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ELECTRIC AND MAGNETIC CHARGE IN EINSTEIN'S UNIFIED FIELD THEORY*

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ABSTRACT

Some new approximate solutions to the field equations of Einstein's unified field theory are constructed, and the physical significance of the theory is examined. According to the conventional physical interpretation of the theory, the singularities of the new solutions should represent magnetically charged point masses. It is found that in order to satisfy the field equations, such solutions must contain not only the usual point-like singularities, but also "string" singularities similar to those in Dirac's theory of magnetic poles. It is possible, nevertheless, to derive equations of motion for the point-like charges at the ends of the "strings," which turn out to be very similar to the Lorentz-Dirac equation. The paths of the "strings" are not arbitrary, but must also satisfy certain constraint conditions. On the basis of the equations of motion obtained, it is suggested that it may be possible to make an alternative physical interpretation of Einstein's theory, in which the point singularities of the new solutions are electric charges, rather than magnetic charges. The conventional interpretation of the theory is based on equations of motion found by Johnson, for some different approximate solutions to the field equations. If either interpretation is accepted, then the equations of motion in each case imply modifications of Maxwell's equations

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for macroscopic electromagnetic fields. The modifications are similar, but not identical, in the two cases. Some observational tests of the modified electromagnetic fields predicted by the theory are discussed, with emphasis on contrasting the two possible interpretations of electric charge. In each case, the field which deviates from the usual Maxwell theory involves a length parameter, an integration constant whose magnitude is not fixed by the theory, at this stage. It is shown that terrestrial tests of Maxwell's equations imply that this length must be greater than about 15 Earth radii, regardless of which interpretation one supposes to be true. It is then observed that the theory does imply an upper bound on the length parameter. Although this theoretical upper limit is not precise, a limit only a few orders of magnitude larger than the current experimental lower limit is suggested. This means that Einstein's theory deviates significantly from Maxwell's theory over astronomical distances. Some astrophysical situations where effects of the theory could be seen in static magnetic dipole fields are mentioned. These offer the possibility of testing whether Einstein's theory is correct, and of determining which type of singularity represents a point electric charge.

I. INTRODUCTION

Einstein's relativistic theory of the nonsymmetric field,[1] which he proposed over 25 years ago[2] as a unified field theory of gravitation and electromagnetism, is rather infrequently discussed in the current literature. Since this paper will present an investigation of some approximate solutions to the field equations of that theory, it is perhaps appropriate to first review the reason for this neglect, and also some recent encouraging results. Initially, the theory received considerable attention, since it was viewed as a particularly simple and elegant generalization of Einstein's general theory of relativity. However, interest in the theory began to wane a few years later, after it was shown that approximate solutions to its field equations gave results which seemed to be unsatisfactory. Specifically, approximate solutions were constructed with singularities which should have described charged point masses, based on what seemed to be the obvious choice for the electromagnetic field in the theory. It was then shown that these singular points satisfied equations of motion which did not resemble the Lorentz force equation, even in the static approximation.[3] In fact, they seemed to behave like uncharged point masses, so the theory was considered to be unsuccessful.

It was noticed from the very beginning, however, that the approximate field equations satisfied by the supposed electromagnetic field were weaker than Maxwell's equations, and hence admitted more general solutions. Recently, Johnson[4] has examined some of these more general approximate solutions. He has shown that there exist solutions with point singularities which satisfy equations of motion containing, among other things, the usual Lorentz force and radiation reaction force of classical electrodynamics. His results generalize some earlier work of other authors,[5] through the development of a Lorentz-covariant approximation technique. On the basis of these equations of motion, it seems possible to interpret the singularities of Johnson's solutions as electrically charged point masses. This belated indication that Einstein's theory might be successful, after all, has stimulated the present investigation.

We shall construct here some new approximate solutions to the field equations of the theory. If one accepts the above-mentioned "conventional" interpretation, that Johnson's solutions describe electric charges, then the point singularities of the new solutions should represent magnetically charged point masses. (Additional singularities the solutions contain are mentioned below.) However, we shall find that the equations of motion satisfied by these "magnetically charged" singularities also contain terms having the structure of the Lorentz force and radiation reaction force, as they usually appear in the Lorentz-Dirac equation.[6] Hence, on the basis of these new results, it seems that it may be possible to make an alternative interpretation of Einstein's theory, in which our solutions represent electric charges and Johnson's solutions represent magnetic charges. The basic difference between the two possible interpretations concerns whether a certain antisymmetric tensor, $\phi_{\mu\nu}$, or the dual tensor, $\epsilon_{\mu\nu\rho\sigma}\phi^{\rho\sigma}$, is identified as the electromagnetic field in the theory.

One of the most interesting aspects of Einstein's theory is that it predicts modifications of the usual laws of electrodynamics. This occurs because the equations of motion, for both types of solutions, contain force terms in addition to those in the Lorentz-Dirac equation. In each case these extra terms involve, in an essential fashion, a parameter with the dimension of length. The two length parameters, which we shall call ℓ, and $\widetilde{\ell}$, for Johnson's solutions and our solutions, respectively, are both integration constants that occur in solving the field equations. If ℓ, or $\widetilde{\ell}$, is a sufficiently large length, then the equations of motion are not necessarily inconsistent with experiment, since the effects of the extra force terms are very small for phenomena occurring on a scale small compared to ℓ, or $\tilde{\ell}$.

We shall examine some observational tests of Einstein's theory, based on the two different assumptions, that Johnson's solutions represent electric charges, and that our solutions represent electric charges. The equations of motion for the point charges allow us to write down, in each case, a set of "modified" Maxwell equations, which we shall suppose can be applied to macroscopic phenomena. It is then shown that if the predictions of these modified equations are to be consistent with experimental tests of Coulomb's law, and with measurements of the static magnetic field of the Earth at the Earth's surface, then one or the other of the parameters ℓ and $\tilde{\ell}$ must be greater than about 10^{10} cm (or about 15 Earth radii). Future observations can decide which of the two interpretations, if either, is correct, since the deviations from the Maxwell theory are different in the two cases. The above empirical result implies that the modifications of electromagnetic fields predicted by Einstein's theory should become significant only over astronomical distances. We therefore discuss briefly some implications of the theory concerning static magnetic dipole fields in astrophysical situations. These effects should be particularly striking, since both possible interpretations imply that at distances r from a dipole source which are large compared to ℓ, or $\tilde{\ell}$, the static dipole field will fall off with increasing distance as $\frac{1}{r}$, rather than the usual $\frac{1}{r^3}$ behavior that follows from Maxwell's equations.

It is of some interest that, even for approximate solutions, the structure of theory implies that the parameters ℓ and $\tilde{\ell}$ are not completely arbitrary. This was noted by Johnson,[7] who showed that the length parameter ℓ in his solutions is related to another, very small length, r_E, which is the characteristic distance from a singular point at which the weak-field approximation for the electromagnetic field begins to fail. In other words, at this distance non-linearities in the theory become important. For gravitational interactions, the analogous characteristic distance has the order of magnitude of the Schwarzschild radius. The relation between ℓ and r_E takes the form

$$r_E^2 \sim \ell\sqrt{\frac{Ge^2}{c^4}} \quad , \tag{1.1}$$

where G is the gravitational constant, e is the electron charge, and c is the speed of light. The origin of this relation will be discussed, and at the same time it will be shown that a relation of exactly the same form applies, involving the parameter $\tilde{\ell}$, if our solutions represent electric charges, instead of Johnson's.

It is evident that (1.1) implies that we cannot suppose that the length ℓ, or the length $\tilde{\ell}$ in the alternative case, is arbitrarily large, for then r_E would have to be, also. That this theoretical upper limit on ℓ, or $\tilde{\ell}$, is a physically significant one, is a point which is best discussed now. We expect that r_E should be, at most, not too much larger than a typical atomic dimension, since it seems reasonable to suppose that the weak-field approximation should be valid at distances where classical electrodynamics is known to be successful. For sake of argument, if we suppose r_E is the classical electron radius, $e^2/4\pi m_e c^2 \approx 3 \times 10^{-13}$ cm, then the corresponding value of ℓ, or $\tilde{\ell}$, is about 10^8 cm, according to (1.1). If instead we take for r_E the Bohr radius, 5×10^{-9} cm, then we find for ℓ, or $\tilde{\ell}$, a value of about 10^{16} cm (roughly one hundreth of a light-year). The theoretical upper limit then should lie somewhere in this range. It was mentioned earlier that the current experimental lower limit is about 10^{10} cm, so the range of allowed values for the length ℓ, or the length $\tilde{\ell}$, is limited. A significant improvement in the experimental lower limit could therefore imply that the theory is untenable. On the other hand, the existence of the upper limit implies, if the theory is not simply wrong, that the modifications of Maxwell's equations will be very important certainly on a galactic scale, if not a smaller one. In particular, the dominant behavior for static magnetic dipole fields at such distances should follow the $\frac{1}{r}$ law, rather than $\frac{1}{r^3}$.

Let us now mention the most significant feature of the new approximate solutions presented in this paper. The conventional interpretation for the electromagnetic field in Einstein's theory is based on one of the exact field equations, which, it is supposed, states that there can be no magnetically charged currents. We are able to construct approximate solutions with "magnetically charged" singularities, and to satisfy the approximate version of the above field equation, only at the cost of introducing singularities of a more complicated type than occur in Johnson's solutions. In fact, at any instant in time, our solutions contain singularities not merely at isolated points in space, but also along lines, or "strings", extending from the singular points either to infinity or to other singular points with opposite charge. It is not possible to satisfy the approximate field equations with solutions of this "magnetic" type if they contain singularities only at isolated points in space; they must contain string singularities as well. These solutions are thus very similar to solutions which Dirac constructed to his theory of magnetic poles.[8] The string singularities arise for essentially the same reason here as they do in Dirac's theory.

In four-dimensional space-time our solutions are thus singular not only on curves, or "world-lines," but also on two-dimensional "sheets" swept out by the strings in time. We may choose these sheets so that the world-lines of the charged point masses lie on their boundaries. It is of considerable importance that the equations of motion we obtain are relations which must be satisfied only on these world-lines, not everywhere on the sheets. It is this fact which allows us to identify the world-lines as trajectories of point charges, in the usual sense. Besides these equations of motion, however, the integrability conditions also imply a second restriction on the solutions, which does have to be satisfied everywhere on the sheets. It seems reasonable to suppose that, physically, this condition determines where the strings must lie in space, and how they develop in time. In other words, this restriction means that the paths of the strings in our solutions may not be chosen arbitrarily, as is possible for the strings in Dirac's theory, but are rather determined by the field equations. If one accepts the conventional interpretation of Einstein's theory, it is natural to conjecture that the existence of the strings is connected with the nonappearance of magnetic charges in nature. However, if the alternative interpretation is made, it turns out that the presence of the strings is not apparent in physical situations where the distance scale is small compared to the length parameter $\tilde{\ell}$ occurring in our solutions. For, as mentioned earlier, in this limit the equations of motion contain only the usual Lorentz force and radiation reaction force, involving interactions only of point singularities.

Section II of this paper contains a brief review of the field equations of Einstein's unified field theory. The treatment emphasizes the symmetry properties of the theory under a group of local gauge transformations, rather than the geometric structure, as in the usual presentations. This makes the simplicity of the theory, as a generalization of general relativity, particularly evident. For, the unified field theory simply enlarges the group of local gauge transformations, which in the special case of general relativity is the Lorentz group,[9] to include unitary transformations in Minkowski space, acting on complex fields. In other words, the transformation group is enlarged from O(3, 1) to U(3, 1).

In Section III the new approximate solutions are constructed. First, the approximation method is stated. Then the new solutions for the "electromagnetic" field are given, and finally the equations of motion are derived from the integrability conditions for solutions for the "gravitational" field to exist. The corresponding results from Johnson's papers[4] are also quoted in this section.

Section IV contains the discussion of observational tests of Einstein's theory. First, the modified Maxwell equations are written down. Then, the two tests which give the lower limit on the length parameters are examined. Next, the implications for cosmic magnetic fields are mentioned. Finally, the theoretical restriction implied by (1. 1) is derived. This section is intentionally written at a less-sophisticated level than the previous two sections, and is reasonably well self-contained. If one is willing to take for granted some of the earlier results that are quoted in this section, then it is possible to skip over Sections II and III.

Section V contains concluding remarks. These explain the necessity of the appearance of modifications of the usual equations of classical electrodynamics, as due to the formal scale invariance of Einstein's theory. Some unresolved problems concerning the new solutions are also pointed out.

In an Appendix certain details concerning self-field terms in the equations of motion are presented.

II. THE FIELD EQUATIONS

The fundamental field in Einstein's unified field theory is a Hermitian[10] second-rank tensor, $h^{\mu\nu}$, which transforms as a contravariant tensor under general coordinate transformations in four-dimensional space-time. We may write it as

$$h^{\mu\nu} = g^{\mu\nu} + i\, f^{\mu\nu} \tag{2.1}$$

in terms of a real, symmetric tensor $g^{\mu\nu}$ and a real, antisymmetric tensor $f^{\mu\nu}$. When $f^{\mu\nu}$ vanishes, the field equations of the theory reduce to those of general relativity, with $g^{\mu\nu}$ as the contravariant form of the metric tensor of a Riemannian geometry. We suppose in the following that $h^{\mu\nu}$ has the same signature as the metric tensor of the flat space-time of special relativity, i.e., (+, -, -, -). Such a Hermitian tensor may always be expressed in terms of a set of four complex vector fields e^{μ}_{a}, (a = 0, 1, 2, 3), as a bilinear form

$$h^{\mu\nu} = \eta^{ab}\, e^{*\mu}_{a}\, e^{\nu}_{b}\,, \tag{2.2}$$

where $e^{*\mu}_{a}$ denotes the set of complex conjugate vector fields. Here $\eta^{00} = 1$, $\eta^{11} = \eta^{22} = \eta^{33} = -1$, $\eta^{ab} = 0$ for $a \neq b$, and repeated indices are summed. To avoid an overabundance of indices, it is convenient to use matrix notation in place of the indices a, b. Thus, let e^{μ} denote, for fixed μ, a four-component column matrix, whose components are e^{μ}_{a}. It is useful to define a four-component row matrix $\bar{e}^{\mu}$, again for fixed μ, whose components are given by $\bar{e}^{\mu,a} = e^{*\mu}_{b}\,\eta^{ba}$. The quantity $e^{\nu}\,\bar{e}^{\mu}$ then denotes, for fixed μ and ν, a four-by-four matrix whose components are $(e^{\nu}\,\bar{e}^{\mu})^{a}_{b} = e^{\nu}_{b}\, e^{*\mu}_{c}\,\eta^{ca}$. In this notation the tensor $h^{\mu\nu}$ is written as

$$h^{\mu\nu} = \mathrm{Tr}\{ e^{\nu} \bar{e}^{\mu} \} \quad , \tag{2.3}$$

where $\mathrm{Tr}\{M\}$ denotes the trace of the matrix M.

Consider now transformations of the basis vectors e^{μ} of the form

$$e^{\mu} \rightarrow e'^{\mu} = U e^{\mu} \quad , \tag{2.4}$$

where U is some four-by-four matrix. If we denote by $\eta M \eta^{-1}$ a matrix whose components are $(\eta M \eta^{-1})^a_b = \eta_{bc} M^c_d \eta^{ad}$, with η_{ab} defined by $\eta_{ab}\eta^{bc} = \delta^c_a$, then the transformation rule for the vectors $\bar{e}^{\mu}$ is

$$\bar{e}^{\mu} \rightarrow \bar{e}'^{\mu} = \bar{e}^{\mu} \, \eta U^* \eta^{-1} \quad , \tag{2.5}$$

in terms of the complex conjugate matrix U*. It is evident from (2.3) that $h^{\mu\nu}$ will be left invariant by such transformations provided that U satisfies

$$\eta U^* \eta^{-1} = U^{-1} \quad , \tag{2.6}$$

where U^{-1} is the inverse of U. If we define $U^{\dagger} \equiv \eta U^* \eta^{-1}$, and let 1 be the unit matrix, then we can write (2.6) as

$$U^{\dagger} U = 1 \quad .$$

In other words, $h^{\mu\nu}$ is invariant under transformations of the basis vectors e^{μ} which are unitary with respect to the metric η. Since the matrix U may, in general, be a function of the space-time coordinates x^{μ}, we shall refer to the transformations

$$e^{\mu}(x) \to e'^{\mu}(x) = U(x)\, e^{\mu}(x) \tag{2.7}$$

as local gauge transformations. A fundamental property of Einstein's theory is that it is invariant under these local unitary gauge transformations.

In order to write down the Lagrangian of the unified field theory we need to consider transformations of derivatives of the vector fields e^{μ}. Let ∂_{μ} denote $\partial/\partial x^{\mu}$. Since

$$\partial_{\mu} e'^{\nu} = U\, \partial_{\mu} e^{\nu} + \partial_{\mu} U\, U^{-1} e'^{\nu} \quad ,$$

it is natural to introduce a new set of vector fields V_{μ}, which, for fixed μ, are represented by a four-by-four matrix transforming as

$$V_{\mu} \to V'_{\mu} = UV_{\mu} U^{-1} - \partial_{\mu} U\, U^{-1} \quad . \tag{2.8}$$

Then the "covariant" derivative defined by

$$D_{\mu} e^{\nu} \equiv \partial_{\mu} e^{\nu} + V_{\mu} e^{\nu} \tag{2.9}$$

transforms according to

$$\begin{aligned} D_{\mu} e^{\nu} \to D'_{\mu} e'^{\nu} &\equiv \partial_{\mu} e'^{\nu} + V'_{\mu} e'^{\nu} \\ &= U\, D_{\mu} e^{\nu} \quad . \end{aligned} \tag{2.10}$$

If we require the matrices V_{μ} to be anti-Hermitian with respect to the metric η, i.e.,

$$V^{\dagger}_{\mu} \equiv \eta V^{*}_{\mu} \eta^{-1} = -V_{\mu} \quad , \tag{2.11}$$

then the "covariant" derivative of the field $\bar{e}^{\nu}$ may be written as

$$\bar{D}_{\mu} \bar{e}^{\nu} \equiv (D_{\mu} e^{\nu})^{\dagger} = \partial_{\mu} \bar{e}^{\nu} - \bar{e}^{\nu} V_{\mu} \quad . \tag{2.12}$$

We make the choice (2.11) so that the derivative of $h^{\mu\nu}$ satisfies

$$\partial_\rho h^{\mu\nu} = \mathrm{Tr}\left\{(D_\rho e^\nu)\bar{e}^\mu + e^\nu(\bar{D}_\rho \bar{e}^\mu)\right\} \quad . \tag{2.13}$$

There is then a one-to-one correspondence between the independent matrix components of V_μ, for fixed μ, and the generators of infinitesimal unitary transformations. For, any local unitary transformation can be written as

$$U(x) = e^{\epsilon^a(x) T_a} \qquad (a = 1, \ldots, 16) \quad , \tag{2.14}$$

where the $\epsilon^a(x)$ are parameters and the sixteen linearly independent matrices T_a are anti-Hermitian with respect to η,

$$T_a^\dagger \equiv \eta T_a^* \eta^{-1} = -T_a \quad , \tag{2.15}$$

and generate the Lie algebra

$$\left[T_a, T_b\right] \equiv T_a T_b - T_b T_a = C^c_{ab} T_c \quad . \tag{2.16}$$

Here the $C^c_{ab} = -C^c_{ba}$ are the structure constants of the group, U(3, 1), of transformations unitary with respect to the pseudo-Euclidean metric η. There are thus sixteen real vector fields $V_{\mu,a}$, defined by

$$V_{\mu,a} \equiv \mathrm{Tr}\left\{T_a V_\mu\right\} \ .$$

Using V_μ, we may define a set of antisymmetric tensor fields $F_{\mu\nu} = -F_{\nu\mu}$, by

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu + \left[V_\mu, V_\nu\right] \quad . \tag{2.17}$$

They transform under the unitary group as

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \quad ,$$

and, like V_μ, are anti-Hermitian matrices

$$F^\dagger_{\mu\nu} \equiv \eta F^*_{\mu\nu} \eta^{-1} = -F_{\mu\nu} \quad .$$

The Lagrangian density of Einstein's unified field theory can be expressed in terms of the fields $F_{\mu\nu}$ and e^{μ} as

$$\mathscr{L} = \frac{1}{\sqrt{-h}} \operatorname{Tr}\left\{ F_{\mu\nu} e^{\nu} \bar{e}^{\mu} \right\} \tag{2.18}$$

where $h = \det(h^{\mu\nu})$. It is evident from (2.18) that $\mathscr{L}$ is invariant under the local unitary gauge transformations U(x). The field equations are obtained by varying $\mathscr{L}$ with respect to V_{μ} and $\bar{e}^{\mu}$ (or e^{μ}). They are

$$\partial_{\nu}\left(\frac{1}{\sqrt{-h}} (e^{\mu}\bar{e}^{\nu} - e^{\nu}\bar{e}^{\mu})\right) + \left[V_{\nu}, \frac{1}{\sqrt{-h}} (e^{\mu}\bar{e}^{\nu} - e^{\nu}\bar{e}^{\mu})\right] = 0 \tag{2.19}$$

and

$$F_{\mu\nu} e^{\nu} - \frac{1}{2} e_{\mu} \operatorname{Tr}\left\{F_{\rho\sigma} e^{\sigma} \bar{e}^{\rho}\right\} = 0 \quad . \tag{2.20}$$

Equation (2.20) is equivalent to the simpler expression

$$F_{\mu\nu} e^{\nu} = 0 \quad . \tag{2.21}$$

In (2.19), we have used the notation $[A, B] = AB-BA$ for two matrices A and B.

Let us now see how these field equations reduce to their usual tensor form. First, we construct a covariant tensor $h_{\mu\nu}$ which is the inverse of $h^{\mu\nu}$, i.e., $h_{\mu\nu}$ satisfies

$$h_{\mu\nu} h^{\nu\rho} = h_{\nu\mu} h^{\rho\nu} = \delta^{\rho}_{\mu} \quad , \tag{2.22}$$

using the Kronecker δ notation. Note the ordering of the indices in (2.22). A set of covariant vector fields e_{μ} is then defined by $e_{\mu} \equiv e^{\nu} h_{\nu\mu}$ and $\bar{e}_{\mu} \equiv h_{\mu\nu} \bar{e}^{\nu}$. Hence, $h_{\mu\nu} = \operatorname{Tr}\left\{e_{\nu}\bar{e}_{\mu}\right\}$. An affine connection $A^{\rho}_{\mu\nu}$ may now be constructed according to the definition

$$A^{\rho}_{\mu\nu} \equiv -\operatorname{Tr}\left\{(\partial_{\mu} e^{\rho} + V_{\mu} e^{\rho})\, \bar{e}_{\nu}\right\} \quad , \tag{2.23}$$

and this definition used to bring the field equation (2.19) into a more standard form. One finds, after some manipulation, that (2.19) implies that the affine connection $A^{\rho}_{\mu\nu}$ must have the form

$$A^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + iS^{\rho}_{\mu\nu} + iB_{\mu}\delta^{\rho}_{\nu} \quad , \tag{2.24}$$

where both $\Gamma^{\rho}_{\mu\nu}$ and $S^{\rho}_{\mu\nu}$ are real, and have the symmetry properties

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} \, , \qquad S^{\rho}_{\mu\nu} = -S^{\rho}_{\nu\mu} \quad . \tag{2.25}$$

In addition, $S^{\rho}_{\mu\nu}$ must satisfy

$$S^{\rho}_{\mu\rho} = 0 \quad . \tag{2.26}$$

The vector B_{μ} in (2.24) is real, but otherwise arbitrary. It is useful to define a second affine connection $B^{\rho}_{\mu\nu}$ by

$$B^{\rho}_{\mu\nu} \equiv \Gamma^{\rho}_{\mu\nu} + iS^{\rho}_{\mu\nu} \quad , \tag{2.27}$$

because the derivative of $h^{\mu\nu}$ then satisfies the equation

$$\partial_{\rho}h^{\mu\nu} + B^{\nu}_{\rho\lambda}h^{\mu\lambda} + B^{\mu}_{\lambda\rho}h^{\lambda\nu} = 0 \quad , \tag{2.28}$$

which is a consequence of the relation (2.24) and the definition (2.23). The set of equations (2.25 - 2.28) are the usual form[1] of those equations of Einstein's theory which determine the affine connection $B^{\rho}_{\mu\nu}$ in terms of the Hermitian tensor $h^{\mu\nu}$ and its derivatives.

The remaining field equations which Einstein proposed follow from (2.21). First, we define a generalized Riemann tensor $R_{\mu\nu\rho}{}^{\sigma}$ by

$$R_{\mu\nu\rho}{}^{\sigma} \equiv \partial_{\mu}B^{\sigma}_{\nu\rho} - \partial_{\nu}B^{\sigma}_{\mu\rho} + B^{\sigma}_{\mu\lambda}B^{\lambda}_{\nu\rho} - B^{\sigma}_{\nu\lambda}B^{\lambda}_{\mu\rho} \quad . \tag{2.29}$$

Then it can be shown that the contracted tensor $R_{\mu\nu}$, given by

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma} \quad , \tag{2.30}$$

is Hermitian, i.e., its symmetric part, $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu}+R_{\nu\mu})$, is real and its antisymmetric part, $R_{[\mu\nu]} \equiv \frac{1}{2}(R_{\mu\nu}-R_{\nu\mu})$, is imaginary. The tensor $R_{\mu\nu\rho}{}^{\sigma}$ is related to $F_{\mu\nu}$ of (2.17) by

$$-\mathrm{Tr}\left\{F_{\mu\nu} e^{\sigma} \bar{e}_{\rho}\right\} = R_{\mu\nu\rho}{}^{\sigma} + i(\partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu})\,\delta_{\rho}^{\sigma} \quad , \tag{2.31}$$

as is easily verified using the definitions (2.23), (2.27), and (2.29), and the relation (2.24) for $A_{\mu\nu}^{\rho}$. Equation (2.21) is now equivalent to

$$R_{\mu\nu} + i(\partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}) = 0 \quad , \tag{2.32}$$

so that the field equations for $R_{\mu\nu}$ are

$$R_{(\mu\nu)} = 0 \tag{2.33}$$

and

$$\partial_{[\rho}R_{\mu\nu]} = 0 \quad . \tag{2.34}$$

The notation $\partial_{[\rho}M_{\mu\nu]} \equiv \partial_{\rho}M_{\mu\nu} + \partial_{\mu}M_{\nu\rho} + \partial_{\nu}M_{\rho\mu}$, which applies for any antisymmetric tensor $M_{\mu\nu}$, has been used here. The equations (2.33 - 2.34) are the field equations given by Einstein.[1]

From the above formalism, Einstein's general theory of relativity is obtained simply by taking the special case in which $f^{\mu\nu} = 0$. Then all the fields in the theory are real, and the group of unitary transformations reduces to the group of orthogonal transformations in Minkowski space, or the homogeneous Lorentz group. In this case the local gauge transformation formalism was first given by Utiyama, and later by Kibble.[9] In general relativity the symmetric tensor $g^{\mu\nu}$ is, of course, associated with the gravitational field, and the Riemannian geometry has the inverse tensor $g_{\mu\nu}$ as its metric tensor. For the unified field theory, in the general case in which $f^{\mu\nu} \neq 0$, the geometry is non-Riemannian, and it does not seem easy to uniquely identify a particular object in the theory as the metric tensor, except perhaps by consideration of characteristic surfaces.

The physical interpretation which Einstein proposed for the antisymmetric tensor field $f^{\mu\nu}$ in his theory is based on the relation

$$\partial_\nu \left(\frac{1}{\sqrt{-h}} f^{\mu\nu} \right) = 0 \quad , \tag{2.35}$$

which is easily verified to be equivalent to (2.26), $S^{\rho}_{\mu\rho} = 0$, by using (2.25), (2.27), and (2.28). Equation (2.35) can also be obtained directly by taking the trace of (2.19). Note that (2.35) resembles the set of Maxwell's equations which imply the nonexistence of magnetically charged currents. If this relation holds at all points, then we can write

$$\frac{1}{\sqrt{-h}} f^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \tag{2.36}$$

for some vector field A_μ. Here $\epsilon^{\mu\nu\rho\sigma}$ is the four-dimensional antisymmetric symbol, with $\epsilon^{0123} = 1$. The vector A_μ is then the natural candidate for the electromagnetic vector potential in Einstein's theory. The approximate solutions to the field equations which have been constructed by Johnson[4] show that such an interpretation is possible, although the relation between A_μ and the usual vector potential of Maxwell's electrodynamics is somewhat more complicated than originally anticipated by Einstein. The approximate solutions we shall construct in the next section suggest that a different interpretation of the theory may be possible, in which, instead, the vector field B_μ appearing in (2.32) is related to the electromagnetic vector potential. This relation is of a similarly complicated type. The field B_μ is essentially the same as the vector field

$\mathrm{Tr}\{V_\mu\}$, which corresponds to the Abelian invariant subgroup U(1) in the decomposition $U(3,1) = SU(3,1) \otimes U(1)$. For, an examination of the definition (2.23) of $A^{\rho}_{\mu\nu}$ and the relation (2.24) shows that B_μ may be absorbed into a redefinition of the field $\mathrm{Tr}\{V_\mu\}$. If $\mathrm{Tr}\{V_\mu\}$ does represent the electromagnetic vector potential, then the U(1) subgroup, the group of local phase transformations, may be identified with the group of local gauge transformations that occurs in Maxwell's electrodynamics.

It should be mentioned that these two possible interpretations for the electromagnetic vector potential are mutually inconsistent. To see this we must now turn to approximate solutions of the field equations, which consist of the set (2.25 - 2.28) and (2.33 - 2.34).

III. APPROXIMATE SOLUTIONS

A. The Approximation Method

To construct approximate solutions to the field equations we suppose that $h^{\mu\nu}$ may be expanded in a power series in an arbitrary parameter λ, with $h^{\mu\nu}$ in zeroth order equal to the Minkowski metric tensor $\eta^{\mu\nu}$. In our notation $\eta^{00} = 1$, $\eta^{11} = \eta^{22} = \eta^{33} = -1$, and $\eta^{\mu\nu} = 0$ for $\mu \neq \nu$. It is most convenient to expand the tensor density $\frac{1}{\sqrt{-h}} h^{\mu\nu}$, so we write

$$\frac{1}{\sqrt{-h}} h^{\mu\nu} = \eta^{\mu\nu} + \sum_{n=1}^{\infty} \lambda^n \left(\gamma^{\mu\nu}_{(n)} + i\,\phi^{\mu\nu}_{(n)}\right) \quad , \tag{3.1}$$

where the $\gamma^{\mu\nu}_{(n)}$ are real and symmetric in μ and ν, and the $\phi^{\mu\nu}_{(n)}$ are real and antisymmetric. The structure of the field equations implies that, without loss of generality, we may assume that the expansion of the symmetric part of $\frac{1}{\sqrt{-h}} h^{\mu\nu}$ contains only even powers of λ, and the expansion of the antisymmetric part only odd powers. Thus, we may suppose that

$$\frac{1}{\sqrt{-h}} h^{\mu\nu} = \eta^{\mu\nu} + \sum_{n\ \text{even}} \lambda^n \gamma^{\mu\nu}_{(n)} + i \sum_{n\ \text{odd}} \lambda^n \phi^{\mu\nu}_{(n)} \quad . \tag{3.2}$$

Any other choice may be brought to this form by a redefinition of fields. The parameter λ is used simply to group terms with the same power of λ in writing down the approximate field equations in each order. Once this is done, we may set λ=1. The approximation method will be useful only if the components of $\gamma^{\mu\nu}_{(n)}$ and $\phi^{\mu\nu}_{(n)}$ are small compared to unity, so we shall suppose this is the case. The quantities $\gamma^{\mu\nu}_{(n)}$ and $\phi^{\mu\nu}_{(n)}$ are tensors under Lorentz transformations of the coordinates x^μ, which leave $\eta^{\mu\nu}$ invariant, but not under general coordinate transformations. In the following the Greek indices μ, ν, etc., will be raised and lowered with $\eta^{\mu\nu}$ and its inverse $\eta_{\mu\nu}$, according to the usual rules. All equations will then have a Lorentz-covariant form, unless mentioned otherwise.

We wish to study the approximate field equations only to second order in powers of λ, so we need consider only the two fields $\phi^{\mu\nu}_{(1)} \equiv \phi^{\mu\nu}$ and $\gamma^{\mu\nu}_{(2)} \equiv \gamma^{\mu\nu}$. Neglecting higher orders means that we are completely ignoring gravitational interactions, and including electromagnetic interactions only in the lowest non-trivial order. It is our purpose in this paper only to show that it is possible to obtain equations of motion for point charges which resemble those of classical electrodynamics, and for this we need consider only the first and second order field equations. For the very strong fields which necessarily occur in the regions close to the point singularities, the approximation method fails, as discussed in Section IV. E.

The derivation of the approximate equations which follow from the exact field equations, (2.25 - 2.28) and (2.33 - 2.34), and from the expansion (3.2), has been given in detail by Johnson (see Ref. 4, especially Papers I and VIII). Here we shall simply quote the results. To first order in λ, one obtains the linear homogeneous differential equations for $\phi^{\mu\nu}$,

$$\partial_\mu \phi^{\mu\nu} = 0 \tag{3.3}$$

and

$$\Box \partial_{[\rho} \phi_{\mu\nu]} = 0 \quad . \tag{3.4}$$

Here $\Box$ is defined by $\Box \psi = \eta^{\mu\nu} \partial_\mu \partial_\nu \psi$, for any ψ. The equations in second order are greatly simplified if we choose the four coordinates so that $\gamma^{\mu\nu}$ satisfies the harmonic condition,

$$\partial_\mu \gamma^{\mu\nu} = 0 \quad . \tag{3.5}$$

One then obtains the linear inhomogeneous differential equation for $\gamma^{\mu\nu}$,

$$\Box\gamma^{\mu\nu} = t^{\mu\nu} \quad . \tag{3.6}$$

The inhomogeneous term $t^{\mu\nu}$ is given by an expression quadratic in $\phi^{\mu\nu}$ and its derivatives,

$$\begin{aligned} t_{\mu\nu} = & -\frac{1}{2}\, \partial_\mu \phi_{\rho\sigma} \partial_\nu \phi^{\rho\sigma} + \partial_\sigma \phi_{\mu\rho} \partial^\sigma \phi_\nu{}^\rho - \partial_\sigma \phi_{\mu\rho} \partial^\rho \phi_\nu{}^\sigma \\ & + \frac{1}{4}\, \eta_{\mu\nu}\, \partial_\tau \phi_{\rho\sigma} \partial^\tau \phi^{\rho\sigma} + \frac{1}{2}\, \eta_{\mu\nu}\, \partial_\tau \phi_{\rho\sigma} \partial^\sigma \phi^{\rho\tau} \\ & - \phi^{\rho\sigma} \left(\partial_\nu \partial_\rho \phi_{\mu\sigma} + \partial_\mu \partial_\rho \phi_{\nu\sigma} \right) + \frac{1}{2}\, \eta_{\mu\nu}\, \phi^{\rho\sigma} \Box \phi_{\rho\sigma} \quad . \end{aligned} \tag{3.7}$$

Note that $t_{\mu\nu}$ is homogeneous of second degree in derivatives of $\phi_{\mu\nu}$. In the following we shall refer to (3.3 - 3.4) as the electromagnetic field equations, and to (3.5 - 3.7) as the gravitational field equations.

If the field equations (3.3) and (3.4) are both satisfied, then it is easily verified from (3.7) that $\partial^\mu t_{\mu\nu} = 0$. However, if only (3.3) holds, but not (3.4), then one finds that

$$\partial^\mu t_{\mu\nu} = \frac{1}{2}\, \phi^{\rho\sigma}\, \Box \partial_{[\nu} \phi_{\rho\sigma]} \; . \tag{3.7'}$$

This equation will be useful in Sections III. B and III. C, since we shall be considering solutions of the approximate field equations which fail at certain singular points. In fact, the relatively simple result (3.7') is the key to understanding the nature of electromagnetic interactions in Einstein's theory. It is of central importance in the derivation of equations of motion, and by comparing it to the divergence of the usual Maxwell electromagnetic energy-momentum tensor, one can guess the appropriate types of solutions to choose for $\phi_{\mu\nu}$, in order to get physically interesting results. A related discussion on this last point is contained in the concluding Section V.

B. The Electromagnetic Field

If equation (3.3) is satisfied at all points, then

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \tag{3.8}$$

for some vector field A_μ. On the other hand, equation (3.4) implies

$$\Box \phi_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{3.9}$$

for some vector field B_μ.[11] Since (3.8) and (3.9) define A_μ and B_μ only up to gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \Lambda^A \qquad B_\mu \to B_\mu + \partial_\mu \Lambda^B \quad ,$$

where Λ^A and Λ^B are arbitrary functions, we may choose the two vector fields to satisfy the Lorentz gauge conditions

$$\partial^\mu A_\mu = 0 \quad , \qquad \partial^\mu B_\mu = 0 \quad . \tag{3.10}$$

Then the field equations (3.3) and (3.4) imply that

$$\Box\Box A_\mu = 0 \quad , \tag{3.11}$$

and

$$\Box B_\mu = 0 \quad . \tag{3.12}$$

As discussed in the previous section, when Einstein proposed his theory, he suggested that the vector field A_μ should be identified with the electromagnetic vector potential. However, the fourth-order differential equation (3.11) is weaker than the familiar wave equation

$$\Box A_\mu = 0 \quad , \tag{3.13}$$

which follows from Maxwell's equations in the Lorentz gauge. Any solution of (3.13) satisfies (3.11), of course, but (3.11) admits more general solutions, as well. Johnson's contribution,[4] which built on the earlier results of others,[5] was to show that there exist physically interesting solutions of (3.11) for A_μ, which do not satisfy the stronger equation (3.13). The fact that his solutions satisfy only the weaker equation is crucial for obtaining equations of motion containing the Lorentz force and radiation reaction force. He thereby avoided the difficulty of Callaway,[3] who used solutions for A_μ which satisfied (3.13) and as a result found no force terms in the equations of motion.

For future reference, let us state here the essential features[12] of Johnson's solutions for A_μ. They have the form

$$A_\mu(x) = \sum_p A_\mu^{(p)}(x) \quad , \tag{3.14}$$

with

$$A_\mu^{(p)}(x) = f^{(p)}\, a_\mu^{(p)}(x) + \frac{f'^{(p)}}{\ell^2}\, a_\mu'^{(p)}(x) \quad . \tag{3.15}$$

Here $a_\mu^{(p)}(x)$ is the usual retarded potential of a point charge moving along a world-line $z_\mu^{(p)}(\tau^{(p)})$, i.e.,

$$a_\mu^{(p)}(x) = \int d\tau^{(p)}\, \frac{dz_\mu^{(p)}}{d\tau^{(p)}}\, D_{ret}(x-z^{(p)}) \quad , \tag{3.16}$$

but $a_\mu'^{(p)}(x)$ is given by

$$a_\mu'^{(p)}(x) = \int d^4x'\, a_\mu^{(p)}(x')\, D_{ret}(x-x') \quad . \tag{3.17}$$

The parameter $\tau^{(p)}$ describes the world-line, and $f^{(p)}$, $f'^{(p)}$, and ℓ^2 are constants. The sum over (p) in (3.14) runs over the various different world-lines $z_\mu^{(p)}$, each with "charges" $f^{(p)}$ and $f'^{(p)}$. The retarded Green's function is

$$D_{ret}(x) = \frac{1}{4\pi\,|\vec{x}|}\, \delta(x_0 - |\vec{x}|) \quad , \tag{3.18}$$

in terms of the Dirac δ-function. Since $\Box D_{ret}(x) = \delta^4(x)$, where $\delta^4(x)$ is the four-dimensional δ-function, we find

$$\Box\Box A_\mu(x) = \sum_p \int d\tau^{(p)} \frac{dz_\mu^{(p)}}{d\tau^{(p)}} \left(\frac{f'^{(p)}}{\ell^2} + f^{(p)}\Box\right) \delta^4(x - z^{(p)}) \quad ,$$

so that (3.11) is indeed satisfied everywhere except on the singular curves $z_\mu^{(p)}$. We must defer a discussion of the derivation of the equations of motion until Section III. C, but suffice it to say here that both terms in (3.15), with coefficients $f^{(p)}$ and $f'^{(p)}$, are essential in obtaining the Lorentz force term in the equations of motion. In Callaway's solutions $f'^{(p)}$ was equal to zero, which led to the unsatisfactory result.

On the basis of Johnson's results[13] it seems possible to interpret the singularities on the world-lines $z_\mu^{(p)}$ in (3.14 - 3.17) as electrically charged point masses, in general agreement with the original interpretation proposed by Einstein. As mentioned in Section II, Einstein also suggested that (2.35), whose approximate version is (3.3), should be interpreted as implying the nonexistence of magnetic charges. However, we wish to show here that it is possible to construct solutions of the approximate equations (3.3 - 3.7) which contain "magnetically charged" point singularities. More precisely, the singularities will represent magnetically charged point masses if the singularities of the solutions for A_μ in (3.14 - 3.17) are assumed to represent electrically charged point masses. Since we shall find that the so-called "magnetic charges" obey equations of motion which also contain terms with the structure of the Lorentz force and radiation reaction force, this conventional interpretation is no longer the only one it is possible to make. In our solutions it is the vector B_μ which resembles the retarded field of a point charge, in contrast to (3.14 - 3.17). That the relation of the two types of singularities is that of electric charge to magnetic charge is evident from the fact that (3.8) and (3.9) imply

$$\epsilon^{\mu\nu\rho\sigma}\Box(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \partial^\mu B^\nu - \partial^\nu B^\mu \quad . \tag{3.19}$$

If the singularities of A_μ are assumed to be electric charges, then the electromagnetic field is associated with the dual tensor $\epsilon_{\mu\nu\rho\sigma}\phi^{\rho\sigma}$. If we assume, alternatively, that the singularities of B_μ are electric charges, then the electromagnetic field is associated with the tensor $\phi_{\mu\nu}$ itself. In each case, however, the physically interesting solutions are those for which this "electromagnetic field" contains a term in addition to the usual Maxwell-Lorentz field.

We are able to find solutions corresponding to "magnetically charged" point masses only by introducing singularities of a more complicated type than occur in the expressions (3.14 - 3.17) for A_μ. To see this, suppose that we take as a solution of equation (3.12) for B_μ, the retarded potential of a point charge on a world-line $z_\mu(\tau)$, i.e.,

$$B_\mu(x) = \int d\tau \; \frac{dz_\mu}{d\tau} \; D_{ret}(x - z) \quad . \tag{3.20}$$

This satisfies (3.12) everywhere except on the world-line z_μ, since

$$\Box B_\mu(x) = \int d\tau \; \frac{dz_\mu}{d\tau} \; \delta^4(x-z) \quad . \tag{3.21}$$

But then (3.9) implies that

$$\Box \partial^\mu \phi_{\mu\nu}(x) = \int d\tau \; \frac{dz_\nu}{d\tau} \; \delta^4(x - z) \quad , \tag{3.22}$$

and so it appears that $\partial^\mu \phi_{\mu\nu} \neq 0$, in contradiction to the field equation (3.3). This problem may be avoided by supposing that equation (3.9) fails on a two-dimensional sheet in space-time. If so, then we can rewrite (3.9) formally as

$$\Box\phi_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + C_{\mu\nu} \quad , \tag{3.23}$$

where $C_{\mu\nu}$ is some function, expressible in terms of Dirac δ-functions, which is nonvanishing only on the sheet (or sheets). It will then be possible to satisfy (3.3) provided that $C_{\mu\nu}$ satisfies

$$\Box B_\nu + \partial^\mu C_{\mu\nu} = 0 \quad . \tag{3.24}$$

If B_μ has the form (3.20), then we may take $C_{\mu\nu}$ to be

$$C_{\mu\nu}(x) = \iint d\tau_0 \, d\tau_1 \left(\frac{\partial y_\mu}{\partial \tau_0} \frac{\partial y_\nu}{\partial \tau_1} - \frac{\partial y_\mu}{\partial \tau_1} \frac{\partial y_\nu}{\partial \tau_0} \right) \delta^4(x - y(\tau_0, \tau_1)) \quad . \tag{3.25}$$

Here $y_\mu(\tau_0, \tau_1)$ is a general point on the sheet, which is described by allowing the two parameters τ_0 and τ_1 to vary. We suppose that

$$y_\mu(\tau_0, 0) = z_\mu(\tau) \quad , \tag{3.26}$$

i.e., $\tau_1 = 0$ on the world-line of the point charge, which lies on the boundary of the sheet. We choose the parameters so that τ_1 ranges from 0 to ∞, and in doing so maps out, for fixed τ_0, a string running from the world-line to infinity. We allow τ_0 to go from $-\infty$ to ∞ as one goes from infinite past to infinite future. It is now easily verified that $C_{\mu\nu}$ satisfies (3.24) by using Stokes' theorem,

$$\iint d\tau_0 d\tau_1 \left(\frac{\partial F}{\partial \tau_0} \frac{\partial G}{\partial \tau_1} - \frac{\partial F}{\partial \tau_1} \frac{\partial G}{\partial \tau_0} \right) = \int F \left(\frac{\partial G}{\partial \tau_0} d\tau_0 + \frac{\partial G}{\partial \tau_1} \partial \tau_1 \right) , \tag{3.27}$$

for any two functions F and G on the sheet, and the relation (3.26). For, the integrand in the second term in (3.24) reduces to a curl, and the only part of the boundary of the sheet not at infinity is the world-line z_μ. This construction of the singular sheets $y_\mu(\tau_0, \tau_1)$ was first given by Dirac[8] in his theory of magnetic poles, where he found a similar problem arose in trying to find solutions to his theory that described both electric and magnetic charges.

In this paper we wish to construct solutions to (3.12) for the vector field B_μ which contain singularities on various different world-lines. The specific form one chooses for B_μ determines the character of the equations of motion that one can derive for the singular points. We shall limit ourselves here to the simplest type of solution which yields physically interesting equations of motion. Let the fundamental solution of (3.12), the retarded potential of a point moving along a world-line $z_\mu^{(p)}(\tau^{(p)})$, be denoted by $b_\mu^{(p)}$, i.e.,

$$b_\mu^{(p)}(x) = \int d\tau^{(p)} \frac{dz_\mu^{(p)}}{d\tau^{(p)}} D_{ret}(x - z^{(p)}) \quad . \tag{3.28}$$

We then take the field B_μ to have the form,

$$B_\mu(x) = \sum_p \left(\frac{g'^{(p)}}{\tilde{\ell}^2} + g^{(p)}\Box\right) b_\mu^{(p)}(x) \quad , \tag{3.29}$$

containing singularities on several world-lines $z_\mu^{(p)}$. Here $g^{(p)}$, $g'^{(p)}$, and $\tilde{\ell}^2$ are arbitrary constants.[14] The length $\tilde{\ell}$ is introduced so that $g^{(p)}$ and $g'^{(p)}$ have the same dimensions. The second term in (3.29) is to be understood by the formal expression

$$\Box b_\mu^{(p)}(x) = \frac{dz_\mu^{(p)}}{d\tau^{(p)}} \delta^4(x-z^{(p)}) \quad ; \tag{3.30}$$

in other words, it is nonvanishing only on the world-lines $z_\mu^{(p)}$. Equation (3.29) implies that we must take for $C_{\mu\nu}$ in (3.23) the expression

$$C_{\mu\nu}(x) = \sum_p \left(\frac{g'^{(p)}}{\tilde{\ell}^2} + g^{(p)}\Box\right) c_{\mu\nu}^{(p)}(x) \quad , \tag{3.31}$$

where

$$c_{\mu\nu}^{(p)}(x) = \iint d\tau_0^{(p)} d\tau_1^{(p)} \epsilon_{\alpha\beta} \frac{\partial y_\mu^{(p)}}{\partial \tau_\alpha^{(p)}} \frac{\partial y_\nu^{(p)}}{\partial \tau_\beta^{(p)}} \delta^4(x - y^{(p)}(\tau_0^{(p)}, \tau_1^{(p)})) \quad . \tag{3.32}$$

Here we have used the two-dimensional antisymmetric symbol $\epsilon_{\alpha\beta}$, with $\epsilon_{01} = -\epsilon_{10} = 1$. We should emphasize that both terms in (3.29), with coefficients

$g^{(p)}$ and $g'^{(p)}$, will be essential if we are to obtain the Lorentz force term in the equation of motion. The analogy between (3.29) for B_μ and (3.15) for A_μ is clear.

Using (3.29) and (3.31) we may integrate equation (3.23) for $\phi_{\mu\nu}$. If we write

$$\phi_{\mu\nu}(x) = \sum_p \left(\frac{g'^{(p)}}{\tilde{\ell}^2} + g^{(p)} \Box \right) \psi_{\mu\nu}^{(p)}(x) \quad , \tag{3.33}$$

then a solution of (3.23) is given by

$$\psi_{\mu\nu}^{(p)}(x) = \int d^4x' \, D_{ret}(x - x') \left\{ \partial'_\mu b_\nu^{(p)}(x') - \partial'_\nu b_\mu^{(p)}(x') + c_{\mu\nu}^{(p)}(x') \right\} . \tag{3.34}$$

Here ∂'_μ denotes $\frac{\partial}{\partial x'^\mu}$. We may, of course, add to (3.33) any solution of the homogeneous equation $\Box\phi_{\mu\nu} = 0$. We have, in effect, already done so in taking for B_μ the expression (3.29). For, the second term in (3.33) is a solution of the homogeneous equation except at points on the sheets $y_\mu^{(p)}$ and the world-lines $z_\mu^{(p)}$ on their boundaries. For simplicity, we do not wish to consider more general solutions. If the point x_μ is not on one of the sheets $y_\mu^{(p)}$, then we may write $\phi_{\mu\nu}(x)$ as

$$\phi_{\mu\nu}(x) = \sum_p \left\{ g^{(p)} \left(\partial_\mu b_\nu^{(p)}(x) - \partial_\nu b_\mu^{(p)}(x) \right) + \frac{g'^{(p)}}{\tilde{\ell}^2} \psi_{\mu\nu}^{(p)}(x) \right\} \quad , \tag{3.35}$$

since $c_{\mu\nu}^{(p)}(x) = 0$ in this case. It is worth noting that $\phi_{\mu\nu}$ then differs from the usual expression for the electromagnetic field of a point charge only by the second term, which involves the length parameter $\tilde{\ell}$. Note also that for such points x_μ,

$$\Box \psi_{\mu\nu}^{(p)}(x) = \partial_\mu b_\nu^{(p)}(x) - \partial_\nu b_\mu^{(p)}(x) \quad . \tag{3.36}$$

By using (3.28) and (3.32) in the solution (3.34) for $\psi^{(p)}_{\mu\nu}$, we may express it in the form

$$\psi_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} (\partial^\rho C^\sigma - \partial^\sigma C^\rho) \quad , \tag{3.37}$$

with

$$C_\mu(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \iint d\tau_0 \, d\tau_1 \, \frac{\partial y^\rho}{\partial \tau_0} \frac{\partial y^\sigma}{\partial \tau_1} \partial^\nu \Delta_{ret}(x-y) \quad , \tag{3.38}$$

if we drop the superscripts (p), for clarity. Δ_{ret} is given by

$$\Delta_{ret}(x-y) = \int d^4x' \, D_{ret}(x-x') \, D_{ret}(x'-y) = \frac{1}{8\pi} \theta(x_0-y_0) \, \theta((x-y)^2) \quad , \tag{3.39}$$

where we have introduced the θ-function, $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$, and used the notation $a^2=a^\mu a_\mu$. The results (3.37 - 3.38) may be obtained by some manipulations involving Stokes' theorem applied to the sheet y_μ. Equation (3.37) explicitly exhibits the fact that $\partial^\mu \psi_{\mu\nu}=0$, which must be true if the field equation (3.3) is to be satisfied. In fact, for our solutions, $\partial^\mu \phi_{\mu\nu}(x)=0$ at all points x_μ, without exception.

It is useful at this point to look at explicit expressions for the integral for $\psi^{(p)}_{\mu\nu}$ in (3.34) or (3.37 - 3.38), in some special cases, in order to see how the presence of the more complicated singularities we have introduced affects the field $\phi_{\mu\nu}$. We shall drop the superscripts (p) in $\psi^{(p)}_{\mu\nu}$ in this discussion, for clarity. First, recall that the integral for the usual retarded potential (3.28) may be written as

$$\begin{aligned} b_\mu(x) &= \int d\tau \, \frac{dz_\mu}{d\tau} \frac{1}{2\pi} \, \theta(x_0-z_0) \, \delta((x-z)^2) \\ &= \frac{1}{4\pi} \left[\frac{dz_\mu}{d\tau} \Big/ \frac{dz}{d\tau} \cdot (x-z) \right]_{ret} \quad . \end{aligned} \tag{3.40}$$

The notation $[\ \ ]_{ret}$ means that the quantities in the bracket are to be evaluated at the "retarded point", $(x-z)^2=0$, $x_0-z_0>0$. Also, $a\cdot b\equiv a^\mu b_\mu$ for any two vectors a_μ and b_μ. We may write (3.34) for the field $\psi_{\mu\nu}$ as

$$\psi_{\mu\nu} = \partial_\mu b'_\nu - \partial_\nu b'_\mu + c'_{\mu\nu} \quad , \tag{3.41}$$

with

$$b'_\mu(x) = \int d\tau \, \frac{dz_\mu}{d\tau} \, \frac{1}{8\pi} \, \theta(x_0-z_0) \, \theta((x-z)^2) \tag{3.42}$$

and

$$c'_{\mu\nu}(x) = \iint d\tau_0 \, d\tau_1 \, \epsilon_{\alpha\beta} \, \frac{\partial y_\mu}{\partial\tau_\alpha} \, \frac{\partial y_\nu}{\partial\tau_\beta} \, \frac{1}{2\pi} \, \theta(x_0-y_0) \, \delta((x-y)^2) \quad . \tag{3.43}$$

Since $\partial_\mu \theta(x^2) = 2x_\mu \delta(x^2)$, we find that

$$\partial_\mu b'_\nu(x) - \partial_\nu b'_\mu(x) = \frac{1}{8\pi} \left[\left((x-z)_\mu \, \frac{dz_\nu}{d\tau} - (x-z)_\nu \, \frac{dz_\mu}{d\tau} \right) \Big/ \frac{dz}{d\tau}\cdot(x-z) \right]_{ret} . \tag{3.44}$$

In (3.43) we use the δ-function to perform the integration over τ_0. Then $c'_{\mu\nu}$ becomes

$$c'_{\mu\nu}(x) = \frac{1}{4\pi} \int d\tau_1 \left[\epsilon_{\alpha\beta} \, \frac{\partial y_\mu}{\partial\tau_\alpha} \, \frac{\partial y_\nu}{\partial\tau_\beta} \, \frac{1}{\frac{\partial y}{\partial\tau_0}\cdot(x-y)} \right]_{ret} . \tag{3.45}$$

Here the integral over τ_1 is to be taken along the curve formed by the intersection of the sheet $y_\mu(\tau_0, \tau_1)$ and the backward light cone from the point x_μ, i.e., the points y_μ which satisfy $(x-y)^2=0$, $x_0-y_0>0$. To evaluate the integral (3.45) we must be given an explicit expression for the sheet $y_\mu(\tau_0, \tau_1)$. Then the retarded condition must be solved for τ_0, so that the integrand is expressed as a function of only τ_1 and x_μ. Hence, this integral will, in general, be very complicated.

To take the simplest example of these formulas, consider a single point charge at rest at the origin in space. Then $z_\mu(\tau)=(\tau, \vec{0})$, and $dz_\mu/d\tau$ is a unit vector pointing along the time axis. Here we have used the notation $a^\mu = (a^0, \vec{a})$ to express the four-dimensional vector a^μ in terms of a time component a^0 and a spatial vector $\vec{a}$. Note that $a_\mu = (a^0, -\vec{a})$. Let us suppose we can choose the sheet $y_\mu(\tau_0, \tau_1)$ so that $\tau_0 = \tau$ for all τ_1, and so that as τ_1 varies from 0 to ∞, with τ_0 fixed, a string is mapped out along a straight line from the origin to infinity in a direction given by the spatial unit vector $\hat{n}$ $(n^\mu n_\mu = -\hat{n}\cdot\hat{n} = -1)$. Furthermore, we suppose the direction $\hat{n}$ is independent of τ_0. Thus, $y_\mu(\tau_0, \tau_1) = (\tau_0, -\tau_1\hat{n})$. It is then obvious that the only nonzero components of $\psi_{\mu\nu}$ will be ψ_{0i}, $(i = 1, 2, 3)$. Using (3.44 - 3.45), or (3.37 - 3.38), and denoting $\psi_{0i} \equiv (\vec{\psi})_i$, we find

$$\vec{\psi}(x) = \frac{1}{8\pi}\int_0^\infty d\tau_1 \left\{(\vec{x} - \tau_1\hat{n})\frac{d}{d\tau_1}\frac{1}{|\vec{x} - \tau_1\hat{n}|} - \frac{1}{|\vec{x} - \tau_1\hat{n}|}\frac{d}{d\tau_1}(\vec{x} - \tau_1\hat{n})\right\} . \tag{3.46}$$

Note that $\vec{\psi}$ is independent of the time coordinate x^0, as expected.

The integral over τ_1 in (3.46) diverges logarithmically as $\tau_1 \to \infty$. Hence, the function $\vec{\psi}$, and therefore also $\phi_{\mu\nu}$, are not well-defined for a single point charge with its associated string. It may be possible to find some way of consistently subtracting off the divergent part of this integral, but we shall avoid the problem in another manner. If we have a solution for $\phi_{\mu\nu}$ of the form (3.33) which contains any number of pairs of singular world-lines $z_\mu^{(p_1)}$ and $z_\mu^{(p_2)}$ with opposite charges, i.e., $g^{(p_1)} = -g^{(p_2)}$ and $g'^{(p_1)} = -g'^{(p_2)}$, then we can always arrange, for fixed time, that the strings run between these pairs of oppositely charged singularities, and as a result have finite lengths. The integrals over the strings will then be well-defined. It is possible to have all strings be of finite length only if the total charge $\sum_p g'^{(p)}$ vanishes. This can be seen by integrating both sides of (3.19),

$$\epsilon^{\mu\nu\rho\sigma} \Box (\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \partial^\mu B^\nu - \partial^\nu B^\mu \quad , \tag{3.19}$$

over any closed surface enclosing all the singularities of B_μ at any fixed time, with B_μ given by (3.29). The left-hand side of (3.19) vanishes because of Stokes' theorem, since it is a curl, but the right-hand side is zero only if $\sum_p g'^{(p)} = 0$, as is easily verified. This means that if $\sum_p g'^{(p)} \neq 0$, then (3.19) must fail at at least one point on the surface; or, in other words, a string (or strings) must pass through the surface. Since the surface may be arbitrarily large, at least one of the strings must then extend to infinity. Of course, if $\sum_p g'^{(p)} = 0$, then no contradiction arises. There is no similar restriction on $\sum_p g^{(p)}$, but we shall see that the physically interesting solutions are those for which $g^{(p)}$ and $g'^{(p)}$ are proportional. In the following we shall consider solutions for $\phi_{\mu\nu}$, of the form (3.33), containing only such pairs of opposite charges, so that $\phi_{\mu\nu}$ will be well-defined. In formal manipulations there will often be no need to make explicit reference to this fact, but the assumption will be implicit throughout.

If we consider, as a second example, a solution which contains a single pair of world-lines $z_\mu^{(1)}$ and $z_\mu^{(2)}$, carrying opposite charges $g^{(1)} = -g^{(2)}$ and $g'^{(1)} = -g'^{(2)}$, then we expect that the charges will, in general, not remain at rest. Since a construction of the sheet $y_\mu(\tau_0, \tau_1)$ lying between the world-lines requires a knowledge of the motion of the charges, we shall limit ourselves here to the static approximation, to make things as simple as possible. We wish then to calculate the finite integral

$$g'^{(1)} \psi_{\mu\nu}^{(1)} + g'^{(2)} \psi_{\mu\nu}^{(2)} = g'^{(1)} \left(\psi_{\mu\nu}^{(1)} - \psi_{\mu\nu}^{(2)} \right) \quad ,$$

for two infinitely massive point charges at the ends of a string. Let us suppose the two charges are at points in space $a\hat{n}$ and $-a\hat{n}$, so that $z_\mu^{(1)}(\tau) = (\tau, -a\hat{n})$ and $z_\mu^{(2)}(\tau) = (\tau, a\hat{n})$, where τ is the common time. By symmetry, the string must run in the direction $\hat{n}$ from $-a$ to a, as τ_1 varies with τ_0 fixed, and its position is of course independent of τ_0 in the static approximation, since we can choose $\tau_0 = \tau$. Thus, the string is given by $y_\mu(\tau_0, \tau_1) = (\tau_0, -\tau_1\hat{n})$, with $-a < \tau_1 < a$. We then find that the only nonzero components of $\psi_{\mu\nu}^{(1)} - \psi_{\mu\nu}^{(2)}$ are $\psi_{0i}^{(1)} - \psi_{0i}^{(2)} \equiv (\vec{\psi}^{(1)} - \vec{\psi}^{(2)})_i$, $(i = 1, 2, 3)$, and are given by the integral

$$\vec{\psi}^{(1)}(x) - \vec{\psi}^{(2)}(x) = -\frac{1}{8\pi}\int_{-a}^{a} d\tau_1 \left\{ (\vec{x} - \tau_1\hat{n}) \frac{d}{d\tau_1} \frac{1}{|\vec{x} - \tau_1\hat{n}|} - \frac{1}{|\vec{x} - \tau_1\hat{n}|} \frac{d}{d\tau_1} (\vec{x} - \tau_1\hat{n}) \right\} \tag{3.47}$$

The integral in (3.47) is easily performed, so we see that

$$-8\pi\,(\vec{\psi}^{(1)}(x) - \vec{\psi}^{(2)}(x)) = \frac{\vec{x} - a\hat{n}}{|\vec{x} - a\hat{n}|} - \frac{\vec{x} + a\hat{n}}{|\vec{x} + a\hat{n}|} + 2\hat{n}\log\frac{-\hat{n}\cdot(\vec{x} - a\hat{n}) + |\vec{x} - a\hat{n}|}{-\hat{n}\cdot(\vec{x} + a\hat{n}) + |\vec{x} + a\hat{n}|}\,. \tag{3.48}$$

The expression (3.48) will prove useful in the discussion in the Appendix concerning self-field terms in the equations of motion. We may, finally, write down an explicit result for the field $\phi_{\mu\nu}$ of two opposite charges, in the static approximation. From (3.35), we see that the nonzero components $\phi_{0i} \equiv (\vec{\phi})_i$ are given by

$$\vec{\phi}(x) = -\frac{g^{(1)}}{4\pi}\left\{ \frac{\vec{x} - a\hat{n}}{|\vec{x} - a\hat{n}|^3} - \frac{\vec{x} + a\hat{n}}{|\vec{x} + a\hat{n}|^3} \right\}$$
$$-\frac{g'^{(1)}}{8\pi\tilde{\ell}^2}\left\{ \frac{\vec{x} - a\hat{n}}{|\vec{x} - a\hat{n}|} - \frac{\vec{x} + a\hat{n}}{|\vec{x} + a\hat{n}|} + 2\hat{n}\log\frac{-\hat{n}\cdot(\vec{x} - a\hat{n}) + |\vec{x} - a\hat{n}|}{-n\cdot(\vec{x} + a\hat{n}) + |\vec{x} + a\hat{n}|} \right\}. \tag{3.49}$$

If the singularities at an and -an are interpreted as electric charges, then the first term in (3.49) is just the usual Coulomb field. The presence of the string is apparent in the second term, because of the logarithmic singularity.

C. The Gravitational Field and The Equations of Motion

If we insert the solution (3.33), of the electromagnetic field equations for $\phi_{\mu\nu}$, into the expression for $t_{\mu\nu}$ given by (3.7), then we may attempt to integrate the inhomogeneous equation (3.6) to obtain the "gravitational" field $\gamma_{\mu\nu}$. The equations of motion arise as consistency conditions for a solution of (3.6), together with the harmonic condition (3.5), to exist. To see this, suppose that we write the formal integral of (3.6) as

$$\gamma_{\mu\nu}(x) = \gamma^{H}_{\mu\nu}(x) + \int d^4x' \, D_{ret}(x-x') \, t_{\mu\nu}(x') \tag{3.50}$$

where $\gamma^{H}_{\mu\nu}$ is a solution of the homogeneous equation

$$\Box \gamma^{H}_{\mu\nu} = 0 \quad . \tag{3.51}$$

The expression (3.50) for $\gamma_{\mu\nu}$ indeed satisfies $\Box\gamma_{\mu\nu} = t_{\mu\nu}$, but it must also satisfy the harmonic condition $\partial^{\mu}\gamma_{\mu\nu} = 0$. Therefore, we require that the equation

$$0 = \partial^{\mu}\gamma^{H}_{\mu\nu}(x) + \int d^4x' \, D_{ret}(x-x') \, \partial'^{\mu} \, t_{\mu\nu}(x') \tag{3.52}$$

must hold. We have here used Gauss' theorem[15] to obtain (3.52). The divergence of $t_{\mu\nu}$ may be calculated from the definition (3.7). The calculation gives

$$\partial^{\mu} t_{\mu\nu} = \frac{1}{2} \, \phi^{\rho\sigma} \Box \partial_{[\nu} \, \phi_{\rho\sigma]} \, , \tag{3.53}$$

if we use the fact that the equation $\partial^\mu \phi_{\mu\nu} = 0$ is satisfied everywhere, as is easily verified from (3.33) and (3.37). From (3.23), we then have

$$\partial^\mu t_{\mu\nu} = \frac{1}{2} \phi^{\rho\sigma} \partial_{[\nu} C_{\rho\sigma]} \quad . \tag{3.54}$$

In other words, $\partial^\mu t_{\mu\nu}$ is nonvanishing only on the sheets $y_\mu^{(p)}$. The δ-functions in (3.31 - 3.32) provide a formal expression of this fact. The δ-function technique we are using is a convenient method for obtaining most of the terms in the equations of motion, but it leads to some ambiguities when considering self-field contributions, as we shall see. The relation (3.54) allows us to write (3.52) in the form

$$\partial^\mu \gamma^H_{\mu\nu}(x) = \sum_p \iint d\tau_0^{(p)} \, d\tau_1^{(p)} \, K_\nu^{(p)}(x;y^{(p)}) \quad , \tag{3.55}$$

where

$$K_\nu(x;y) = \frac{1}{2} \epsilon_{\alpha\beta} \frac{\partial y_{[\rho}}{\partial \tau_\alpha} \frac{\partial y_\sigma}{\partial \tau_\beta} \partial^y_{\nu]} \left(\frac{g'}{\gamma^2} + g\Box_y \right) \left\{ \phi^{\rho\sigma}(y) \, D_{ret}(x-y) \right\} \quad , \tag{3.56}$$

by using (3.31 - 3.32). Here $\partial^y_\mu = \frac{\partial}{\partial y^\mu}$, $\Box_y = \eta^{\mu\nu} \partial^y_\mu \partial^y_\nu$, and we have suppressed the superscripts (p), for clarity, in (3.56). As usual, the square brackets imply antisymmetrization with respect to ρ, σ, and ν.

We shall permit solutions $\gamma^H_{\mu\nu}$ of the homogeneous equation (3.51) which are singular on the sheets $y_\mu^{(p)}$. Thus, we suppose $\gamma^H_{\mu\nu}$ has the general form

$$\gamma^H_{\mu\nu}(x) = \sum_p \iint d\tau_0^{(p)} \, d\tau_1^{(p)} \, a^{(p)}_{\mu\nu}(y^{(p)}) \, D_{ret}(x-y^{(p)})$$

$$+ \sum_{i=1}^{\infty} \sum_p \iint d\tau_0^{(p)} \, d\tau_1^{(p)} \, a^{(p)}_{\mu\nu,\rho_1\cdots\rho_i}(y^{(p)}) \, \partial^{\rho_1} \cdots \partial^{\rho_i} D_{ret}(x-y^{(p)}) \quad , \tag{3.57}$$

where $a^{(p)}_{\mu\nu}$ and $a^{(p)}_{\mu\nu,\rho_1\cdots\rho_i}$ are both arbitrary functions of $y_\mu^{(p)}(\tau_0^{(p)}, \tau_1^{(p)})$, except that both are symmetric in μ and ν, and the latter is also completely symmetric in $\rho_1 \ldots \rho_i$. We are interested in finding the restrictions on the sheets $y_\mu^{(p)}(\tau_0^{(p)}, \tau_1^{(p)})$, and on the world-lines $z_\mu^{(p)}(\tau^{(p)})$ on their boundaries, which are imposed by requiring that the harmonic condition, in the form of equation (3.55), be satisfied. If this condition can be satisfied, we have then constructed a solution to the complete set of approximate field equations (3.3 - 3.7). Now, many terms on the right-hand side of (3.55) may be expressed in the same form as the terms on the left-hand side, if we understand $\gamma_{\mu\nu}^{H}$ to have the general form (3.57). Since the functions $a_{\mu\nu}^{(p)}$ and $a_{\mu\nu,\rho_1\ldots\rho_i}^{(p)}$ are arbitrary, we may combine such terms so as to define new arbitrary functions $a_{\mu\nu}^{\prime(p)}$ and $a_{\mu\nu,\rho_1\ldots\rho_i}^{\prime(p)}$ (which must, however, have the same symmetry properties). This then defines a new solution to the homogeneous equation $\gamma_{\mu\nu}^{\prime H}$. We shall obtain nontrivial restrictions only if such manipulations do not suffice to eliminate all the terms on the right-hand side of (3.55). By carrying out the above steps,[16] we can bring equation (3.55) into the form,

$$\partial^\mu \gamma_{\mu\nu}^{\prime H}(x) = \sum_p \iint d\tau_0^{(p)} d\tau_1^{(p)} \, K_\nu^{\prime(p)}(x;y^{(p)}) \ , \tag{3.58}$$

where

$$K_\nu^{\prime(p)}(x;y^{(p)}) = \epsilon_{\alpha\beta} \left\{ \frac{1}{2}\, \partial_{[\nu}\chi_{\rho\sigma]}^{(p)}(y^{(p)}) \, \frac{\partial y^{(p)\rho}}{\partial \tau_\alpha^{(p)}} \, \frac{\partial y^{(p)\sigma}}{\partial \tau_\beta^{(p)}} \, D_{ret}(x-y^{(p)}) \right.$$
$$\left. - \frac{\partial y^{(p)\rho}}{\partial \tau_\alpha^{(p)}} \, \frac{\partial}{\partial \tau_\beta^{(p)}} \left[\chi_{\nu\rho}^{(p)}(y^{(p)}) \, D_{ret}(x-y^{(p)}) \right] \right\} \ . \tag{3.59}$$

Here we have defined $\chi_{\mu\nu}^{(p)}$ by

$$\chi_{\mu\nu}^{(p)} = \left(\frac{g^{\prime(p)}}{\tilde{\ell}^2} + g^{(p)} \Box \right) \phi_{\mu\nu} \ . \tag{3.60}$$

The newly-defined $\gamma_{\mu\nu}^{\prime H}$ now contains those terms on the right-hand side of (3.55) which can be expressed as the divergence of a solution of the homogeneous equation (3.51). The remaining terms, in $K'^{(p)}_{\nu}$, will yield the "interaction" terms in the equations of motion. The second term in (3.59) has the form of a curl, so we may use Stokes' theorem to write its contribution to (3.58) as an integral over the boundary of the sheet. This brings the harmonic condition into its final form,

$$\partial^{\mu} \gamma^{H}_{\mu\nu} = \sum_{p} \int d\tau^{(p)} F^{(p)}_{\nu}(z^{(p)}) D_{ret}(x-z^{(p)}) + \sum_{p} \iint d\tau_0^{(p)} d\tau_1^{(p)} G^{(p)}_{\nu}(y^{(p)}) D_{ret}(x-y^{(p)}) \quad , \tag{3.61}$$

where

$$F^{(p)}_{\nu}(z^{(p)}) = \frac{dz^{(p)\rho}}{d\tau^{(p)}} \chi^{(p)}_{\nu\rho}(z^{(p)}) \tag{3.62}$$

and

$$G^{(p)}_{\nu}(y^{(p)}) = \frac{1}{2} \epsilon_{\alpha\beta} \frac{\partial y^{(p)\rho}}{\partial \tau^{(p)}_{\alpha}} \frac{\partial y^{(p)\sigma}}{\partial \tau^{(p)}_{\beta}} \partial_{[\nu} \chi^{(p)}_{\rho\sigma]}(y^{(p)}) \quad . \tag{3.63}$$

In the above expressions the field $\phi_{\mu\nu}$ is evaluated at a point on the sheet $y^{(p)}_{\mu}(\tau_0^{(p)}, \tau_1^{(p)})$, or on the world-line $z^{(p)}_{\mu}(\tau^{(p)})$ on its boundary. At such points we may separate $\phi_{\mu\nu}$ into two parts,

$$\phi_{\mu\nu} = \phi^{(p)ext}_{\mu\nu} + \phi^{(p)self}_{\mu\nu} \quad , \tag{3.64}$$

with

$$\phi^{(p)ext}_{\mu\nu} = \sum_{q \neq p} \left\{ g^{(q)} \left(\partial_{\mu} b^{(q)}_{\nu} - \partial_{\nu} b^{(q)}_{\mu} \right) + \frac{g'^{(q)}}{\tilde{\ell}^2} \psi^{(q)}_{\mu\nu} \right\} \tag{3.65}$$

and

$$\phi^{(p)self}_{\mu\nu} = \left(\frac{g'^{(p)}}{\tilde{\ell}^2} + g^{(p)} \Box \right) \psi^{(p)}_{\mu\nu} \quad . \tag{3.66}$$

The sum in $\phi_{\mu\nu}^{(p)ext}$ runs over all sheets and world-lines (q) except that of "particle" (p). In the case of pairs of opposite charges each connected by a single string, so that both world-lines of a pair lie on the boundaries of the same sheet, we must consider the two charges together in the field $\psi_{\mu\nu}^{(p)}$ in (3.66). Therefore the sum over (q) in (3.65) runs only over the other pairs of charges. This will be implicit in the following formal manipulations. The form (3.65) for $\phi_{\mu\nu}^{(p)ext}$ applies if the sheets $y_\mu^{(p)}$ and $y_\mu^{(q)}$, and the world-lines $z_\mu^{(p)}$ and $z_\mu^{(q)}$, have no point in common, as we shall assume. By inserting these expressions into (3.62) and (3.63), we may define similar decompositions for $F_\mu^{(p)}$ and $G_\mu^{(p)}$,

$$F_\mu^{(p)} = F_\mu^{(p)ext} + F_\mu^{(p)self} \tag{3.67}$$

and

$$G_\mu^{(p)} = G_\mu^{(p)ext} + G_\mu^{(p)self} \quad , \tag{3.68}$$

which we need not write out explicitly.

The field $\phi_{\mu\nu}^{(p)self}$ is not well-defined at the points $y_\mu^{(p)}$ or $z_\mu^{(p)}$, so its contribution to (3.61), the harmonic condition, is ambiguous. To evaluate this contribution we must use a method other than the formal technique given above. For example, we may exclude from the volume integration in (3.50) regions containing the sheets $y_\mu^{(p)}$ and world-lines $z_\mu^{(p)}$. Then (3.52) is replaced by

$$0 = \partial^\mu \gamma_{\mu\nu}^H + \sum_p \int d^3 S'_{(p)}\, n'^\mu_{(p)}\, t_{\mu\nu}(x')\, D_{ret}(x-x') \quad , \tag{3.69}$$

where $d^3 S'_{(p)}$ is an element, at the point x'_μ, of the surface $S'_{(p)}$ surrounding the sheet $y_\mu^{(p)}$ and its boundary curve $z_\mu^{(p)}$, and $n'^\mu_{(p)}$ is the normal vector to the surface. This follows from Gauss' theorem, since $\partial^\mu t_{\mu\nu} = 0$ except on the sheets, from (3.54). Consideration of the singularities that arise in this surface integral as the surface $S'_{(p)}$ is allowed to approach the sheet $y_\mu^{(p)}$, shows that in the self-field terms in $F_\mu^{(p)}$ and $G_\mu^{(p)}$ we may use the formal expressions,

$$\Box\,\psi_{\mu\nu}^{(p)}(y^{(p)}) = \partial_\mu b_\nu^{(p)}(y^{(p)}) - \partial_\nu b_\mu^{(p)}(y^{(p)}) \tag{3.70}$$

and

$$\Box\,\Box\,\psi_{\mu\nu}^{(p)}(y^{(p)}) = 0 \quad , \tag{3.71}$$

and similarly for points $z_\mu^{(p)}$. In other words, ill-defined terms involving δ-functions on the right-hand sides of (3.70) and (3.71) may be ignored.[17] As a result, $F_\mu^{(p)\mathrm{self}}$ and $G_\mu^{(p)\mathrm{self}}$ take the form,

$$F_\mu^{(p)\mathrm{self}} = \frac{dz^{(p)\nu}}{d\tau^{(p)}} \left\{ 2\,\frac{g^{(p)}g'^{(p)}}{\tilde{\ell}^2} \left(\partial_\mu b_\nu^{(p)} - \partial_\nu b_\mu^{(p)}\right) + \left(\frac{g'^{(p)}}{\tilde{\ell}^2}\right)^2 \psi_{\mu\nu}^{(p)} \right\} \tag{3.72}$$

and

$$G_\mu^{(p)\mathrm{self}} = \frac{1}{2}\left(\frac{g'^{(p)}}{\tilde{\ell}^2}\right)^2 \epsilon_{\alpha\beta}\, \frac{\partial y^{(p)\rho}}{\partial \tau_\alpha^{(p)}}\, \frac{\partial y^{(p)\sigma}}{\partial \tau_\beta^{(p)}}\, \partial_{[\mu}\psi_{\rho\sigma]}^{(p)} \quad , \tag{3.73}$$

if we use (3.70 - 3.71) in (3.62 - 3.63), remembering that $\chi_{\mu\nu}^{(p)}$ is defined by (3.60).

Now, note that the term in $F_\mu^{(p)\mathrm{self}}$ with coefficient $2\,\frac{g^{(p)}g'^{(p)}}{\tilde{\ell}^2}$ has the same form as would have been obtained if we had used for $t_{\mu\nu}$, instead of (3.7), the Maxwell electromagnetic energy-momentum tensor, with the electromagnetic field given by the usual retarded field of a point charge. Hence, we may use the results of Dirac,[18] who examined the problem of the self-fields in this case and found that, except for the radiation reaction term,

$$4\pi\left(\partial_\mu b_\nu^{(p)} - \partial_\nu b_\mu^{(p)}\right)_{\mathrm{rad}} = \frac{2}{3}\left(\frac{d^3 z_\mu^{(p)}}{d\tau^{(p)3}}\,\frac{dz_\nu^{(p)}}{d\tau^{(p)}} - \frac{d^3 z_\nu^{(p)}}{d\tau^{(p)3}}\,\frac{dz_\mu^{(p)}}{d\tau^{(p)}}\right) \quad , \tag{3.74}$$

the contributions to $F_\mu^{(p)self}$ could be absorbed into the inertial term in the equation of motion. This is equivalent to saying that such contributions can be absorbed into $\partial^\mu \gamma'^{H}_{\mu\nu}$ in (3.61). The familiar expression (3.74) comes from half the difference of the retarded and advanced fields.

We have finally to consider the contributions to $F_\mu^{(p)self}$ and $G_\mu^{(p)self}$ with coefficients $\left(\frac{g'^{(p)}}{\tilde{\ell}^2}\right)^2$. Since the function $\psi_{\mu\nu}^{(p)}$ appears in them, they involve the sheets $y_\mu^{(p)}$ in an essential fashion. In an Appendix we show that in the special case of two opposite charges connected by a straight string, in the static approximation, these self-field terms vanish. We have not shown that this result is generally true, but we shall nevertheless assume, in the following, that we can ignore any self-field contributions of this type.

With this proviso, $F_\mu^{(p)}$ and $G_\mu^{(p)}$ now become

$$F_\mu^{(p)}(z^{(p)}) = F_\mu^{(p)ext} + \frac{2g^{(p)}g'^{(p)}}{4\pi\tilde{\ell}^2}\,\frac{2}{3}\left(\ddot{u}_\mu^{(p)} + \dot{u}^{(p)2}\,u_\mu^{(p)}\right) \qquad (3.75)$$

and

$$G_\mu^{(p)}(y^{(p)}) = G_\mu^{(p)ext}(y^{(p)}) \quad , \qquad (3.76)$$

if we introduce the notation

$$\frac{dz_\mu}{d\tau} = u_\mu \quad , \qquad \frac{d^2 z_\mu}{d\tau^2} = \dot{u}_\mu \quad , \qquad \frac{d^3 z_\mu}{d\tau^3} = \ddot{u}_\mu \quad ,$$

and use the fact that $u^\mu u_\mu = 1$. Furthermore, we have

$$F_\mu^{(p)ext} = u^{(p)\nu} \sum_{q\neq p} \left\{ \frac{g^{(p)}g'^{(q)}+g^{(q)}g'^{(p)}}{\tilde{\ell}^2}\left(\partial_\mu b_\nu^{(q)} - \partial_\nu b_\mu^{(q)}\right) + \frac{g'^{(p)}g'^{(q)}}{\tilde{\ell}^2}\,\frac{1}{\tilde{\ell}^2}\,\psi_{\mu\nu}^{(q)} \right\}$$

(3.77)

and

$$G_{\mu}^{(p)\mathrm{ext}} = \frac{1}{2}\,\epsilon_{\alpha\beta}\,\frac{\partial y^{(p)\rho}}{\partial\tau_{\alpha}^{(p)}}\,\frac{\partial y^{(p)\sigma}}{\partial\tau_{\beta}^{(p)}}\,\sum_{q\neq p}\,\frac{g'^{(p)}g'^{(q)}}{\tilde{\ell}^{4}}\,\partial_{[\mu}\psi_{\rho\sigma]}^{(q)} \quad . \qquad (3.78)$$

To obtain equations of motion we must now specify the arbitrary functions $a'^{(p)}_{\mu\nu}$ and $a'^{(p)}_{\mu\nu,\rho_1\ldots\rho_i}$ in the solution $\gamma'^{H}_{\mu\nu}$ of the homogeneous equation, which appears on the left-hand side of (3.58). These functions characterize the structure of the singularities on the sheets and on the world-lines on their boundaries. They may thus be thought of as relating to the internal structure of the point charges and their associated strings. We wish to consider here only the simplest possible type of point charge, characterized by a single parameter, $\mu^{(p)}$, related to its mass. Therefore, we shall suppose that the $a'^{(p)}_{\mu\nu,\rho_1\ldots\rho_i}$ are equal to zero for $i \geq 2$, and that $a'^{(p)}_{\mu\nu}$ and $a'^{(p)}_{\mu\nu,\rho}$ are such that

$$a'^{(p)}_{\mu\nu}(y^{(p)})\, D_{\mathrm{ret}}(x-y^{(p)}) + a'^{(p)}_{\mu\nu,\rho}(y^{(p)})\, \partial^{\rho}\, D_{\mathrm{ret}}(x-y^{(p)})$$

$$= -\frac{1}{2}\,\epsilon_{\alpha\beta}\,\frac{\partial y_{\mu}^{(p)}}{\partial\tau_{\alpha}^{(p)}}\,\frac{\partial}{\partial\tau_{\beta}^{(p)}}\left[\mu^{(p)}\,\frac{dy_{\nu}^{(p)}}{d\tau^{(p)}}\,D_{\mathrm{ret}}(x-y^{(p)})\right] + (\mu \leftrightarrow \nu) \quad , \qquad (3.79)$$

where $+ (\mu \leftrightarrow \nu)$ means that we must add the first term with μ and ν interchanged. For, by using Stokes' theorem on the sheet $y_{\mu}^{(p)}$, we then have

$$\gamma'^{H}_{\mu\nu}(x) = \sum_{p} \int d\tau^{(p)}\,\mu^{(p)}\,u_{\mu}^{(p)}\,u_{\nu}^{(p)}\,D_{\mathrm{ret}}(x-z^{(p)}) \quad . \qquad (3.80)$$

Here the parameter $\mu^{(p)}$ may, in general, be a function of $\tau^{(p)}$. If $\gamma'^{H}_{\mu\nu}$ has the form (3.80), we conclude that

$$\partial^{\mu}\gamma'^{H}_{\mu\nu}(x) = \sum_{p} \int d\tau^{(p)} \frac{d}{d\tau^{(p)}} \left(\mu^{(p)} u^{(p)}_{\nu}\right) D_{ret}(x-z^{(p)}) \quad , \tag{3.81}$$

after an integration by parts.

We are finally in a position to write down the equations of motion. In order that the initial expression, (3.50), for $\gamma_{\mu\nu}$ be a solution of the field equations, we must demand that $\partial^{\mu}\gamma_{\mu\nu}(x) = 0$, for all x_{μ}. From the above results, we see that this will be the case if we require that, on each world-line $z^{(p)}_{\mu}(\tau^{(p)})$, the following equation hold,

$$\frac{d}{d\tau^{(p)}} \left(\mu^{(p)} u^{(p)}_{\mu}\right) = F^{(p)ext}_{\mu} + \frac{2g^{(p)} g'^{(p)}}{4\pi\tilde{\ell}^{2}} \; \frac{2}{3} \left(\ddot{u}^{(p)}_{\mu} + \dot{u}^{(p)2} \; u^{(p)}_{\mu}\right) \quad , \tag{3.82}$$

and that on each sheet $y^{(p)}_{\mu}(\tau^{(p)}_{0}, \tau^{(p)}_{1})$, the equation

$$G^{(p)ext}_{\mu} = 0 \tag{3.83}$$

be satisfied. Here $F^{(p)ext}_{\mu}$ and $G^{(p)ext}_{\mu}$ are given by (3.77) and (3.78), respectively. Note that it follows from (3.82) that $\frac{d}{d\tau^{(p)}}\,\mu^{(p)} = 0$, i.e., the mass parameter is independent of the proper time $\tau^{(p)}$. Note also that the terms in (3.82) and (3.77) with coefficients $\frac{gg'}{\tilde{\ell}^{2}}$ have the same structure as the usual Lorentz force and radiation reaction force in the equations of motion of charged particles in classical electrodynamics.

A significant feature of the above result is that the consistency conditions can be separated into two parts, one being an equation which must be satisfied only on the world-lines $z^{(p)}_{\mu}$, while the other equation must be satisfied everywhere on the sheets $y^{(p)}_{\mu}$. The first condition, (3.82), may then be interpreted as an equation of motion for point masses, in the usual sense. The second condition (3.83), may be written in the form

$$\epsilon_{\alpha\beta} \frac{\partial y^{(p)\rho}}{\partial \tau_{\alpha}^{(p)}} \frac{\partial y^{(p)\sigma}}{\partial \tau_{\beta}^{(p)}} \sum_{q \neq p} g'^{(q)} \partial_{[\mu} \psi_{\rho\sigma]}^{(q)} (y^{(p)}) = 0 \quad . \tag{3.84}$$

Since $\partial^{\mu} \psi_{\mu\nu}^{(q)} = 0$ at all points x_{μ}, without exception, it follows that $\partial_{[\mu} \psi_{\rho\sigma]}^{(q)}$ cannot vanish identically; for this would then imply $\Box \psi_{\mu\nu}^{(q)} = 0$, in contradiction to (3.36). Therefore, the condition (3.84) is a non-trivial restriction on the sheets $y_{\mu}^{(p)}$. Since the field $\psi_{\mu\nu}^{(q)}$ involves an integral over the sheets $y_{\mu}^{(q)}$, this restriction is a quite complicated one. If it is to make any sense, it must be interpreted physically as a condition which determines where the strings must lie in space, and how they move in time to form the sheets. We have not been able to show that this condition can always be satisfied. However, it is easy to find some simple cases where this is possible. Suppose, for example, that we have two pairs of oppositely charged particles, lying on a common straight line in space, and moving along that line. We suppose the charges are arranged so that the two strings run along this same line between members of each pair, without overlapping. Then, since in space-time both sheets lie on the same two-dimensional flat surface, the condition (3.84) is trivially satisfied. In the static approximation, many other examples may be found. In view of this, it seems reasonable to assume that the condition $G_{\mu}^{(p)\text{ext}} = 0$ can be satisfied for some choice of the sheets $y_{\mu}^{(p)}$, even in more complicated situations.

In the following, we suppose that, by choosing the positions of the strings appropriately, the condition $G_{\mu}^{(p)\text{ext}} = 0$ will be satisfied. This situation should be contrasted with that in Dirac's theory of magnetic poles,[8] where the positions of the singular strings is completely arbitrary. This is clearly not the case in Einstein's theory. How restrictive the above condition is, and whether it also constrains the types of motions which are allowed on the world-lines $z_\mu^{(p)}$ on the boundaries of the sheets, remains as a subject for future study. If it should turn out that the presence of the sheets $y_\mu^{(p)}$ severely restricts the motions along the world-lines $z_\mu^{(p)}$, then this could be viewed as support for Einstein's original conjecture that magnetically charged currents do not exist in his theory, at least in a form resembling electrically charged currents.

However, if we suppose that the presence of the sheets, satisfying (3.84), does not restrict the world-lines $z_\mu^{(p)}$, then, on the basis of the equations of motion (3.82) for the singular points moving along these lines, it now seems possible to make a physical interpretation of Einstein's theory which is different from the conventional one exhibited in Johnson's approximate solutions. In fact, let us now suppose that the singularities of our solutions represent electrically charged point masses, rather than magnetically charged point masses. We wish to choose the various integration constants appearing in (3.82), and (3.77), so that (3.82) resembles the Lorentz-Dirac equation for particles with electric charges $e^{(p)}$ and masses $m^{(p)}$. According to the usual convention, the parameter $\mu^{(p)}$ is related to the mass $m^{(p)}$ by

$$\frac{\mu^{(p)}}{4\pi} = \frac{4G\,m^{(p)}}{c^2} \qquad , \tag{3.85}$$

where G is the gravitational constant and c is the speed of light. To have the Coulomb force be attractive for opposite charges and repulsive for like charges, we must choose [19]

$$g^{(p)} = g'^{(p)} \qquad . \tag{3.86}$$

Then, if we define the electric charge $e^{(p)}$ by

$$\frac{e^{(p)}}{c} = \sqrt{\frac{c^2}{2G}}\ \frac{g^{(p)}}{4\pi\tilde{\ell}} \quad , \tag{3.87}$$

the equation of motion (3.82) takes the form

$$m^{(p)}\ \dot{u}^{(p)}_{\mu} = \frac{e^{(p)}}{c}\ \tilde{F}^{(p)ext}_{\mu\nu}\ u^{(p)\nu} + \frac{2}{3}\left(\frac{e^{(p)}}{c}\right)^2 \left(\ddot{u}^{(p)}_{\mu} + \dot{u}^{(p)2}\ u^{(p)}_{\mu}\right) \quad . \tag{3.88}$$

Here we have defined $\tilde{F}^{(p)ext}_{\mu\nu}$ by

$$\tilde{F}^{(p)ext}_{\mu\nu} = \sum_{q\neq p} 4\pi\ \frac{e^{(q)}}{c}\left[\partial_{\mu}b^{(q)}_{\nu} - \partial_{\nu}b^{(q)}_{\mu} + \frac{1}{2\tilde{\ell}^2}\ \psi^{(q)}_{\mu\nu}\right] \quad . \tag{3.89}$$

Equations (3.88 - 3.89) differ from the usual Lorentz-Dirac equation[6] only through the extra term in $\tilde{F}^{(p)ext}_{\mu\nu}$ containing the field $\psi^{(q)}_{\mu\nu}$. The coefficient $\frac{1}{2\tilde{\ell}^2}$ reflects the fact that $\psi^{(q)}_{\mu\nu}$ and $\partial_{\mu}b^{(q)}_{\nu} - \partial_{\nu}b^{(q)}_{\mu}$ have different dimensions. In fact, the two fields are related by (3.36), which in this case is

$$\Box\psi^{(q)}_{\mu\nu}(z^{(p)}) = \partial_{\mu}b^{(q)}_{\nu}(z^{(p)}) - \partial_{\mu}b^{(q)}_{\nu}(z^{(p)}) \quad , \tag{3.90}$$

since we have assumed that the sheets $y^{(q)}_{\mu}$ and the world-lines $z^{(q)}_{\mu}$ do not intersect the world-line $z^{(p)}_{\mu}$.

Equations (3.84 - 3.90) contain the important new results of this paper. We should mention again that we have assumed that additional self-field contributions to (3.88), due to the string singularities, vanish, which we have not been able to prove true, in general.

To conclude this section, we state here the equations of motion Johnson obtained[13] using the solutions (3.14 - 3.17) for A_{μ}. We immediately note that resemblance between the expression for $\Box A_{\mu}$ obtained from (3.14 - 3.17) and the expression for B_{μ} of (3.29). If we now make the "conventional" assumption, that it is the singularities in A_{μ}, rather than B_{μ}, which represent electrically charged point masses, then the electric charge $e^{(p)}$ is defined in terms of the constants $f^{(p)}$, $f'^{(p)}$, and ℓ^2 in (3.15) by relations[20] analogous to (3.86 - 3.87), i.e.,

$$f^{(p)} = f'^{(p)} \tag{3.91}$$

and

$$\frac{e^{(p)}}{c} = \sqrt{\frac{c^2}{2G}\,\frac{f^{(p)}}{4\pi\ell}} \quad . \tag{3.92}$$

Then Johnson's equations of motion take the form

$$m^{(p)}\ \dot{u}_\mu^{(p)} = \frac{e^{(p)}}{c}\ F_{\mu\nu}^{(p)\mathrm{ext}}\ u^{(p)\nu} + \frac{2}{3}\left(\frac{e^{(p)}}{c}\right)^2 \left(\ddot{u}_\mu^{(p)} + \dot{u}^{(p)2}\ u_\mu^{(p)}\right) , \tag{3.93}$$

with $F_{\mu\nu}^{(p)\mathrm{ext}}$ given by

$$F_{\mu\nu}^{(p)\mathrm{ext}} = \sum_{q\neq p} 4\pi\,\frac{e^{(q)}}{c}\left[\partial_\mu a_\nu^{(q)} - \partial_\nu a_\mu^{(q)} + \frac{1}{2\ell^2}\ \chi_{\mu\nu}^{(q)}\right] . \tag{3.94}$$

Here $a_\mu^{(q)}$ is given by (3.16), and the field $\chi_{\mu\nu}^{(q)}$ has the form

$$\chi_{\mu\nu}^{(q)} = \partial_\mu a'^{(q)}_\nu - \partial_\nu a'^{(q)}_\mu \tag{3.95}$$

in terms of $a'^{(q)}_\mu$ of (3.17), and thus satisfies

$$\Box\chi_{\mu\nu}^{(q)} = \partial_\mu a_\nu^{(q)} - \partial_\nu a_\mu^{(q)} \quad . \tag{3.96}$$

Equation (3.96) is analogous to (3.90), but it holds at all points x_μ, unlike (3.90). By performing the integral in (3.17), one finds that (3.95) takes a form similar to (3.44),

$$\chi_{\mu\nu}(x) = \frac{1}{8\pi}\left[\left((x-z)_\mu \frac{dz_\nu}{d\tau} - (x-z)_\nu \frac{dz_\mu}{d\tau}\right) \Big/ \frac{dz}{d\tau}\cdot(x-z)\right]_{\mathrm{ret}} , \tag{3.97}$$

if we drop the superscripts (q). We should also mention that, for Johnson's solutions, it can easily be shown that there are no extra radiation reaction terms in (3.93), so no assumption is involved, as there is in (3.88).

The similarity between (3.86 - 3.89) in one case, and (3.91 - 3.94) in the other, is evident. The crucial difference between the two physical interpretations is that $\psi^{(p)}_{\mu\nu}$ satisfies

$$\partial^{\mu} \psi^{(p)}_{\mu\nu} = 0 \quad , \qquad \partial_{[\rho} \psi^{(p)}_{\mu\nu]} \neq 0 \quad , \tag{3.98}$$

but $\chi^{(p)}_{\mu\nu}$ satisfies

$$\partial^{\mu} \chi^{(p)}_{\mu\nu} \neq 0 \quad , \qquad \partial_{[\rho} \chi^{(p)}_{\mu\nu]} = 0 \quad . \tag{3.99}$$

The implications of these equations for experiment is the subject of the next section.

IV. OBSERVATIONAL TESTS OF THE THEORY

A. Modified Maxwell Equations

We have just seen that there are two sets of approximate solutions to the field equations of Einstein's theory, each containing singular points satisfying equations of motion that resemble the Lorentz-Dirac equation for charged point masses in classical electrodynamics. The structure of the theory implies that the relation of the two types of charges should be that of electric charge to magnetic charge. However, in view of the close similarity of the equations of motion in the two cases, it is not immediately obvious which type corresponds to electric charge. Since it is electric charge that is of physical interest, we shall here consider the two different interpretations for electric charge implied by the two sets of equations (3.86 - 3.89) and (3.91 - 3.94).

Both equations of motion, (3.88) and (3.93), differ from the Lorentz-Dirac equation in that the electromagnetic fields $\tilde{F}^{(p)ext}_{\mu\nu}$, in (3.88), and $F^{(p)ext}_{\mu\nu}$, in (3.93), each contain a term in addition to the usual field of Maxwell-Lorentz electrodynamics, due to the curl of the retarded potential of the "external" point charges. It is the additional terms which are of special interest here, because they imply that electromagnetic fields behave somewhat differently from the predictions of the Maxwell-Lorentz theory. We begin by writing down two sets of "modified" Maxwell equations, deduced from the equations of motion for the point charges, corresponding to the two different physical interpretations. We then suppose that these modified Maxwell equations in each case may be applied to the description of fields produced by macroscopic distributions of electric charge and current. This makes it possible to find predictions of the theory to test against observation. It is important, however, to keep in mind the "microscopic" origin of these equations, in choosing the proper solutions.

Consider first the case of Johnson's approximate solutions, which yield the equations of motion (3.93), when their singularities are assumed to be electric charges. In this section we shall refer to the field $F^{(p)ext}_{\mu\nu}$ appearing in (3.93) simply as $F_{\mu\nu}$. From (3.94), $F_{\mu\nu}$ can be written as the sum of two parts,

$$F_{\mu\nu} = F^{M}_{\mu\nu} + F^{E}_{\mu\nu} \tag{4.1}$$

where $F^{M}_{\mu\nu}$ is the usual electromagnetic field of Maxwell and Lorentz, and $F^{E}_{\mu\nu}$ gives the modifications due to Einstein's theory. (The labels "M" and "E" are intended to stand for "Maxwell" and "Einstein", and should not be confused with magnetic and electric.) From (3.94 - 3.95) and (3.16 - 3.17), we find that $F^{M}_{\mu\nu}$ and $F^{E}_{\mu\nu}$ satisfy the equations

$$\partial^{\mu} F^{M}_{\mu\nu} = 0 \qquad\qquad \partial_{[\rho} F^{M}_{\mu\nu]} = 0 \tag{4.2}$$

and

$$\Box F^{E}_{\mu\nu} = \frac{1}{2\ell^2} F^{M}_{\mu\nu} \qquad\qquad \partial_{[\rho} F^{E}_{\mu\nu]} = 0 \quad , \tag{4.3}$$

at all points except those on the singular world-lines. Note that, in general, $\partial^{\mu} F^{E}_{\mu\nu} \neq 0$. For, if both $\partial_{[\rho} F^{E}_{\mu\nu]} = 0$ and $\partial^{\mu} F^{E}_{\mu\nu} = 0$ held, then we would conclude that $\Box F^{E}_{\mu\nu} = 0$, which is not possible unless $F^{M}_{\mu\nu} = 0$. If we apply these equations to macroscopic phenomena, then according to the physical interpretation for electric charge assumed in this case, F^{M}_{0i} (i=1, 2, 3) is the usual Maxwell electric field, and F^{M}_{ij} (i, j=1, 2, 3) the usual Maxwell magnetic field, as observed in laboratory experiments. It is clear from (4.3) that the modifications of electromagnetic fields due to $F^{E}_{\mu\nu}$ will become significant only over distances of the magnitude of the length ℓ, or larger. We expect that, to avoid obvious disagreement with experiment, the length ℓ must be quite large on a terrestrial scale. In Sections IV.B and IV.C, we confirm this expectation.

Secondly, let us suppose that the point-like singularities of the new approximate solutions, found in Section III, represent electric charges. The equations of motion in this case are given by (3.88). Let $\tilde{F}^{(p)ext}_{\mu\nu}$ in (3.88) here be denoted by $\tilde{F}_{\mu\nu}$. Equation (3.89) tells us that we can write $\tilde{F}_{\mu\nu}$ in a form analogous to (4.1) for $F_{\mu\nu}$, i.e.,

$$\tilde{F}_{\mu\nu} = \tilde{F}^{M}_{\mu\nu} + \tilde{F}^{E}_{\mu\nu} \quad . \tag{4.4}$$

Now, from (3.89), (3.28), (3.32), and (3.34), we deduce equations for $\tilde{F}^{M}_{\mu\nu}$ and $\tilde{F}^{E}_{\mu\nu}$,

$$\partial_{[\rho}\tilde{F}^{M}_{\mu\nu]} = 0 \qquad\qquad \partial^{\mu}\tilde{F}^{M}_{\mu\nu} = 0 \tag{4.5}$$

and

$$\Box\tilde{F}^{E}_{\mu\nu} = \frac{1}{2\tilde{\ell}^{2}}\,\tilde{F}^{M}_{\mu\nu} \qquad\qquad \partial^{\mu}\tilde{F}^{E}_{\mu\nu} = 0 \quad , \tag{4.6}$$

which hold at all points except those on the singular world-lines and on the two-dimensional sheets associated with them. In this case, $\partial_{[\rho}\tilde{F}^{E}_{\mu\nu]} \neq 0$ unless $\tilde{F}^{M}_{\mu\nu} = 0$. For macroscopic phenomena, the alternative physical interpretation for electric charge applicable here implies that $\tilde{F}^{M}_{0i}$, (i=1, 2, 3), is the usual Maxwell electric field and $\tilde{F}^{M}_{ij}$, (i, j=1, 2, 3), the usual Maxwell magnetic field, as observed in laboratory experiments. The effect of $\tilde{F}^{E}_{\mu\nu}$ is significant only over distances with a scale at least as large as the length $\tilde{\ell}$. We shall find that $\tilde{\ell}$, likewise, must be large on a terrestrial scale, if the equations (4.4 - 4.6) are to be consistent with experiment. Note that ℓ and $\tilde{\ell}$ are two different parameters; both occur as integration constants in the approximate solutions.

We can emphasize the difference between the two possible interpretations by pointing out that $F^{E}_{\mu\nu}$ and $\tilde{F}^{E}_{\mu\nu}$ satisfy, respectively,

$$\partial^{\mu}F^{E}_{\mu\nu} \neq 0 \qquad\qquad \partial_{[\rho}F^{E}_{\mu\nu]} = 0 \quad , \tag{4.7}$$

and

$$\partial^{\mu}\tilde{F}^{E}_{\mu\nu} = 0 \qquad \partial_{[\rho}\tilde{F}^{E}_{\mu\nu]} \neq 0 \quad , \tag{4.8}$$

which are consequences of (3.99) and (3.98), respectively, expressed in the notation of this section. It is evident that, if we apply the two sets of modified Maxwell equations to a given physical situation and assume that $F^{M}_{\mu\nu}$ and $\tilde{F}^{M}_{\mu\nu}$ have the same form, then $F^{E}_{\mu\nu}$ and $\tilde{F}^{E}_{\mu\nu}$ will, in general, be different. Since experiments presumably measure the total field, $F_{\mu\nu}$ or $\tilde{F}_{\mu\nu}$, the predictions for the two cases will be different and can be tested. Thus, it is possible to decide from such tests which interpretation, if either, is correct.

An important remark must be made concerning the meaning of the inhomogeneous equations

$$\Box F^{E}_{\mu\nu} = \frac{1}{2\ell^{2}} F^{M}_{\mu\nu} \quad , \qquad \Box\tilde{F}^{E}_{\mu\nu} = \frac{1}{2\tilde{\ell}^{2}} \tilde{F}^{M}_{\mu\nu} \quad .$$

Formally, we can add to any particular solutions of these equations any solutions of the corresponding homogeneous equations, which are the same as the equations satisfied by $F^{M}_{\mu\nu}$ and $\tilde{F}^{M}_{\mu\nu}$, i.e.,

$$\Box F^{M}_{\mu\nu} = 0 \quad , \qquad \Box\tilde{F}^{M}_{\mu\nu} = 0 \quad .$$

To know which particular solutions of the inhomogeneous equations we should choose in a given situation, it is necessary to refer to their "microscopic" origin in the equations of motion. The general rule is that, if we choose for $F^{M}_{\mu\nu}$ and $\tilde{F}^{M}_{\mu\nu}$ solutions which are singular at certain points, then we must take particular integrals for $F^{E}_{\mu\nu}$ and $\tilde{F}^{E}_{\mu\nu}$ which are "less singular" at these points. What this means will become clear in the examples below.

B. A Laboratory Test of Static Electric Fields

The classic experimental test of Coulomb's law exploits the fact that the electrostatic field inside any closed, charged conducting surface vanishes, according to the usual Maxwell equations. In other words, the electrostatic potential should be constant in the interior region. By applying a large potential to the surface, a sensitive test of this prediction is possible, mainly because it is a null-type experiment. Many types of modifications of Maxwell's equations imply that the potential should vary in the interior region, which means that non-null potential differences should be seen, if the theories are correct.

For Einstein's theory, we analyze this experiment first using the equations (4.1 - 4.3) for $F_{\mu\nu}$, which come from Johnson's solutions. If we have only a static electric field $(\vec{E})_i = F_{0i}$, then they are

$$\vec{E} = \vec{E}^M + \vec{E}^E \tag{4.9}$$

$$\vec{\nabla} \cdot \vec{E}^M = 0 \qquad \vec{\nabla} \times \vec{E}^M = 0 \tag{4.10}$$

$$\vec{\nabla}^2 \vec{E}^E = - \frac{1}{2\ell^2} \vec{E}^M \qquad \vec{\nabla} \times \vec{E}^E = 0 \quad . \tag{4.11}$$

We can write these equations in terms of an electrostatic potential $\Phi = \Phi^M + \Phi^E$, as

$$\vec{E}^M = - \vec{\nabla} \Phi^M \qquad \vec{\nabla}^2 \Phi^M = 0 \tag{4.12}$$

$$\vec{E}^E = - \vec{\nabla} \Phi^E \qquad \vec{\nabla}^2 \Phi^E = - \frac{1}{2\ell^2} \Phi^M \quad . \tag{4.13}$$

To simplify matters, suppose the conducting surface is a spherical shell of radius R, held at a potential V_0. The charge distribution on the surface may be thought of, from a microscopic point of view, as due to many point charges q_i. The potential Φ^M at a point $\vec{r}$ due to charges q_i at points $\vec{r}_i$ on the shell is

$$\Phi^M(\vec{r}) = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} \quad .$$

From (4.13), we see that the potential Φ^E is then

$$\Phi^E(\vec{r}) = -\frac{1}{2\ell^2} \sum_i \frac{1}{2} q_i |\vec{r} - \vec{r}_i| \quad ,$$

if we take only the particular integral which is implied by the origin of (4.9 - 4.13) in the equations of motion (3.93 - 3.94). We go over to the macroscopic limit by assuming a uniform charge distribution. The total potential Φ is then given by

$$\Phi(\vec{r}) = \frac{Q}{4\pi} \int d\Omega' \left(\frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{1}{4\ell^2} |\vec{r} - \vec{r}\,'| \right) \quad ,$$

where $d\Omega'$ is an element of the spherical surface at the point $\vec{r}\,'$, and Q is the total charge on the surface. For a point $\vec{r}$ inside the sphere, the integral gives

$$\Phi(\vec{r}) = \frac{Q}{R} \left[1 - \frac{1}{4\ell^2} \left(R^2 + \frac{1}{3} r^2 \right) \right] \approx V_0 \left(1 - \frac{r^2 - R^2}{12\ell^2} \right) \quad , \tag{4.14}$$

where in the last expression we have neglected terms of higher order in $(R/\ell)^2$, which we suppose is small compared to unity. Hence, Johnson's solutions predict,[21] if the particles they describe are electric charges, that there will be variations in the electrostatic potential inside the sphere of the order of magnitude $V_0 \left(\frac{R}{\ell} \right)^2$.

Let us now consider the prediction of the equations (4.4 - 4.6) for $\tilde{F}_{\mu\nu}$, obtained from the new solutions presented in this paper. The static electric field $(\vec{\tilde{E}})_i = \tilde{F}_{0i}$ satisfies,

$$\vec{\tilde{E}} = \vec{\tilde{E}}^M + \vec{\tilde{E}}^E \tag{4.15}$$

$$\vec{\nabla} \times \vec{\tilde{E}}^M = 0 \qquad \vec{\nabla} \cdot \vec{\tilde{E}}^M = 0 \tag{4.16}$$

$$\vec{\nabla}^2 \vec{\tilde{E}}^E = -\frac{1}{2\tilde{\ell}^2} \vec{\tilde{E}}^M \qquad \vec{\nabla} \cdot \vec{\tilde{E}}^E = 0 \quad . \tag{4.17}$$

As in the other case, we may write for $\vec{\tilde{E}}^{M}$

$$\vec{\tilde{E}}^{M} = -\vec{\nabla}\,\tilde{\Phi}^{M} \qquad \vec{\nabla}^{2}\,\tilde{\Phi}^{M} = 0 \quad , \tag{4.18}$$

so that $\tilde{\Phi}^{M}$ is constant in the interior of the sphere. To calculate $\vec{\tilde{E}}^{E}$ we must again consider the "microscopic" origin of the equations (4.15 - 4.17), in the equations of motion (3.88 - 3.89). Here the situation is more complicated because we must consider the string singularities as well as the point charges on the surface. For simplicity, let us suppose that there is an outer concentric spherical shell at ground potential, and therefore oppositely charged. The strings run from the charges q_i on the inner shell along radial lines to charges $-q_i$ on the outer shell, because of the spherical symmetry. We shall argue below that, in the limit of a uniform charge distribution on the two shells, the field $\vec{\tilde{E}}^{E}$ due to the strings and the point charges at their ends is such that $\vec{\nabla} \times \vec{\tilde{E}}^{E} = 0$ inside the inner shell. If this is so, then $\vec{\tilde{E}}^{E}$ may be derived from a potential $\tilde{\Phi}^{E}$; but in this case $\vec{\nabla} \cdot \vec{E}^{E} = 0$, so that we have

$$\vec{\tilde{E}}^{E} = -\vec{\nabla}\,\tilde{\Phi}^{E} \qquad \vec{\nabla}^{2}\,\tilde{\Phi}^{E} = 0 \quad , \tag{4.19}$$

in contrast to (4.13) for Φ^{E}. Hence, we would conclude that $\tilde{\Phi}^{E}$, like $\tilde{\Phi}^{M}$, is constant in the interior of the inner shell, at potential V_0. As a result, the prediction of Einstein's theory, if this alternative interpretation for electric charge is accepted, is that no deviation from the prediction of the usual Maxwell equations is expected for this particular experiment.

The argument that $\vec{\nabla} \times \vec{\tilde{E}}^{E} = 0$ inside the inner shell is rather lengthy. It involves the use of condition (3.84), which must be satisfied on the strings if a solution of the field equations is to exist. We suppose that (3.84) can be satisfied by placing point charges on the two spherical shells appropriately, with strings running along radial lines between them. In the static approximation (3.84) becomes

$$\frac{d\vec{y}}{d\tau_1} \times \left[\vec{\nabla} \times \vec{\tilde{E}}^{E}_{ext}(\vec{y})\right] = 0 \quad , \tag{4.20}$$

where $\vec{y}(\tau_1)$ is some particular string, described by a parameter τ_1, and $\vec{\tilde{E}}^{E}_{ext}$ is the field due to all other strings except that one. Since for the case at hand $\frac{d\vec{y}}{d\tau_1}$ is in the radial direction, (4.20) states that the components of $\vec{\nabla} \times \vec{\tilde{E}}^{E}_{ext}$ transverse to the radial lines, say $\left(\vec{\nabla} \times \vec{\tilde{E}}^{E}_{ext}\right)_T$, must vanish on any string. By symmetry, they must also vanish on the extension of these radial lines inside the inner shell. As for the self-field contributions, it is easily verified from (3.48) that although $\left(\vec{\nabla} \times \vec{\tilde{E}}^{E}_{self}\right)_T$ diverges as one approaches the appropriate string, it does vanish on the extension of this radial line into the interior region. Hence, $\left(\vec{\nabla} \times \vec{\tilde{E}}^{E}\right)_T = 0$ on these radial lines in the interior region, so that in the limit of a uniform charge distribution on the two shells, the equation holds everywhere in the interior. (This limit cannot be taken seriously, since it implies the absurdity of the strings filling all space between the two shells. But a harmonic function which vanishes on many closely-spaced lines must be small everywhere, so it is a good approximation.) We may next argue that the component of $\vec{\nabla} \times \vec{\tilde{E}}^{E}$ in the radial direction, say $\left(\vec{\nabla} \times \vec{\tilde{E}}^{E}\right)_R$, must vanish in the interior region if we impose the physical requirement that no current flow in the conducting surfaces. For, $\left(\vec{\nabla} \times \vec{\tilde{E}}^{E}\right)_R$ must then vanish on the surface of the inner shell, and being a harmonic function, it is then zero in the interior also. Hence, we have shown that $\vec{\nabla} \times \vec{\tilde{E}}^{E} = 0$ inside the inner shell, as claimed. It is reasonable to suppose that this argument can be generalized to geometries other than spherical.

A recent experiment testing these predictions is that of Williams <u>et al.</u>[22] The geometry of the conducting shells was not precisely spherical, but that is unimportant unless a non-null result is found. The voltage applied to one of the shells was not static, but oscillating in time. However, the modulation frequency ω was such that $\left(\frac{\omega R}{c}\right)^2 \ll 1$, where R was a typical dimension of the apparatus, so nonstatic effects could be ignored. The voltage difference between two conducting shells inside the charged shell was found to be zero to an accuracy of better than $10^{-12}\, V_0$, where V_0 was the applied voltage. Their results imply that, for the prediction for the electrostatic potential Φ given in (4.14) to be consistent with experiment, the length ℓ must satisfy

$$\ell \geq 2 \times 10^9 \text{ cm} \quad . \tag{4.21}$$

Of course, the null result is consistent with the prediction of the alternative set of equations, so no information concerning the corresponding length $\tilde{\ell}$ is obtained.

C. The Static Magnetic Field of the Earth

A second test of Maxwell's equations turns out to imply a lower limit on either the length ℓ, if one interpretation is accepted, or the length $\tilde{\ell}$, if the other interpretation holds. This test involves the static magnetic dipole field of the Earth, measured at the Earth's surface, and was first proposed by Schrödinger.[23] It gives the best limit for the accuracy of the usual Maxwell equations known to date, according to Goldhaber and Nieto.[24]

Let us analyze this test first using the equations (4.1-4.3) for $F_{\mu\nu}$. If we have only a static magnetic field $(\vec{H})^i = \epsilon^{ijk} F_{jk}$, then these equations become

$$\vec{H} = \vec{H}^M + \vec{H}^E \tag{4.22}$$

$$\vec{\nabla} \times \vec{H}^M = 0 \qquad \vec{\nabla} \cdot \vec{H}^M = 0 \tag{4.23}$$

$$\vec{\nabla}^2 \vec{H}^E = -\frac{1}{2\ell^2} \vec{H}^M \qquad \vec{\nabla} \cdot \vec{H}^E = 0 \quad . \tag{4.24}$$

Equations (4.23 - 4.24) imply that we can write

$$\vec{H}^M = \vec{\nabla} \times \vec{A}^M \qquad \vec{\nabla} \cdot \vec{A}^M = 0 \tag{4.25}$$

$$\vec{H}^E = \vec{\nabla} \times \vec{A}^E \qquad \vec{\nabla} \cdot \vec{A}^E = 0 \quad , \tag{4.26}$$

which defines vector potentials $\vec{A}^M$ and $\vec{A}^E$. Of course, choosing them to be divergenceless is not necessary, but it is convenient. These vector potentials then satisfy

$$\vec{\nabla}^2 \vec{A}^M = 0 \tag{4.27}$$

$$\vec{\nabla}^2 \vec{A}^E = -\frac{1}{2\ell^2} \vec{A}^M \quad . \tag{4.28}$$

Now, suppose that $\vec{A}^M$ is given by the usual vector potential of a static magnetic dipole at the origin of coordinates,

$$\vec{A}^M(\vec{r}) = \vec{\nabla} \times \frac{\vec{d}}{r} \quad . \tag{4.29}$$

Here the constant vector $\vec{d}$ gives both the direction and the magnitude of the dipole. Then, from (4.28), we must take for $\vec{A}^E$,

$$\vec{A}^E(\vec{r}) = -\frac{1}{2\ell^2} \vec{\nabla} \times \left(\frac{1}{2} r\vec{d}\right) \quad , \tag{4.30}$$

which gives the modifications of a dipole field due to Einstein's theory. Once again, the particular integral (4.30) of equation (4.28) was chosen on the basis of the "microscopic" equations of motion (3.93 - 3.94). The total magnetic field $\vec{H}$ now becomes[25]

$$\begin{aligned} \vec{H} &= \vec{\nabla} \times \left\{ \vec{\nabla} \times \left[\left(\frac{1}{r} - \frac{r}{4\ell^2} \right) \vec{d} \right] \right\} \\ &= \left(\frac{1}{r^3} + \frac{1}{12\ell^2} \frac{1}{r} \right) \left(\frac{3(\vec{r} \cdot \vec{d})\vec{r}}{r^2} - \vec{d} \right) + \frac{1}{3\ell^2} \frac{\vec{d}}{r} \end{aligned} \tag{4.31}$$

for $r \neq 0$. At the Earth's surface, idealized as spherical, the second term in the last expression will give a constant magnetic field parallel to the direction of the dipole $\vec{d}$. Such a field can be distinguished from the usual dipole field and from any linear combination of higher spherical harmonics, which is important because the Earth's field is not a perfect dipole and contains such higher harmonics. If we were to use the usual Maxwell equations to interpret (4.31), the contribution of $\vec{H}^E$ would appear to be due to an effective "external" current. In fact, one finds that

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{H}^E = -\frac{1}{2\ell^2} \frac{\vec{r} \times \vec{d}}{r^3} \quad . \tag{4.32}$$

On the other hand, remember that $\vec{\nabla} \cdot \vec{H} = 0$.

Secondly, let us consider the modifications of a static magnetic dipole field due to equations (4.4 - 4.6) for $\tilde{F}_{\mu\nu}$. Defining $(\vec{\tilde{H}})^i = \epsilon^{ijk} \tilde{F}_{jk}$, they become for this example,

$$\vec{\tilde{H}} = \vec{\tilde{H}}^M + \vec{\tilde{H}}^E \tag{4.33}$$

$$\vec{\nabla} \cdot \vec{\tilde{H}}^M = 0 \qquad \vec{\nabla} \times \vec{\tilde{H}}^M = 0 \tag{4.34}$$

$$\vec{\nabla}^2 \vec{\tilde{H}}^E = -\frac{1}{2\tilde{\ell}^2} \vec{\tilde{H}}^M \qquad \vec{\nabla} \times \vec{\tilde{H}}^E = 0 \quad . \tag{4.35}$$

In this case, we may then write

$$\vec{\tilde{H}}^M = -\vec{\nabla} \tilde{\Psi}^M \qquad \vec{\nabla}^2 \tilde{\Psi}^M = 0 \tag{4.36}$$

$$\vec{\tilde{H}}^E = -\vec{\nabla} \tilde{\Psi}^E \qquad \vec{\nabla}^2 \tilde{\Psi}^E = -\frac{1}{2\tilde{\ell}^2} \tilde{\Psi}^M \quad , \tag{4.37}$$

for some scalar potentials $\tilde{\Psi}^M$ and $\tilde{\Psi}^E$. Since

$$\vec{\nabla} \times \left(\vec{\nabla} \times \frac{\vec{d}}{r}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \frac{\vec{d}}{r}\right)$$

for $r \neq 0$, we suppose that the dipole potential $\tilde{\Psi}^M$ has the form

$$\tilde{\Psi}^M = -\vec{\nabla} \cdot \left(\frac{\vec{d}}{r}\right) \quad . \tag{4.38}$$

It follows that $\tilde{\Psi}^E$ is given by

$$\tilde{\Psi}^E = \frac{1}{2\tilde{\ell}^2} \vec{\nabla} \cdot \left(\frac{1}{2} r \vec{d}\right) \quad , \tag{4.39}$$

by using (4.37), keeping in mind its "microscopic" origin. Therefore, the total magnetic field $\vec{\tilde{H}}$ in this case is

$$\begin{aligned} \vec{\tilde{H}} &= \vec{\nabla} \left\{ \vec{\nabla} \cdot \left[\left(\frac{1}{r} - \frac{r}{4\tilde{\ell}^2} \right) \vec{d} \right] \right\} \\ &= \left(\frac{1}{r^3} + \frac{1}{12\tilde{\ell}^2} \frac{1}{r} \right) \left(\frac{3(\vec{r} \cdot \vec{d})\vec{r}}{r^2} - \vec{d} \right) - \frac{1}{6\tilde{\ell}^2} \frac{\vec{d}}{r} \quad . \end{aligned} \tag{4.40}$$

The second term in the last expression for (4.40) differs from the corresponding term in (4.31) by a factor of $-\frac{1}{2}$. Most importantly, this means that the extra constant magnetic field at the Earth's surface will be antiparallel to the direction of the dipole, in contrast to the other case. Also significant is the fact that $\vec{\tilde{H}}$ satisfies

$$\vec{\nabla} \cdot \vec{\tilde{H}} = \vec{\nabla} \cdot \vec{\tilde{H}}^E = \frac{1}{2\tilde{\ell}^2} \frac{\vec{r} \cdot \vec{d}}{r^3} \quad , \tag{4.41}$$

but $\vec{\nabla} \times \vec{\tilde{H}} = 0$. Equations (4.32) and (4.41) illustrate the essential difference between the two assumptions for electric charge.

The predictions of (4.31) and (4.40) can be tested by measurements of the magnetic field of the Earth at various points on the Earth's surface. These measurements have been analyzed recently by Goldhaber and Nieto,[24] as a test of Maxwell's equations. They find no evidence, as yet, for the existence of such modifications of the static dipole field. There are a number of uncertainties in this analysis, but they obtain a limit which, when applied to the predictions above, implies that

$$\ell \text{ (or } \tilde{\ell}) \gtrsim 10^{10} \text{ cm} \quad ,$$

if either of (4.31) or (4.40) is to be consistent with the observations.

D. Cosmic Magnetic Fields

The results of Sections IV.B and IV.C show that, if Einstein's theory is correct, then either the length ℓ, or the length $\tilde{\ell}$, must be greater than about 15 Earth radii. Thus, we need to look to effects on a cosmic scale for the most likely possibilities for testing the theory. In this section we shall consider predictions deduced from the modifications of static magnetic dipole fields, exhibited, for the two different physical interpretations of the theory, in (4.31) and (4.40). Electrostatic fields do not generally exist over such large distances, and, as for magnetic fields, the static dipole gives the longest-range field where deviations from the usual Maxwell theory can be clearly seen. We limit ourselves to a qualitative description of these fields. In astrophysical situations, magnetic fields are rarely so simple, but our examination provides a basis for study of more realistic problems.

First, let us consider, in general terms, possibilities for testing the theory using cosmic magnetic fields. Within the Solar system, the Solar wind plasma introduces a complicating factor which makes tests difficult. The Earth's magnetic field beyond 10 Earth radii or so is dominated by its interaction with the Solar wind,[26] so a test here appears very unlikely. The interplanetary field in the ecliptic plane appears to be due to streaming with the high conductivity plasma,[27] a dynamic effect which masks any static fields. However, if the length ℓ, or $\tilde{\ell}$, is not more than a few Solar radii, effects may be detectable out of the ecliptic plane, in particular in the polar regions of the Solar magnetic field. The best possibility for a test in the near future would appear to be the magnetic field of Jupiter. Here effects could be seen if ℓ, or $\tilde{\ell}$, is less than the distance where the Solar wind begins to dominate. As for larger distances, we shall see in Section IV. E that there are theoretical reasons for believing that ℓ, or $\tilde{\ell}$, should be bounded by an upper limit which, conservatively, is the order of magnitude of a light-year. Hence, on a galactic scale the modifications due to Einstein's theory should dominate the behavior of electromagnetic fields. In principle, then, galactic fields offer the best possibilities for testing the theory. However, plasma effects are again an important complication here, since a plasma can affect magnetic fields over long ranges, even though it is inherently a relatively small-scale phenomenon.

Let us now look at the structure of the modified magnetic dipole field (4.31), which follows from the equations of motion for electric charges of Johnson's solutions. Once again writing $\vec{H} = \vec{H}^M + \vec{H}^E$, we have

$$\vec{H}^M = \frac{1}{r^3}\left(\frac{3(\vec{r}\cdot\vec{d})\vec{r}}{r^2} - \vec{d}\right) \tag{4.43}$$

$$\vec{H}^E = \frac{1}{4\ell^2}\frac{1}{r}\left(\frac{(\vec{r}\cdot\vec{d})\vec{r}}{r^2} + \vec{d}\right) \quad . \tag{4.44}$$

Suppose that $\vec{d} = d\hat{z}$, i.e., the dipole vector is in the direction of the z-axis and has magnitude d. We choose the usual polar coordinates r, θ, ϕ with respect to that axis, and centered at the origin. In the equatorial plane $\theta = \frac{\pi}{2}$, perpendicular to the dipole axis, the field is aligned along the axis, as usual. However,

$$\vec{H}\cdot\hat{z} = -d\left(\frac{1}{r^3} - \frac{1}{4\ell^2}\frac{1}{r}\right), \quad \left(\theta = \frac{\pi}{2}\right) \quad , \tag{4.45}$$

so the field, although it behaves like a normal dipole for $r \ll \ell$, goes to zero at $r^2 = 4\ell^2$, and changes direction for larger distances. The magnitude of $\vec{H}\cdot\hat{z}$ then goes through a maximum at $r^2 = 12\ell^2$, and decreases as $\frac{d}{4\ell^2}\frac{1}{r}$ for distances

$r \gg \ell$. The most striking aspect of this behavior is the $\frac{1}{r}$ fall-off at large distances, rather than the $\frac{1}{r^3}$ dependence Maxwell's theory implies. The existence of a circle of null points in the field is another very characteristic feature, which is easily seen to occur only in the equatorial plane.

To see what the field looks like out of the equatorial plane, consider the lines of magnetic force at very large distances, $r \gg \ell$, where $\vec{H}^M$ can be ignored. They are given by curves lying in surfaces X^E = constant, where the function X^E satisfies $\vec{H}^E \cdot \vec{\nabla} X^E = 0$. From (4.44), one finds that these surfaces are described by $r \sin^2 \theta$ = constant. Such surfaces have an hour-glass shape, with the neck lying in the equatorial plane. At large distances above and below the plane, the surfaces approximate paraboloids of revolution about the dipole axis, opening out to infinity in both directions. The fact that the field lines are open at large distances is of considerable importance, since it means that charged particles spiralling about the lines can escape to infinity, in contrast to the usual dipole field. More precisely, those field lines which pass through the equatorial plane outside of the circle of null points at $r=2\ell$ will be open, while those inside this circle will be closed. This means that particles emerging from near the dipole source close to the polar directions will escape, while those closer to the equator will be trapped.

The field in the polar regions is of particular interest. Let $\hat{\rho}$ be a unit vector outward from the z-axis in a perpendicular direction. Then for small polar angles, i.e., $\theta \ll 1$, we find

$$\vec{H} \cdot \hat{z} \approx 2d\left(\frac{1}{r^3} + \frac{1}{4\ell^2}\,\frac{1}{r}\right) \qquad (4.46)$$

$$\vec{H} \cdot \hat{\rho} \approx \theta d \left(\frac{3}{r^3} + \frac{1}{4\ell^2}\,\frac{1}{r}\right) \quad , \qquad (4.47)$$

keeping only the leading terms. So the field is nearly parallel to the dipole axis in this region, as usual, but it again falls off as $\frac{1}{r}$ at large distances, much more slowly than a normal dipole field. This pattern of slowly-diverging, nearly parallel field lines could produce effects of astrophysical significance. In particular, it could align randomly-directed high-energy charged particles into a beam moving outward along the dipole axis. The radiation emitted by such a beam could perhaps explain some observed phenomena, but it seems very unlikely that this could be used as anything more than very qualitative evidence in support of Einstein's theory.

Consider now the modified magnetic dipole field (4.40), which follows from assuming that electric charges are described by the equations of motion obtained from the new solutions of Section III. With $\vec{\tilde{H}} = \vec{\tilde{H}}^M + \vec{\tilde{H}}^E$, we have

$$\vec{\tilde{H}}^M = \frac{1}{r^3}\left(\frac{3(\vec{r}\cdot\vec{d})\,\vec{r}}{r^2} - \vec{d}\right) \tag{4.48}$$

$$\vec{\tilde{H}}^E = \frac{1}{4\tilde{\ell}^2}\,\frac{1}{r}\left(\frac{(\vec{r}\cdot\vec{d})\,\vec{r}}{r^2} - \vec{d}\right) \tag{4.49}$$

The only difference from (4.43 - 4.44) is the sign in the second term in (4.49). For the field in the equatorial plane in this case, we find

$$\vec{\tilde{H}}\cdot\hat{z} = -d\left(\frac{1}{r^3} + \frac{1}{4\tilde{\ell}^2}\,\frac{1}{r}\right) \quad , \qquad \left(\theta = \frac{\pi}{2}\right) \quad , \tag{4.50}$$

so there is no circle of null points, as there is in the other case. The dominant $\frac{1}{r}$ behavior at distances $r >> \tilde{\ell}$ is found once again, but the direction of the field at such distances is the same as the usual dipole field, rather than opposite. To see the field pattern for $r >> \tilde{\ell}$ out of the equatorial plane, observe that

$$\vec{\tilde{H}}^E\cdot\vec{r} = 0 \quad . \tag{4.51}$$

Hence, the lines of magnetic force due to $\vec{\tilde{H}}^E$ are circles about the origin. This seemingly peculiar field pattern becomes more understandable if we realize that $\vec{\tilde{H}}^E$ dominates the field, for $r \gg \tilde{\ell}$, only away from the polar regions. In fact, for polar angles $\theta \ll 1$, near the dipole axis, $\vec{\tilde{H}}^E$ is very small, viz.,

$$\vec{\tilde{H}}^E \cdot \hat{z} \approx -\theta^2 \frac{d}{8\tilde{\ell}^2} \frac{1}{r} \tag{4.52}$$

$$\vec{\tilde{H}}^E \cdot \hat{\rho} \approx \theta \frac{d}{4\tilde{\ell}^2} \frac{1}{r} \tag{4.53}$$

Hence, it is the usual dipole field which dominates if the angle is small enough. The most significant difference between this field, and the one derived from the other physical interpretation, is that in this case all the field lines are closed, even at distances $r \gg \tilde{\ell}$. Therefore all charged particle orbits are trapped ones, in contrast to the other case.

E. Theoretical Restrictions

In Section IV. C we found a lower limit of about 10^{10} cm on either ℓ, or $\tilde{\ell}$, if Einstein's theory is to be consistent with terrestrial observations of electromagnetic fields. Since these length parameters arise as integration constants in the approximate solutions, it might appear at first sight that we could suppose them to be arbitrarily large. If so, then it would never be possible to rule out Einstein's theory, as being incorrect, no matter how accurately the usual Maxwell equations were observed to hold true. We do expect, for exact solutions to Einstein's unified field theory, that the values of the integration constants appearing in the solutions should be fixed by some general principles, yet to be discovered. However, it is very uncertain whether such constraints on the integration constants could be obtained by considering only approximate solutions. In view of these facts, it is of some interest than one can argue that the length

ℓ, or $\widetilde{\ell}$, must have an upper bound, on the basis of the structure of the approximate solutions to the field equations, together with certain physical requirements.

This argument was presented by Johnson,[7] for the approximate solutions he constructed. The extension to the case of the new solutions presented here is an obvious one. The essential point is as follows. The approximation method in Section III.A supposes that, to second order in powers of the arbitrary parameter λ, we may write

$$\frac{1}{\sqrt{-h}} h^{\mu\nu} = \eta^{\mu\nu} + i\lambda\phi^{\mu\nu} + \lambda^2\gamma^{\mu\nu} \quad . \tag{4.54}$$

The parameter λ may be absorbed in the integration constants of the solutions, or, as we have done, set equal to unity after writing down the field equations to each order. It is then supposed, if the approximation method is to be useful, that the components of $\phi^{\mu\nu}$ and $\gamma^{\mu\nu}$ are small compared to unity; for then neglected nonlinear terms should be even smaller. However, the approximate solutions for $\phi^{\mu\nu}$ and $\gamma^{\mu\nu}$ constructed in this paper and in Johnson's papers contain singularities at certain points. If one approaches close enough to these points, certain components of $\phi^{\mu\nu}$ and $\gamma^{\mu\nu}$ become arbitrarily large. Hence, at these small distances the approximation method fails, i.e., it ceases to be useful because terms nonlinear in $\phi^{\mu\nu}$ and $\gamma^{\mu\nu}$ are at least as important as linear ones. One can make an order of magnitude estimate of the characteristic distances where this breakdown occurs by calculating the distances from a typical singular point where the magnitudes of appropriate components of $\phi^{\mu\nu}$ and $\gamma^{\mu\nu}$, as given by the approximate solutions, become equal to unity. This allows one to express these characteristic distances in terms of integration constants in the solutions, and hence in terms of physical constants, using identifications made on the basis of the equations of motion. One must then check to see if the relations so obtained make sense physically. This is the origin of the above-mentioned restriction on ℓ, or $\tilde{\ell}$.

First, let us consider Johnson's approximate solutions, in the special case of a single point charge at rest at the origin of coordinates. Using (3.15 - 3.18) and (3.8), we find for $\phi^{\mu\nu}$

$$\begin{aligned} \phi^{ij} &= -\epsilon^{ijk}\,\partial_k\left(\frac{f}{4\pi}\,\frac{1}{r} - \frac{f'}{4\pi}\,\frac{r}{4\ell^2}\right) \\ \phi^{0i} &= 0 \quad , \end{aligned} \tag{4.55}$$

where i, j, k run from 1 to 3, and $\epsilon^{ijk} \equiv \epsilon^{0ijk}$. The field $\gamma^{\mu\nu}$ is the same for both types of solutions in the linear approximation, so, from (3.80), we have

$$\gamma^{00} = \frac{\mu}{4\pi}\,\frac{1}{r} \quad , \qquad \gamma^{0i} = \gamma^{ij} = 0 \quad . \tag{4.56}$$

The physical mass, m, and charge, e, are related to the integration constants by

$$\frac{\mu}{4\pi} = \frac{4Gm}{c^2} \tag{4.57}$$

$$\frac{f}{4\pi} = \ell\sqrt{\frac{2G\,e^2}{c^4}} \quad , \qquad f' = f \quad , \tag{4.58}$$

from (3.85) and (3.91 - 3.92). A characteristic gravitational radius r_G may be defined as that value of r at which $|\gamma^{00}| = 1$. From (4.56 - 4.57), we find

$$r_G = 4\,\frac{Gm}{c^2} \quad . \tag{4.59}$$

This length r_G gives an order of magnitude estimate of the distance from the singularity at the origin at which the weak-field approximation for the gravitational field breaks down. Of course, it is the same as the Schwarzschild radius, within a factor of two. A characteristic electromagnetic radius r_E may be defined as that value of r at which $|\phi^{ij}| = \epsilon^{ijk}\,\frac{x_k}{r}$. We get from (4.55) and

and (4.58), keeping only the most singular term is ϕ^{ij},

$$r_E^2 \approx \sqrt{2}\ \ell \sqrt{\frac{Ge^2}{c^4}} \quad . \tag{4.60}$$

The length r_E gives an order of magnitude estimate of the distance from the singularity at which the weak-field approximation for the electromagnetic field breaks down in Einstein's theory. This is the relation quoted in (1.1), ignoring the inessential factor $\sqrt{2}$.

Secondly, consider the new solutions constructed in Section III, for the special case of a single pair of oppositely charged point masses, connected by a string. If we are interested only in the most singular terms in $\phi^{\mu\nu}$ near the point charges, we can ignore the string, as is easily verified from (3.49). Near one of the two charges, supposed to be at the origin, we find from (3.49),

$$\phi^{0i} \approx -\frac{g}{4\pi}\frac{x^i}{r^3} \quad , \qquad \phi^{ij} = 0 \quad , \tag{4.61}$$

keeping only the most singular term in ϕ^{0i}. From (3.87), g is given by

$$\frac{g}{4\pi} = \tilde{\ell} \sqrt{\frac{2Ge^2}{c^4}} \quad , \tag{4.62}$$

if the charges are electric ones. Defining a characteristic electromagnetic radius $\tilde{r}_E$ by that value of r at which $|\phi^{0i}| = \frac{x^i}{r}$, we find a relation

$$\tilde{r}_E^2 \approx \sqrt{2}\ \tilde{\ell} \sqrt{\frac{Ge^2}{c^4}} \quad , \tag{4.63}$$

of exactly the same form as (4.60). We note from (3.49) that ϕ^{0i} also contains a much weaker, logarithmic singularity as one approaches the string, away from the end points. If we call ρ_E the characteristic distance from the string at which appropriate components of ϕ^{0i} have magnitude unity, then it is easily seen that ρ_E has the order of magnitude

$$\rho_E \sim L \exp\left(-k\tilde{\ell}^2/\tilde{r}_E^2\right) \tag{4.64}$$

in terms of the length L of the string. Here k is a constant of order of magnitude unity. As we discuss below, $\tilde{\ell}$ is a very large length and $\tilde{r}_E$ is a very small one, so ρ_E is so small that for practical purposes the logarithmic singularity may be ignored.

Let us now review the discussion in Section I of numerical values for r_E and ℓ, or $\tilde{r}_E$ and $\tilde{\ell}$. From (4.60), we see that if r_E has an upper bound, $(r_E)_{max}$, then one obtains an upper bound on ℓ of the form

$$\ell_{max} \sim \sqrt{\frac{c^4}{Ge^2}}\left(r_E^2\right)_{max} \quad . \tag{4.65}$$

All our remarks here will apply as well to $\tilde{r}_E$ and $\tilde{\ell}$, if we make the alternative interpretation, in view of (4.63). Now, classical electrodynamics is known to correctly describe phenomena on a terrestrial scale down to distances approaching atomic dimensions, where, of course, it begins to fail. Johnson's results show that the weak-field approximation to Einstein's theory gives the usual equations of classical electrodynamics on this same scale, provided that the length parameter ℓ is greater than 10^{10} cm. Hence, r_E should not be too much larger than a typical atomic dimension, for the nonlinear terms in Einstein's theory can be expected to give significant deviations from these results at distances of the order of r_E. It is difficult to say precisely what $(r_E)_{max}$ should be. The classical electron radius is the natural choice, but it could probably be as large as the Bohr radius. If one could argue that the modified Maxwell equations of Section IV. A should be carried over naively to quantum electrodynamics, then it might be possible to argue that $(r_E)_{max}$ is so small that ℓ_{max} must be less than 10^{10} cm, which would imply that Einstein's theory is wrong. However, such an argument seems highly questionable. The numbers given in Section I indicate that, to be conservative, we should say that ℓ, or $\tilde{\ell}$, should be less than about a light-year. It is then possible that observational tests of Einstein's theory on a galactic scale could be used to rule the theory out, if the usual Maxwell equations are found to be correct.

V. CONCLUSION

The investigations of Johnson, as well as those presented in this paper, show that it is quite possible that Einstein's unified field theory may successfully describe the behavior of electrically charged particles, at least where quantum effects can be ignored. However, regardless of which of the two possible interpretations for electric charge one accepts, the theory necessarily implies that Maxwell-Lorentz electrodynamics is modified significantly on the scale of astronomical distances. We shall discuss here, in terms of the general structure of the theory, the reason why these modifications must occur. We shall see that this property is related to one of the features of Einstein's theory which makes it most attractive from a theoretical point of view.

From the presentation of the field equations in Section II, it is evident that the unified field theory, like general relativity, contains no "fundamental" dimensional parameters. In other words, the Lagrangian of the theory contains no arbitrary constants which have length dimensions. (In fact, it contains no arbitrary parameters at all.) As a result, the field equations transform homogeneously under an arbitrary change of scale of the coordinates. This formal scale invariance of Einstein's theory is a strong argument in its favor, on the basis of simplicity and elegance. Of course, this by no means implies that physical fields, as described by solutions of the equations, are invariant under such a change of scale.[28]

Consider now the approximate field equations (3.6) for the "gravitational" field $\gamma^{\mu\nu}$, i.e.,

$$\Box\gamma^{\mu\nu} = t^{\mu\nu} \quad . \tag{5.1}$$

The expression for $t^{\mu\nu}$ is given in (3.7), and is quadratic in the "electromagnetic" field $\phi^{\mu\nu}$ and its derivatives. Furthermore, it is homogeneous of second degree in derivatives of $\phi^{\mu\nu}$. It must have this latter property, in view of the formal scale invariance of the theory, because $\Box\gamma^{\mu\nu}$ does also. Now, the term $t^{\mu\nu}$ in the differential equation (5.1) plays the role of the "electromagnetic energy-momentum tensor" of the theory, within the context of approximate solutions. To see this, recall the familiar Einstein-Maxwell equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu} \quad , \tag{5.2}$$

where $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the gravitational energy-momentum tensor of general relativity, $T_{\mu\nu}$ is the Maxwell electromagnetic energy-momentum tensor, and k is a constant. Keeping only the linear approximation for the gravitational field, (5.2) becomes

$$\Box\gamma^{\mu\nu} = 2k \left(F^{\mu\sigma} F^{\nu}{}_{\sigma} - \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) \quad , \tag{5.3}$$

if the harmonic condition, $\partial_\mu \gamma^{\mu\nu} = 0$, is chosen. Here $\gamma^{\mu\nu}$ is the same as in (5.1), to this approximation, and $F^{\mu\nu}$ is, of course, the usual electromagnetic field, satisfying Maxwell's equations. The similarity in structure of (5.1) and (5.3) is evident, even though their right-hand sides are quite different in detail.

It is well-known[29] that the consistency conditions for a solution of (5.3) to exist, satisfying the harmonic condition, yields the usual Lorentz-Dirac equation for point singularities of $F^{\mu\nu}$ and $\gamma^{\mu\nu}$. It is far from obvious, at first sight, that a similar result can be obtained from (5.1), which is the main reason why Einstein's theory has received so little attention over the past 25 years. It is clear that the relation between the two antisymmetric tensors, $F^{\mu\nu}$ and $\phi^{\mu\nu}$, cannot be a simple one. Dimensional considerations show that the only possibility for producing terms similar to those in (5.3) is to try to find solutions for $\phi^{\mu\nu}$ of the form (3.35), or (3.14 - 3.15) and (3.8), containing a linear combination of two fields with inherently different dimensions. That such solutions do allow one to obtain equations of motion containing the Lorentz force, from such an apparently complicated expression as (3.7) for $t_{\mu\nu}$, is quite remarkable. (The key relation is (3.7') for $\partial^{\mu} t_{\mu\nu}$. This leads directly to (3.54) for our solutions, and to a similar equation for Johnson's solutions.[30]) It is evident that this linear combination of two fields necessarily implies that the equations of motion must also contain other terms, with inherent dimensions different from that of the Lorentz force. This, then, is the basic reason why Einstein's theory predicts modified Maxwell equations, containing the length parameter ℓ, or $\tilde{\ell}$. It is intimately related to the formal scale invariance of the theory.

The lack of any arbitrary parameters in the Lagrangian of Einstein's theory is one formal argument in favor of it. A second attractive property is its invariance under the group of local unitary gauge transformations, discussed in Section II. In what manner this symmetry is reflected in solutions of the field equations is a question that promises to bring a deeper understanding of the theory. The major argument in favor of the theory, of course, is that it is a very simple and natural generalization of the general theory of relativity, a theory which has a degree of logical completeness unlike any other physical theory we know.

In this paper we have argued that it may be possible to interpret the new solutions we have constructed as representing electric charges, rather than, in accord with the conventional interpretation, as magnetic charges. In a way, this recognition of the possibility of two alternative physical interpretations is a step backwards for the theory. The usual free-space Maxwell equations are symmetric under an interchange of electric and magnetic fields, but nature clearly is not. Thus, it was another argument in support of Einstein's unified field theory that it has no such symmetry, and furthermore, that one field equation, (2.35), seemed to rule out the existence of magnetically charged currents, in agreement with observation. While the existence of such currents in the theory now remains an open question, it is certainly true that the theory lacks symmetry under interchange of electric and magnetic fields. One needs only refer to the modified Maxwell equations of Section IV. A. However, it is worth noting that the lack of symmetry becomes apparent only at astronomical distances, for macroscopic fields.

It must be admitted that the new solutions presented in this paper are much less natural than are Johnson's solutions, since they involve singularities of a considerably more complicated type. We have by no means shown that they are completely consistent. It is quite possible that the string singularities which must be introduced may lead to physical consequences which rule out our alternative interpretation. At the moment, we do not have any understanding, on the basis of the theory, of why magnetic charges do not appear in nature. It is attractive to suppose that this is somehow connected with the existence of the strings extending between oppositely charged point masses. We observed that the paths of the strings are not arbitrary, as they are in Dirac's theory of magnetic poles, but must satisfy certain constraint conditions, if the solutions are to exist. Because of these constraints, a string may sufficiently restrict the motion of the two charges at the ends, that they will not behave like ordinary charged particles. Einstein frequently argued[1] that the only exact solutions to the field equations which should be permitted are those which are nonsingular. Precisely what this means, and what its relevance is to our approximate solutions, is an intriguing question.

In closing, let us mention the question of electromagnetic radiation, which we have not yet discussed in this paper. In view of the complexity of equation (3.7) for $t_{\mu\nu}$, it might appear that radiation fields in Einstein's theory could be quite different from those in the Maxwell-Lorentz theory. However, for Johnson's approximate solutions we can argue that this is not the case. We observed that for these solutions the only radiation reaction term in the equations of motion is the usual one in the Lorentz-Dirac equation. Hence, all conclusions concerning radiation which are deduced from the energy-momentum conservation law $\partial^{\mu} t_{\mu\nu} = 0$, will be the same as in the Maxwell-Lorentz theory. Since we have not been able to show that the additional radiation reaction terms necessarily vanish in the case of the new solutions, it is possible that they might lead to modified radiation fields. A better understanding of the string singularities will be needed in order to decide this question.

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APPENDIX

We wish to show here that the contribution to $F_\mu^{(p)\text{self}}$ from the second term in (3.72), and also $G_\mu^{(p)\text{self}}$ in (3.73), both vanish in the special case of two opposite charges connected by a straight string, in the static approximation. The proof in this very simple case is almost trivial, but it serves as a useful illustration of effects due to the presence of string singularities. The ill-defined formal expressions in (3.72 - 3.73) may be calculated by using the surface integral in (3.69),

$$\sum_p \int d^3 S'_{(p)}\, n'^{\mu}_{(p)}\, t_{\mu\nu}(x')\, D_{\text{ret}}(x-x') \quad .$$

This means that we must show that the appropriate contribution to this surface integral may be combined with the first term in (3.69) so that the sum has the form $\partial^\mu \gamma'^{H}_{\mu\nu}$, for some $\gamma'^{H}_{\mu\nu}$ of the form (3.57). From (3.7), $t_{\mu\nu}$ is quadratic in $\phi_{\mu\nu}$, and we see that only those terms that are quadratic in the functions $\psi^{(p)}_{\mu\nu}$, involving integrals over the strings, are of interest here. Let us call these terms $\bar{t}_{\mu\nu}$. In our special case we have only a single surface S', whose element d^3S' may be written as $d\tau'\, d^2S'$, where $d\tau'$ is a time element and d^2S' is an element of the two-dimensional surface surrounding the string at fixed time. The integral we need to consider is therefore

$$\int d^2S'\, n'^{\mu}\, \bar{t}_{\mu\nu}(\vec{x}')\, \frac{1}{4\pi|\vec{x}-\vec{x}'|} \quad ,$$

if we perform the time integration first. The static approximation of course implies that $\bar{t}_{\mu\nu}$ is independent of x^0. We may expand $\dfrac{1}{|\vec{x}-\vec{x}'|}$ about some fixed point on the string, and then it is easily seen that, in order to prove the desired result, it is sufficient to show that the integral I_ν, defined by

$$I_\nu = \int d^2S' \, n'^\mu \, \bar{t}_{\mu\nu}(\vec{x}') \quad , \tag{A.1}$$

vanishes in the limit that the integration surface approaches the string.

For the case at hand, we have for $\psi_{\mu\nu}^{(p)}$ only the single finite integral $\psi_{\mu\nu}^{(p_1)} - \psi_{\mu\nu}^{(p_2)} \equiv \psi_{\mu\nu}$, formed by the difference of two formally infinite integrals, $\psi_{\mu\nu}^{(p_1)}$ and $\psi_{\mu\nu}^{(p_2)}$, due to the two opposite charges, (p_1) and (p_2). Let the two charges be at points in space, a and -a, on the z-axis. Then the nonzero components of $\psi_{\mu\nu}$ are given by (3.48), with $\hat{n}$ a unit vector in the z-direction. Choosing the usual cylindrical coordinates ρ, z, and ϕ, we find

$$\psi_{0z} = \frac{1}{8\pi}\left[\frac{z-a}{\sqrt{\rho^2+(z-a)^2}} - \frac{z+a}{\sqrt{\rho^2+(z+a)^2}} + 2\log\frac{z+a+\sqrt{\rho^2+(z+a)^2}}{z-a+\sqrt{\rho^2+(z-a)^2}}\right] , \tag{A.2}$$

$$\psi_{0\rho} = \frac{1}{8\pi}\left[\frac{\rho}{\sqrt{\rho^2+(z-a)^2}} - \frac{\rho}{\sqrt{\rho^2+(z+a)^2}}\right] , \tag{A.3}$$

and $\psi_{0\phi} = 0$. We suppose, in the following, that the two-dimensional surface S' is described by the coordinates ρ, z, and ϕ. We may take it to consist of a cylindrical surface of radius δ, centered on the z-axis and extending from $z = -(a+\epsilon)$ to $z = a+\epsilon$, with $\epsilon > 0$, $a > 0$, together with discs perpendicular to the z-axis at the two ends of the cylinder. After calculating the integral, we shall take the limits $\delta \to 0$, and $\epsilon \to 0$, in that order. Because of the cylindrical symmetry and the static approximation, the only component of I_ν which is not obviously zero in this limit is I_z. It is given by

$$-I_z(\delta, \epsilon) = 2\pi\delta \int_{-(a+\epsilon)}^{a+\epsilon} dz \ \bar{t}_{\rho z}(\delta, z)$$

$$+ 2\pi \int_0^{\delta} d\rho \ \rho \left[\bar{t}_{zz}(\rho, a+\epsilon) - \bar{t}_{zz}(\rho, -(a+\epsilon)) \right] \quad . \tag{A. 4}$$

The extra minus sign comes from the fact that the normal vector n^μ points into the cylinder.

The expression for $\bar{t}_{\mu\nu}$ to be used in (A. 4) has the same form as (3. 7), except that $\phi_{\mu\nu}$ is replaced by $\frac{g'}{\ell^2} \psi_{\mu\nu}$, where $g' = g'^{(p_1)} = -g'^{(p_2)}$. By using the results (A. 2 - A. 3) for $\psi_{\mu\nu}$, one finds that $\bar{t}_{\rho z}(\delta, z) = -\bar{t}_{\rho z}(\delta, -z)$, so the first integral in (A. 4) vanishes. It is also easily verified that $\bar{t}_{zz}(0, \pm (a+\epsilon))$ are both finite, so that the second term vanishes in the limit $\delta \to 0$. Hence, we have that

$$\lim_{\epsilon \to 0} \ \lim_{\delta \to 0} \ I_\nu(\delta, \epsilon) = 0 \quad , \tag{A. 5}$$

as was to be shown. Gauss' theorem implies that this result is independent of the particular choice we have made for the surface S'.

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6. For a discussion of the Lorentz-Dirac equation for charged point masses in classical electrodynamics, see, for example, F. Rohrlich, Classical Charged Particles (Addison-Wesley, Reading, Massachusetts, 1965).
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10. There is a second version of the theory in which $h^{\mu\nu}$ is a real, nonsymmetric tensor. Most of our results concerning approximate solutions to the field equations of the "Hermitian version" of the theory may be applied to the case of the "real version" by simply supposing that $f^{\mu\nu}$ in equation (2.1) is imaginary, rather than real. However, the discussion of local unitary

transformations in Section II applies only for a Hermitian tensor $h^{\mu\nu}$. For a real tensor, the corresponding analysis is more involved. In the papers ~~of~~ Johnson (Reference 4), the "real version" of the theory is considered. Where results are quoted from Johnson's work in Sections III and IV, they have been converted so as to apply to the "Hermitian version", as is pointed out in subsequent footnotes.

11. This vector B_μ differs from the B_μ of Section II by a factor of 2.
12. Johnson actually considers solutions which are considerably more general than (3.14 - 3.17). We exhibit here only the simplest type which contains the essential features needed for derivation of equations of motion containing the Lorentz force.
13. For Johnson's derivation of the equations of motion for these charged singularities, see Johnson, Papers II and VII.
14. We could have chosen $g^{(p)}$ and $g'^{(p)}$ to be functions of $\tau^{(p)}$, but the gauge condition $\partial^\mu B_\mu = 0$ would then imply that they are independent of $\tau^{(p)}$. Since we may always choose this gauge, we may always choose the "charges" $g^{(p)}$ and $g'^{(p)}$ so that they are constants. This means that for our solutions "charge" may be defined so that it is conserved. This is a special case of a general result discussed in Johnson, Paper IV.
15. We have here ignored the surface integral "at infinity". This cannot be justified without a study of radiation fields, which is difficult without having an explicit expression for the sheets $y_\mu^{(p)}$, which occur in $\psi_{\mu\nu}^{(p)}$ of (3.34). However, we can always modify the asymptotic behavior of the Green's function to suit any boundary conditions we wish to choose. This asymptotic behavior is irrelevant to the derivation of equations of motion for the point singularities, so we shall not consider it further.

16. We shall not detail these lengthy manipulations since they are very similar to steps Johnson has written out in deriving the equations of motion which follow from his solutions. For this, see Johnson, Paper VII, especially pages 2841 - 2842.

One important point deserves comment, however. If we permit solutions $\gamma^{H}_{\mu\nu}$ of the homogeneous equation (3.51) which are singular not merely on world-lines, but also on two-dimensional sheets, as in (3.57), then we shall not obtain nontrivial equations of motion for any spin moments the point masses may have. By this, we mean that any terms on the right-hand side of (3.55) which could contribute to such equations of motion may always be expressed in the form $\partial^{\mu}\gamma'^{H}_{\mu\nu}$, for some choice of $a'^{(p)}_{\mu\nu}$ and $a'^{(p)}_{\mu\nu,\rho_1 \cdots \rho_i}$. Thus, we may always define spin moments in this case so thay they satisfy "free particle" equations of motion, with no "interaction" terms. This is a generalization of a well-known result in general relativity, which states that, for solutions containing singularities only on world-lines, the equations of motion for all moments with spin greater than one are "trivial" in the above sense. For this latter result, see, for example, R. P. Kerr, Nuovo Cimento 13, 469 (1959). Johnson shows that this result also applies for his solutions of Einstein's unified field theory in Johnson, Paper I, Appendix A. With the more complicated singularities of our solutions, the result holds for spin one, as well.

17. This result becomes evident if we realize that calculating the integral in (3.69) is essentially equivalent to integrating the field equations in the region near the singular sheets. It is then easy to see that, for dimensional reasons, these highly singular terms may always be absorbed into $\partial^{\mu}\gamma'^{H}_{\mu\nu}$ in (3.58).

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19. It is at this point that the difference between the "Hermitian version" and the "real version" of Einstein's theory becomes important. (See Reference 10.) In the "real version" of the theory the charges $g^{(p)}$ and $g'^{(p)}$ as we have defined them would be imaginary, rather than real, and it then turns out that the choice $g^{(p)} = -g'^{(p)}$ is the one which gives the correct Coulomb force.

20. Johnson, in Paper II, makes the choice $f^{(p)} = -f'^{(p)}$ because he considers the "real version" of Einstein's theory. (See References 10 and 19.) This difference in sign between the two versions of the theory is important for the discussion of observational tests in Section IV. If the sign is chosen so that the usual Lorentz force has the correct sign for either version, then this leads, for the two different versions, to opposite signs for the extra force terms. As a result, the extra field in the "modified" Maxwell equations of Section IV will have the opposite sign if one chooses to consider the "real version" of the theory, instead of the "Hermitian version" given in this paper.

21. This result was noted in Johnson, Paper II, Appendix C. Johnson considers the "real version" of Einstein's theory, which leads to a difference in the sign of the extra term in the potential. (See References 10, 19, and 20.)

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