fixed if sequential measurements are made). This text is by no means alone in the omission that for measurements relative to the earth, the acceleration is proportional to the sum of the masses.

IV. CONCLUSION

The analysis shows that both the relative velocity and acceleration of two gravitating bodies are functions of the sum of the masses of the two bodies. As measured from the earth, a heavier body in vacuum will in principle fall slightly faster than a light body (provided that the rest of the earth's mass remains fixed if sequential experiments are performed). However, in practice the difference in speed is negligible for most test bodies, due to the large mass of the earth.

Although Galileo was not rigorously correct, we must credit him with tremendous insight. Our analysis is in no way intended to detract from Galileo's revolutionary contribution which led to a deep understanding of motion.

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