# SOLU'ION OF THE ON-SHELL FADDEEV EQUATIONS* 

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#### Abstract

The on-shell Faddeev equations provide a consistent and convergent phenomenology for the description of non-relativistic three-hadron systems. A finite matrix approximation retains the complicated energy dependence of the interference between two-body channels in three-body observables even though only a small number of terms are retained.


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[^0]The on-shell limit of the Faddeev equations ${ }^{1}$ is mathematically ambiguous in the sense that the three-body bound state spectrum depends on how this limit is taken. ${ }^{2}$ But the original equations are physically ambiguous. There is no unique experimental or theoretical way to construct the dynamical input to the equations - the "off-shell" t-matrices assumed by Faddeev to be given by "local potentials" bounded by const $/ \mathrm{r}^{3 / 2+\epsilon}$. Nevertheless, for a three-hadron system which scatters via the materialization and exchange of additional particles of finite mass, the on-shell Faddeev equations provide a unique phenomenological non-relativistic limit. This interpretation suggests a practical method for solving the equations.

The full three-particle transition matrix from a state of three free particles of specified momenta to another such state is determined by nine two-variable functions (times appropriate angular functions) called, in the notation of Bollé and Osborn, ${ }^{3} \mathscr{M}_{\alpha \beta ; \ell}^{J} \lambda_{\alpha} ; i_{\beta}^{\prime} \lambda_{\beta}^{\prime}\left(\mathrm{p}_{\alpha}, \mathrm{q}_{\alpha} ; \mathrm{p}_{\beta}^{\prime}, \mathrm{q}_{\beta}^{\prime} ; \mathrm{z}\right)$. The replacement of the off-shell t-matrices $\mathrm{t}_{\ell_{\alpha}}^{\alpha}\left(\mathrm{q}_{\alpha}, \mathrm{q}_{\alpha}^{\prime} ; \mathrm{z}-\mathrm{p}_{\alpha}^{2}\right)$ by their on-shell values: $\tau_{\ell_{\alpha}}^{\alpha}\left(\mathrm{q}_{\alpha}^{2}\right) / 2 \pi^{2}=-\mathrm{e}^{\mathrm{i} \delta} \sin \delta / 2 \pi^{2} \mathrm{q}_{\alpha}$ in the equations for $\mathscr{M}_{\alpha \beta}^{\mathrm{J}}$ reduces these equations to the one-variable set
where

$$
\begin{gather*}
\mathrm{K}_{\alpha \gamma, \ell_{\alpha} \lambda_{\alpha} ; \ell_{\gamma} \lambda_{\gamma}}\left(\mathrm{p}_{\alpha}, \mathrm{p} ; \mathrm{z}\right)=\frac{\left(1-\delta_{\alpha \gamma}\right)}{8 \pi^{2}} \int_{\mathrm{q}_{-}}^{2_{+}^{2}} \mathrm{dq}^{2} \frac{\tau_{\ell}^{\gamma}\left(\mathrm{q}^{2}\right)}{\widetilde{\mathrm{p}}^{2}+\widetilde{\mathrm{q}}^{2}-\mathrm{z}} \mathrm{~V}_{\ell_{\alpha} \lambda_{\alpha} ; \ell_{\gamma} \lambda_{\gamma}\left(\mathrm{p}_{\alpha} ; \mathrm{p}, \mathrm{q}\right) \quad(2)}^{\mathrm{p}^{2}=2 \mathrm{n}_{\gamma} \widetilde{\mathrm{p}}^{2} ; \quad \mathrm{q}^{2}=2 \mu_{\gamma} \widetilde{\mathrm{q}}^{2} ; \quad \mathrm{n}_{\gamma}=\mathrm{m}_{\gamma}\left(\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}\right) /\left(\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}+\mathrm{m}_{\gamma}\right) ; \quad \mu_{\gamma}=\mathrm{m}_{\alpha} \mathrm{m}_{\beta} /\left(\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}\right)}  \tag{2}\\
\mathrm{q}_{ \pm}=\left|\mathrm{p}_{\alpha} \pm \mathrm{m}_{\alpha} \mathrm{p} /\left(\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}\right)\right|
\end{gather*}
$$

and

$$
\begin{equation*}
\chi_{\alpha \beta ; \ell_{\alpha} \lambda_{\alpha} ; \lambda_{\beta}^{\prime}}^{\mathrm{J}}\left(\mathrm{p}_{\alpha}^{2}\right)=\frac{2\left(1-\delta_{\alpha \beta}\right)}{\mathrm{p}_{\beta}^{\prime}} \sum_{\ell_{\beta}^{\prime}} \mathrm{K}_{\alpha \beta ; \ell_{\alpha} \lambda_{\alpha ;}^{J} \ell_{\beta}^{\prime} \lambda_{\beta}^{\prime}}\left(\mathrm{p}_{\alpha^{\prime}} \mathrm{p}_{\beta}^{\prime} ; \mathrm{z}\right) \tag{3}
\end{equation*}
$$

The three-body T-matrix is then given in terms of the solution of this equation by

$$
\begin{align*}
\mathscr{M}_{\alpha \beta ; \ell_{\alpha}}^{\mathrm{J}} \alpha_{\alpha} ; \mathrm{l}_{\beta}^{\prime} \lambda_{\beta}^{\prime} & =-\frac{1}{2 \pi^{2}} \tau_{\ell_{\alpha}}^{\alpha}\left(\mathrm{q}_{\alpha}^{2}\right)\left\{\delta_{\alpha \beta^{\delta} \lambda_{\alpha} \lambda_{\beta}^{\prime} \delta\left(\mathrm{p}_{\alpha}-\mathrm{p}_{\alpha}^{\mathrm{t}}\right) / \mathrm{p}_{\alpha}^{2}}\right. \\
& \left.+\mathrm{N}_{\alpha \beta ; \ell}^{\mathrm{J}} \lambda_{\alpha} ; \mathrm{l}_{\beta}^{\prime} \lambda_{\beta}^{\prime}\left(\mathrm{p}_{\alpha}^{2} ; \mathrm{p}_{\beta}^{\prime}, \mathrm{q}_{\beta}^{\prime} ; \mathrm{z}\right)\right\} \tag{4}
\end{align*}
$$

Alternatively, we could have used the equation for the dependence on $p_{\beta}^{\prime 2}$ of some function $\overline{\mathrm{N}}$ which satisfies a similar equation. The existence of these two alternative equations for $\mathscr{M}_{\alpha \beta}^{J}$ is all that is required to establish unitarity, as has been shown by Freedman, Lovelace and Namyslowski. ${ }^{4}$

It is obvious that the non-relativistic Faddeev equations cannot correctly describe any three-hadron system above production threshold for a fourth hadron, but it is often ignored that they also cannot be expected to be accurate at energies (negative in the non-relativistic sense) corresponding to four (virtual) hadrons in the system. This is a very severe limitation in the three-nucleon system, as this "anomalous threshold" corresponds to $3 N+\pi$ at an energy $E \approx-\left(\epsilon_{d}+m_{\pi}^{2} / 3 M_{N}\right) \approx$ -9.2 MeV only $10 \%$ below the -8.48 MeV binding energy of the triton! ${ }^{5,6}$ This pion can be expected to generate "three-body forces" and "off-shell effects" just as significant as the "off-shell effects" coming from any phenomenological potential or off-shell t-matrix constructed to describe two-nucleon scattering experiments. The simplest and most consistent way to construct a nonrelativistic phenomenology for three-hadron systems is to limit the description to energies greater than the first four-hadron "anomalous threshold". This
physical cutoff removes the ambiguity in the three-particle bound state spectrum mentioned above. ${ }^{2}$

Once we have excluded from consideration values of the three-body energy $\mathrm{z} \rightarrow \mathrm{E}+\mathrm{i} 0^{+}$less than some finite threshold $-\epsilon_{0}$ (determined a priori from the hadronic mass spectrum), the one-variable on-shell Faddeev equations can be reduced to matrix equations by projection onto some set of functions $\phi_{n}\left(p^{2}\right)$ complete on the interval $0 \leq \mathrm{p}^{2} \leq \infty$ (i.e., $\left.\int_{0}^{\infty} \operatorname{dp}^{2} \phi_{\mathrm{n}}\left(\mathrm{p}^{2}\right) \phi_{\mathrm{n}},\left(\mathrm{p}^{2}\right)=\delta_{\mathrm{nn}}{ }^{\prime}\right)$ and well defined for $\mathrm{p}^{2}$ greater than this negative threshold. An obvious choice is

$$
\begin{equation*}
\phi_{\mathrm{n}}^{\gamma}\left(\mathrm{p}^{2}\right)=\frac{\sqrt{2 \mathrm{n}_{\gamma} \epsilon_{0}(2 \mathrm{n}+1)}}{\left(\mathrm{p}^{2}+2 \mathrm{n}_{\gamma} \epsilon_{0}\right)} \mathrm{P}_{\mathrm{n}}\left(\frac{2 \mathrm{n}_{\gamma} \epsilon_{0}-\mathrm{p}^{2}}{2 \mathrm{n}_{\gamma} \epsilon_{0}+\mathrm{p}^{2}}\right) \tag{5}
\end{equation*}
$$

The convergence of the on-shell equations has already been proved, ${ }^{7}$ and is reconfirmed here by the convergence of the integrals in the matrix approximation to the integral equation. Since the geometrical factors $\mathrm{V}^{\mathrm{J}}$ are simply functions of angles in their physical range, ${ }^{3}$ we need only prove convergence for $\mathrm{J}=0=\ell=\lambda$, in which case $V_{\alpha \gamma}^{J}=\left(m_{\alpha}+\mathrm{m}_{\beta}\right) / 2 \mathrm{~m}_{\alpha} \mathrm{p}_{\alpha}$.

If we define the leading term in the kernel for this case

$$
\begin{equation*}
\mathrm{k}_{\alpha \gamma}^{\mathrm{o}}(\mathrm{z})=\int_{0}^{\infty} \mathrm{dp}_{\alpha}^{2} \phi_{0}^{\alpha}\left(\mathrm{p}_{\alpha}^{2}\right) \int_{0}^{\infty} \mathrm{dp}^{2} \phi_{0}^{\gamma}\left(\mathrm{p}^{2}\right) \mathrm{K}_{\alpha \gamma ; 00 ; 00}^{\mathrm{o}}\left(\mathrm{p}_{\alpha}, \mathrm{p} ; \mathrm{z}\right) \tag{6}
\end{equation*}
$$

then this constant becomes

$$
\begin{equation*}
\mathrm{k}_{\alpha \gamma}^{\mathrm{o}}(\mathrm{z})=\int_{0}^{\infty} \mathrm{dq} \tau_{0}^{\gamma}\left(\mathrm{q}^{2}\right) \mathrm{I}_{\alpha \gamma}^{\mathrm{o}}\left(\mathrm{q}^{2}\right) \tag{7}
\end{equation*}
$$

where the purely kinematical quantity $\mathrm{I}_{\alpha \gamma}^{\mathrm{O}}\left(\mathrm{q}^{2}\right)$ is given by

$$
\begin{align*}
\mathrm{I}_{\alpha \gamma}^{\mathrm{o}}\left(\dot{\mathrm{q}}^{2}\right)= & \frac{\left(1-\delta_{\alpha \gamma}\right) \epsilon_{0}}{\pi^{2} \sqrt{2 \mathrm{n}_{\gamma}}}\left(\frac{\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}}{\mathrm{m}_{\alpha}}\right) \sec ^{2} \theta \csc \theta \int_{0}^{\infty} d x \int_{0}^{\csc \theta} d y(\mathrm{x}+\mathrm{y}) \\
& \cdot \mathrm{q}^{4}\left\{\mathrm{q}^{2}\left[(\mathrm{x}+\mathrm{y})^{2}+\sec ^{2} \theta / 2 \mu_{\gamma}\right]-\mathrm{z} \sec ^{2} \theta\right\}^{-1} \\
& \cdot\left[\mathrm{q}^{2}(\mathrm{x}+\mathrm{y})^{2}+\epsilon_{0} \sec ^{2} \theta\right]^{-1}\left[\mathrm{q}^{2}(\mathrm{x}-\mathrm{y}+\csc \theta)^{2}+\epsilon_{0} \csc ^{2} \theta\right]^{-1} \tag{8}
\end{align*}
$$

The auxiliary variables x and y were introduced to remove the absolute value sign in the limits on the original $q^{2}$ variable (cf. Eq. (2)) coming from the use of radial variables in momentum space. The transformation is

$$
\begin{align*}
\mathrm{p}_{\alpha} & =\sqrt{2 \mathrm{n}_{\alpha}} \mathrm{q}[1+\sin \theta(\mathrm{x}-\mathrm{y})] \\
\mathrm{p} & =\sqrt{2 \mathrm{n}_{\gamma}} \mathrm{q} \cos 0(\mathrm{x}+\mathrm{y})  \tag{9}\\
\tan \theta & =\mathrm{m}_{\alpha} /\left(\mathrm{m}_{\alpha}+\mathrm{m}_{\beta}\right)
\end{align*}
$$

Note that the kernel only depends on knowing the two-body phase shifts for real physical energies, and that the only singularity in the integrand is the physical branch cut generated by on-shell kinematics in the transformation from one twobody channel to another. This same branch cut occurs in the usual Faddeev equations, but with variable limits; here the constant limits on the variables of integration make it easy to handle numerically. In fact, if we use a representation for $\tau$ with the physical property of a branch cut starting at $-\mu^{2} / 4$ (where $\mu^{2}$ is given in terms of the masses of the two particles and the mass of the lightest third particle which can be exchanged between them), the integral can be changed to an integral over the discontinuity across this cut; for the effective range approximation, this cut reduces to poles, and the integral can be evaluated explicitly. One of the remaining integrals is then elementary leaving at most a single non-singular integral for numerical evaluation. All of this remains true if any of the indices $\left(J \ell \lambda_{\mathbf{n}}\right)$ differ from zero. Further, the driving term $\chi_{\alpha \beta}^{J}$ requires only two integrations, so can be evaluated analytically for pole representations of $\tau$.

If we look closely at Eq. (2), we see that both the single scattering terms and the dynamical terms $\mathbb{N}_{\alpha \beta}^{\mathrm{J}}$ carry the same two-body phase shifts as a factor. If only S-waves ( $\ell=0$ ) are included, the full three-body T-matrix will then contain only three terms, and the entire content of the solution of the dynamical equations
which can show up in three-body observables are three complex numbers which multiply the two-body scattering amplitudes (phase shifts). For instance in a ${ }^{2} \mathrm{~S}_{1 / 2}$ three-nucleon final state (which differs from the spinless model given here only by numerical factors) the energy spectrum of the outgoing proton will be dominated by the $n-n ~{ }^{1} \mathrm{~S}_{0}$ virtual state at the upper end, but the kinematics smears out the ${ }^{1} S_{0}$ and ${ }^{3} S_{1}{ }^{n}-p$ peaks producing another complicated structure at about a third of the total energy available to the proton. This means that the complicated $n$-d breakup spectrum can be described (once the $S$-wave phase shifts are put in this way) by a very small number of parameters, as has been noted previously. ${ }^{8,9}$ This suggests that even the simplest approximation to the onshell equations can have interesting physical content at low energy. Taking only the nine numbers given by Eq. (6) (and the corresponding driving terms $\chi_{\alpha \beta}^{0}$ ) then gives an explicit approximation to the full three-body T-matrix (Eq. (2)) through the numbers

$$
\begin{align*}
N_{\alpha \beta}^{o}(\mathrm{z})= & \left(1+\mathrm{k}_{\alpha+1, \alpha-1}^{\mathrm{o}} \mathrm{k}_{\alpha-1, \alpha+1}^{\mathrm{o}}\right) \chi_{\alpha} \\
& +\left(\mathrm{k}_{\alpha, \alpha+1}^{o}+\mathrm{k}_{\alpha, \alpha-1}^{\mathrm{o}} \mathrm{k}_{\alpha-1, \alpha+1}^{\mathrm{o}}\right) \chi_{\alpha+1}  \tag{10}\\
& +\left(\mathrm{k}_{\alpha, \alpha-1}^{\mathrm{o}}+\mathrm{k}_{\alpha, \alpha+1}^{\mathrm{o}} \mathrm{k}_{\alpha-1, \alpha+1}^{\mathrm{o}}\right) \chi_{\alpha-1} \\
\chi_{\mathrm{i}}= & \left(1-\delta_{i \beta}\right) / \chi_{i \beta}^{\mathrm{o}} \Delta \\
\Delta= & 1+\mathrm{k}_{12}^{\mathrm{o}} \mathrm{k}_{21}^{\mathrm{o}}+\mathrm{k}_{13}^{\mathrm{o}} \mathrm{k}_{31}^{\mathrm{o}}+\mathrm{k}_{23}^{\mathrm{o}} \mathrm{k}_{32}^{\mathrm{o}}-\mathrm{k}_{12}^{\mathrm{o}} \mathrm{k}_{23}^{\mathrm{o}} \mathrm{k}_{31}^{\mathrm{o}}-\mathrm{k}_{21}^{\mathrm{o}} \mathrm{k}_{31}^{\mathrm{o}} \mathrm{k}_{13}^{\mathrm{o}}
\end{align*}
$$

It will also be of interest to see whether this approximation generates a pole in $\mathscr{M}$ (three-body bound state) for negative real values of z greater than the anomalous threshold $-\epsilon_{0}$.

Although we have so far assumed no two-body bound states, the same formalism can be readily extended to that case simply by using a representation
for $\tau$ which has the appropriate poles, i.e., $\tau^{\alpha}\left(q^{2}\right)=-N_{\alpha}^{2} /\left(q^{2}+\kappa_{\alpha}^{2}\right)+\hat{\tau}^{\alpha}\left(q^{2}\right)$. For the boundary conditions used above, the terms in $\mathscr{M}_{\alpha \beta}$ proportional to $\tau$ still represent $3-3$ scatterings, while the pole term will give the probability that two of the three particles end up as a bound pair, as can readily be seen by using Eq. (4) for $\mathscr{M}_{\alpha \beta}$ to calculate the configuration space wave function. Of course, if we evaluate $\mathrm{K}_{\alpha \beta}^{\mathrm{J}}$ and $\chi_{\alpha \beta}^{\mathrm{J}}$ by contour integration, we must take care to pick up the pole contributions coming from the bound states. Further, by taking an appropriate weighted average over the initial state parameters $\underline{p}_{\beta}^{\prime}, \underline{q}_{\beta}^{\prime}$, we can also calculate the amplitudes for elastic scattering, rearrangement collisions and breakup starting from one particle incident on some specific bound pair. Note that, formally speaking, the kernel in the equation still involves an integral over the two-body phase shifts only over positive energies; in order to generate the correct singularity structure we must require that our representation for $\tau$ extrapolate to the correct bound state poles at negative energies. The residues at these poles are physical parameters whose relative values can be observed in three-body systems that contain two or more bound two-body subsystems, and which indicate how much of the bound state is an "elementary particle", and how much is to be attributed to a composite structure which can be scparated into two particles using energies less than their rest energies. Only if we require that $N_{\alpha}^{2}=2 \kappa_{\alpha}$ will the number of particles remain constant in the conventional non-relativistic sense. Thus, in general our formalism is "unitary" only if interpreted consistently from a hadronic point of view, a statement which is obviously also true of the Amado model ${ }^{10}$ when the "renormalization constant" at the bound state pole is treated as a free parameter.

We emphasize in closing that the equations developed above allow us to make unique predictions for all three-body observables in the energy range
$-\epsilon_{0}<\mathrm{E}<\infty$ knowing only the two-body phases shifts in the physical region, and the positions and residues of their bound-state poles. Of course we cannot believe these predictions as we approach the physical threshold for the production of a fourth hadron, or the anomalous threshold $-\epsilon_{0}$ but within these restrictions we have a consistent non-relativistic on-shell phenomenology. The range of validity of the theory can be extended, or the accuracy of the predictions within the specified range improved, only by explicitly introducing a fourth hadron and solving the four-body on-shell equations in the energetic region where only three or fewer particles can be present. Since these equations will also be reducible to one-variable integral equations, this task may not prove to be too formidable. Such equations would then provide the equivalent of "off-shell" effects in the two-body subsystems and "three-body forces" in a consistent way. But we could also mock up these effects phenomenologically simply by changing the $\chi_{\alpha \beta}^{\mathrm{J}}$ without changing the way the two-body phase shifts enter the equations. This would generate a unitary three-body amplitude with parameters which, if experimentally required to be non-zero, ${ }^{11}$ would unambiguously indicate three-body effects not explicable in terms of the on-shell properties of the two-body subsystems. ${ }^{12}$

## FOOTNOTES

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