# $\pi-\pi$ SCATTERING AND $\pi^{+} d$ INTERACTIONS AT $7 \mathrm{GeV} / \mathrm{c}^{*}$ 

J．T．Carroll and J．A．J．Matthews

Stanford Linear Accelerator Center Stanford University，Stanford，Calif． 94305

W．D．Walker<br>Duke University，Durham，N。C． 27706

M．W．Firebaugh
University of Wisconsin，Parkside， Kenosha，Wisconsin 53140

J．D．Prentice and T．S．Yoon
University of Toronto， Toronto，Ontario，Canada


#### Abstract

Using bubble chamber data on the reactions $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p} \pi^{0} \pi^{0}$, $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-}$and $\pi^{-} \mathrm{p} \rightarrow \mathrm{n}^{+} \pi^{-}$at $7 \mathrm{GeV} / \mathrm{c}$ incident $\pi$ momentum， $\pi-\pi$ phase shifts are determined for $0.6<\mathrm{M}(\pi \pi)<1.5 \mathrm{GeV} / \mathrm{c}^{2}$ 。An $I=0 \mathrm{~S}$－wave resonance is obscrved in the $\mathrm{f}^{0}$ peak region of $\mathrm{M}(\pi \pi)$ 。 Constructive $\rho-\omega$ interference is found in the reaction $\pi^{+} \mathrm{n} \rightarrow \mathrm{p} \pi^{+} \pi^{-}$ and evidence is presented for some specifically deuteron effects in the data with large spectator proton momentum．


（Submitted to Phys．Rev．）

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## I. INTRODUCTION

In the last decade there has been great interest in studies of the reaction $\pi \pi \rightarrow \pi \pi$. The elastic scattering of identical spinless particles attracts inquiry in part due to its ultimate simplicity. In this paper we report on a study of both elastic and inelastic $\pi-\pi$ scattering.

We study the reactions

$$
\begin{align*}
& \pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}+\text { Neutrals }  \tag{1}\\
& \pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-} \tag{2}
\end{align*}
$$

as observed in a deuterium filled bubble chamber exposed to a $7 \mathrm{GeV} / \mathrm{c} \pi^{+}$beam. The feature of reaction (1) in which we are most interested is the reaction

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{\mathrm{o}} \pi^{\mathrm{o}} \tag{3}
\end{equation*}
$$

This reaction allows one to study the $\pi-\pi$ system in a state restricted to even values of spin $\ell$ and isotopic spin I. We are able to reconstruct reaction (3) using gammas from the $\pi^{\circ}$ decay which were converted in two $\frac{1}{8}$ inch tantalum plates mounted at the downstream end of the chamber. The dominant feature of the missing mass spectrum from reaction (1) is the $\mathrm{f}^{\mathrm{o}}$ meson. In addition to a $\mathrm{f}^{\mathrm{o}}$ signal, reaction (2) shows strong $\rho^{\circ}$ and $\mathrm{g}^{\circ}$ resonance production.

Reactions (1) and (2) have previously been studied in bubble chamber experiments with an incident $\pi^{+}$beam momentum of $2.7,{ }^{1} 3.65,{ }^{2} 4.5,{ }^{3} 5.1,{ }^{4} 6.0,{ }^{5}$ and $9.0 \mathrm{GeV} / \mathrm{c} .{ }^{6}$ In addition there have been some $\pi^{+} \mathrm{d}$ experiments with $\mathrm{P}_{\mathrm{LAB}} \simeq 2 \mathrm{GeV} / \mathrm{c}$ searching for the $\epsilon^{\circ}$ meson. ${ }^{7}$ The charge conjugate reaction to reaction (3),

$$
\begin{equation*}
\pi^{-} \mathrm{p} \rightarrow \mathrm{n} \pi^{\mathrm{o}} \pi^{\mathrm{o}} \tag{4}
\end{equation*}
$$

has been studied using spark chambers to measure the $\gamma$ directions from the $\pi^{0}$ decay. ${ }^{8}$

The outline of this paper is as follows. In Section II we discuss our data on reactions (1) and (3). Our experimental procedure is outlined and results on the elastic charge exchange reaction $\pi^{+} n \rightarrow p \pi^{\circ}$ are presented. We discuss the missing mass spectrum from reaction (1) and determine the cross section for $f^{\circ} \rightarrow$ all neutrals. Our procedure for reconstructing the $2 \pi^{\circ}$ system using the measured gamma directions is then introduced (see also Appendix A) and the fitted $2 \pi^{\circ}$ events are used to study the $\mathrm{M}\left(\mathrm{p} \pi^{\circ}\right)$ mass spectrum.

In Section III we discuss our data on reaction (2). Cross sections and resonance parameters for $\rho^{\circ}, f^{\circ}$, and $g^{\circ}$ production are obtained and compared with the data of B.Y.Oh et al. ${ }^{9}$ for the reaction

$$
\begin{equation*}
\pi^{-} \mathrm{p} \rightarrow \mathrm{n} \pi^{+} \pi^{-} \tag{5}
\end{equation*}
$$

at $7 \mathrm{GeV} / \mathrm{c}$. In Section IV we determine $\pi-\pi$ phase shift parameters for $0.6<\mathrm{M}(\pi \pi)<1.5 \mathrm{GeV} / \mathrm{c}^{2}$ by fitting the $\pi^{+} \pi^{-}$angular distributions using an Absorption-modified One-Pion-Exchange (AOPE) model. For this purpose we combine our data from reaction (2) with that of B.Y. Oh et al. ${ }^{9}$ for reaction (5). We discuss the inelasticity of the $I=0$-wave using data on the non $-2 \pi$ decay modes of $f^{0}$ meson. In Section $V$ we present evidence for constructive $\rho-\omega$ interference in the reaction (2) 4 -prong data. The $\pi$-nucleon mass spectra in reactions (2) and (5) are examined in Section VI, and some effects arising from our use of a deuteron target are discussed in Section VII.

$$
\text { II. } \pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}+\text { NEUTRALS }
$$

The $7 \mathrm{GeV} / \mathrm{c} \pi^{+} \mathrm{d}$ film analyzed in this experiment was obtained in a 650 K picture exposure of the Midwestern Universities Research Association ARGONNE National Laboratory 30 -inch bubble chamber using the $7^{\circ}$ separated beam from the ZGS. ${ }^{10-11}$ Each roll of film was scanned once within a specified fiducial volume for 2 -prong events for which both tracks were identifiable protons. All events were examined by an editor (an experienced scanner with special training) who checked the identification of the tracks, checked for stopping tracks, and estimated the proton ionization. The editor also checked the tantalum plates for associated $\gamma \rightarrow \mathrm{e}^{+} \mathrm{e}^{-}$pairs, checked the alignment of the gammas with the vertex of the event, and estimated a lower limit on the gamma energy. The editor took a 35 mm photograph of each good event, which was used to locate the gammas for measuring. The events were measured manually on film plane digitizers and all 2-prong events were processed using the DIANA ${ }^{12}$ spatial reconstruction and kinematic fitting program. The fits were checked for consistency between the calculated and scanner-estimated ionization.

For all events with (Missing Mass) ${ }^{2}<0.5 \mathrm{GeV}^{2}$ we tried the 1 -constraint (1C) hypothesis

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{\mathrm{o}} \tag{6}
\end{equation*}
$$

Figure $1(a, b)$ shows the missing mass and $\chi^{2}$ distributions for 2 -prong events accepted as fitting this reaction. Of the 250 events which fit reaction (6), $36 \%$ had a 3 -constraint fit using the measured gamma directions. This implies an overall $\gamma$ detection efficiency of 0.6 for $\gamma^{\prime}$ s which hit the plates with enough energy ( $\gtrsim 0.2 \mathrm{GeV}$ ) to produce a visible shower. For these single $\pi^{\circ}$ events the average $\pi^{\circ}$ momentum is $6.8 \mathrm{GeV} / \mathrm{c}$ and the average $\gamma \gamma$ opening angle is $3.0^{\circ}$,
so the fraction of $\gamma^{\prime}$ s missing the plates is negligible. Figure $1(\mathrm{~d})$ is a plot of $\mathrm{M}(\gamma \gamma)$ from the 2 C fit $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \gamma_{1} \gamma_{2}$; the $\pi^{o}$ peak is 10 MeV wide. In Fig. $1(\mathrm{c})$ we have plotted the ratio $\mathrm{E}_{\gamma_{1}} / \mathrm{E}_{\pi^{o}}$. There is a slight deviation from the expected flat distribution.

We find the elastic charge exchange cross section at $7 \mathrm{GeV} / \mathrm{c}$ to be $67 \pm 10 \mu \mathrm{~b}$ as shown in Table 1. This cross section is corrected for 1 -prong events (spectator proton unseen) but no allowance has been made for our upper cutoff of $1.3 \mathrm{GeV} / \mathrm{c}$ on proton momentum (apart from Fermi motion smearing this corresponds to $\left.t \simeq-1.25(\mathrm{GeV} / \mathrm{c})^{2}\right)$. In Fig. 2(a) we have plotted the elastic charge exchange cross section for reaction (6) vs $\mathrm{P}_{\text {LAB }}$. Our value seems to be consistent with measurements of the charge conjugate reaction $\pi^{-} p \rightarrow \pi^{0}{ }^{0} .^{13}$ Also shown are measurements of $\sigma\left(\pi^{+} \mathrm{n} \rightarrow \pi^{\circ} \mathrm{p}\right)$ at $4.5,{ }^{3}$ and $6 \mathrm{GeV} / \mathrm{c} .{ }^{5}$ Figure 2(b) is a plot of $\mathrm{d} \sigma / \mathrm{dt}$ for our elastic events. Fitting the data from 0.12 to 0.6 $(\mathrm{GeV} / \mathrm{c})^{2}$, we find an exponential falloff with slope $\simeq 10(\mathrm{GeV} / \mathrm{c})^{-2}$ in agreement with the results of Wahlig et $\underline{\mathrm{al}}^{14}$ for $\pi^{-} \mathrm{p} \rightarrow \pi^{\mathrm{O}} \mathrm{n}$ at both 6 and $10 \mathrm{GeV} / \mathrm{c}$. The turnover in the distribution at small $t$ is also seen in $\pi^{-} p \rightarrow \pi^{\circ} n$. Also shown in Fig. 2(b) are the low $t$ data points corrected for Pauli exclusion assuming all spin non-flip and using the Hulthen wave function to describe the deuteron form factor. ${ }^{15}$ The differential cross section (with no correction for Pauli exclusion) is given in Table 2.

In Fig. 3 we show the Missing Mass (MM) spectrum with single $\pi^{\circ}$ events excluded and with spectator momentum (a) $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right| \leq 0.3 \mathrm{GeV} / \mathrm{c}$, and (b) $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|>0.3$ $\mathrm{GeV} / \mathrm{c}$. The latter plot shows little evidence of the resonant structure so prominent in the former distribution. (A similar effect in the $\pi^{+} \mathrm{d} \rightarrow \mathrm{pp} \pi^{+} \pi^{-}$data is shown in Figs. 28, 29) 。 In agreement with the impulse approximation we will discard events with $\left|\vec{p}_{\mathrm{s}}\right|>0.3 \mathrm{GeV} / \mathrm{c}$, and this cut will always be understood
unless there is an explicit statement to the contrary. The dominant feature of the missing mass plot is the $\mathrm{f}^{\mathrm{o}}$ at $1.26 \mathrm{GeV} / \mathrm{c}^{2}$ 。Below the $\mathrm{f}^{\mathrm{O}}$ there is an $\eta^{\circ}$ signal at $0.548 \mathrm{GeV} / \mathrm{c}^{2}$ and a slight enhancement between $0.7-0.85 \mathrm{GeV} / \mathrm{c}^{2}$ from the neutral decay of the $\omega^{0}$. There is also some indication of structure above the $f^{\circ}$ at $1.65 \mathrm{GeV} / \mathrm{c}^{2}$. We estimate our mass resolution near the $\mathrm{f}^{\mathrm{o}}$ to be $0.05 \mathrm{GeV} / \mathrm{c}^{2}$.

The Chew-Low plot for these missing mass events is shown in Fig. 4. Most of the events, especially at the $f^{0}$, are concentrated at low $t\left(\pi^{+} \rightarrow\right.$ missing mass $)$, while the flatter $t$ distributions in the $\eta$ and $\omega$ mass regions are quite apparent. In the missing mass distribution, Fig. 3, we observe that demanding $|t|<0$. 2 $(\mathrm{GeV} / \mathrm{c})^{2}$ removes most of the $\eta$ and $\omega$ peaks but leaves a strong $\mathrm{f}^{\mathrm{o}}$ signal. In the Chew-Low plot there seems to be an excess of events at larger $|t|>0.2$ $(\mathrm{GeV} / \mathrm{c})^{2}$ just above the $\mathrm{f}^{\mathrm{o}}$ peak, probably from $\mathrm{A}_{2}^{\circ} \rightarrow \eta^{\circ} \pi^{\circ}$. Since we have only measured protons up to $1.3 \mathrm{GeV} / \mathrm{c}$, there is an effective upper cutoff at $|t| \simeq 1.25(\mathrm{GeV} / \mathrm{c})^{2}$. Because we are working in deuterium, the Chew-Low boundary is not sharp.

The total cross section for reaction (1) with $M M \geq 2 \mathrm{~m}_{\pi}$ is $620 \pm 60 \mu \mathrm{~b}$ as shown in Table 1. Comparing this result to other $\pi^{+} d$ experiments, ${ }^{3-5}$ we find the 2 -prong missing mass cross section is falling as $\sim \mathrm{P}_{\mathrm{LAB}}^{-0.9} \quad$ at our energy. For MM $\lesssim 1.0 \mathrm{GeV} / \mathrm{c}^{2}$ the $3 \pi^{\circ}$ phase space is negligible, thus to estimate cross sections for reaction (3) we need only to correct for the neutral decay modes of the $\eta^{\circ}$ and $\omega^{\circ}$. These corrections have been made by using known branching ratios together with our determination of $\eta^{\circ}$ and $\omega^{\circ}$ production cross sections in the reaction

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-} \pi^{\mathrm{o}} \tag{7}
\end{equation*}
$$

The dashed line below $0.9 \mathrm{GeV} / \mathrm{c}^{2}$ in Fig. 3(a) shows the result of making these corrections.

We have determined the $\mathrm{f}^{0}$ cross section by fitting the missing mass distribution above $0.9 \mathrm{GeV} / \mathrm{c}^{2}$ using a Breit-Wigner resonance form for the $f^{\mathrm{o}} .^{16}$ The best fit, as shown in Fig. 3(a), used a background of roughly equal amounts of peripheral $2 \pi^{\circ}$ and $3 \pi^{\circ}$ phase space. We also made a subtraction for $\mathrm{A}_{2} \rightarrow \eta^{\circ} \pi^{\circ}$ based on data from reaction (7), 17,18 The most obvious failure of the fit is the inability to fit the high side of the $f^{\circ}$. We find:

$$
\begin{aligned}
& \mathrm{M}\left(\mathrm{f}^{\mathrm{o}}\right)=1.26 \pm 0.01 \mathrm{GeV} / \mathrm{c}^{2} \\
& \Gamma\left(\mathrm{f}^{\mathrm{o}}\right)=0.18 \pm 0.03 \mathrm{GeV} / \mathrm{c}^{2} \\
& \sigma\left(\mathrm{f}^{\mathrm{o}}\right)=120 \pm 20 \mu \mathrm{~b}
\end{aligned}
$$

In missing mass data at $5.1 \mathrm{GeV} / \mathrm{c}$ Armenise et al..$^{4}$ found $\mathrm{M}\left(\mathrm{f}^{\mathrm{o}}\right)=1.27$ and $\Gamma\left(f^{\mathrm{O}}\right)=0.188 \mathrm{GeV} / \mathrm{c}^{2} .^{19}$ Our cross section of $120 \mu \mathrm{~b}$ for $\pi^{+} \mathrm{n} \rightarrow \mathrm{pf}^{\mathrm{O}}$ is for $\mathrm{f}^{\mathrm{o}} \rightarrow$ all neutrals. The cross section for $\pi^{+}{ }_{\mathrm{n}} \rightarrow \mathrm{pf}^{\mathrm{O}}$ with $\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{\mathrm{o}} \pi^{\mathrm{o}}$ is somewhat smaller since the $\mathrm{f}^{\mathrm{O}}$ has other all neutral decay modes (see Section IV).

Figure 5 shows momentum transfer (-t) distributions for mass intervals below, at, and above the $f^{\circ}$. We have fit the events in the $f^{\circ}$ region to an exponential distribution of the form $\mathrm{e}^{\beta t}$. Fitting the data with $0.04<|t|<0.5$ $(\mathrm{GeV} / \mathrm{c})^{2}$ we find a slope $\beta=8.0 \pm 1.3(\mathrm{GeV} / \mathrm{c})^{-2}$ as shown in Fig. 5(d). The CERN experiment at $5.1 \mathrm{GeV} / \mathrm{c},{ }^{4}$ found a corresponding slope of $\beta=8.8 \pm 1.7$ $(\mathrm{GeV} / \mathrm{c})^{-2}$ (they fit the t interval $\left.0.04-0.28\right)$. For the $\mathrm{f}^{\mathrm{o}}$ in reaction (2) we find an exponential slope of $10.0 \pm 1.0(\mathrm{GeV} / \mathrm{c})^{-2}$ for the $t$ distribution in the mass interval $1.18<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.34 \mathrm{GeV} / \mathrm{c}^{2}$. The lower value of the slope in the missing mass data can come fron non $-2 \pi^{\circ}$ events, for example: $\mathrm{A}_{2}^{\mathrm{O}} \rightarrow \eta^{0} \pi^{0}$, or $3 \pi^{\circ}$ 。

If we observe all four gammas from the $2 \pi^{\circ}$ decay in reaction (3), we have an ordinary 2C fit. Because of our limited gamma detection efficiency (see Appendix A) most $2 \pi^{\circ}$ events do not yield four observed gammas. In Fig. 6 the missing mass is plotted according to the number of $\gamma^{\prime}$ s measured. An upper limit of 6 was imposed on the number of measured gammas per event ( $\approx .003$ of the events had more than 6 associated gammas). Only $10 \%$ of the events have no observed gammas while $32 \%$ have two gammas. The $\mathrm{f}^{\mathrm{O}}$ is quite apparent in all five categories of gammas measured in Fig. 6.

In order to fit the events with less than $4 \gamma^{\prime}$ s we must make some approximations, since ordinarily these events would be underconstrained. The opening angle for the decay of a particle of mass $\mu$ and momentum $p_{\pi}$ into two gammas satisfies the inequality $\tan \frac{1}{2} \theta \geq \mu / \mathrm{p}_{\pi} . \quad$ As $\mathrm{p}_{\pi}$ increases $\theta_{\text {MIN }}$ decreases and the opening angle distribution becomes sharply peaked near $\theta_{\text {MIN• }}$ Our procedure has been to construct artificial $\pi^{0}$ tracks constrained to lie on cones of half-angle $\approx 1.25 \theta_{\mathrm{MIN}} / 2$ about the measured gamma direction. For the 2 and $3 \gamma$ events the fitting was done using these artificial $\pi^{0}$ tracks. For $1 \gamma$ events we simply point $\pi_{1}^{o}$ in the $\gamma$ direction and calculate the direction of $\pi_{2}^{0}$. This fitting procedure and our $\gamma$ detection efficiency are discussed in Appendix A.

Figure 7(b) shows the fitted $M\left(\pi^{0} \pi^{0}\right)$ for events with $1,2,3$ or 4 gammas. Except for a more rapid fall-off at large $M(\pi \pi)$ the structure of the spectrum is basically the same as the missing mass plot. The fitting procedure with less than $4 \gamma^{\prime}$ s does not improve the mass resolution. Peaks at the $\eta^{\circ}$ and $\omega^{\circ}$ are apparent. Most of the $\eta^{\circ}$ events come from $\eta^{\circ} \rightarrow 3 \pi^{\circ}$, since we have extracted the $2 \gamma$ fits. The fitting procedure yields no discrimination against $3 \pi^{\circ}$ events for $M\left(3 \pi^{\circ}\right) \lesssim 1.0 \mathrm{GeV} / \mathrm{c}^{2}$. In order to achieve such discrimination the
individual $\pi^{\mathrm{O}_{1}} \mathrm{~s}$ must have sufficient transverse momentum to be well separated in the Ta plates. Our method offers no hope of distinguishing between $\pi^{\mathrm{o}} \gamma$ and $2 \pi^{\mathrm{o}}$ unless we see all $3 \gamma^{\prime}$ s from the $\pi^{\mathrm{o}} \gamma$ decay. Consequently, the structure from $0.7-0.85 \mathrm{GeV} / \mathrm{c}^{2}$ is consistent with $\omega^{\circ} \rightarrow \pi^{0} \gamma$.

We have chosen to include the $1 \gamma$ fits even though these events provide no discrimination against $3 \pi^{\circ}$ and have an angular resolution slightly worse than the 2, 3 and $4 \gamma$ events. There are two factors motivating this decision. First, the details of the fitted $1 \gamma$ mass and angular distributions are nearly the same as for the 2,3 and $4 \gamma$ fits. Although the $1 \gamma$ mass plot, Fig. 7(a), has more events above $\mathrm{M}(\pi \pi)=1.5 \mathrm{GeV} / \mathrm{c}^{2}$, this is not a serious drawback since we are mostly interested in the region at and below the $\mathrm{f}^{\mathrm{O}}$. Secondly, by using the $1 \gamma$ fits we avoid having to correct the fitted distributions for the absence of these events. We can also ignore, to a first approximation, any biases introduced by the fitting program efficiency - a genuine $2 \pi^{\circ}-2 \gamma$ event which failed to fit would probably fit with only $1 \gamma$.

Using our fitted $2 \pi^{\circ}$ events we examine the $M\left(p \pi^{\circ}\right)$ mass spectrum. $M\left(p \pi_{1}^{\circ}\right)$ and $\mathrm{M}\left(\mathrm{p} \pi_{2}^{0}\right)$ defined such that $\left|t\left(\pi^{+} \rightarrow \pi_{2}^{0}\right)\right|<\left|t\left(\pi^{+} \rightarrow \pi_{1}^{0}\right)\right|$ are shown in Fig. 8 (a,b). In $\mathrm{M}\left(\mathrm{p} \pi_{1}^{0}\right)$ we see what appears to be the $\Delta^{+}(1236)$, particularly in the plot with $\left|t\left(\pi^{+} \rightarrow \pi_{2}^{0}\right)\right|<0.2 \mathrm{GeV} / \mathrm{c}^{2}$ (see for comparison Fig. 23, 24). While the peak position is slightly high, a Monte Carlo study of our fitting procedure showed no systematic shift in $M\left(p \pi^{\circ}\right)$ 。In the insert, Fig. 8(c), we plot $M\left(\pi^{\circ} \pi^{\circ}\right)$ for events in the $\Delta^{+}$band. There is a peak at $1.3 \mathrm{GeV} / \mathrm{c}^{2}$ from the overlap of the $f^{\circ}$ and $\Delta^{+}$, otherwise the structure is quite smooth. Correcting for the fitting program efficiency (see Appendix A), we estimate a cross section of $13 \pm 5 \mu \mathrm{~b}$ for $\Delta^{+}$production in reaction (3). This compares favorably with a predicted cross section for $\pi^{+} \mathrm{n} \rightarrow \Delta^{+} \pi^{\circ}$ in reaction (3) of $20 \pm 10 \mu \mathrm{~b}$ from data on $\pi^{-} \mathrm{p} \rightarrow \mathrm{p} \pi^{-} \pi^{\circ}$ at $7 \mathrm{GeV} / \mathrm{c}{ }^{9}$

$$
\text { III. } \pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-}
$$

The film was scanned for all three prong events with one proton and four prong events with one or two protons identifiable by ionization ( $|\overrightarrow{\mathrm{p}}| \lesssim 1.5 \mathrm{GeV} / \mathrm{c}$ ). This scanning selection implies that the four prong events are essentially unbiased as regards target proton momentum, whereas the three prong events have an upper cutoff on target proton momentum at $|\overrightarrow{\mathrm{p}}| \simeq 1.5 \mathrm{GeV} / \mathrm{c}$. For the 4 -prongs $1.6 \%$ of the events have a proton with momentum larger than $1.5 \mathrm{GeV} / \mathrm{c}$. The 3 and 4 prong events were processed with the BRAVE-TVGP-SQUAWARROW system of programs. For the three prong events we used the standard constraint on the unseen spectator proton as provided by SQUAW. The optical data, beam constraint, etc. were the same as used for the two prong events. For most of these events the best fit was selected on the basis of highest constraint class and lowest $\chi^{2}$. For the three prongs we also demanded that the fit spectator momentum projected onto the $x-y$ plane ( $z$ is along the optic axis) be less than $0.1 \mathrm{GeV} / \mathrm{c}$. A detailed discussion of the experimental procedure can be found in Ref. 10 and 11.

We find a total cross section of $0.95 \pm 0.07 \mathrm{mb}$ for reaction (2) with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$ (see Table 3). This cross section agrees well with what one would predict from lower energy $\pi^{+}$d experiments. ${ }^{3-5}$ However, our result is $30 \%$ smaller than the cross section found for the charge conjugate reaction (5) and the difference is too large to be accounted for simply by Glauber screening; a correction of $3 \%$ is used for screening. In fact, our analysis suggests that a substantial part of this discrepancy may be attributed to three sources. Firstly, the Pauli exclusion principle at small momentum transfers suppresses the $\pi^{+} \mathrm{n}$ cross section by $\gtrsim 6 \%$ (a lower bound obtained by assuming pure spin flip at the nucleon vertex). Secondly, the $\pi^{+} d$ scanning criteria cause the high momentum
transfer events to be lost. A direct comparison of $\pi^{+} n$ and $\pi^{-} p$ data indicates that the $\pi^{+} \mathrm{n}$ cross section should be scaled up by the factor $1.06 \pm 0.02$. Finally a correction factor of $1.07 \pm 0.01$ is required to account for the abnormally large number of deuterium events with spectator momenta $\geq 0.3 \mathrm{GeV} / \mathrm{c}$ (see Table 3), possibly a result of secondary interactions with the spectator nucleon.

As shown in Fig. 9 the $\pi^{+} \pi^{-}$mass spectrum is dominated by $\rho^{\circ}, \mathrm{f}^{0}$ and $\mathrm{g}^{0}$ production. For the 4 prong events we have demanded $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$. The relative heights of the $\rho^{0}$ and $f^{\circ}$ peaks are the same for the 3 and 4 prong sets of data. Our mass resolution is 10 MeV near the $f^{\circ}$ peak. We have fit the combined 3 and 4 prong data with $\mathrm{M}\left(\pi^{+} \pi^{-}\right)<2.1 \mathrm{GeV} / \mathrm{c}^{2}$ to $2 \pi$ phase space and Breit-Wigner resonance forms for the $\rho$, f and g (see Ref. 18 for a description of the fit procedure). The resulting fit is shown in Fig. 9 with $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$in 20 MeV bins. The high mass sides of the $\rho$ and f are not fit very well. The mass plot shows what appear to be shoulder-like structures on the high side of these peaks. The fit to the $\mathrm{g}^{\circ}$ is poor, mostly because the mass and width of the g are not well determined by the data.

Resonance parameters and cross sections as determined by this fit are given in Table 4. Also shown in Table 4 are the corresponding parameters for the $7 \mathrm{GeV} / \mathrm{c} \pi^{-} \mathrm{p}$ data using the same fitting procedure. Our $\rho^{0}$ cross section is consistent with what one would predict from lower energy $\pi^{+}$d experiments assuming a $P_{\text {LAB }}^{-2}$ energy dependence. We find the ratio

$$
\sigma\left(\mathrm{f}^{0} \rightarrow \text { all neutrals }\right) / \sigma\left(\mathrm{f}^{0} \rightarrow \pi^{+} \pi^{-}\right)=0.47 \pm .09
$$

in good agreement with lower energy $\pi^{+}$d experiments at $3.65,{ }^{2}$ and $5.1 \mathrm{GeV} / \mathrm{c} .{ }^{4}$
In Fig. 10 we plot the momentum transfer, t , from the beam to the $\pi^{+} \pi^{-}$ system for various $\pi-\pi$ mass intervals in the $\pi^{+} d$ data. The forward differential
cross sections for $\rho^{0}$ and $f^{0}$ production in the combined $\pi^{-} p$ and $\pi^{+} d$ data are given in Table 5. Both the $\pi^{-} p$ and $\pi^{+} d$ data show a break in the $t$ distribution at $|t| \simeq 0.25(\mathrm{GeV} / \mathrm{c})^{2}$. Fitting the $\pi^{+} \mathrm{d}$ distributions for $|\mathrm{t}|<0.24(\mathrm{GeV} / \mathrm{c})^{2}$ to an exponential of the form $\mathrm{e}^{\beta \mathrm{t}}$ we find a slope $\beta$ in the range $11-14(\mathrm{GeV} / \mathrm{c})^{-2}$ for all mass intervals shown in Fig. 10. The corresponding exponential slopes from the $7 \mathrm{GeV} / \mathrm{c} \pi^{-} \mathrm{p}$ data are with one exception within errors of the $\pi^{+} \mathrm{d}$ values. The exception is for $1.34<\mathrm{M}(\pi \pi)<1.42 \mathrm{GeV} / \mathrm{c}^{2}$ where the $\pi^{-} \mathrm{p}$ data gives $\beta=7.4 \pm 2.9$. From comparison with the $\pi^{-} p$ data we can make a rough estimate of the number of small $t$ events missing in the $\pi^{+} d$ data because of Pauli Exclusion. It appears that for $\mathrm{M}\left(\pi^{+} \pi^{-}\right) \lesssim 0.9 \mathrm{GeV} / \mathrm{c}^{2}$ we lose from $30-$ $45 \%$ of the events with $|\mathrm{t}|<0.02(\mathrm{GeV} / \mathrm{c})^{2}$ while the loss at larger t is negligible. This loss is consistent with Pauli Exclusion assuming approximately half spin-flip and half spin-non-flip. In the $\mathrm{f}^{\circ}$ region the loss of events with $|t|<0.02(\mathrm{GeV} / \mathrm{c})^{2}$ is $5-10 \%\left(\left|\mathrm{t}_{\mathrm{MIN}}\right| \simeq 0.014\right.$ at the $\left.\mathrm{f}^{\mathrm{o}}\right)$.

$$
\text { IV. } \pi-\pi \text { SCATTERING IN } \pi N \rightarrow N \pi \pi
$$

## A. Procedure

The $\pi-\pi$ scattering is usually parameterized in terms of phase shifts $\delta_{\ell}^{\mathrm{I}}$ and inelasticities $\eta_{\ell}^{\mathrm{I}}$ ( $\ell=$ angular momentum and $\mathrm{I}=$ isotopic spin of the $\pi-\pi$ system). For $\mathrm{M}(\pi \pi) \lesssim 1.0 \mathrm{GeV} / \mathrm{c}^{2}$ the $\pi-\pi$ phase shifts have been studied by many authors. ${ }^{20-22}$ Recent experiments have clarified the behavior of the phase shifts in the $\rho^{\circ}$ mass region and provided data in and above the $\mathrm{f}^{\mathrm{o}}$ mass region. ${ }^{23-27}$

Since the reaction $\pi \pi \rightarrow \pi \pi$ cannot be studied directly, one is always dependent on a model to extract $\pi-\pi$ scattering data from some other reaction. With the OPE model as first developed by Goebel ${ }^{28}$ and by Chew and Low ${ }^{29}$ the idea was to extract $\pi-\pi$ phase shifts from the reaction $\pi N \rightarrow N \pi \pi$ by extrapolating in the variable $t$ from the physical (off mass shell) to the unphysical (on mass shell) point at $t=\mathrm{m}_{\pi}^{2}$. Other than direct extrapolation procedures the methods for using the OPE model to study $\pi-\pi$ scattering fall into two general categories. The first approach, as used by Ferrari and Selleri, ${ }^{30}$ attempts to allow for off mass shell effects by introducing form factor functions of $t$ at the upper and lower verticies of the OPE diagram. Dürr and Pilkuhn modified this procedure by adding angular momentum barrier penetration factors to the vertex function, and Benecke and Dürr did a relativistic Dürr-Pilkuhn treatment. ${ }^{31}$

The second method of modifying the simple OPE model is to adjust the formalism so as to take into account the strong absorption of the low partial waves in the entrance and exit channels of the reaction. In the Absorptionmodified One-Pion-Exchange model (AOPE) as originally developed by Gottfried and Jackson ${ }^{32}$ the absorption in the initial and final states is appoximated as being similar to elastic $\pi$-nucleon scattering. In our analysis we have used the AOPE formalism of Durand and Chiu. ${ }^{33}$ For a detailed discussion of our analysis see Ref. 9 and 10 .

For the purpose of making a phase shift analysis of the $\pi^{+} \pi^{-}$system we have combined the $\pi^{+}$d data of this experiment with the $6.93 \mathrm{GeV} / \mathrm{c} \pi^{-} \mathrm{p}$ data ${ }^{9}$ to obtain a total sample of $10845 \pi^{+} \pi^{-}$events. For this study we use only the data with a momentum transfer $|\mathrm{t}|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$. In Fig. 11 we plot $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$for the 8039 events which survive this $t$ cut. We confine our study to the mass range $0.58<\mathrm{M}(\pi \pi)<1.5 \mathrm{GeV} / \mathrm{c}^{2}$. In Fig. 12-13 we plot the $\pi-\pi$ scattering angles, $\cos \theta_{\pi \pi}$ and the azimuthal angle $\varphi$, in the Jackson frame. We use 40 MeV bins except for the mass range $1.22<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.34 \mathrm{GeV} / \mathrm{c}^{2}$ where we use 20 MeV bins. The angular distributions plotted separately for the $\pi^{-} p$ and $\pi^{+} d$ data (not shown) are in good agreement. In the $\rho^{\circ}$ region of Fig. 12 we see the forward peaking in $\cos \theta$ resulting from the $S-P$ wave interference. Around $1.0 \mathrm{GeV} / \mathrm{c}^{2}$ there is peaking in the backward direction and above $1.14 \mathrm{GeV} / \mathrm{c}^{2}$ there is a strong D-wave signal characteristic of the $f^{\circ}$. Near the $\rho^{o}$ the azimuthal distributions (Fig. 13) tend to peak near $\varphi=0^{\circ}$. This peaking is well described by the absorption model. Above $1.0 \mathrm{GeV} / \mathrm{c}^{2}$ the data is consistent with isotropic distributions in $\varphi$. The curves in Fig. 12-13 are the result of fitting these angular distributions to determine $\pi-\pi$ phase shifts.

The differential cross section for $\pi N \rightarrow N \pi \pi$ can be written in the form ${ }^{9}$

$$
\begin{equation*}
\left.\frac{\partial^{4} \sigma}{\partial \mathrm{~m}_{\pi \pi^{\partial t} \partial \cos \theta \partial \varphi}^{\partial}}=\mathrm{C} \frac{\mathrm{k}}{\mathrm{P}_{\mathrm{LAB}}^{2}} \quad \frac{1}{2} \sum_{\mu \lambda^{\prime} \lambda}\left|\left\langle\mu \lambda^{\prime}\right| \mathrm{T}\right| \lambda\right\rangle\left.\right|^{2} \tag{8}
\end{equation*}
$$

$C=$ normalization constant
$P_{\text {LAB }}=$ incident laboratory beam momentum,
$\mathrm{k}=$ momentum of outgoing $\pi$ in $\pi \pi$ C. M. ,
$\mu=$ helicity of the dipion system,
$\lambda^{\prime}, \lambda=$ helicity of the outgoing and incoming nucleons.

The amplitudes $\left\langle\mu \lambda^{\prime}\right| T \mid \lambda>$ can be expanded in terms of spherical harmonics and the $\pi-\pi$ phase shifts. Appendix B of Ref. 9 gives explicit forms of these amplitudes with the absorption modifications as used in this analysis. ${ }^{34}$ The only free parameters in (8) are the $\pi \pi$ phase shifts, inelasticities, and normalization constant, C.

For each $\pi-\pi$ mass interval in Fig. 12 we made a least-squares fit to the $\cos \theta$ and $\varphi$ distributions simultaneously with the phase shifts and inelasticities as the only free parameters. Normally we used bins of 0.1 in $\cos \theta$ and $18^{\circ}$ in $\varphi$, i.e. a $20 \times 10$ matrix. For a given trial set of phase shift parameters we performed a numerical integration of equation (8) over the $t$ interval $|t|_{\mathrm{MIN}}<|t|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$ for each point of the $20 \times 10$ matrix in $\cos \theta$ and $\varphi$ (the integration was carried out in terms of $\cos \theta \mathrm{CM}^{\prime}$ ). A fit with 27 degrees of freedom typically yielded a $\chi^{2}$ of 29 to 35 .

The overall normalization in equation (8) was fixed so as to maximize the agreement between the fit values for $\delta_{\mathrm{D}}^{0}$ and the predictions of a Breit-Wigner resonance form for the $\mathrm{f}^{\mathrm{O}}$ in the mass range $1.25<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.32 \mathrm{GeV} / \mathrm{c}^{2}$. Good agreement can only be obtained on the low mass side since we find the $I=0 D$-Wave to be significantly inelastic at and above the $f^{0}$ peak. We estimate that this procedure allows the normalization to be determined to $\approx 10 \%$. In order to obtain values of $\delta_{\mathrm{P}}^{1}$ at the $\rho^{0}$ peak in good agreement with a BreitWigner resonance form we had to use a normalization $13 \%$ larger than that found at the $f^{\circ}$ peak. For $\mathrm{M}\left(\pi^{+} \pi^{-}\right)<0.98 \mathrm{GeV} / \mathrm{c}^{2}$ we have used the larger normalization found at the $\rho^{\circ}$ peak. Occasionally we have constrained a particular parameter so as to maintain reasonable continuity from one mass bin to the next, usually this was not necessary and the solutions were unique.
B. $I=2$ Phase Shifts

The $\pi^{+} \pi^{-}$data is rather insensitive to the $I=2$ phase shifts and inelasticities: $\eta_{\mathrm{S}}^{2} \delta_{\mathrm{S}}^{2}, \eta_{\mathrm{D}}^{2}$, and $\delta_{\mathrm{D}}^{2}$. To fit the $\pi^{+} \pi^{-}$angular distributions we have fixed the $\mathrm{I}=2$
parameters at values determined from the reaction

$$
\begin{equation*}
\pi^{-} p \rightarrow p \pi^{-} \pi^{\circ} \tag{9}
\end{equation*}
$$

For $M\left(\pi^{+} \pi^{-}\right)<1.2 \mathrm{GeV} / \mathrm{c}^{2}$ we used the $\mathrm{I}=2$ phase shifts of J.P. Baton et al. ${ }^{22}$ as shown in Fig. 14. Our $\pi^{+} \pi^{-}$data in this mass region supports these results. Studies of the reaction ${ }^{35}$

$$
\pi^{-} \mathrm{d} \rightarrow \mathrm{pp} \pi^{-} \pi^{-}
$$

also support the general features of the $I=2$ analysis of Baton et al. In Fig. 15 we plot $\pi-\pi$ angular distributions for reaction (9) from the data of B.Y. Oh et al. ${ }^{9}$ with $P_{\text {LAB }}=6.93 \mathrm{GeV} / \mathrm{c}$. Fitting this data in 80 MeV bins for $0.98<\mathrm{ML}\left(\pi^{-} \pi^{\mathrm{O}}\right)<$ $<1.22 \mathrm{GeV} / \mathrm{c}^{2}$, we find that the resulting $\delta_{\mathrm{S}}^{2}$ and $\delta_{\mathrm{D}}^{2}$ also agrees with the phase shifts determined by Baton et al.

To determine the $\mathrm{I}=2$ parameters above $1.2 \mathrm{GeV} / \mathrm{c}^{2}$ we have fit the $\pi^{-} \pi^{\circ}$ angular distributions from the $6.93 \mathrm{GeV} / \mathrm{c}$ data as shown in Fig. 15. The overall normalization has been adjusted so as to maximize the agreement between our fit results and those of Baton et al. in the mass range $.98-1.22 \mathrm{GeV} / \mathrm{c}^{2}$. The P-wave parameters were fixed so as to agree with the $\pi^{+} \pi^{-}$fit results. The resulting fits to the $\pi^{-} \pi^{\circ}$ angular distributions are shown in Fig. 15, and the fit parameters $\eta_{\mathrm{S}}^{2}$, $\delta_{\mathrm{S}}^{2}$, and $\delta_{\mathrm{D}}^{2}$ are plotted with error bars in Fig. 14. Although the data is consistent with $\eta_{\mathrm{D}}^{2}=1.0$ for $\mathrm{M}(\pi \pi)<1.58 \mathrm{GeV} / \mathrm{c}^{2}$, we cannot rule out the possibility that $\eta_{\mathrm{D}}^{2}$ is somewhat smaller than 1.0 above $1.46 \mathrm{GeV} / \mathrm{c}^{2}$. Above $1.2 \mathrm{GeV} / \mathrm{c}^{2}$ we find $\delta_{\mathrm{S}}^{2}$ is falling steadily and gradually becoming inelastic. The errors are large since in addition to limited statistics we must contend with a low S-wave unitarity bound. The D -wave, $\delta_{\mathrm{D}}^{2}$, is relatively constant near $-16^{\circ}$ for $1.25<\mathrm{M}(\pi \pi)<1.55 \mathrm{GeV} / \mathrm{c}^{2}$. For the purpose of fitting the $\pi^{+} \pi^{-}$data we have used the smooth curves drawn through the fit results above $1.2 \mathrm{GeV} / \mathrm{c}^{2}$ in Fig. 14.

## C. The $\rho^{0}$ Region: $\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.0 \mathrm{GeV} / \mathrm{c}^{2}$

For the purpose of discussing the $\pi^{+} \pi^{-}$fit results we consider the data above and below 1.0 GeV/c ${ }^{2}$ separately. This division is prompted by the dominance of the resonant parameters $\delta_{\mathrm{P}}^{1}$ and $\delta_{\mathrm{D}}^{0}$ for $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$below and above $1.0 \mathrm{GeV} / \mathrm{c}^{2}$. With the $\mathrm{I}=2$ phase shifts fixed as discussed above we fit the $\pi^{+} \pi^{-}$ angular distributions in the $\rho^{0}$ mass region with the $I=0,1$ phase shifts and inelasticities as the only free parameters. The resulting best fit parameters are shown in Fig. 14 and 16. The AOPE model fits well the general deviation from isotropy in the azimuthal angle distributions. Some details of the $\cos \theta$ distributions such as the sharp forward peaking near $\cos \theta_{\pi \pi}=1.0$ are poorly fit. A previous analysis which included nucleon-pole terms in the production amplitude was also unable to fit this forward peaking. ${ }^{9}$

The fit $I=0,1$ phase shift parameters are tabulated in Table 6. The quoted errors for $\delta_{\ell}^{I}$ are usually taken from the least-squares fit. Occasionally the fitting program has trouble determining the error for a particular fit parameter, e.g. $\delta_{\mathrm{D}}^{0}$ near $0^{0}$ in the $0.8-1.0 \mathrm{GeV} / \mathrm{c}^{2}$ region. In such cases the errors have been estimated from fits with the particular parameter in question fixed at various trial values. The error estimates for $\eta_{\mathrm{D}}^{0}$ have usually been found in the same manner, i.e. by trial and error. Because $\eta_{\ell}^{I}$ and $\delta_{\ell}^{I}$ are usually highly correlated the fitting program has difficulty determining reasonable errors for these two parameters simultaneously. The quoted errors do not include the $\approx 10 \%$ uncertainty in the overall normalization; however, we note that a change in the normalization of $\approx 12 \%$ at the $f^{0}$ peak moves $\delta_{S}^{0}$ and $\delta_{D}^{0}$ by $5-6^{0}$.

The various families of $I=0 \mathrm{~S}$-wave phase shifts in the $\rho^{\circ}$ mass region have been the subject of considerable controversy in the literature. The up-up set of phase shifts as shown in Fig. 17(a) was originally proposed by Hagopian and

Selove. ${ }^{36}$ This solution received support in the work of Malamud and Schlein, ${ }^{37}$ and L.J. Gutay et al. ${ }^{38}$ The now accepted up-down family was first proposed by W.D. Walker et al. ${ }^{39}$ Several experiments on the $2 \pi^{\circ}$ system including this one now clearly indicate that the up-down solution is correct in the $\rho^{\circ}$ region (e.g. J.R. Bensinger et al. ${ }^{7}$ ). The Berkeley experiment of Protopopescu et al. ${ }^{24}$ which showed a sudden jump in $\delta_{\mathrm{S}}^{0}$ in the $900-950 \mathrm{MeV}$ mass range, finished any controversy regarding the S-wave in the $\rho^{0}$ region. In Fig. 14 and 16 we have indicated that $\delta_{S}^{0}$ rises rapidly through $90^{\circ}$ by the break in the data for $0.9<\mathrm{M}(\pi \pi)<1.0 \mathrm{GeV} / \mathrm{c}^{2}$, although our $\pi^{+} \pi^{-}$data cannot resolve this behavior. Our $2 \pi^{\circ}$ data from reaction (3) also favors the down solution for $\delta_{S}^{0}$ above the $\rho^{\circ}$ peak. In Fig. 17(b) we plot $\mathrm{d} \sigma / \mathrm{dm}_{\pi \pi}$, corrected for $\eta^{\mathrm{o}}$ and $\omega^{\mathrm{o}}$ contamination, along with the prediction of Malamud and Schlein ${ }^{37}$ for "down-up", "up-down", and "up-up" solutions for $\delta_{S}^{0}$. For the cross section curves of Fig. 17(b) we have used the $I=2 \mathrm{~S}$-wave phase shifts of J.P. Baton et al. ${ }^{22}$ The Malamud and Schlein predictions give absolute cross sections and are not renormalized for our data. For $M\left(\pi^{\circ} \pi^{\circ}\right) \leq 0.5 \mathrm{GeV} / \mathrm{c}^{2}$, our $2 \pi^{\circ}$ cross section is systematically larger than the Malamud and Schlein predictions and is inconclusive with regard to the various solutions for $\delta_{S}^{0}$. In the mass region from 0.7 to $0.9 \mathrm{GeV} / \mathrm{c}^{2}$ our data definitely favors the "down" branch of the "up-down" or "down-down" solutions. In the interval $0.6-0.9 \mathrm{GeV} / \mathrm{c}^{2}$ the "down-up", "updown", and "up-up" solutions have a $\chi^{2}$ of $12.3,0.1$, and 10.1 respectively (3 degrees of freedom). The upward curving branches of the "up-down" and "down-up" solutions above $0.9 \mathrm{GeV} / \mathrm{c}^{2}$ in Fig. 17(b) show the effect of including a D-wave as found in our $\pi^{+} \pi^{-}$analysis. Below $0.9 \mathrm{GeV} / \mathrm{c}^{2}$ the D -wave correction is negligible. Apparently the D -wave below $1.0 \mathrm{GeV} / \mathrm{c}^{2}$ is not enough to account for the failure of the "up" branch for $\delta_{S}^{0}$.

As an additional check on the consistency of the $\pi^{\circ} \pi^{0}$ and $\pi^{+} \pi^{-}$data we compare $\sigma\left(\pi^{0} \pi^{0}\right)$ with $\sigma\left(\pi^{+} \pi^{-}\right)$in the $\rho^{0}$ peak region. Using the data for
$0.7<\mathrm{M}(\pi \pi)<0.8 \mathrm{GeV} / \mathrm{c}^{2}$ we find

$$
\frac{\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{p} \pi^{+} \pi^{-}\right)}{\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{p} \pi^{o} \pi^{o}\right)}=13.1 \pm 4.1
$$

The large error results from the low statistics of the $2 \pi^{\circ}$ data。Nevertheless this ratio is consistent with that expected for $P$-wave to $S$-wave at the unitarity limit

$$
\frac{\sigma_{+-}}{\sigma_{00}}=\frac{\left(12+\frac{16}{9}\right) \pi \hbar^{2}}{\frac{8}{9} \pi \hbar^{2}}=15.5
$$

The $2 \pi^{\circ}$ angular distributions are plotted in Fig. 18 for events which fit reaction (3) using the measured $\gamma$ directions. The data shown have $|t|<0.3$ $(\mathrm{GeV} / \mathrm{c})^{2}$. We have fit these angular distributions with $\delta_{\mathrm{S}}^{0}$ and $\delta_{\mathrm{D}}^{0}$ as free parameters and the overall normalization adjusted so as to maximize the agreement with $\delta_{D}^{0}$ as determined from our $\pi^{+} \pi^{-}$data in the $f^{o}$ peak region (i.e., at the $f^{0}$ peak only $\delta_{S}^{0}$ is being determined by the fit). Low statistics demanded the use of large mass bins and consequently the fit is often averaging over an interval where one of the parameters is known to vary rapidly. The AOPE model fit results for $\delta_{\mathrm{S}}^{0}$ and $\delta_{\mathrm{D}}^{0}$ are given in Table 7 and the curves in Fig. 18 show the resulting fits to the $2 \pi^{\circ}$ angular distributions. For $0.8<\mathrm{M}(\pi \pi)<1.0 \mathrm{GeV} / \mathrm{c}^{2}$ we find $\delta_{\mathrm{D}}^{0}$ larger than our $\pi^{+} \pi^{-}$results (see Table 6). However, in this mass interval $\delta_{\mathrm{S}}^{0}$ agrees with the data of Protopopescu et al. ${ }^{24}$
D. The $f^{0}$ Region

With the $I=2$ phase shifts fixed there are six free parameters to be determined: $\eta_{\mathrm{S}}^{0}, \delta_{\mathrm{S}}^{0}, \eta_{\mathrm{P}}^{1}, \delta_{\mathrm{P}}^{1}, \eta_{\mathrm{D}}^{0}$, and $\delta_{\mathrm{D}}^{0}$. Of these the $\mathrm{I}=0$ S-wave parameters are most difficult to fit because of the low S-wave unitarity bound. We find that the $\ell=3$ partial wave, $\delta_{F}^{1}$, becomes important only for $M(\pi \pi)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$.

As shown in Fig. 16 the $I=0 \mathrm{D}$-wave, $\delta_{D}^{0}$, rises steadily from $10^{\circ}$ to $45^{\circ}$ from $1.0-1.2 \mathrm{GeV} / \mathrm{c}^{2}$ while $\delta_{\mathrm{P}}^{1}$ holds in the interval $150^{\circ}$ to $160^{\circ}$. In the mass range $0.98-1.14 \mathrm{GeV} / \mathrm{c}^{2}$ the $\cos \theta$ distributions of Fig. 12 become sharply peaked near $\cos \theta=-1$. This results in the negative $Y_{3}^{0}$ moment of the $\pi-\pi$ angular distribution in this mass interval between the $\rho^{\circ}$ and $f^{\circ}$ peak regions. ${ }^{40}$ In a plot of $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$for $\cos \theta<-0.8$ we see no statistically significant structure. This backward peaking in $\cos \theta$ must be produced by the interference of two or more states of opposite parity, e.g. S-P, P-D, or S-P-D interference. Our results would favor $P-D$ interference, i.e., a fairly constant $P$-wave interfering with a rising $D$-wave ( $\mathrm{B}, \mathrm{Y}$. Oh et al. ${ }^{9}$ arrived at this same conclusion). For $0.9<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.2 \mathrm{GeV} / \mathrm{c}^{2}$ both the S and P-wave have inelasticity $\eta<1.0$. Above $1.2 \mathrm{GeV} / \mathrm{c}^{2} \eta_{\mathrm{P}}^{1}$ stays mostly in the interval $0.8-0.9$ (see Fig. 14), while the S-wave inelasticity reaches a minimum near $1.16 \mathrm{GeV} / \mathrm{c}^{2}$ (somewhat above $K \bar{K}$ threshold) and is consistent with $\eta_{S}^{0}=1.0$ near the $f^{0}$ peak. Near $1.1 \mathrm{GeV} / \mathrm{c}^{2}$ the inelastic S -wave is associated with the $\mathrm{S}^{*}(1060)$ resonance decaying into $K \bar{K}$. From data on

$$
\begin{equation*}
\pi^{-} \mathrm{p} \rightarrow \mathrm{nK}^{\mathrm{o}} \overline{\mathrm{~K}}^{\mathrm{o}} \tag{10}
\end{equation*}
$$

at 4 and $6.2 \mathrm{GeV} / \mathrm{c}, \mathrm{W}$. Beusch et al. ${ }^{41}$ estimated $0<\eta_{\mathrm{S}}^{0} \lesssim 0.6$ and $\delta_{\mathrm{S}}^{0}$ near $90^{\circ}$ or $180^{\circ}$ at the $S^{*}$ peak. From a compilation of data on reactions (10) and

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{pK}^{+} \mathrm{K}^{-} \tag{11}
\end{equation*}
$$

R. Diamond et al. ${ }^{42}$ found that the $K \overline{\mathrm{~K}}$ system is dominated by the $\mathrm{I}=0 \mathrm{~S}$-wave in the $1.0-1.3 \mathrm{GeV} / \mathrm{c}^{2}$ mass range. As shown in Fig. 14, we find $\eta_{\mathrm{S}}^{0}=0.74 \pm 0.2$ for $M\left(\pi^{+} \pi^{-}\right) \simeq 1.06 \mathrm{GeV} / \mathrm{c}^{2}$ 。

In the lower half of the $\mathrm{f}^{\mathrm{o}}$ peak region, $1.14-1.28 \mathrm{GeV} / \mathrm{c}^{2}$, the $\cos \theta$ distributions are nearly symmetric about $\cos \theta=0$ and sharply peaked at $\cos \theta= \pm 1$, indicating the dominance of the $Y_{2}^{0}$ term. At larger $\mathbb{M}(\pi \pi)$ there is a stronger peaking in the forward direction at $\cos \theta=+1$. The azimuthal angle distributions (Fig. 13) are relatively isotropic throughout the $f^{0}$ mass region, especially in comparison to the $\rho$. Apparently absorptive effects are more important for the $\rho$ than for $\mathrm{f}^{\mathrm{O}}$ production. ${ }^{25}$ In the $\mathrm{f}^{\mathrm{o}}$ peak region we find $\delta_{\mathrm{S}}^{0}$ rising slowly through $270^{\circ}$, i.e., an S-Wave resonance. ${ }^{23}$ From a study of $\pi^{+} p \rightarrow \Delta^{++} \pi^{+} \pi^{-}$at $8 \mathrm{GeV} / \mathrm{c}$, J.V. Beaupre et al..$^{43}$ found $\delta_{S}^{0}$ near $90^{\circ}$ at the $f^{\circ}$ peak, and in a $\pi-\pi$ phase shift analysis using data on reaction (5) at $17.2 \mathrm{GeV} / \mathrm{c}$, P. Estabrooks et al. ${ }^{25}$ also observe a large $S$-wave phase at the $f^{0}$ peak. This large $S$-wave phase accounts for the near absence of events near $\cos \theta=0$ at the $f^{\circ}$ peak.

The $2 \pi^{\circ}$ data (Fig. 18) agree reasonably well with the results for $\delta_{S}^{0}$ and $\delta_{D}^{0}$ in the $f^{\circ}$ peak region. As shown in Table 7, the AOPE model fits to the $2 \pi^{\circ}$ angular distributions also yield an S-wave phase shift passing through $90^{\circ}$ near the $\mathrm{f}^{\mathrm{O}}$ peak. The errors given in Table 7 do not include an uncertainty in the normalization of $\approx 25 \%$. The $\cos \theta$ distribution for $1.15<\mathrm{M}\left(\pi^{\circ} \pi^{\circ}\right)<1.25$ $\mathrm{GeV} / \mathrm{c}^{2}$ has some peculiar structure and the fit $\delta_{\mathrm{D}}^{0}$ is $\approx 15^{\circ}$ too small. Otherwise the $2 \pi^{\circ}$ results for $\delta_{\mathrm{D}}^{0}$ agree within errors with the $\pi^{+} \pi^{-}$data.
E. Non $-2 \pi$ Decay Modes of the $f^{0}$ and $\eta_{D}^{0}$

With regard to the inelasticity of the $\mathrm{I}=0 \mathrm{D}$-wave, $\eta_{\mathrm{D}}^{0}$, it is interesting to look for non $-2 \pi$ decay modes of the $f^{0}$ 。 In Fig. 19(a) we plot the missing mass from reaction (1) for events with one or more gammas which failed to fit $2 \pi^{\circ}$. In addition to a definite signal at the $\mathrm{f}^{\mathrm{O}}$ there is a broad structure around 1.7 $\mathrm{GeV} / \mathrm{c}^{2}$ and a general background suggestive of $3 \pi$ phase space. The number of events at the $f^{\circ}$ is larger than expected from the fitting program inefficiency for
reconstructing the $2 \pi^{\circ}$ system. A Monte Carlo study of the $2 \pi^{\circ}$ fitting procedure (see Appendix A) predicts $\approx 8$ events above background per 50 MeV bin in the $\mathrm{f}^{\mathrm{O}}$ region of Fig。19(a). From our $\pi^{+} \pi^{-} \pi^{\circ}$ data (reaction (7)) we estimate that the reaction $\pi^{+} n \rightarrow \mathrm{pA}_{2}^{\mathrm{o}}$ with $\mathrm{A}_{2}^{\mathrm{o}} \rightarrow \eta^{\circ} \pi^{\circ}$ should contribute $\approx 6$ events to Fig. 19(a). ${ }^{10,17}$ Another source of structure in Fig. 19(a) is the $K \bar{K}$ decay mode of the $\mathrm{f}^{\mathrm{o}}$. R. Diamond et al..$^{42}$ have estimated this branching ratio to be

$$
\mathrm{R}=\frac{\Gamma\left(\mathrm{f}^{\mathrm{O}} \rightarrow \mathrm{~K} \overline{\mathrm{~K}}\right)}{\Gamma\left(\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{+} \pi^{-}\right)}=0.035 \pm 0.007
$$

Events with associated $V$ 's have been excluded from the reaction (1) data. Assuming that the structure observed near $1.3 \mathrm{GeV} / \mathrm{c}^{2}$ in Fig. 19(a) results from an all neutral $f^{0}$ decay mode other than $2 \pi^{\circ}$, we estimate the cross section to be

$$
\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{pf}^{\mathrm{o}}, \mathrm{f}^{\mathrm{O}} \rightarrow \text { all neutrals } \neq 2 \pi^{\circ} \text { or } \mathrm{K}^{\mathrm{O}} \overline{\mathrm{~K}}^{\mathrm{O}}\right)=7.5 \pm 4 \mu \mathrm{~b} .
$$

This cross section includes the above mentioned corrections for $\mathrm{A}_{2}^{\mathrm{o}} \rightarrow \eta^{\circ} \pi^{\circ}$, $\mathrm{f}^{\mathrm{O}} \rightarrow \mathrm{K}^{\mathrm{O}} \widetilde{\mathrm{K}}^{\mathrm{O}}$, and inefficiency in the $2 \pi^{\mathrm{o}}$ fitting.

Other possible all-neutral $\mathrm{f}^{\mathrm{O}}$ decay modes are $\mathrm{f}^{\mathrm{O}} \rightarrow \eta^{\mathrm{o}} \eta^{\circ}$ and $\mathrm{f}^{\mathrm{O}} \rightarrow 4 \pi^{\circ}$.
Reaction (1) events with four or more measured $\gamma^{\prime}$ s were fit to the hypothesis

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}^{\circ} \eta^{o} \tag{12}
\end{equation*}
$$

with $\eta^{\circ} \rightarrow \gamma \gamma$ (a two constraint fit). In Fig. 19(b) we plot $M\left(\eta^{\circ} \eta^{\circ}\right)$ for the 15 events which fit reaction (12) - this plot includes 1-prong events. The accumulation of these events near the $f^{\circ}$ suggests the possibility of a $\eta^{\circ} \eta^{\circ}$ decay mode. Assuming that all of the remaining $7.5 \mu \mathrm{~b}$ in the all-neutral topology (discussed above) results from $\mathrm{f}^{\circ} \rightarrow \eta^{o} \eta^{\circ}$ yields a cross section of $\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{f}^{\circ} \rightarrow \eta^{\circ} \eta^{\circ}\right)=$ $=15 \pm 8 \mu \mathrm{~b}$. This cross section for $\mathrm{f}^{\circ} \rightarrow \eta^{\circ} \eta^{\circ}$ is corrected for a branching ratio of . 711 for $\eta^{\circ} \rightarrow$ all neutrals. However, we cannot exclude the possibility that all or part of the $7.5 \mu \mathrm{~b}$ in the all-neutral topology results from $\mathrm{f}^{\mathrm{O}} \rightarrow 4 \pi^{\circ}$.

In addition to $f^{\circ} \rightarrow 4 \pi^{\circ}$ there are two other possible $4 \pi$ decay modes, $\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{+} \pi^{+} \pi^{-} \pi^{-}$and $\mathrm{f}^{\mathrm{O}} \rightarrow \pi^{+} \pi^{-} \pi^{\mathrm{o}} \pi^{\mathrm{o}}$. From a study of the reaction

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{+} \pi^{-} \pi^{-} \tag{13}
\end{equation*}
$$

at $6 \mathrm{GeV} / \mathrm{c}, \mathrm{J} . \mathrm{C}$. Anderson et al. ${ }^{44}$ estimated a branching ratio of ( $5.5 \pm 1.0$ ) \% for $\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{+} \pi^{+} \pi^{-} \pi^{-}$(see Table 8). In this experiment we have studied the reaction

$$
\begin{equation*}
\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-} \pi^{\mathrm{o}} \pi^{\mathrm{o}} \tag{14}
\end{equation*}
$$

by reconstructing the $2 \pi^{\circ}$ system using the measured $\gamma$ directions. The $2 \pi^{\circ}$ fitting procedure is basically the same as that used to study reaction (3) (see Appendix A) except that TVGP and SQUAW were used for the reconstruction and fitting. In Fig. 20 we plot $M\left(\pi^{+} \pi^{-} \pi^{0} \pi^{0}\right.$ ) for events with 2 , 3 , or 4 gammas which fit reaction (14). There appears to be some structure above background at $\sim 1.25 \mathrm{GeV} / \mathrm{c}^{2}$, especially in the data with $\left|\mathrm{t}_{\mathrm{np}}\right|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$. There is also some structure in the g meson region. In addition to $\mathrm{f}^{\mathrm{O}} \rightarrow \pi^{+} \pi^{-} \pi^{0} \pi^{0}$ there is the possibility of $\mathrm{B}^{0}(1235) \rightarrow \omega^{0} \pi^{0}$ contributing to this low mass structure. Correcting for $\mathrm{B}^{\mathrm{O}} \rightarrow \omega^{\mathrm{O}} \pi^{\mathrm{o}}$ and $\mathrm{A}_{2}^{\mathrm{O}} \rightarrow \eta^{\mathrm{O}} \pi^{\mathrm{O}}$ we estimate ${ }^{10}$

$$
\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{pf}^{\mathrm{o}}, \quad \mathrm{f}^{\mathrm{o}} \rightarrow \pi^{+} \pi^{-} \pi^{\mathrm{o}} \pi^{\mathrm{o}}\right)=6 \pm 3 \mu \mathrm{~b} .
$$

This cross section must be taken as a lower limit since we have not corrected for $\gamma$-conversion and $2 \pi^{\circ}$ fitting program inefficiencies. However, a plot of $\mathrm{M}\left(\pi^{+} \pi^{-}+\right.$Missing Mass) for events with 0 or $1 \gamma$ observed shows no evidence for structure near the $f^{\circ}$.

Using the estimates of non $-2 \pi \mathrm{f}^{\mathrm{O}}$ decay modes as summarized in Table 8, we estimate $\eta_{D}^{0}=0.79 \pm 0.04$ in the $\mathrm{f}^{\mathrm{O}}$ peak region, ${ }^{45}$ with $\sigma\left(\mathrm{f}^{\mathrm{O}} \rightarrow \pi^{+} \pi^{-}\right)=$ $=258 \pm 25 \mu \mathrm{~b}$ and $\sigma\left(\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{\mathrm{o}} \pi^{\mathrm{o}}\right)=110 \pm 20 \mu \mathrm{~b}$. If the $\mathrm{f}^{\mathrm{o}}$ peak in Fig. 19(a) is
interpreted as a $4 \pi^{0}$ decay mode instead of $\mathrm{f}^{\mathrm{O}} \rightarrow \eta^{\mathrm{O}} \eta^{\mathrm{o}}$, then we obtain $\eta_{\mathrm{D}}^{0}=$ $=0.82 \pm 0.03$. The error for $\eta_{\mathrm{D}}^{0}$ is simply statistical and does not allow for any systematic error in our cross section estimates. The $\pi-\pi$ phase shift analysis gave a smaller value of $\eta_{\mathrm{D}}^{0}=0.70 \pm 0.15$ at $\mathrm{M}(\pi \pi)=1.27 \mathrm{GeV} / \mathrm{c}^{2}$ (see Table 6 and Fig. 14). Our calculation of $\eta_{D}^{0}$ could be in error if we have neglected or overestimated $\sigma\left(\mathrm{f}^{\mathrm{O}} \rightarrow \pi \pi\right)$. It is also possible that some final states are more readily absorbed by the nucleon or deuteron than others. The above value for $\eta_{\mathrm{D}}^{0}$ does agree within errors with the results of our $\pi-\pi$ phase shift analysis. We conclude that the $I=0 \mathrm{D}$-wave is significantly inelastic near the $\mathrm{f}^{\mathrm{o}}$ peak.

## V. $\rho-\omega$ INTERFERENCE

Clear evidence for $\rho-\omega$ interference has been seen in several high resolution, lârge statistics experiments. ${ }^{46}$ In this section we discuss the observation of $\rho-\omega$ interference in reaction (2). In Fig. 21 we plot $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$in the $\rho^{0}$ mass region and observe a four standard deviation peak for $0.78<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<0.8 \mathrm{GeV} / \mathrm{c}^{2}$. Only 4-prong events with $\left|\vec{p}_{S}\right|<0.3 \mathrm{GeV} / \mathrm{c}$ are plotted. The 3 -prong data (i.e. spectator proton unseen) show no evidence for a sharp peak near the $\omega$ mass. This difference between the 3 and 4 -prong samples is compatible with the $\pi-\pi$ mass resolution which we estimate to be 25 and $16 \mathrm{MeV} / \mathrm{c}^{2}$ for the 3 and 4prong data near the $\rho^{\circ}$. As shown by the shaded events in Fig. 21, most of the $\omega$ peak comes from the data with $|\mathrm{t}|>0.1(\mathrm{GeV} / \mathrm{c})^{2}$. The $\pi-\pi$ decay angular distributions show no statistically significant differences for mass intervals below, at, and above the $\omega$ peak.

The surprising feature of Fig. 21 is the presencc of a poaly rather than a dip, since a Regge model based on $\pi-\mathrm{B}$ exchange degeneracy ${ }^{47}$ predicts that we should observe destructive interference in reaction (2). While our 3-prong data does not show a peak, there is no evidence for a dip in the $\omega$ region. In a 2. $15 \mathrm{GeV} / \mathrm{c} \pi^{+} \mathrm{d}$ experiment J. Bensinger and A.R. Erwin ${ }^{48}$ also observed no indication of a dip in $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$. However, D.S. Ayres et al. ${ }^{49}$ have observed destructive interference in a study of reaction (2) using the Argonne Effective Mass Spectrometer.

We have fit the mass spectrum in Fig. 21 to a distribution similar to that used by Hagopian et al. and Allison et al..$^{46}$, i.e., two Breit-Wigner resonance forms for the $\rho$ and $\omega$ with a relative phase $\varphi$ between them:

$$
\begin{gathered}
\frac{d N}{d m}=f_{p s}(m)\left\{A_{\omega}^{2}\left|f_{B W-\omega}(m)\right|^{2}+A_{\rho}^{2}\left|f_{B W-\rho}(m)\right|^{2}\right. \\
\left.+2 \alpha A_{\rho} A_{\omega} \operatorname{Re}\left[e^{i \varphi} f_{B W-\omega}(m) f_{B W-\rho}^{*}(m)\right]+C\right\} \\
-25-
\end{gathered}
$$

Here $m=M\left(\pi^{+} \pi^{-}\right), f_{p s}(m)$ is a phase space factor, and the functions $f_{B W-\omega}$ and $\mathrm{f}_{\mathrm{BW}-\rho}$ are P-wave Breit-Wigner amplitudes for the $\omega$ and $\rho$ respectively. ${ }^{16}$ $\mathrm{A}_{\omega}$ and $\mathrm{A}_{\rho}$ are the (real) amplitudes for decay into $\pi^{+} \pi^{-}, \alpha$ is a coherence factor $\left(0 \leq \alpha^{*} \leq 1\right)$, and $C$ is a constant for the phase space background.

The $\rho^{0}$ resonance parameters were fixed at $\mathrm{M}_{\rho}=0.780 \mathrm{GeV} / \mathrm{c}^{2}$ and $\Gamma_{\rho}=$ $0.18 \mathrm{GeV} / \mathrm{c}^{2}$ as found from a fit to the total $\pi^{+} \pi^{-}$mass spectrum (including 3prong events). For the $\omega$ we used $M_{\omega}=0.790 \mathrm{GeV} / \mathrm{c}^{2}$ and $\Gamma_{\omega}=0.012 \mathrm{GeV} / \mathrm{c}^{2}$. This slightly high value of $\mathrm{M}_{\omega}$ improved the fit to the peak in Fig. 21; in reaction (7) we found $\mathrm{M}_{\omega}=0.784 \pm 0.014 \mathrm{GeV} / \mathrm{c}^{2} .{ }^{50}$ We must also make some choice for the parameter $\alpha$. Using $\alpha=1$ corresponding to complete coherence, yields a lower limit on $A_{\omega}$ and is the usual procedure. The phase angle $\varphi$ is rather insensitive to $\alpha$; changing $\alpha$ from 1.0 to 0.2 changed $\varphi$ by only $10^{\circ}$. With the resonance parameters and $\alpha$ fixed there are four free parameters: $A_{\omega}, A_{\rho}$, $\varphi$ and C. Performing a least-squares fit to the data from $0.5-1.0 \mathrm{GeV} / \mathrm{c}^{2}$ we found a best fit with $\chi^{2}=58$ for 46 degrees of freedom and a phase $\varphi=-1^{\circ} \pm 39^{\circ}$. This result together with the $\omega$ cross section in the $\pi^{+} \pi^{-} \pi^{0}$ channel ${ }^{50}$ yields a branching ratio $R(\omega \rightarrow 2 \pi / \omega \rightarrow 3 \pi)=(3.9 \pm 3.5) \%$. With $\varphi$ fixed at $180^{\circ}$ and all other parameters at their best fit values we found $\chi^{2}=103$, while $A_{\omega}=0$ yielded $\chi^{2}=67$. The phase does not depend critically on the $\rho$ resonance parameters, e.g. using $M_{\rho}=0.787 \mathrm{GeV} / \mathrm{c}^{2}$ gave $\varphi=+8^{\circ}$, while $\Gamma_{\rho}=.17 \mathrm{GeV} / \mathrm{c}^{2}$ yielded $\varphi=-8^{\circ}$. Both of these results are well within errors of the best fit value of $\varphi=-1^{\mathrm{O}} \pm 39^{\circ}$. With $\mathrm{M}_{\omega}$ fixed at $0.784 \mathrm{GeV} / \mathrm{c}^{2}$ the fit results are $\varphi=-43^{\circ}$ and $R=2.3 \%$ with a fit $\chi^{2}=60$.

These results are quantitatively similar to previous observations of $\rho-\omega$ interference except that we find constructive rather than the expected destructive interference. As pointed out by C. Quigg ${ }^{51}$ this anomaly could be explained by a strong natural parity exchange contribution to $p-\omega$ production. The original prediction of Goldhaber et al. ${ }^{47}$ of destructive interference in

$$
\begin{equation*}
\pi^{+} p \rightarrow \Delta^{++} \pi^{+} \pi^{-} \tag{15}
\end{equation*}
$$

assumed unnatural parity exchange. For natural parity exchange the Regge pole exchange degeneracy arguments imply constructive interference for reactions (2) and (15). This would show up in the $\rho_{11}+\rho_{1-1}$ combination of the density matrix elements. While we have found a natural parity exchange contribution to $\rho^{0}$ production with $|\mathrm{t}| \geq 0.3(\mathrm{GeV} / \mathrm{c})^{2},{ }^{18}$ the $\rho-\omega$ peak in Fig. 21 comes mostly from lower $t$ events. In Fig. 22(a, b) we plot $M\left(\pi^{+} \pi^{-}\right)$weighted by $\rho_{\text {oo }}$ and $\rho_{11}+\rho_{1-1}$ as a function of mass. Essentially all of the $\rho-\omega$ peak structure is associated with the $\rho_{\mathrm{oo}}$ component. It is interesting to note that Protopopescu et al. ${ }^{24}$ have reported observation of constructive $\rho-\omega$ interference in reaction (15) at small momentum transfer and in the $\rho_{\mathrm{oo}}$ state. This would be quite consistent with our results.

## VI. THE $\pi$-NUCLEON SYSTEM

Although $\rho$ and f production account for roughly $\frac{2}{3}$ of the reaction (2) events it is also interesting to consider baryon resonance production. The $M\left(p \pi^{+}\right)$ spectrum shows little or no evidence of any low mass resonant structure. With the other $\pi$-nucleon combination we observe considerable structure at small $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$in Fig. 23(a). On top of a rapidly falling OPE background there are peaks at approximately $1.24,1.38,1.5$ and $1.65 \mathrm{GeV} / \mathrm{c}^{2}$. The location of the first peak is consistent with the $\Delta(1236)$, while the last two peaks are near the $N^{*}(1520)$ and $N^{*}(1690)$ respectively. The explanation of the peak at $1.38 \mathrm{GeV} / \mathrm{c}^{2}$ is not clear since this is slightly below the usual location of the "Roper" $\mathrm{P}_{11}$ resonance at $\approx 1.47 \mathrm{GeV} / \mathrm{c}^{2},{ }^{52}$ (this could be the result of interference).

Much of the broad structure at small $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$is a reflection of OPE in the $\rho$ and fregions of $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$and perhaps the result of interference with these amplitudes. As shown in the shaded portion of Fig. 23(a), when we demand $\mathrm{M}\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ we get a rather smoothly falling low mass structure in $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$with a shoulder in the $1.65 \mathrm{GeV} / \mathrm{c}^{2}$ region. Momentum transfer cuts on $t_{\pi \pi}$, the four-momentum transfer from $\pi_{\text {in }}^{+}$to $\pi_{\text {out }}^{+}$, do not help to disentangle the overlap. It seems likely that some peculiar features of the $\pi-\pi$ angular distributions (e.g. a sharp spike in the forward direction of $\cos \theta \pi \pi$ near the $\rho$ ) are related to the overlap between $N^{*}$ and $\rho$ or f production. ${ }^{53}$

Using simple non-relativistic Breit-Wigner resonance forms plus a handdrawn background, we have fit the mass distribution of Fig. 23(a) for $M\left(p \pi^{-}\right)<$ $<2.0 \mathrm{GeV} / \mathrm{c}^{2}$. The resonance widths were fixed at $60-80 \mathrm{MeV} / \mathrm{c}^{2}-$ these rather small widths were necessary in order to reproduce the observed structure. The fitted resonance masses and cross sections are shown in Table 9 (the fit $\chi^{2}=34$ for 37 degrees of freedom). The large errors reflect both the statistical
uncertainty of the observed peaks and the uncertainty in the background shape. Our cross sections for $\Delta(1236)$ and $N^{*}(1520)$ agree with those of Anderson et al. 54 who observe $\pi^{-} p \rightarrow \pi^{-} N^{*}$ in a missing mass spectrum at $8 \mathrm{GeV} / \mathrm{c}$.

In Fig. 23(b) we plot $M^{*}=M\left(n \pi^{+}\right)+M\left(p \pi^{-}\right)$from the $7 \mathrm{GeV} / \mathrm{c} \pi^{-} p$ and $\pi^{+} d$ data respectively. The peaks at 1.24 and $1.4 \mathrm{GeV} / \mathrm{c}^{2}$ are poorly defined in the $\mathrm{M}\left(\mathrm{n} \pi^{+}\right)$distribution. The $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$mass resolution of $\sim 8 \mathrm{MeV} / \mathrm{c}^{2}$ in the low mass region is probably somewhat better than for $M\left(n \pi^{+}\right)$. In Fig. 24(a) we plot $\mathbb{M}^{*}$ for $\mathrm{M}\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ and $\left|\mathrm{t}_{\pi \pi^{\prime}}\right|<0.2(\mathrm{GeV} / \mathrm{c})^{2}$, and observe a broad low mass structure suggesting a diffractive production process. For $\left|t_{\pi \pi}\right|>0.2$ $(\mathrm{GeV} / \mathrm{c})^{2}$ in Fig. 24(b) there are possible enhancements at $\approx 1.5$ and $1.65 \mathrm{GeV} / \mathrm{c}^{2}$. The structure from $1.6-1.7 \mathrm{GeV} / \mathrm{c}^{2}$ is most definite since it persists either as a peak or a shoulder for most of the cuts that we have tried. This probably indicates that we are observing more than one resonance in this mass region. The $t_{\pi \pi}$ distributions in both experiments are well fit by exponentials of the form $e^{\alpha \mathrm{t}}$ where $\alpha$ depends on $\mathrm{M}^{*}$, the $\pi$-nucleon mass. In Fig. 25(a) we plot $\alpha$ vs $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$and find a variation of $\alpha$ with mass in good agreement with the $\mathrm{M}\left(\mathrm{n} \pi^{+}\right)$ data (see Ref. 9). The slope is roughly a factor of two smaller in the 1.5-1.7 $\mathrm{GeV} / \mathrm{c}^{2}$ region as compared to the $1.2-1.4 \mathrm{GeV} / \mathrm{c}^{2}$ region, in agreement with the data of Anderson et al. ${ }^{54}$ In Fig. 25(b) we plot $\alpha$ for the combined $7 \mathrm{GeV} / \mathrm{c}$ $\pi^{-} \mathrm{p}$ and $\pi^{+} \mathrm{d}$ data with $\mathrm{M}\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ to eliminate the overlap with $\rho$ and f production. This cut is seen to reduce $\alpha$ slightly while leaving the same general dependence on $M^{*}$.

To study further the low mass $\mathrm{M}^{*}$ system we examine the nucleon-nucleon scattering angle, $\cos \theta$ NN, as defined in Fig. 26(a). We must demand $\mathrm{M}\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ if we want to observe features of the angular distributions which may be characteristic of the $\pi$-nucleon rather than the $\pi-\pi$ system.

The $\cos \theta$ NN distributions for $\mathrm{M}\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ are shown in Fig. 27. For $\left|t_{\pi \pi}\right|<0.2(\mathrm{GeV} / \mathrm{c})^{2}$ the distributions are almost flat, especially in comparison to the large $\left|t_{\pi \pi}\right|$ data of Fig. $27(\mathrm{~b})$. The small $t_{\pi \pi}$ data appear to be consistent with production via diffraction dissociation; ${ }^{55} \cos \theta$ NN seems to be mostly Swave with the exception of $M^{*}(1.58-1.70)$. For $\left|t_{\pi \pi}\right|>0.2(\mathrm{GeV} / \mathrm{c})^{2} \cos \theta \mathrm{NN}$ is strongly peaked in the forward direction, and the data with $\mathrm{M}^{*}<1.4 \mathrm{GeV} / \mathrm{c}^{2}$ is suggestive of S-P wave interference. In the mass interval $1.4-1.46 \mathrm{GeV} / \mathrm{c}^{2}$ the forward peak has become sharper indicating that $D$-wave is becoming important. The distribution from $1.58-1.7 \mathrm{GeV} / \mathrm{c}^{2}$ is most unusual, since in the backward direction it looks like a spin-flipped D-wave and is fit rather well by $\cos \theta \sin \theta \propto Y_{2}^{1}$. This situation is reminiscent of the $A_{2}$ which is also produced in a spin-flipped state. ${ }^{17}$ Like the $A_{2}$ the structure from $1.58-1.7 \mathrm{GeV} / \mathrm{c}^{2}$ also lies on the falling edge of a large diffractive-like background and is enhanced by discarding the small momentum transfer events.

## VII. DEUTERON EFFECTS

Deuterium is often used in experiments as a means of obtaining a neutron target. Ordinarily one imagines that either the proton or the neutron is struck by the high energy projectile and then escapes without further interaction. This picture of the interactions is probably moderately accurate. Looking at the momentum spectrum of the spectator protons, one can account for about 80 $90 \%$ of the spectrum by means of the Hulthén wave function of the deuteron. Beyond a spectator momentum of $\approx 200 \mathrm{MeV} / \mathrm{c}$ other processes probably constitute a modest fraction of the nominally neutron events.

As an example of an effect that we have observed we show Figs. 28 and 29 in which the dipion mass spectrum from reaction (2) has been plotted for different cuts on the spectator momentum. For the case of the invisible spectator the mass spectrum shows $\rho^{\circ}, f^{0}$ and $\mathrm{g}^{0}$ peaks (Fig. 28(a)). For visible spectator protons ( $\left|\overrightarrow{\mathrm{p}}_{\mathrm{s}}\right| \gtrsim 80 \mathrm{MeV} / \mathrm{c}$ ) the $\mathrm{g}^{\circ}$ has disappeared and the $\mathrm{f}^{\circ}$ is slightly diminished as shown in Fig. 28(b). Figure 29(a,b) shows $M\left(\pi^{+} \pi^{-}\right)$and the $\rho^{0}$ decay angular distributions for events with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|>0.3 \mathrm{GeV} / \mathrm{c}$. In this case the $\mathrm{f}^{\mathrm{O}}$ is practically gone and only the $\rho^{0}$ is clearly visible. The angular distribution of the decay of the $\rho^{\circ}$ shows a dominance of the $\rho_{\mathrm{oo}}$ moment of the distribution which is characteristic of the OPE process. The problem is then how to account for an OPE dominated process and a high energy spectator.

A possible explanation of the effects observed can be given by considering the diagram shown in Fig. 26(b). In this case the virtual $\pi^{+}$from the upper vertex is absorbed by the deuteron producing a diproton state. The cross section at the pion pole is given by the usual expression for the OPE process:

$$
\frac{\mathrm{d}^{3} \sigma}{\mathrm{dm} \mathrm{~d}^{*} \mathrm{M}^{*} \mathrm{dt}}=\frac{1}{4 \pi^{3} \mathrm{P}_{\mathrm{o}}^{2} \mathrm{E}^{2}}\left(\mathrm{Km}^{*^{2}} \sigma_{\pi-\pi}\right) \frac{1}{\left(\mathrm{t}-\mathrm{m}_{\pi}^{2}\right)^{2}}\left(\mathrm{PM}^{*} \sigma_{\pi-\mathrm{d}}\right)
$$

where
$P_{o}, E=$ center of mass momentum and energy,
$\mathrm{m}^{*}, \mathrm{M}^{*}=\pi-\pi$ and $\mathrm{p}-\mathrm{p}$ invariant mass,
$\mathrm{K}, \mathrm{P}=$ virtual $\pi$ momentum in $\pi^{+} \pi^{-}$and $\pi \mathrm{d}$ center of mass, and $\sigma_{\pi-\pi}, \sigma_{\pi-\mathrm{d}}$ are the "on the mass shell" cross sections for $\pi-\pi$ and $\pi \mathrm{d} \rightarrow \mathrm{pp}$ interactions. Figure 30 shows our observed diproton distribution. We show also the distribution expected on the basis of the Hulthén distribution and the distribution calculated on the basis of the OPE cross section. The interesting feature of the process $\pi^{+} \mathrm{d} \rightarrow \rho^{\circ} \mathrm{pp}$ is that the reaction can go over a wide range of the diproton mass distribution with the virtual pion very close to the real pion, i.e., close to the pion pole. The curve shown on Fig. 30 is close to an absolute prediction. It was calculated using OPE and then normalized with respect to the observed process $\pi^{+} \mathrm{p} \rightarrow \rho^{0} \Delta^{++}$. Beyond a diproton mass of $2.10 \mathrm{GeV} / \mathrm{c}^{2}$ our experimental distribution is $\approx 30-40 \%$ low because of cuts made at the scanning level - correcting for this would tend to make the agreement better.

In Section V on $\rho-\omega$ interference we found a large difference in the mass spectrum depending on whether or not one observed a spectator. This effect is perhaps larger than can be accounted for by differences in resolution. It is possible that here we are also observing specifically dinucleon effects.

## VIII. CONCLUSIONS

From AOPE model fits to the $\pi^{+} \pi^{-}$angular distributions we find strong evidence of resonant behavior in the $I=0 \mathrm{~S}$-wave near the $\mathrm{f}^{\circ}$ peak. There is a rapid change in $\eta_{\mathrm{S}}^{0}$ for $1.0<\mathrm{M}(\pi \pi)<1.2 \mathrm{GeV} / \mathrm{c}^{2}$ while near the $\mathrm{f}^{\mathrm{o}}$ peak $\eta_{\mathrm{S}}^{0} \simeq 1$ and $\delta_{\mathrm{S}}^{0} \simeq 270^{\circ}$ implying a large imaginary S-wave amplitude. Our $2 \pi^{\circ}$ data is consistent with this behavior. The $\mathrm{I}=0 \mathrm{D}$-wave is significantly inelastic at the $\mathrm{f}^{\mathrm{O}}$ peak ( $\eta_{\mathrm{D}}^{0}=0.70$ ) and this observation is supported by estimates of non $-2 \pi \mathrm{f}^{\mathrm{o}}$ decay modes. We find evidence for $\eta^{\mathrm{o}} \eta^{\circ}$ and $\pi^{+} \pi^{-} \pi^{\circ} \pi^{\circ}$ decay modes of the $f^{\circ}$. The branching ratio for this $4 \pi$ decay mode is not consistent with that one would predict from $\mathrm{f}^{\circ} \rightarrow \pi^{+} \pi^{+} \pi^{-} \pi^{-}$assuming the decay proceeds through a $\rho \rho$ intermediate state.

We observe constructive $\rho-\omega$ interference in $\pi^{+} \mathrm{n} \rightarrow \mathrm{p} \pi^{+} \pi^{-}$in disagreement with most current theories. In the $\pi$-nucleon mass spectra there are small signals from $\mathrm{N}^{*}$ production superimposed on a OPE background. Finally the events with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{s}}\right| \gtrsim 0.3 \mathrm{GeV} / \mathrm{c}$ show evidence of specifically deuteron effects in the OPE process.

## ACKNOWLEDGMENTS

It is a pleasure to thank the operating staff of the ANL-MURA 30 -inch bubble chamber under the direction of Dr. L. Voyvodic. We sincerely appreciate the fine work of the Toronto and Wisconsin scanning and measuring staffs. We are most grateful for the assistance of Dr. R.N. Diamond and Dr. J.T. Lynch on this experiment, and we thank Professors L. Durand, A.R. Erwin and M.A. Thompson at the University of Wisconsin for helpful discussions. Finally, we thank Mr. W. M. Yeager at Duke University for his assistance with the data presented in this paper.

## APPENDIX A

## $2 \pi^{\circ}$ Fitting With Gammas

In this appendix we discuss the procedure we used to fit the reaction $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{\circ} \pi^{\circ}$ using the measured gamma directions. Our method is basically the same as that used by R. Morse to study the reaction $\pi^{-} p \rightarrow p \pi^{-} \pi^{o} \pi^{o}$ at $7 \mathrm{GeV} / \mathrm{c} .{ }^{56}$ We also discuss an analysis of this fitting procedure using Monte Carlo generated events.

As part of our normal scanning procedure all events were checked for associated $\gamma^{\prime} \mathrm{s}$ in the two tantalum (Ta) plates and in the deuterium. All 2-prong reaction (1) events were also checked a second time by gamma editors, who were experienced scanners with special training concerning the use of the Ta plates to detect $\gamma^{\prime}$ s. All gammas were classified as either "definite" or "questionable" depending on how well the $\gamma$ pointed at the vertex of the event, whether or not the origin of the $\gamma$ was ambiguous between two or more verticies, etc. Our procedure in fitting was never to discard definite $\gamma^{\prime}$ s, e.g., a $3 \gamma$ event with 2 definite gammas would be fit to $\pi^{\circ} \pi^{\circ}$ using all $3 \gamma^{\prime}$ s and using only the 2 definite $\gamma^{\prime}$ s. Approximately $54 \%$ of the gammas measured were definite.

Since we have only measured the $\gamma$ directions as two point tracks we loose one constraint for each $\gamma$ produced. Consequently unless we observe all 4 gammas from the decay of $2 \pi^{0}$ 's we must make some approximations in order to fit the event. The approximations we make are based on well known kinematic features of the decay $\pi^{0} \rightarrow \gamma \gamma$. The opening angle $\theta$ from the decay of a $\pi^{\circ}$ of mass $\mu$ and momentum $p_{\pi}$ satisfies the inequality

$$
\tan \frac{1}{2} \theta \geq \mu / p_{\pi}
$$

From the opening angle distribution, ${ }^{57}$

$$
\mathrm{W}(\theta)=\frac{1-\mathrm{v}^{2}}{2 \mathrm{v}} \frac{\cos \frac{1}{2} \theta}{\sin ^{2} \frac{1}{2} \theta}\left(\mathrm{v}^{2}-\cos ^{2} \frac{1}{2} \theta\right)^{-1 / 2}
$$

we compute the probability $P(\theta)$ that a $\pi^{0}$ of velocity $v$ will decay into 2 gammas with an opening angle $\theta$ or larger,

$$
P(\theta)=1-\frac{\left(v^{2}-\cos ^{2} \frac{1}{2} \theta\right)^{1 / 2}}{v \sin \frac{1}{2} \theta}
$$

As shown in Fig. A1(a), $P(\theta)$ is sharply peaked towards the minimum opening angle $\theta_{\text {MIN }}$, and consequently if we know the $\pi^{\circ}$ momentum we can make a good estimate of the probable $\gamma-\gamma$ opening angle. Conversely, if we have a measure of the opening angle for a particular $\pi^{\circ} \rightarrow \gamma \gamma$ decay we can estimate the $\pi^{\circ}$ momentum. We define $P\left(p_{\pi}\right)$ to be the probability that the $\pi^{0}$ momentum will be $p_{\pi}$ or larger for a given minimum momentum $p_{\text {MIN }}$. In Fig. A1(b) we have plotted $\mathrm{P}\left(p_{\pi}\right)$ as determined from Monte Carlo generated $2 \pi^{\circ}$ events. The sharp peaking results partly from the kinematic restrictions of this experiment, i.e., as $p_{\text {MIN }}$ increases the allowed values of $p_{\pi}$ are restricted by the finite amount of missing momentum ( $\approx 6.6 \mathrm{GeV} / \mathrm{c}$ ). With this introduction we proceed to discuss the $2 \pi^{\circ}$ fitting procedure for events with 1,2 , or 3 measured gammas; see Ref. 10, 48 and 56 for additional details.

## A. $1 \gamma$ Events

For events with 1 measured $\gamma$ we made a 0 -constraint fit by pointing $\pi_{1}^{o}$ in the direction of the measured $\gamma$ and allowing the fitting program to calculate the momentum of $\pi_{1}^{\mathrm{o}}$ and the direction and momentum of $\pi_{2}^{0}$. A diagram of this situation is shown in Fig. A2(a). Since this is only a calculation there is little or no
discrimination against events with more than $2 \pi^{\circ} \mathrm{s}$, however the following factors speak in favor of this procedure. For events with $\mathrm{MM} \leqq 1.0 \mathrm{GeV} / \mathrm{c}^{2}$ the only appreciable $3 \pi^{\circ}$ contribution comes from $\eta^{\circ} \rightarrow 3 \pi^{\circ}$. Secondly, if we see only $1 \gamma$ from a $2 \pi^{\circ}$ event it is likely to have come from the more energetic of the two $\pi^{\circ}{ }^{\circ} \mathrm{s}$. A $4 \mathrm{GeV} / \mathrm{c} \pi^{\circ}$ has $\frac{1}{2} \theta_{\text {MIN }} \simeq 1.93^{\circ}$ and the probability is only 0.13
 be a good approximation. In analyzing these events we have demanded that the calculated momentum of $\pi_{1}^{0}$ be larger than the lower limit imposed by our estimate of the $\gamma$ energy. Finally the most convincing argument for using these $1 \gamma$ events is that their fitted $M\left(\pi^{\circ} \pi^{0}\right)$ and angular distributions are similar to the 2-3-4 $\gamma$ events. The most obvious difference between the $1 \gamma$ and the 2-3-4 $\gamma$ fits is that the former have slightly more events with $\mathrm{M}\left(\pi^{\mathrm{o}} \pi^{\circ}\right)>1.5 \mathrm{GeV} / \mathrm{c}^{2}$ and show a somewhat stronger $D$-wave at the $f^{\circ}$ peak.
B. $2 \gamma$ and $3 \gamma$ Events

For $2 \gamma$ events there are two possibilities to consider: type 1 - one $\gamma$ from each $\pi^{\circ}$ is observed (see Fig. A2(b)), type $2-$ two $\gamma^{\prime}$ s from one $\pi^{\circ}$ and none from the other are detected (Fig. A2(c)).

For type 1 events we began by taking the $\pi^{0}$ directions to be the same as the $\gamma$ directions and solve for the $\pi^{o}$ momenta using the angles $\theta_{1}$ and $\theta_{2}$ between the $\pi^{\circ}$ directions and the missing momentum (=PM):

$$
\begin{aligned}
& p_{1} \cos \theta_{1}+p_{2} \cos \theta_{2}=P M \\
& p_{1} \sin \theta_{1}-p_{2} \sin \theta_{2}=0
\end{aligned}
$$

Using these estimates of the $\pi^{0}$ momenta we determine the corresponding minimum opening angle $\theta_{\text {MIN }}$ for each $\pi^{\circ}$ and constrain the $\pi^{\circ}$ directions to lie
within cones of half angle $1.25 \theta{ }_{\text {MIN }} / 2$ ．Actually it might be more correct to constrain the directions to lie within conical shells but the standard bubble chamber kinematic fitting programs are not amenable to this type of a constraint． We also constrain the $\pi^{0}$ momentum to be $p_{\text {int }} \pm 0.2 p_{\text {int }}$ where $p_{\text {int }}$ is the initial guess．

For type 2 events，two gammas from one $\pi^{\circ}$ ，we take the initial $\pi_{1}^{0}$ direction along the bisector of the $2 \gamma^{\prime} \mathrm{s}$ and calculate the minimum $\pi_{1}^{\mathrm{o}}$ momentum from the $\gamma-\gamma$ opening angle，

$$
\mathrm{p}_{\mathrm{MIN}}=\frac{\mu}{\tan \frac{1}{2} \theta}
$$

As shown Fig．A1（b）the actual $\pi^{\circ}$ momentum is usually only slightly larger than the minimum，especially for fast $\pi^{\mathrm{O}^{\prime}} \mathrm{s}$ ．The $\pi_{1}^{\mathrm{o}}$ momentum was constrained to be $\approx 1.1 \mathrm{p}_{\text {MIN }}{ }^{ \pm} 0.2 \mathrm{p}_{\text {MIN }}$ ．The $\pi_{1}^{\circ}$ direction was constrained to lie in the plane of the two gammas and allowed to vary between the $\gamma$ directions in this plane．As an aid in discriminating between $2 \gamma$ events of type 1 and type 2 ，we defined the angles $\alpha$ and $\beta$ as follows：

$$
\begin{aligned}
& \text { PM }=\text { Missing Momentum, } \\
& \alpha=\cos ^{-1}\left[\left(\hat{\gamma}_{1} \times \hat{\gamma}_{2}\right) \cdot \mathrm{PM}\right] \quad-\quad 0 \leq \alpha \leq 180^{\circ} \\
& \beta=\cos ^{-1}\left[\left(\hat{\gamma}_{1} \times \hat{\mathrm{PM}}\right) \cdot\left(\hat{\gamma}_{2} \times \hat{\mathrm{PM}}\right)\right] \quad 0 \leq \beta \leq 180^{\circ}
\end{aligned}
$$

If $\gamma_{1}$ and $\gamma_{2}$ are actually associated with the event in question then $\alpha$ should be close to $90^{\circ}$ 。Distributions of $\alpha$ for both fitted $2 \pi^{\circ}$ and Monte Carlo generated events are sharply peaked at $\alpha=90^{\circ}$ ．The $2 \gamma$ events of type 1 are all within $\alpha=90^{\circ} \pm 12^{\circ}$ ，and $94 \%$ of the type 2 events are within $\alpha=90^{\circ} \pm 12^{\circ}$ 。The azimuthal angle $\beta$ is very useful in deciding between $2 \gamma$ fits of type 1 and 2 。

Type 1 events are peaked towards $\beta=180^{\circ}$ while type 2 events peak near $\beta=0^{\circ}$ with $\beta$ usually less than $90^{\circ}$ 。Our Monte Carlo studies indicate that this angle alone is sufficient to distinguish between type 1 and 2 for 0.94 of the fake events. This agrees well with a ratio of 0.93 for real events. We found these angles to be just as useful as the $\chi^{2}$ for selecting the correct fit.

For $3 \gamma$ events there are three possible permutations of the gammas corresponding to which pair of $\gamma^{\prime}$ s is assumed to come from $\pi_{1}^{0}$. Consider the case in which $\gamma_{1}$ and $\gamma_{2}$ come fron $\pi_{1}^{\mathrm{o}}$ and $\gamma_{3}$ from $\pi_{2}^{\mathrm{o}}$. Initially we take $\pi_{1}^{\mathrm{o}}$ along the bisector of $\gamma_{1}$ and $\gamma_{2}$ and point $\pi_{2}^{o}$ in the direction of $\gamma_{3}$. Now we can solve for the $\pi^{o}$ momenta and find errors for the direction and momentum of $\pi_{2}^{o}$ just as for type 1 of the $2 \gamma$ events. In fitting this hypothesis we demand $M\left(\gamma_{1} \gamma_{2}\right)$ $=\mu$ and use the artificially constructed track for $\pi_{2}^{o}$

## C. $\gamma$ Detection Efficiency

We have checked our $\gamma$ fitting procedure with Monte Carlo generated events. There are two points of interest: (1) what is our $\gamma$ detection efficiency, and (2) what is the program efficiency for fitting the $2 \pi^{\circ}$ events? We should also consider the effect of the non $-2 \pi^{\circ}$ background in the missing mass spectrum. We will assume that this background is predominantly $3 \pi^{\circ}$ for $M M \gtrsim \mathrm{M}_{\omega}$.

Initially we consider the problem of our $\gamma$ detection efficiency. There are two efficiencies of interest, the "Geometrical Detection Efficiency" (GDE), and the "Actual Detection Efficiency" (ADE)。 The GDE is a measure of the effective solid angle subtended by the plates and depends simply on the fraction of gammas which hit the plates. The ADE is the probability for detecting $n$ out of $N$ produced gammas from a given reaction. In addition to geometry it depends on the $\gamma$-conversion probability (which is a function of the incident photon energy) and the "Survival Probability" (SP) of the $e^{ \pm}$pairs. SP is the probability that the $e^{ \pm}$ will escape from the Ta plates with enough energy to be detected.

To determine these efficiencies we generated Monte Carlo events of the type $\pi^{+} d \rightarrow p_{s} p+n \pi^{o}$ with $n=2$ or 3 . The events were generated with $t$ distributions of the form $e^{\beta \mathrm{t}}$ with $\beta=4.0$ and $2.2(\mathrm{GeV} / \mathrm{c})^{-2}$ for $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events respectively. In Fig. $\mathrm{A} 3(\mathrm{a}, \mathrm{c})$ we plot the GDE for $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events. Since the multi $\pi^{\circ}$ system has a net momentum of $6-7 \mathrm{GeV} / \mathrm{c}$ our GDE is very good. Here we have neglected the strong D-wave in the $\pi-\pi$ system above $1.0 \mathrm{GeV} / \mathrm{c}^{2}$ 。

For each $\gamma$ which hits the plates we decide in a random fashion whether or not it converts and if it converts whether or not it produces a visible $e^{ \pm}$shower (see Appendix B of Ref. 56 for details). The ADE for $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events is shown in Fig. A3(b, d) 。 Obviously we cannot hope to study the $2 \pi^{\circ}$ system using only $4 \gamma$ events. "The problem is a low $\gamma$ conversion probability; the SP is of secondary importance since most of the gammas are fast and $S P=1$ for $\mathrm{E}_{\mathrm{e}} \pm \geq 0.3 \mathrm{GeV}$. The two $\frac{1}{8}$-inch Ta plates provide $2 \times 0.76$ radiation lengths yielding a conversion probability of $\approx 0.69$ for $\mathrm{E}_{\gamma} \gtrsim 1.0 \mathrm{GeV}(0.03$ of the measured $\gamma^{\prime}$ s converted in the deuterium). The increase in conversion probability for photons not incident normal to the plates is somewhat compensated for by a decrease in the SP.

The encouraging feature of Fig. A3(b) is that most of the $2 \pi^{\circ}$ events produce at least one observable gamma. The fraction of events with 0 gammas is less than 0.1 out to $\mathrm{M}(\pi \pi)=1.7 \mathrm{GeV} / \mathrm{c}^{2}$. For comparison, $10 \%$ of the 2 -prong missing mass events had no measured gammas (see Section II). In Table A1 we list the fraction of reaction (1) events with 1, 2, 3, and 4 gammas for various $\mathrm{ML}\left(\pi^{0} \pi^{0}\right)$ intervals (these ratios have not been corrected for $0 \gamma$ events). For comparison we list the corresponding ratios for the total sample as predicted from our Monte Carlo study. The agreement is good considering that we have neglected the fitting program efficiency and the $3 \pi^{\circ}$ background. The largest discrepancy,
which is for the $3 \gamma$ events，will be seen to be related to the fitting program efficiency．

D．Efficiency and Resolution of $2 \pi^{\circ}$ Fitting
To check our procedure for reconstructing $2 \pi^{\circ}$ events with less than $4 \mathrm{ob}-$ served gammas we have used the Monte Carlo generated events as input to the reconstruction and kinematic fitting program．The events were processed through the $2 \pi^{\circ}$ gamma fitting programs using the same procedure as for real events．The program efficiency is defined to be the fraction of events which yield a good fit by the same criteria as used for real events．The efficiency for fake $2 \gamma$ and $3 \gamma$ events is given in Table A2．While the recovery rate for the $2 \pi^{\circ}$ data is quite satisfactory，the discrimination against $3 \pi^{\circ}$ events is low．The statistical uncertainty in the efficiency within the various mass intervals is typically $\pm 0.2$ ．The reason for the rather low $3 \gamma$ efficiency is not known．The $\chi^{2}$ distributions for the fake $2 \pi^{\circ}$ events are similar to those for real events whereas the $3 \pi^{\circ} \chi^{2}$ distributions are rather flat．This would indicate a small $3 \pi^{\circ}$ contamination in the real fitted events。

For $1 \gamma$ events the program recovery rate is $\approx 1.0$ for both $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events，i．e．，no discrimination against $3 \pi^{\circ}$ 。 The advantage of the fitting pro－ cedure for $1 \gamma$ events is that we get some information on the $\pi^{\circ}$ directions．For $\mathrm{M}(\pi \pi) \lesssim 1.4 \mathrm{GeV} / \mathrm{c}^{2}$ phase space alone yields some discrimination against $3 \pi^{\circ}$ ． The fitting also gives little or no increase in mass resolution since the direction and momentum of the proton is usually much more tightly constrained than are the $\pi^{0}{ }^{\prime}$ s。

In Fig．A4 we compare the $\pi^{0}$ directions and momenta as reconstructed by the fitting program，with the actual Monte Carlo generated $\pi^{\circ}$ directions（for $2 \pi^{\circ}$ events）．Figure A4（a）shows the angular resolution－the angle between the

Monte Carlo generated and fitted $\pi^{\circ}$ direction for each event - for the various $\gamma$ topologies. For the worst case of the $1 \gamma$ events the resolution is $\sim 2^{\circ}$ in the laboratory frame of reference. For $3 \gamma$ events the resolution is $1^{\circ}$ or better. In Fig. A4(b) we plot the fractional difference between the Monte Carlo and fitted $\pi^{\circ}$ momentum. Again we see the progressive improvement as we go from 1 to 4 measured gammas. Two $\pi^{\circ}$ events of type $2\left(2 \gamma^{\prime} \mathrm{s}\right.$ from $\left.1 \pi^{\circ}\right)$ are seen to be more tightly constrained than type 1 events ( $1 \gamma$ from each $\pi^{\circ}$ )。 Finally in Fig. A4(c) we plot the difference in $\cos \theta \pi \pi$ as calculated for the Monte Carlo and fitted $\pi^{\circ}$ directions in the $\pi \pi$ center of mass. For $1 \gamma$ and $2 \gamma-$ type 1 the resolution is $\approx 0.2$ while for $2 \gamma-$ type 2 and $3 \gamma$ the resolution is at least 0.1 . For $4 \gamma$ events the resolution is smaller than the binning in the plot.

The above results indicate that our $2 \pi^{\circ}$ fitting procedure allows us to recover useful information on the $\pi^{\circ}$ directions. While there is little if any improvement in mass resolution there is some discrimination against $3 \pi^{\circ}$, especially for $\mathrm{M}(3 \pi) \gtrsim 1.2 \mathrm{GeV} / \mathrm{c}^{2}$. The fitting procedure does not produce any systematic shift of $\mathrm{M}(\pi \pi)$ with respect to the missing mass. For $\mathrm{M}(\pi \pi)>1.0 \mathrm{GeV} / \mathrm{c}^{2}$ we find $\mathrm{M}(\pi \pi)$ is always within $0.025 \mathrm{GeV} / \mathrm{c}^{2}$ of the missing mass. The fitting procedure is rather insensitive to small changes in the error assignments for the $\pi^{0}$ directions and momenta.

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TABLE 1

| Reaction | Spectator Momentum ( $\mathrm{GeV} / \mathrm{c}$ ) | $\sigma(\mu \mathrm{b})$ |
| :---: | :---: | :---: |
| $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p} \pi^{\mathrm{o}}$ | $\left\|\vec{p}_{s}\right\| \leq 0.3$ | $67 \pm 10$ |
|  | $>0.3$ | $12 \pm 5$ |
| $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p}+\mathrm{MM}\left(\geq 2 \mathrm{~m}_{\pi}^{\mathrm{o}}\right)$ | $\left\|\vec{p}_{s}\right\| \leq 0.3$ | $620 \pm 60$ |
|  | $>0.3$ | $60 \pm 10$ |
| $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{pf}^{\mathrm{o}}\left(\mathrm{f}^{\mathrm{o}} \rightarrow\right.$ all neutrals $)$ | $\left\|\vec{p}_{S}\right\| \leq 0.3$ | $120 \pm 20$ |
| $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{S} \Delta^{+}(1236) \pi^{\circ}, \Delta^{+} \rightarrow \mathrm{p} \pi^{0}$ | $\left\|\overrightarrow{p_{S}}\right\| \leq 0.3$ | $13 \pm 5$ |

Forward Differential Cross Section for $\pi^{+} n \rightarrow p \pi^{\circ}$ at $6.95 \mathrm{GeV} / \mathrm{c}$

| $\|\mathrm{t}\|$ <br> $\left(\mathrm{GeV}^{2}\right)$ | $\frac{\mathrm{d} \sigma}{\mathrm{dt}}\left(\mu \mathrm{b} / \mathrm{GeV}^{2}\right)$ |
| :---: | :---: |
| $1 \mathrm{t}_{\min } \mid-0.04$ | $190 \pm 40$ |
| $0.04-0.08$ | $290 \pm 55$ |
| $0.08-0.12$ | $280 \pm 50$ |
| $0.12-0.16$ | $370 \pm 55$ |
| $0.16-0.20$ | $220 \pm 40$ |
| $0.20-0.24$ | $120 \pm 30$ |
| $0.24-0.28$ | $120 \pm 30$ |
| $0.28-0.36$ | $64 \pm 16$ |
| $0.36-0.44$ | $43 \pm 13$ |
| $0.44-0.60$ | $12 \pm 5$ |
| $0.60-1.00$ | $9 \pm 3$ |
| $1.00-1.30$ | $5 \pm 2$ |

## TABLE 3

| $\left(\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}^{+} \pi \pi^{-}\right)$at $6.95 \mathrm{GeV} / \mathrm{c}$ |  |  |
| :--- | :---: | :---: |
| Topology | Events | $\sigma(\mu \mathrm{b})$ |
| 3-prongs | 4122 | 589 |
| 4-prongs $\left(\left\|\overrightarrow{p_{s}}\right\|<0.3 \mathrm{GeV} / \mathrm{c}\right)$ | 2532 | 361 |
| 4-prongs $\left(\left\|\overrightarrow{p_{s}}\right\|>0.3 \mathrm{GeV} / \mathrm{c}\right)$ | 477 | $68 \pm 6$ |
| Total $\quad\left(\left\|\overrightarrow{p_{s}}\right\|<0.3 \mathrm{GeV} / \mathrm{c}\right)$ | 6654 | $950 \pm 70$ |

TABLE 4
Resonance Parameters and Production Cross Sections in $\pi N \rightarrow N \pi^{+} \pi^{-}$

| Expt. | Resonance | $\begin{gathered} \mathrm{Mass}_{2} \\ \left(\mathrm{GeV} / \mathrm{c}^{2}\right) \end{gathered}$ | $\stackrel{\Gamma}{\left(\mathrm{GeV} / \mathrm{c}^{2}\right)}$ | $\begin{gathered} \sigma \\ (\mu \mathrm{b}) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\pi^{+} \mathrm{d}$ | $\rho^{\circ}$ | $0.780 \pm 0.003$ | $0.165 \pm 0.010$ | $352 \pm 70$ |
| $\pi^{+} \mathrm{d}$ | $\mathrm{f}^{0}$ | $1.264 \pm 0.004$ | $0.194 \pm 0.015$ | $258 \pm 25$ |
| $\pi^{+} \mathrm{d}$ | $\mathrm{g}^{\text {o }}$ | $1.68 \pm 0.01$ | $0.16 \pm 0.04$ | $50 \pm 20$ |
| $\pi^{-} \mathrm{p}$ | $\rho^{\circ}$ | $0.783 \pm 0.003$ | $0.145 \pm 0.010$ | $387 \pm 40$ |
| $\pi^{-} \mathrm{p}$ | $\mathrm{f}^{0}$ | $1.274 \pm 0.005$ | $0.170 \pm 0.020$ | $231 \pm 30$ |
| $\pi{ }^{-} \mathrm{p}$ | $\mathrm{g}^{0}$ | $1.65 \pm 0.02$ | $0.07 \pm 0.02$ | $28 \pm 10$ |

TABLE 5
Forward Differential Cross Sections for $\pi N \rightarrow \rho^{\circ} N^{\prime}$ and $\pi \mathrm{N} \rightarrow \mathrm{f}^{\mathrm{O}} \mathrm{N}^{\prime}$ at $6.95 \mathrm{GeV} / \mathrm{c}^{\text {. }}{ }^{\text {(a) }}$

| $\pi \mathrm{N} \rightarrow \rho^{\mathrm{o}_{\mathrm{N}^{\prime}}}{ }^{(\mathrm{b})}$ |  | $\pi \mathrm{N} \rightarrow \mathrm{f}^{\mathrm{O}} \mathrm{N}^{(\mathrm{c})}$ |  |
| :---: | :---: | :---: | :---: |
| $\|t\|\left(\mathrm{GeV}^{2}\right)$ | $\frac{\mathrm{d} \sigma}{\mathrm{dt}} \mathrm{mb} / \mathrm{GeV}^{2}$ | $\|t\|\left(\mathrm{GeV}^{2}\right)$ | $\frac{\mathrm{d} \sigma}{\mathrm{dt}} \mathrm{mb} / \mathrm{GeV}^{2}$ |
| $\mathrm{t}_{\text {min }}-0.02$ | $2.78 \pm 0.19$ | $t_{\text {min }}-0.02$ | $2.05 \pm 0.27$ |
| 0.02-0.04 | $3.56 \pm 0.21$ | 0.02-0.04 | $2.45 \pm 0.15$ |
| 0.04-0.06 | $2.36 \pm 0.14$ | 0.04-0.06 | $1.94 \pm 0.11$ |
| 0.06-0.08 | $1.90 \pm 0.12$ | 0.06-0.08 | $1.43 \pm 0.09$ |
| 0.08-0.10 | $1.44 \pm 0.11$ | 0.08-0.10 | $1.09 \pm 0.08$ |
| 0.10-0.15 | $0.90 \pm 0.05$ | $0.10-0.12$ | $0.82 \pm 0.07$ |
| 0.15-0.20 | $0.49 \pm 0.05$ | 0.12-0.14 | $0.75 \pm 0.07$ |
| 0.20-0.25 | $0.31 \pm 0.03$ | 0.14-0.16 | $0.58 \pm 0.06$ |
| 0.25-0.30 | $0.22 \pm 0.03$ | 0.16-0.18 | $0.44 \pm 0.05$ |
| 0.30-0.35 | $0.15 \pm 0.02$ | 0.18-0.20 | $0.40 \pm 0.05$ |
| 0.35-0.40 | $0.11 \pm 0.02$ | 0.20-0.22 | $0.33 \pm 0.04$ |
| 0.40-0.50 | $0.078 \pm 0.011$ | 0.22-0.26 | $0.26 \pm 0.03$ |
| 0.50-0.60 | $0.080 \pm 0.011$ | 0.26-0.30 | $0.14 \pm 0.02$ |
| 0.60-0.80 | $0.047 \pm 0.006$ | 0.30-0.34 | $0.13 \pm 0.02$ |
| 0.80-1.00 | $0.024 \pm 0.004$ | 0.34-0.40 | $0.107 \pm 0.014$ |
|  |  | 0.40-0.50 | $0.068 \pm 0.009$ |
|  |  | 0.50-0.60 | $0.044 \pm 0.007$ |
|  |  | 0.60-0.80 | $0.020 \pm 0.003$ |
|  |  | 0.80-1.00 | $0.014 \pm 0.003$ |

## TABLE 5 (cont'd)

(a) The $\pi^{+} n$ data have been corrected for Pauli exclusion effects assuming pure spin flip at the nucleon vertex.
(b) The $\rho^{0}$ differential cross section is normalized to an integrated cross section of $360 \mu \mathrm{~b}$ for $|\mathrm{t}| \leq 1.0 \mathrm{GeV}^{2}$.
(c) The $f^{0}$ differential cross section is normalized to an integrated cross section of $250 \mu \mathrm{~b}$ for $|\mathrm{t}| \leq 1.0 \mathrm{GeV}^{2}$ 。

TABLE 6
$\pi-\pi$ Phase Shifts and Inelasticities ( $\delta$ in Degrees)

| $\begin{aligned} & \mathrm{M}\left(\pi^{+} \pi^{-}\right) \\ & \mathrm{GeV} / \mathrm{c}^{2} \end{aligned}$ | $\delta_{S}^{0}$ | $\eta_{\mathrm{S}}^{0}$ | $\delta^{1}$ | $\eta_{\mathrm{P}}^{1}$ | $\delta_{\text {D }}^{0}$ | $\eta_{\mathrm{D}}^{0}$ | $\delta_{\mathrm{F}}^{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 60 | $44 \pm 20$ |  | $18 \pm 9$ |  |  |  |  |
| . 64 | $47 \pm 25$ |  | $20 \pm 9$ |  |  |  |  |
| . 68 | $59 \pm 25$ |  | $34 \pm 9$ |  |  |  |  |
| . 72 | $65 \pm 20$ |  | $52 \pm 7$ |  |  |  |  |
| . 76 | $55 \pm 15$ |  | $69 \pm 6$ |  | 0 |  |  |
| . 80 | $62 \pm 15$ |  | $98 \pm 10$ | $1.0 \pm .05$ | $1 \pm 3$ |  |  |
| . 84 | $67 \pm 15$ | $1.0 \pm .05$ | $128 \pm 5$ | . $95 \pm .1$ | $2 \pm 5$ |  |  |
| . 88 | $86 \pm 20$ | . $95 \pm .05$ | $138 \pm 5$ | . $90 \pm .1$ | $3 \pm 7$ |  |  |
| . 92 |  |  | $145 \pm 5$ | . $80 \pm .1$ | $3 \pm 5$ |  |  |
| . 96 |  |  | $151 \pm 5$ | $.75 \pm .1$ | $6 \pm 4$ |  |  |
| 1.00 |  |  | $158 \pm 10$ | $.70 \pm .1$ | $21 \pm 10$ |  |  |
| 1.04 | $222 \pm 20$ | . $77 \pm .2$ | $149 \pm 10$ | . $68 \pm .1$ | $14 \pm 11$ |  |  |
| 1.08 | $245 \pm 18$ | . $71 \pm .2$ | $158 \pm 11$ | $.75 \pm .1$ | $25 \pm 9$ |  |  |
| 1.12 | $242 \pm 25$ | . $62 \pm .15$ | $156 \pm 12$ | $.79 \pm .1$ | $28 \pm 10$ |  |  |
| 1.16 | $248 \pm 30$ | $.38 \pm .35$ | $157 \pm 12$ | $.85 \pm .1$ | $38 \pm 9$ | $1.0 \pm .05$ |  |
| 1.20 | $256 \pm 20$ | $1.0 \pm .2$ | $159 \pm 10$ | $.88 \pm .1$ | $44 \pm 6$ | . $93 \pm .1$ |  |
| 1.23 | $257 \pm 40$ | $1.0 \pm .2$ | $166 \pm 20$ | $.89 \pm .1$ | $59 \pm 15$ | $.85 \pm .15$ |  |
| 1.25 | $263 \pm 30$ | $1.0 \pm .2$ | $177 \pm 7$ | . $94 \pm .15$ | $72 \pm 10$ | . $73 \pm .15$ |  |
| 1.27 | $274 \pm 35$ | $1.0 \pm .2$ | $176 \pm 10$ | $.89 \pm .15$ | $93 \pm 10$ | $.70 \pm .15$ |  |
| 1.29 | $288 \pm 35$ | . $95 \pm .2$ | $176 \pm 10$ | . $82 \pm .15$ | $100 \pm 15$ | . $65 \pm .15$ |  |
| 1.31 | $290 \pm 35$ | $1.0 \pm .2$ | $174 \pm 8$ | $.70 \pm .15$ | $117 \pm 12$ | . $58 \pm .15$ |  |

TABLE 6 (cont 'd)

| $\mathrm{M}\left(\pi^{+} \pi\right)$ <br> $\mathrm{GeV} / \mathrm{c}^{2}$ | ${ }^{\delta_{\mathrm{S}}^{0}}$ | $\eta_{\mathrm{S}}^{0}$ | $\delta_{\mathrm{P}}^{1}$ | $\eta_{\mathrm{P}}^{1}$ | $\delta_{\mathrm{D}}^{0}$ | $\eta_{\mathrm{D}}^{0}$ | $\delta_{\mathrm{F}}^{1}$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.33 | $296 \pm 40$ | $1.0 \pm .2$ | $178 \pm 9$ | $.81 \pm .15$ | $123 \pm 14$ | $.59 \pm .15$ |  |  |
| 1.36 | $310 \pm 17$ | $.80 \pm .2$ | $176 \pm 5$ | $.85 \pm .15$ | $136 \pm 7$ | $.65 \pm .2$ |  |  |
| 1.40 | $308 \pm 34$ | $.96 \pm .2$ | $175 \pm 6$ | $.87 \pm .15$ | $147 \pm 10$ | $.54 \pm .2$ | 0 | $\pm 5$ |
| 1.44 | $291 \pm 30$ | $.82 \pm .2$ | $178 \pm 12$ | $.85 \pm .15$ | $152 \pm 8$ | $.60 \pm .15$ | $1.2 \pm 5$ |  |
| 1.48 | $330 \pm 10$ | $.68 \pm .2$ | $177 \pm 4$ | $.86 \pm .15$ | $158 \pm 4$ | $.64 \pm .15$ | $2.4 \pm 5$ |  |

TABLE 7
$\mathrm{I}=0 \pi-\pi$ Phase Shifts in $\pi^{+} \mathrm{n} \rightarrow \mathrm{p} \pi^{\circ} \pi^{\circ}$ at 6.95 GeV/c

| $\mathrm{M}\left(\pi^{0} \pi^{0}\right)$ <br> $\mathrm{GeV} / \mathrm{c}^{2}$ | Events | $\delta_{\mathrm{S}}^{0}$ <br> (Degrees) | $\delta_{\mathrm{D}}^{0(\mathrm{a})}$ <br> (Degrees) |
| :---: | :---: | :---: | :---: |
| $0.8-1.0$ | 38 | $119 \pm 38$ | $16 \pm 10$ |
| $1.0-1.15$ | 65 | $256 \pm 30$ | $20 \pm 4$ |
| $1.15-1.25$ | 84 | $269 \pm 26$ | $31 \pm 9$ |
| $1.25-1.35$ | 81 | $319 \pm 12$ | $104 \pm 10$ |

(a) Partially constrained to $\delta_{\mathrm{D}}^{0}$ results of Table 6,

## TABLE 8

| Non-2 $\mathrm{f}^{\mathrm{O}}$ Decay Modes |  |  |
| :---: | :---: | :---: |
| Decay Mode (xx) | Cross Section ( $\mu \mathrm{b}$ ) | $\mathrm{R}=\frac{\Gamma\left(\mathrm{f}^{\mathrm{o}} \rightarrow \mathrm{xx}\right)}{\Gamma\left(\mathrm{f}^{\mathrm{o}} \rightarrow \pi^{+} \pi^{-}\right)}$ |
| $K \overline{\mathrm{~K}}$ |  | $.035 \pm .007^{(a)}$ |
| $\eta^{\circ} \eta^{\circ}$ | $15 \pm 8$ | $.06 \pm .03$ |
| $\pi^{+} \pi^{+} \pi^{-} \pi^{-}$ |  | $.055 \pm .010^{(b)}$ |
| $\pi^{+} \pi^{-} \pi^{\circ} \pi^{\circ}$ | $6 \pm 3$ | $.02 \pm .01$ |

(a) Reference 42 .
(b) Reference 44.

TABLE 9

$$
\sigma\left(\pi^{+} \mathrm{n} \rightarrow \mathrm{~N}^{*} \pi^{+}\right) \text {at } 6.95 \mathrm{GeV} / \mathrm{c}
$$

| $\Delta$ or $\mathrm{N}^{*} \operatorname{Mass}\left(\mathrm{GeV} / \mathrm{c}^{2}\right)$ | $\sigma\left(\mathrm{N}^{*} \rightarrow \mathrm{p} \pi^{-}\right)(\mu \mathrm{b})$ |
| :---: | :---: |
| $1.26 \pm .015$ | $32 \pm 15$ |
| $1.37 \pm .02$ | $28 \pm 18$ |
| $1.50 \pm .02$ | $26 \pm 16$ |
| $1.65 \pm .02$ | $33 \pm 15$ |

## TABLE A1

Fraction of Fitted and Monte Carlo $2 \pi^{\circ}$
Events in Different $\gamma$ Topologies

| $\mathrm{M}_{\pi \pi}\left(\mathrm{GeV} / \mathrm{c}^{2}\right)$ | $1 \gamma$ | $2 \gamma$ | $2 \gamma$ <br> Type 1 | $2 \gamma$ <br> Type 2 | $3 \gamma$ | $4 \gamma$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Min - 0.7 | .20 | .50 | .30 | .20 | .25 | .05 |
| $0.7-1.1$ | .30 | .53 | .25 | .28 | .16 | .01 |
| $1.1-1.4$ | .34 | .49 | .23 | .26 | .11 | .06 |
| $1.4-1.8$ | .41 | .43 | .24 | .19 | .16 | .00 |
| $1.8-\mathrm{Max}$ | .62 | .37 | .14 | .23 | .01 | .00 |
| Total | .38 | .46 | .23 | .23 | .13 | .03 |
| Monte <br> Carlo | .35 | .41 | --- | --- | .20 | .04 |

TABLE A2

Fraction of $2 \pi^{\circ}$ and $3 \pi^{\circ}$ Monte Carlo Events
With an Acceptable $2 \pi^{\circ}$ Fit

| $\mathrm{M}\binom{2 \pi^{\mathrm{o}}}{3 \pi^{\mathrm{o}}}$ | $2 \pi^{\mathrm{O}}$ Events |  | $3 \pi^{0}$ Events |  |
| :---: | :---: | :---: | :---: | :---: |
| $\left(\mathrm{GeV} / \mathrm{c}^{2}\right)$ | $2 \gamma$ | $3 \gamma$ | $2 \gamma$ | $3 \gamma$ |
| Min - 0.7 | . 95 | . 59 | --- | - |
| 0.7-1.1 | . 95 | . 80 | 1.0 | . 71 |
| 1.1-1.4 | . 90 | . 88 | . 71 | . 24 |
| 1.4-1.8 | . 92 | . 79 | . 61 | . 24 |
| 1.8-Max | . 83 | . 81 | . 39 | . 06 |
| Total | . $90 \pm .1$ | $.79 \pm .1$ | $.56 \pm .1$ | . $30 \pm .07$ |

1. Event characteristics for $\pi^{+} d \rightarrow p_{s} p \pi^{0}$. (a) $\chi^{2}$ distributions for $1-$ constraint (1C) and 3-constraint (3C) fits; (b) Missing mass distribution;
(c) the ratio $\mathrm{E}_{\gamma_{1}} / \mathrm{E}_{\pi^{\mathrm{o}}}$ for 3 C fits where $\gamma_{1}$ is the first of 2 measured $\gamma^{\prime} \mathrm{s}$;
(d) $\mathrm{M}(\gamma \gamma)$ for 3 C fits.
2. (a) Elastic charge exchange cross section vs. laboratory beam momentum $\left(P_{L}\right)$. The $\pi^{-} p$ data is from Ref. 13 and the 4.5 and $6.0 \mathrm{GeV} / \mathrm{c} \pi^{+} \mathrm{n}$ cross sections are from Ref. 3 and 5. (b) Differential cross section for $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{\mathrm{o}}$ with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{s}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$. Pauli exclusion correction assuming all spin non-flip is indicated by x .
3. Missing Mass (MM) from $\pi^{+} d \rightarrow p_{s} p+$ neutrals with single $\pi^{0}$ events excluded.
(a) MM for $\left|\vec{p}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$;
(b) MM for $\left|\vec{p}_{\mathrm{S}}\right|>0.3 \mathrm{GeV} / \mathrm{c}$.
4. Chew-Low plot of $\Delta^{2}\left(\pi^{+} \rightarrow\right.$ missing mass) vs. missing mass for 2-prong events with $\left|\vec{p}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$.
5. Momentum transfer ( t ) distributions for $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}+$ Missing Mass.
(a) $0.45<\mathrm{MM}<0.65 \mathrm{GeV} / \mathrm{c}^{2}$;
(b) $0.7<\mathrm{MM}<0.85 \mathrm{GeV} / \mathrm{c}^{2}$;
(c) $\mathrm{MM}<1.15 \mathrm{GeV} / \mathrm{c}^{2}$ with $\eta^{\circ}$ and $\omega^{\circ}$ regions excluded; (d) $1.15<\mathrm{MM}<$ $1.35 \mathrm{GeV} / \mathrm{c}^{2}$; (e) $1.35<\mathrm{MM}<1.6 \mathrm{GeV} / \mathrm{c}^{2}$.
6. Missing mass from $\pi^{+} d \rightarrow p_{s} p+$ neutrals for different $\gamma$ topologies (no. of associated $\gamma^{\prime}$ s observed in the tantalum plates). (a) $0 \gamma$; (b) $1 \gamma$; (c) $2 \gamma^{\prime} \mathrm{s}$; (d) $3 \gamma^{\prime} \mathrm{s}$; (e) 4 or more $\gamma^{\prime} \mathrm{s}$.
7. $M\left(\pi^{o} \pi^{o}\right)$ from $\pi^{+} d \rightarrow p_{S} p \pi^{0} \pi^{0}$ fitted events. (a) $1 \gamma$ events ( 0 C fit); (b) $1-4$ $\gamma$ events.
8. (a) $M\left(p \pi_{1}^{o}\right)$ for $\pi^{+} d \rightarrow p_{s} p \pi_{1}^{o} \pi_{2}^{o}$ with $\pi_{1}^{o}$ defined by a momentum transfer cut;
(b) $\mathrm{M}\left(\mathrm{p} \pi_{2}^{\mathrm{o}}\right)$; (c) $\mathrm{M}\left(\pi_{1}^{\mathrm{o}} \pi_{2}^{\mathrm{o}}\right)$ for events with $\mathrm{M}\left(\mathrm{p} \pi_{1}^{\mathrm{o}}\right)$ near the $\Delta^{+}$(1236) and $\left|\mathrm{t}\left(\pi^{+} \rightarrow \pi_{2}^{\mathrm{o}}\right)\right|<0.2(\mathrm{GeV} / \mathrm{c})^{2}$.
9. $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$from $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p} \pi^{+} \pi^{-}$with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$. The curve results from a fit using Breit-Wigner resonance forms for the $\rho^{\circ}, \mathrm{f}^{\mathrm{o}}$ and $\mathrm{g}^{\circ}$.
10. $\overrightarrow{\text { Momentum transfer }(t) \text { distributions in } \pi^{+} d \rightarrow p_{s} p \pi^{+} \pi^{-} \text {for various } M\left(\pi^{+} \pi^{-}\right), ~(1) ~}$ intervals. The curves result from fits to the data with $.04<|t|<.24$ $(\mathrm{GeV} / \mathrm{c})^{2}$ yielding exponential slopes, $\beta$, as shown.
11. $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$for compilation of $\pi^{-} \mathrm{p}$ and $\pi^{+} \mathrm{d}$ data with $\left|\mathrm{t}_{\mathrm{np}}\right|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$.
12. $\operatorname{Cos} \theta_{\pi \pi}$ (Jackson angle) distributions for combined $\pi^{+} n$ and $\pi^{-} p$ data with $|\mathrm{t}|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$. Central values are shown for the $40 \mathrm{MeV} / \mathrm{c}^{2} \mathrm{bins}$ in $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$. The bin size is 0.1 in $\cos \theta$ except for $0.90<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.06 \mathrm{GeV} / \mathrm{c}^{2}$ where the bin size is 0.2 . The solid curves show the AOPE model fit results.
13. Azimuth (Treiman-Yang) angle for $40 \mathrm{MeV} / \mathrm{c}^{2}$ bins in $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$. The bin size is $18^{\circ}$ in $\varphi$ except for $0.90<\mathrm{M}\left(\pi^{+} \pi^{-}\right)<1.06 \mathrm{GeV} / \mathrm{c}^{2}$ where the bin size is $36^{\circ}$. The solid curves show the AOPE model fit results.
14. $\pi-\pi$ phase shifts and inelasticities $\delta_{\mathrm{S}}^{2}, \delta_{\mathrm{D}}^{2}, \eta_{\mathrm{S}}^{2}, \eta_{\mathrm{S}}^{0}, \eta_{\mathrm{P}}^{1}$ and $\eta_{\mathrm{D}}^{0}$ from the AOPE model fits to the $\pi^{+} \pi^{-}$and $\pi^{-} \pi^{\circ}$ angular distributions. The data of Baton et al. is from Ref. 22. The shaded bands indicate roughly the area between the upper and lower limits. The break in $\eta_{S}^{0}$ for $0.9<M(\pi \pi)<1.0$ $\mathrm{GeV} / \mathrm{c}^{2}$ indicates the region where $\delta_{\mathrm{S}}^{0}$ has been shown to rise rapidly through $90^{\circ} .24$
15. $\pi-\pi$ angular distributions for $\pi^{-} \mathrm{p} \rightarrow \mathrm{p} \pi^{-} \pi^{\mathrm{o}}$ with $|t|<0.3(\mathrm{GeV} / \mathrm{c})^{2}$. Central values are shown for the $80 \mathrm{MeV} / \mathrm{c}^{2}$ bins in $\mathrm{M}\left(\pi^{-} \pi^{\mathrm{o}}\right)$ and the curves show the results of the AOPE model fits.
16. $\pi-\pi$ phase shifts $\delta_{\mathrm{S}}^{0}, \delta_{\mathrm{P}}^{1}$ and $\delta_{\mathrm{D}}^{0}$ from the AOPE model fits to the $\pi^{+} \pi^{-}$angular distributions. The shaded bands indicate roughly the area between the upper and lower limits. The break in $\delta_{\mathrm{S}}^{0}$ for $0.9<\mathrm{M}(\pi \pi)<1.0 \mathrm{GeV} / \mathrm{c}^{2}$ indicates the region in which $\delta_{S}^{0}$ rises rapidly through $90^{\circ}$ (see Ref. 24).
17. (a) Diagram of four possible solutions for $\delta_{S}^{0}$ in the $\rho^{0}$ mass region. (b) $\mathrm{M}\left(\pi^{\mathrm{o}} \pi^{\mathrm{o}}\right)$ with curves showing the predictions of Malamud and Schlein (see Ref. 37). The alternate upward curving branches near $1.0 \mathrm{GeV} / \mathrm{c}^{2}$ show the effect of including a small $D$-wave contribution.
18. $\operatorname{Cos} \theta{ }_{\pi \pi}$ and azimuthal angle distributions for $\pi^{+} n \rightarrow p \pi^{\circ} \pi^{\circ}$. The curves show the AOPE model fit results with $\delta_{\mathrm{S}}^{0}$ and $\delta_{\mathrm{D}}^{0}$ as given in Table 7.
19. (a) Missing mass for 2 -prong events with one or more associated $\gamma^{\prime}$ s which failed to fit $\pi^{+} d \rightarrow p_{s} p \pi^{o} \pi^{o}$ 。(b) $\mathrm{M}\left(\eta^{o} \eta^{o}\right)$ for $4 \gamma$ events which fit $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \eta^{\mathrm{o}} \eta^{\mathrm{o}}$ with $\eta^{\mathrm{o}} \rightarrow \gamma \gamma$. This plot includes 1-prong events (spectator proton unseen).
20. $\mathrm{M}\left(\pi^{+} \pi^{-} \pi^{\mathrm{O}} \pi^{\mathrm{o}}\right)$ for 4-prong events which fit $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-} \pi^{\mathrm{o}} \pi^{\mathrm{o}}$ with two or more measured $\gamma^{\prime}$ s. Gammas in this final state were measured and fit for roughly $1 / 2$ of the 650 K picture exposure.
21. $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$for 4 -prong events which fit $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p} \pi^{+} \pi^{-}$with $\left|\overrightarrow{\mathrm{p}}_{\mathrm{S}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$. The curve shows the result of a fit with interfering Breit-Wigner amplitudes for the $\rho$ and $\omega$.
22. $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$for 4-prong events which fit $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-}$with $\left|\overrightarrow{p_{s}}\right|<0.3 \mathrm{GeV} / \mathrm{c}$. (a) $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$weighted by $\rho_{00}$; (b) $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$weighted by $\rho_{11}+\rho_{1-1}$. See Ref. 18 for density matrix elements.
23. (a) $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$for $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p} \pi^{+} \pi^{-}$. The curve results from a fit using a handdrawn background and Breit-Wigner resonance forms for the $\Delta^{\circ}$ and $N^{*}$ peaks. (b) $M\left(n \pi^{+}\right)$and $M\left(p \pi^{-}\right)$for $\pi^{-} p \rightarrow n \pi^{+} \pi^{-}$and $\pi^{+} n \rightarrow p \pi^{+} \pi^{-}$.
24. $M\left(n \pi^{+}\right)$and $M\left(p \pi^{-}\right)$for combined $\pi^{-} p$ and $\pi^{+} d$ data with $M\left(\pi^{+} \pi^{-}\right)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$ 。 (a) $\left|\mathrm{t}_{\pi \pi}\right|<0.2(\mathrm{GeV} / \mathrm{c})^{2}$; (b) $\left|\mathrm{t}_{\pi \pi}\right|>0.2(\mathrm{GeV} / \mathrm{c})^{2}$.
25. Exponential slope parameter, $\alpha$, from fits to the $t_{\pi \pi}$ distributions (fitting the data with $\left.\left|t_{\pi \pi}\right| \lesssim 0.4(\mathrm{GeV} / \mathrm{c})^{2}\right)$. (a) Variation of $\alpha$ with $\mathrm{M}\left(\mathrm{p} \pi^{-}\right)$
for $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{S}} \mathrm{p} \pi^{+} \pi^{-}$; (b) Variation of $\alpha$ for combined $\pi^{-} \mathrm{p}$ and $\pi^{+} \mathrm{d}$ data with $M(\pi \pi)>1.4 \mathrm{GeV} / \mathrm{c}^{2}$.
26. (a) Diffractive scattering diagram showing the definition of $t_{\pi \pi}$ and the nucleon-nucleon scattering angle, ${ }^{\theta}{ }_{\mathrm{NN}}$, defined in the center of mass of the $\pi$-nucleon system. (b) OPE diagram for $\pi d \rightarrow p p \pi \pi$.
27. $\operatorname{Cos} \theta_{\mathrm{NN}}$ distributions for the $\pi^{-} p$ and $\pi^{+} \mathrm{d}$ data in intervals of $\mathrm{M}(\pi$-nucleon) as shown. (a) Data with $\left|t_{\pi \pi}\right|<0.2(\mathrm{GeV} / \mathrm{c})^{2}$; (b) Data with $\left|\mathrm{t}_{\pi \pi}\right|>0.2$ $(\mathrm{GeV} / \mathrm{c})^{2} .{ }^{\theta}{ }_{\mathrm{NN}}$ and $t_{\pi \pi}$ are defined in Fig. 26(a). The curves show $\cos \theta \sin \theta$ distributions.
28. $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$from $\pi^{+} \mathrm{d} \rightarrow \mathrm{p}_{\mathrm{s}} \mathrm{p}^{+} \pi^{-}$. (a) 3-prong events; (b) 4-prong events with $\left|\vec{p}_{\mathrm{S}}\right|<0.3(\mathrm{GeV} / \mathrm{c})$.
29. The reaction $\pi^{+} d \rightarrow p_{s} \pi^{+} \pi^{-}$with $\left|\vec{p}_{S}\right|>0.3 \mathrm{GeV} / \mathrm{c}$. (a) $\mathrm{M}\left(\pi^{+} \pi^{-}\right)$; (b) $\operatorname{Cos}$ $\theta_{\pi \pi}$ and azimuthal angle distributions in the $\rho^{\circ}$ mass region.
30. The diproton mass spectrum, $M(p p)$, for the reaction $\pi^{+} d \rightarrow p p \rho^{\circ}$. The curve is calculated using the OPE model.
A1. (a) Probability that the $\gamma \gamma$ opening angle $\theta$ for $\pi^{0} \rightarrow \gamma \gamma$ will be larger than the minimum opening angle $\theta_{\text {MIN }}$ for various $\pi^{\circ}$ momenta ( $\mu=$ pion mass). (b) Probability that the $\pi^{\circ}$ momentum $P$ for $\pi^{\circ} \rightarrow \gamma \gamma$ will be larger than the minimum momentum $P_{\text {MIN }}$ for a given $\gamma \gamma$ opening angle $\theta$. The curves are Monte Carlo results for $\mathrm{P}_{\text {MIN }}$ in the indicated intervals.
A2. $\pi^{\circ}$ direction assignments for kinematic fitting using measured $\gamma$ directions. (a) one $\gamma$ observed from $\pi_{1}^{0}$ and none from $\pi_{2}^{0}$; (b) one $\gamma$ observed from each $\pi^{\circ}$; (c) two $\gamma^{\prime}$ s observed from $\pi_{1}^{o}$ and none from $\pi_{2}^{o}$. MM denotes the missing momentum in $\pi^{+} d \rightarrow p p+n e u t r a l s$.

A3. Monte Carlo results for $\gamma$ detection efficiencies. (a), (c) Geometrical Detection Efficiency (GDE) for $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events; (b), (d) Actual Detection

Efficiency (ADE) for $2 \pi^{\circ}$ and $3 \pi^{\circ}$ events. Error bars are shown for a few of the data points.
 $\gamma^{\prime}$ s. $\overrightarrow{\mathrm{P}}_{\mathrm{MONTY}}=$ Monte Carlo generated $\pi^{\circ}$ momentum, $\overrightarrow{\mathrm{P}}_{\mathrm{FIT}}=$ momentum found by $2 \pi^{\circ}$ fitting programs. (a) Laboratory angle between the Monte Carlo and fitted $\pi^{\circ}$ directions; (b) The fractional error in the magnitude of the fitted $\pi^{\circ}$ momentum; (c) Difference in the $\pi-\pi$ scattering angle, $\cos \theta_{\pi \pi}$ in the $\pi-\pi$ center of mass, between the Monte Carlo and fitted $2 \pi^{\circ}$ events.


Fig. 1


Fig. 2a


Fig. 2b



Fig. 3


Fig. 4


Fig. 5


Fig. 6


Fig. 7


Fig. 8


Fig. 9


Fig. 10


Fig. 11


Fig: 12


$$
\cos \left(\theta_{\pi \pi}\right)
$$

Fig. 12 (conv' $\left.)^{\prime}\right)$


Fig. 13


Fig. 13 (CONT'D)


Fig. 14


Fig. 15


Fig. 16



Fig. 17


Fig. 18



Fig. 19


Fig. 20


Fig. 21


Fig. 22


Fig. 23


Fia. 24


Fig. 25


Fig. 26


Fig. 27


Fig. 28


Fig. 29


Fig. 30


Fig. AI


## $\pi^{\circ}$ DIRECTION ASSIGNMENT

Fig. A2


Fig. A3


Fia. A4


[^0]:    ＊Work supported in part by the U．S．Atomic Energy Commission under Con－ tracts No．AT（11－1）－881，No．$A T(40-1)-3065$ ，and by the National Research Council of Canada．

