# FIXED MASS AND SCALING SUM RULES: GENERALIZED TRUNCATION AND CURRENT ALGEBRA CONSTRAINTS ON REGGE RESIDUES 

B.F.L. Ward

## Errata

The author would like to call attention to the following errors in the preprint:
Page 7 - In Eq. (2.10), the factor $\sqrt{1+\overrightarrow{\mathrm{q}}^{2}+\frac{\mu^{2}-\mathrm{M}^{2}}{p_{0}^{2}}}$ should be $\sqrt{1+\frac{-\vec{q}^{2}+\mu^{2}-\bar{M}^{2}}{p_{0}^{2}}}$.

Page 15. - Equations (3.17): In the first equation, the subscript i should be 1.
Page 18 - In Eq. (3.22): The lower limit on the first integral over s should be $q^{2}\left(1+\eta^{2} M^{2} / p_{0}^{2} q^{2}\right)$ instead of $q^{2}$.
Page 20 - Equation (3.26): $\mathrm{F}_{1}^{\bar{\nu}+\nu}$ should be $\widetilde{\mathrm{F}}_{1}^{\bar{\nu}+\nu}$.
Page 28, Line 5 - Fristsch should be Fritzsch.

The author humbly begs everyone's pardon.

# FIXED MASS AND SCALING SUM RULES: GENERALIZED TRUNCATION 

 AND CURRENT ALGEBRA CONSTRAINTS ON REGGE RESIDUES*B. F. L. Ward<br>Stanford Linear Accelerator Center Stanford University, Stanford, California 94305


#### Abstract

Using the correspondence principle (continuity in dynamics), we extend the approach of Keppel-Jones-Ward-Taha to fixed mass and scaling current algebraic sum rules so as to consider explicitly the contributions of all classes of intermediate states. A natural, generalized formulation of the truncation ideas of Cornwall, Corrigan, and Norton is introduced as a by-product of this extension. The formalism is illustrated in the familiar case of the spin independent Schwinger term sum rule. New sum rules are derived which relate the Regge residue functions of the respective structure functions to their fixed hadronic mass limits for $q^{2} \rightarrow \infty$ 。


(Submitted to Phys.Rev. D)

[^0]
## I. INTRODUCTION

Keppel-Jones, ${ }^{1}$ Ward, ${ }^{2}$ and Taha ${ }^{3}$ have introduced a new, systematic approach to current algebraic sum rules, both at fixed $q^{2}$ and in the $\mathrm{lim}_{b j}$, which takes as its starting point the familiar quark equal-time current algebra and Bjorken scaling. This approach is to be compared with that of Dicus, Jackiw, and Teplitz ${ }^{4}$ (DJT), which takes as its starting point the more model dependent quark light-cone algebra. ${ }^{5,6}$ Let us remark that both of these approaches represent an improvement over the naive infinite momentum method for deriving sum rules in the sense that both consider the contributions of a wider class of intermediate states than does the naive method.

Specifically, while the approach of DJT does not handle the fixed-mass class II states of Adler and Dashen, ${ }^{7}$ it does, for example, handle the scaling Z graphs. As is well-known, the naive $\mathrm{p}_{0} \rightarrow \infty$ method neglects both Z graphs and class II states.

On the other hand, the approach of Refs. $1,2,3$ in principle permits the inclusion of all kinds of intermediate states. However, in the form in which it was introduced, a convergence presumption (Eq. (2.7) below) about the inclusion of states near $x=-q^{2} / 2 \mathrm{M} \nu=0,-1$ was made. ${ }^{8}$ As a result of this assumption, certain possible contributions are not systematically considered. Below, we shall argue that this assumption can be relaxed somewhat by invoking continuity in dynamics (the correspondence principle recently discussed by Bjorken and Kogut ${ }^{9}$ ) and that the resulting formalism explicitly considers all kinds of states.

This extended formalism will also be seen to provide a natural, generalized formulation of the truncation theory of Cornwall, Corrigan, and Norton (CCN). ${ }^{10}$ In addition, the formalism will be seen to imply sum rules relating the residue
functions of the respective structure functions to their $q^{2} \rightarrow \infty$ fixed hadronic mass limits.

Our ideas will be illustrated in the case of the spin independent Schwinger term sum rule of CCN to facilitate comparison with the work of these authors. However, it will be apparent from the discussion that the ideas pertain to all components of the quark equal-time current algebra and, hence, represent a complete, systematic discussion of all current algebraic sum rules, both at fixed $q^{2}$ and in the $\lim _{b j}$, which considers all classes of intermediate states. A discussion in this connection of the other components of the equal-time algebra will appear elsewhere. ${ }^{11}$

This paper is organized as follows. In Section II we extend the formalism of Refs. $1,2,3$ so that it systematically treats the intermediate states near $x=0,-1$. Section III is devoted to illustration of the resulting formalism in the familiar case of the spin independent Schwinger term sum rule of CCN. In this section, generalized truncation is explicitly demonstrated and several new constraints relating the Regge residue functions of the respective structure functions to their fixed-hadronic mass limits as $q^{2} \rightarrow \infty$ are obtained. Section IV contains some concluding remarks.

## II. CONSIDERATION OF STATES NEAR $\mathrm{x}=0,-1$

In this section we shall show how one extends the approach of Refs. 1, 2, 3 to current algebraic sum rules so that it treats systematically the intermediate states near $x=0,-1$. For the sake of completeness, we shall begin in the spirit of a review of this approach. We define the weak hadronic tensor by

$$
\begin{align*}
\mathrm{w}_{\mu \nu}^{\mathrm{ab}}= & \left.\frac{1}{2 \pi} \int \mathrm{~d}^{4} \mathrm{y} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{y}}<\mathrm{p}\left|\left[\mathrm{~J}_{\mu}^{\mathrm{a}}(\mathrm{y}), \mathrm{J}_{\nu}^{\mathrm{b}}(0)\right]\right| \mathrm{p}\right\rangle \\
= & -\left(\mathrm{g}_{\mu \nu}-\mathrm{q}_{\mu} \mathrm{q}_{\nu} / \mathrm{q}^{2}, \mathrm{w}_{1}^{\mathrm{ab}}+\frac{1}{\mathrm{M}^{2}}\left(\mathrm{p}_{\mu}-\frac{\mathrm{q} \cdot \mathrm{p}}{\mathrm{q}^{2}} \mathrm{q}_{\mu}\right)\left(\mathrm{p}_{\nu}-\frac{\mathrm{q} \cdot \mathrm{p}}{\mathrm{q}^{2}} \mathrm{q}_{\nu}\right) \mathrm{w}_{2}^{\mathrm{ab}}\right. \\
& +\frac{\mathrm{q}_{\mu} \mathrm{q}_{\nu}}{\mathrm{M}^{2}} \mathrm{~W}_{4}^{\mathrm{ab}}+\frac{\mathrm{p}_{\mu} \mathrm{q}_{\nu}+\mathrm{p}_{\nu} \mathrm{q}_{\mu}}{\mathrm{M}^{2}} \mathrm{w}_{5}^{\mathrm{ab}}+\ldots \tag{2.1}
\end{align*}
$$

where $\mid p>$ is a nucleon state of 4 -momentum $p$ (we suppress spin labels), the $J_{\mu}^{a}$ are the full V-A currents, and the $W_{i}$ are the by now familiar structure functions of $\mathrm{q}^{2}$ and $\nu=\mathrm{q} \cdot \mathrm{p} / \mathrm{M}$ ( M denotes the nucleon rest mass). We shall always ignore the possibility of time reversal violation in (2.1). Now, as is well known, Eq. (2.1) and standard equal-time current algebra give, for $\vec{q} \cdot \vec{p}=0$,

$$
\begin{equation*}
\int_{-\infty}^{\infty} \mathrm{d} \nu \frac{\mathrm{M}}{\mathrm{p}_{0}} \mathrm{w}_{\mu \nu}^{\mathrm{ab}}=\mathrm{C}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{~g}_{\mathrm{V}}, \mathrm{~g}_{\mathrm{A}}\right)+\mathrm{P}_{\mu \nu}^{\mathrm{ab}}(\overrightarrow{\mathrm{q}}) \tag{2.2}
\end{equation*}
$$

where $C_{\mu \nu}^{\mathrm{ab}}$ is a definite linear combination of the vector and axial vector couplings $g_{V}, g_{A}$, and $P_{\mu \nu}^{a b}(\vec{q})$ is the Schwinger term polynomial in $\vec{q}$ and satisfies

$$
\begin{equation*}
\mathrm{P}_{\mu \nu}^{\mathrm{ab}}(0)=0 \tag{2.3}
\end{equation*}
$$

Isolating the kinematically independent portions of (2.2) we have in general equations of the type

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \nu \mathrm{I}(\mathrm{q}, \mathrm{p})=\mathrm{B} \tag{2.4}
\end{equation*}
$$

where $B$ is a number which is known from equal-time current algebra and $\mathrm{I}(\mathrm{q}, \mathrm{p})$ is a linear combination of structure functions with known functions of $q, p \overrightarrow{a s}$ coefficients. Here, let us remark that in the naive $p_{0} \rightarrow \infty$ approach, one interchanges the integration over $\nu$ in (2.4) with the limit $p_{0} \rightarrow \infty$. Of course, it is well known that this inter change is suspect-although it leads to Adler 's ${ }^{12}$ sum rule for $W_{2}$ when $\mu=0=\nu$ in (2.2), it leads to nonsense ( $0=1$ ) when $\mu=i \neq j=\nu \quad(i, j=1,2,3)$. In Refs. $1,2,3$, it is shown that, for $\vec{q} \cdot \vec{p}=0$,

$$
\begin{align*}
\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{0}^{\infty} \mathrm{d} \nu \mathrm{I}(\mathrm{q}, \mathrm{p})= & \lim _{\eta \rightarrow \infty}\left[\int_{0}^{\eta} \mathrm{d} \nu \lim _{\mathrm{p}_{0} \rightarrow \infty} \mathrm{I}(\mathrm{q}, \mathrm{p})-\int_{-1}^{-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \mathrm{dx} \frac{\mathrm{p}_{0} \rightarrow \infty}{\mathrm{x}}\right. \\
& \left.+\lim _{\mathrm{p}_{0} \rightarrow \infty}\left\{\int_{-\mathfrak{q}^{-}}^{\mathrm{q}^{2} / 2 \eta \mathrm{M} \mathrm{M}} \mathrm{dx} \frac{\mathscr{I}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}+\int_{-1-\mathrm{q}^{2} / 2 \eta \mathrm{M}}^{-1} \mathrm{dx} \frac{\mathscr{F}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}\right\}\right] \tag{2.5}
\end{align*}
$$

where

$$
\begin{align*}
& \mathscr{I}^{-}=\nu^{-} I\left(\mathrm{q}_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right) \\
& \mathrm{q}_{0}^{\ddagger}=-\mathrm{x} \mp\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2} \mathrm{p}_{0},  \tag{2.6}\\
& \nu^{-}=\mathrm{q}_{0}^{-} \mathrm{p}_{0} / \mathrm{M}
\end{align*}
$$

and

$$
\epsilon=|\overrightarrow{\mathrm{q}}| / \mathrm{p}_{0}
$$

This result follows from the following assumptions:
Assumption II(a): The structure functions scale in the sense of Bjorken ${ }^{13}$ :

$$
\lim _{\mathrm{bj}}(\nu / \mathrm{M}){ }^{\mathrm{n}_{\mathrm{i}}} \mathrm{MW}_{\mathrm{i}}=\mathrm{F}_{\mathrm{i}}(\mathrm{x})
$$

for some integer $n_{i} \geq 0$.

Assumption II(b): In summing the contributions to the LHS of (2.5) of those intermediate states which correspond to the Bjorken limit âs $\mathrm{p}_{0} \rightarrow \infty$, we may commute the summation operation with taking the $\lim _{b j}$.
In Refs. $1,2,3$, it was further assumed ${ }^{8}$ that

$$
\begin{equation*}
\lim _{\eta \rightarrow \infty} \lim _{\mathrm{p}_{0} \rightarrow \infty}\left[\int_{-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}}^{\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \mathrm{dx} \frac{\mathscr{I}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}+\int_{-1-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}}^{-1} \mathrm{dx} \frac{\mathscr{I}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}\right]=0 \tag{2.7}
\end{equation*}
$$

This is the (additional) convergence assumption to which we referred above. It may be relaxed somewhat if we invoke the correspondence principle. ${ }^{9}$ By this principle, we may identify the limits of Bjorken and Regge near $\mathrm{x}=0$. From this correspondence and assumption II(b) we have (see the appendix)

$$
\begin{align*}
\lim _{\eta \rightarrow \infty} \lim _{\mathrm{p}_{0} \rightarrow \infty} & \int_{-\vec{q}^{2} / 2 \eta \mathrm{M}}^{\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \frac{\mathscr{F}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}} \\
& =\lim _{\eta \rightarrow \infty} \lim _{p_{0} \rightarrow \infty} \int_{-\vec{q}^{2}\left(1-\eta^{2} \mathrm{M}^{2} / \overrightarrow{\mathrm{q}}^{2} \mathrm{p}_{0}^{2}\right)}^{\frac{\vec{q}^{4} \mathrm{p}_{0}^{2} / \eta^{2} \mathrm{M}^{2}}{\mathrm{p}_{0} \mathrm{I}(\mathrm{q}, \mathrm{p})}} 2  \tag{2,8}\\
& =\lim _{\eta \rightarrow \infty} \lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{-\vec{q}^{2}\left(1-\eta^{2} \mathrm{M}^{2} / \overrightarrow{\mathrm{q}}^{2} \mathrm{p}_{0}^{2}\right)}^{\mathrm{dq}_{0}^{2} / \eta^{2} \mathrm{M}^{2}} \mathrm{dq}^{2}
\end{align*}
$$

where $R$ is the Regge limit of $p_{0} I$.
We may treat the other term in (2.7) as follows. Note that any delicacies near $x=-1$ should be related by crossing to delicacies near $x=1$. A possible meaningful asymptotic limit near $\mathrm{x}=1$ is the fixed hadronic mass limit $\nu \rightarrow \infty$
with $2 \mathrm{M} \nu+\mathrm{q}^{2}+\mathrm{M}^{2} \equiv \mu^{2}$ fixed. ${ }^{10}$ However, the function $q_{0}^{-}$near $\mathrm{x}=-1$ is evidently not appropriate for discussing this limit. For this reason, we use crossing and change variables from $x$ on $q_{0}^{-}$near $x=-1$ to $\mu^{2}$ on $q_{0}^{+}$near $x=1$, where $q_{0}^{+}$ is given by (2.6). Let $H$ denote the fixed $\mu^{2}$ limit of $I\left(q_{0}^{+},-\vec{q}, p\right)$ as $p_{0} \rightarrow \infty$. Then, we have for $\eta>0$

$$
\begin{equation*}
\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{-1-\vec{q}^{2} / 2 \eta \mathrm{M}}^{1} \mathrm{dx} \frac{\mathscr{g}^{-}}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}=\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{\mu_{0}^{2}}^{2 \mathrm{p}_{0}^{2} \overrightarrow{\mathrm{q}}^{2} / \eta \mathrm{M}} \mathrm{~d} \mu^{2} \frac{\mathrm{H}}{2 \mathrm{M} \sqrt{1+\epsilon \epsilon^{2}+\frac{\mu^{2}-\mathrm{M}^{2}}{2}}} \tag{2.9}
\end{equation*}
$$

where we have used correspondence (see the appendix). Here, $\mu_{0}^{2}$ is the threshold and is clearly $M^{2}$. Thus, the general sum rule equation following from (2.4) is now

$$
\begin{align*}
& \lim _{\eta \rightarrow \infty}\left[\int_{0}^{\eta} d \nu \lim _{0 \rightarrow \infty} I(q, p)-\int_{-1}^{-\vec{q}^{2} / 2 \eta M} d x \lim _{p_{0} \rightarrow \infty} \frac{\left.\nu^{-I\left(q_{0}\right.}, \vec{q}, p\right)}{x}\right. \\
& \lim _{0 \rightarrow \infty}\left\{\frac{1}{2 M} \int_{\mu_{0}}^{2 p_{0}^{2} \vec{q}^{2} / \eta M} d \mu^{2} \frac{\mathrm{H}}{\sqrt{1+\frac{\overrightarrow{\mathrm{q}}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}}\right. \\
& \left.\left.+\int_{-\vec{q}^{2}\left(1-\eta^{2} M^{2} / \vec{q}^{2} p_{0}^{2}\right)}^{\vec{q}^{4} p_{0}^{2} / \eta^{2} M^{2}} \frac{R}{2 M \sqrt{q^{2}+\vec{q}^{2}}}\right\}\right] \\
& =\mathrm{B} \tag{2.10}
\end{align*}
$$

The integral over $\nu$ in this last equation is at fixed $q^{2}=-\vec{q}^{2}$ and is the familiar contribution of the naive $p_{0} \rightarrow \infty$ method to the LHS of (4) as $p_{0} \rightarrow \infty$. The second term, a scaling integral, is discussed in Refs. 1, 2, 3; it is also given by the formalism of Ref. 4. Finally, the terms in curly brackets, which are not discussed in Refs. 1, 2, 3, 4, represent the contributions of intermediate states at $x=0,-1$ such as class II fixed mass states, u-channel fixed hadronic mass states, etc. Equation (2.10) represents, therefore, an approach to all current algebraic sum rules which systematically considers the contributions of all classes of intermediate states. Embodied in it is also a natural generalized formulation of the truncation theory of CCN as well as general constraints on Regge residues. These last two statements will now be illustrated by a discussion of the spin-independent Schwinger term sum rule of CCN.

## III. ILLUSTRATION OF THE EXTENDED APPROACH

We choose to illustrate the application of (2.10) in the particular case of the spin independent Schwinger term sum rule of CCN in order to facilitate comparison with the truncation theory of these authors. We shall first derive the sum rule in the fashion of references $1,2,3$ (Eq. 2.6) making the convergence assumption (2.7). Then we shall present the truncated version due to CCN. Finally, we shall see what (2.10) (our extension of the approach of references $1,2,3$ ) has to say concerning this sum rule. We turn now to the derivation based on (2.6) and (2.7).

The sum rule under discussion obtains from the $\mu=0, \nu=\mathbf{i}, \mathbf{i}=1,2,3$ aspect of (2.2) and (2.4). Considering this aspect, we find

$$
\begin{align*}
\sigma_{1}+\int_{0}^{\infty} \mathrm{d} \nu \frac{\mathrm{M}}{\mathrm{p}_{0}}\left[\frac{\mathrm{Mq}_{0}}{\mathrm{q}^{2}} \mathrm{~W}_{1}^{\bar{\nu}+\nu}\right. & +\frac{1}{2 \mathrm{Mx}}\left(\mathrm{p}_{0}+\frac{\mathrm{q}_{0}}{2 \mathrm{x}}\right) \mathrm{w}_{2}^{\bar{\nu}+\nu} \\
& \left.+\frac{\mathrm{q}_{0}}{\mathrm{M}} \mathrm{w}_{4}^{\bar{\nu}+\nu}+\frac{\mathrm{p}_{0}}{\mathrm{M}} \mathrm{~W}_{5}^{\bar{\nu}+\nu}\right]=0 \tag{3.1}
\end{align*}
$$

where the notation $\mathrm{W}_{\mathrm{i}}^{\bar{\nu}}+\nu$ is the obvious: $\mathrm{W}_{\mathrm{i}}^{\bar{\nu}}+\nu=\mathrm{W}_{\mathrm{i}}^{\bar{\nu}}+\mathrm{W}_{\mathrm{i}}^{\nu}$, and $\sigma_{1}$ is the spin independent Schwinger term defined in Eq. (59a) of reference 2 (for example). Taking the limit $p_{0} \rightarrow \infty$ according to (2.6) and (2.7), we obtain the result of references $1,2,3,4$

$$
\begin{equation*}
\sigma_{1}+\int_{0}^{\infty} \mathrm{d} \nu\left[\frac{\mathrm{M} \nu}{-\mathrm{q}^{2}} \mathrm{~W}_{2}^{\bar{\nu}+\nu}+\left.\mathrm{W}_{5}^{\bar{\nu}+\nu} \mathrm{q}^{2}\right|_{\text {fixed }}-\int_{0}^{\mathrm{l}} \mathrm{dx} \frac{\mathrm{~F}_{1}^{\bar{\nu}+\nu}(\mathrm{x})}{\mathrm{x}}=0\right. \tag{3,2}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{F}_{1}(\mathrm{x})=\lim _{\mathrm{bj}} \mathrm{MW} W_{1} \tag{3,3}
\end{equation*}
$$

and we have taken the usual ${ }^{14}$ scaling behavior for $W_{2}, W_{4}, W_{5}$ : in the $\lim _{b j}$

$$
\begin{align*}
& \nu \mathrm{W}_{2} \rightarrow \mathrm{~F}_{2}(\mathrm{x}) \\
& \frac{\nu^{2}}{\mathrm{M}} \mathrm{~W}_{\mathrm{i}} \rightarrow \mathrm{~F}_{\mathrm{i}}(\mathrm{x}) \quad \mathrm{i}=4,5 \tag{3,4}
\end{align*}
$$

(Of course, to derive (3.2), we only need $\lim _{b j} W_{2}, \nu \mathrm{~W}_{\mathrm{i}}=0, \mathrm{i}=4,5$. ) The integrals in (3.2) diverge if the respective structure functions receive contributions from Regge poles with intercepts $J \geq 0$ 。 It is obvious that the assumption (2.7) is precisely the statement that no such poles contribute to (3.2). In the event that such poles do contribute, what does one do ?

The dispersive approach of CCN allows one to proceed as follows. One presumes only (fixed) Regge poles with $J \neq 0$ occur in the $W_{i}$ so that in the Regge limit

$$
\begin{align*}
& \mathrm{W}_{\mathrm{i}} \rightarrow \sum_{\alpha>0} \mathrm{C}_{\mathrm{i}, \alpha}\left(\mathrm{q}^{2}\right) \nu^{\alpha} \quad \mathrm{i}=1,4 \\
& \mathrm{~W}_{2} \rightarrow \sum_{\alpha>0} \mathrm{C}_{2, \alpha}\left(\mathrm{q}^{2}\right) \nu^{\alpha-2}  \tag{3.5}\\
& \mathrm{~W}_{5} \rightarrow \sum_{\alpha>0} \mathrm{C}_{5, \alpha}\left(\mathrm{q}^{2}\right) \nu^{\alpha-1}
\end{align*}
$$

In general, as we have already observed, due to these poles, Eq. (3.2) may not obtain. However, consider the truncated functions ${ }^{10}$

$$
\begin{align*}
& \overline{\mathrm{W}}_{\mathrm{i}}=\mathrm{W}_{\mathrm{i}}-\sum_{\alpha>0} \mathrm{C}_{\mathrm{i}, \alpha\left(\mathrm{q}^{2}\right) \nu^{\alpha} \quad \mathrm{i}=1,4}^{\overline{\mathrm{W}}_{2}=\mathrm{W}_{2}-\sum_{\alpha>0} \mathrm{C}_{2, \alpha^{\left(\mathrm{q}^{2}\right)} \nu^{\alpha-2}}} \begin{array}{l}
\overline{\mathrm{W}}_{5}=\mathrm{W}_{5}-\sum_{\alpha>0} \mathrm{C}_{5, \alpha^{\left(\mathrm{q}^{2}\right) \nu}} \nu^{\alpha-1} \\
\overline{\mathrm{~F}}_{1}=\theta(1-\mathrm{x}) \mathrm{F}_{1}(\mathrm{x})-\mathrm{M} \sum_{\alpha>0} \mathrm{f}_{1, \alpha} \mathrm{x}^{-\alpha}
\end{array}, l \text {. }
\end{align*}
$$

where the $f_{1, \alpha}$ are determined by correspondence

$$
\begin{equation*}
\mathrm{C}_{1, \alpha}\left(\mathrm{q}^{2}\right) \xrightarrow{\mathrm{q}^{2} \rightarrow \pm \infty} \mp\left|\frac{2 \mathrm{M}}{\mathrm{q}^{2}}\right|^{\alpha} \mathrm{f}_{1, \alpha} \tag{3.7}
\end{equation*}
$$

These functions obviously do not possess the "bad" asymptotic behavior indicated by (3.5) and CCN show dispersively that the analogue of Eq。(3.2) pertains ${ }^{15}$ :

$$
\begin{equation*}
\left.\int_{0}^{\infty} \mathrm{d} \nu\left[\frac{\mathrm{M} \nu}{-\mathrm{q}^{2}} \overline{\mathrm{~W}}_{2}^{\bar{\nu}+\nu}+{\overline{\mathrm{W}_{5}}}^{\bar{\nu}+\nu}\right]\right|_{\mathrm{q}^{2}}-\int_{0}^{\infty} \mathrm{dxxed} \frac{\overline{\mathrm{~F}}_{1} \bar{\nu}+\nu}{\mathrm{x}}+\sigma_{1}=0 \tag{3,8}
\end{equation*}
$$

For comparison and illustration, we shall turn next to Eq. (2.10) for this situation. We shall see that, in addition to the generalization of (3.8), Eq. (2.10) will allow us to derive new current algebra constraints on the Regge residue functions $\mathrm{C}_{\mathrm{i}, \alpha}$.

In using (2.10) we may relax the restriction $\alpha \neq 0$ in (3.5), i.e., we allow poles with $J=0$. Thus, when we apply (2.10) in the situation under discussion, we find the following corrected version of (3.2): for $q^{2}<0$,

$$
\begin{align*}
& \lim _{\eta \rightarrow-\infty}\left\{\left.\int_{0}^{\eta} \mathrm{d} \nu\left(\frac{\mathrm{M} \nu}{-\mathrm{q}^{2}} \mathrm{w}_{2}^{\bar{\nu}+\nu}+\mathrm{W}_{5}^{\bar{\nu}}+\nu \mathrm{q}^{2}\right)\right|_{\text {fixed }}-\int_{-q^{2} / 2 \eta \mathrm{M}}^{1} \mathrm{dx} \frac{\mathrm{~F}_{1}(\mathrm{x})}{\mathrm{x}}\right. \\
& +\lim _{0 \rightarrow \infty}\left(\frac{1}{2 M} \int_{\mu_{0}^{2}}^{-2 p_{0}^{2} q^{2} / \eta M} d \mu^{2} \frac{H}{\sqrt{1+\frac{\mu^{2}-M^{2}-q^{2}}{2}}}+\frac{1}{2} \int_{q^{2}}^{\left.p_{\left(1+\eta^{2}\right.}^{2} M^{2} / p_{0}^{2} q^{2}\right)}{ }^{4} p_{0}^{2} / \eta^{2} M^{2}{ }^{2}{ }^{2} R_{1}\left(s, q^{2}, p_{0}\right)\right. \\
& \left.\left.\left.+\frac{1}{M} R_{4}\left(s, q^{2}, p_{0}\right)+\frac{R_{2}\left(s, q^{2}, p_{0}\right)+R_{5}\left(s, q^{2}, p_{0}\right)}{\sqrt{s-q^{2}}}\right]\left.\right|_{q^{\text {fixed }}}\right)\right\}=-\sigma_{1} \tag{3.9}
\end{align*}
$$

where ${ }^{16}$

$$
\begin{align*}
\mathrm{R}_{1}\left(\mathrm{~s}, \mathrm{q}^{2}, \mathrm{p}_{0}\right)=\lim _{\nu \rightarrow \infty} \mathrm{W}_{1}^{\bar{\nu}+\nu}(\nu, \mathrm{s}) & =\operatorname{Tsixed} \mathrm{C}_{1, \alpha}^{\bar{\nu}+\nu}(\mathrm{s}) \nu^{\alpha} \\
& =\sum_{\alpha \geq 0} \mathrm{C}_{1, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})\left(\frac{\sqrt{\mathrm{s}-\mathrm{q}^{2}} \mathrm{p}_{0}}{\mathrm{M}}\right)^{\alpha}, \\
\mathrm{R}_{2}\left(\mathrm{~s}, \mathrm{q}^{2}, \mathrm{p}_{0}\right)= & \frac{-\mathrm{q}^{2}}{\mathrm{~s}^{2}} \sum_{\alpha \geq 0} \mathrm{C}_{2, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})\left(\frac{\sqrt{\mathrm{s}-\mathrm{q}^{2}}}{\mathrm{M}}\right)^{\alpha-1} \mathrm{p}_{0}^{\alpha}, \\
\mathrm{R}_{4}\left(\mathrm{~s}, \mathrm{q}^{2}, \mathrm{p}_{0}\right)= & \sum_{\alpha \geq 0} \mathrm{C}_{4}^{\bar{\nu}+\nu}(\mathrm{s})\left(\frac{\sqrt{\mathrm{s}-\mathrm{q}^{2}} \mathrm{p}_{0}}{\mathrm{M}}\right)^{\alpha}, \tag{3.10}
\end{align*}
$$

and

$$
\mathrm{R}_{5}\left(\mathrm{~s}, \mathrm{q}^{2}, \mathrm{p}_{0}\right)=\sum_{\alpha \geq 0} \mathrm{C}_{5, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})\left(\sqrt{\mathrm{s}-\mathrm{q}^{2}}\right)^{\alpha-1}\left(\frac{\mathrm{p}_{0}}{\mathrm{M}}\right)^{\alpha}
$$

The LHS of (3.9) would appear to be ill-defined. However, we shall give it an explicit meaning momentarily.

To proceed, we require the form of $H$, the limit of $I\left(q_{0}^{+},-\vec{q}, p\right)$ as $p_{0} \rightarrow \infty$ with $\mu^{2}$ fixed. This limit was first discussed by CCN in the context of the DGS ${ }^{17}$ representation for forward current-hadron scattering amplitudes. ${ }^{18}$ These authors showed that if the imaginary part $V$ of such an amplitude scales in the sense

$$
\begin{equation*}
\lim _{b j} V=F(x) \tag{3.11}
\end{equation*}
$$

then it follows from the DGS representation that $V$ has the following form ${ }^{10}$

$$
\begin{equation*}
\mathrm{V}=\sum_{\mathrm{m}=1}^{\mathrm{N}} \nu^{\mathrm{m}} \mathrm{G}_{\mathrm{m}}\left(2 \mathrm{M} \nu+\mathrm{q}^{2}\right)-\sum_{\mathrm{m}=0}^{1} \nu \nu^{\mathrm{m}-1} \int_{0}^{2 \mathrm{M} \nu+\mathrm{q}^{2}} \mathrm{~d} \sigma \mathrm{~h}_{\mathrm{m}}\left(\sigma,-\mathrm{x}-\frac{\sigma}{2 \mathrm{M} \nu}\right) \tag{3.12}
\end{equation*}
$$

where the $h_{m}$ are spectral functions and the $G_{m}(z)$ decrease faster than $z^{-m}$ for large $z$ and are not involved in the scaling (3.11). If any of the $G_{m}$ are non-zero, then $V$ grows like $\nu^{\mathrm{n}}(\mathrm{n} \geq 1) \nu \rightarrow \infty$ with $\mu^{2}$ fixed. The data do not (to quote CCN) "unequivocally rule out the possibility of small terms of this kind in the cross sections." However, these terms were considered to be sufficiently implausible that they were suppressed in the discussion of CCN. In what follows, for the sake of generality, we shall treat the $G_{m}$ explicitly and employ the complete representation (3.12) in describing the form of $V$ as $\nu \rightarrow-\infty$ with $\mu^{2}$ fixed. Obviously, the behavior of a general V in the latter limit determines the structure of H .

Now it is apparent from (3.12) that the form of V in the $\mu^{2}$ fixed, $\nu \rightarrow-\infty$ limit is determined by the $\mathrm{G}_{\mathrm{m}}$ and by the behavior of $\mathrm{h}_{0}(\sigma, \beta)$ and $\mathrm{h}_{1}(\sigma, \beta)$ near $|\beta|=1$. For example, suppose, for V odd under crossing, $\mathrm{h}_{1}(\sigma, \beta)$ behaves
near $|\beta|=1$ as

$$
\begin{equation*}
h_{1}(\sigma, \beta)=\sum_{\gamma} h_{1 \gamma^{(\sigma)}}(1-|\beta|)^{-\gamma} \epsilon(\beta)+\widetilde{h}_{1}(\sigma, \beta) \tag{3.13}
\end{equation*}
$$

where $\epsilon(\rho) \equiv \rho /|\rho|$, and $h_{1}(\sigma, \beta)$ is regular as $|\beta| \rightarrow 1$. From (3.12) it follows that (3.13) makes the following contribution to the fixed $-\mu^{2}, \nu \rightarrow-\infty$ limit of V:

$$
\begin{equation*}
\sum_{\gamma}|2 \mathrm{M} \nu|^{\gamma} \int_{\mu}^{\infty} \frac{\mathrm{d} \sigma}{-\mathrm{M}^{2}} \frac{\mathrm{~h}_{1 \gamma}(\sigma)}{\left(\sigma-\mu^{2}+\mathrm{M}^{2}\right)^{\gamma}}+\int_{\mu}^{\infty}-\mathrm{M}^{2} \mathrm{~d} \sigma \widetilde{\mathrm{~h}}_{1}(\sigma,+1) \tag{3.14}
\end{equation*}
$$

We shall employ this form (3.14) in what follows. The generalization of our results to an arbitrary behavior for $V$ in the fixed $-\mu^{2}, q^{2} \rightarrow \infty$ limit will be immediate. Hence, we write

$$
\begin{equation*}
\mathrm{V} \rightarrow \sum_{\gamma}|2 \mathrm{M} \nu|^{\gamma} \mathrm{g}_{\gamma}\left(\mu^{2}\right) \tag{3.15}
\end{equation*}
$$

as $\nu \rightarrow-\infty$ with $\mu^{2}$-fixed, $\mathrm{g}_{\gamma}(\mathrm{z}) \rightarrow 0$ faster than $\mathrm{z}^{-\gamma}$ as $\mathrm{z} \rightarrow \infty$ for all $\gamma$. We note that the $g_{\gamma}(z)$ may be non-zero even when all the $G_{m}=0$ in (3.12). However, we suspect that both the $G_{m}$ and the $g_{\gamma}$ are trivial. ${ }^{10,11}$ But, this does not concern us here. When we use (3.15) in conjunction with (3.3) and (3.4) we have: as $\nu \rightarrow-\infty$ with $\mu^{2}$-fixed

$$
\begin{align*}
& \mathrm{W}_{1} \rightarrow+\sum_{\gamma}|2 \mathrm{M} \nu|^{\gamma} \mathrm{g}_{1, \gamma}\left(\mu^{2}\right) \\
& \mathrm{W}_{2} \rightarrow-\sum_{\gamma}(2 \mathrm{M})^{\gamma}|\nu|^{\gamma-1} \mathrm{~g}_{2, \gamma}\left(\mu^{2}\right)  \tag{3.16}\\
& \mathrm{W}_{\mathrm{i}} \rightarrow+\sum_{\gamma}(2 \mathrm{M})^{\gamma}|\nu|^{\gamma-2} \mathrm{~g}_{\mathrm{i}, \gamma}\left(\mu^{2}\right) \quad \mathrm{i}=4,5
\end{align*}
$$

 define for $\mathrm{p}_{0} \rightarrow \infty$

$$
\begin{align*}
& \left.\frac{-\mathrm{Mq}_{0}^{+}}{\mathrm{p}_{0} q_{+}^{2} \sqrt{1+\frac{\overrightarrow{\mathrm{q}}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}} \sum_{\gamma}(2 \mathrm{M})^{\gamma} \mathrm{g}_{1, \gamma}^{\bar{\nu}+\nu}\left(\mu^{2}\right) \right\rvert\, \nu_{+}, \gamma \equiv \sum_{\rho} \hat{\mathrm{g}}_{1, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \\
& \frac{1}{q_{+}^{4} \sqrt{1+\frac{\vec{q}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}} \sum_{\gamma}(2 \mathrm{M})^{\gamma} \mathrm{g}_{2, \gamma}^{\bar{\nu}+\nu}\left(\mu^{2}\right)\left|\nu_{+}\right|^{\gamma} \equiv \sum_{\rho} \hat{\mathrm{g}}_{2, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \\
& \frac{-\mathrm{q}_{0}^{+}}{\mathrm{p}_{0} \sqrt{1+\frac{\overrightarrow{\mathrm{q}}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}} \sum_{\gamma}(2 \mathrm{M})^{\gamma} \mathrm{g}_{4, \gamma}^{\bar{\nu}+\nu}\left(\mu^{2}\right)\left|\nu_{+}\right|^{\gamma-2} \equiv \sum_{\rho} \hat{\mathrm{g}}_{4, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \\
& \left.\frac{1}{\sqrt{1+\frac{\overrightarrow{\mathrm{q}}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}} \sum_{\gamma}(2 \mathrm{M})^{\gamma} \mathrm{g}_{5, \gamma}^{\bar{\nu}+\nu}\left(\mu^{2}\right) \right\rvert\, \nu_{+},{ }^{\gamma-2} \equiv \sum_{\rho} \hat{\mathrm{g}}_{5, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \tag{3.17}
\end{align*}
$$

H is then determined by

$$
\begin{align*}
\frac{\mathrm{H}\left(\overrightarrow{\mathrm{q}}^{2}, \mu^{2}, \mathrm{p}_{0}\right)}{\sqrt{1+\frac{\overrightarrow{\mathrm{q}}^{2}+\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}}= & -\sum_{\rho} \hat{\mathrm{g}}_{1, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho}+\overrightarrow{\mathrm{q}}^{2} \sum_{\rho} \hat{\mathrm{g}}_{2, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \\
& -\sum_{\rho} \hat{\mathrm{g}}_{4, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho}+\sum_{\rho} \hat{\mathrm{g}}_{5, \rho}^{\bar{\nu}+\nu}\left(\mu^{2},-\overrightarrow{\mathrm{q}}^{2}\right) \mathrm{p}_{0}^{\rho} \tag{3.18}
\end{align*}
$$

In (3.17) and (3.18), we have used

$$
\begin{align*}
& \mathrm{q}_{0}^{+}=\mathrm{q}_{0}^{+}\left(\mu^{2}, \overrightarrow{\mathrm{q}}, \mathrm{p}_{0}\right)=\left(-1-\left(1+\frac{\mu^{2}-\mathrm{m}^{2}+\overrightarrow{\mathrm{q}}^{2}}{\mathrm{p}_{0}^{2}}\right)^{1 / 2}\right) \mathrm{p}_{0}, \\
& \dot{\nu}_{+}=\nu_{+}\left(\mu^{2}, \overrightarrow{\mathrm{q}}, \mathrm{p}_{0}\right)=\mathrm{q}_{0}^{+} \mathrm{p}_{0} / \mathrm{M} \tag{3.19}
\end{align*}
$$

and

$$
q_{+}^{2}=q_{+}^{2}\left(\mu^{2}, \vec{q}, p_{0}\right)=\mu^{2}-\mathrm{m}^{2}-2 q_{0}^{+} p_{0}
$$

We may now turn to the interpretation of the LHS of (3.9).
When we introduce (3.18) into (3.9) we see that the LHS of the latter equation still generally appears to be ill-defined. However, one may interpret it by using the asymptotic behavior of the $C_{i, \alpha}$. Specifically, in analogy with $\mathrm{C}_{1, \alpha}$, it follows from Eq. (3.4) and correspondence ${ }^{9,10}$ that as $\mathrm{s} \rightarrow \pm \infty$ we have

$$
\begin{align*}
& \mathrm{C}_{2, \alpha}(\mathrm{~s}) \rightarrow\left|\frac{2 \mathrm{M}}{\mathrm{~s}}\right|^{\alpha-1} \mathrm{f}_{2, \alpha} \\
& \mathrm{C}_{4, \alpha}(\mathrm{~s}) \rightarrow \mp\left|\frac{2 \mathrm{M}}{\mathrm{~s}}\right|^{\alpha+2} \mathrm{f}_{4, \alpha}  \tag{3.20}\\
& \mathrm{C}_{5, \alpha}(\mathrm{~s}) \rightarrow\left|\frac{2 \mathrm{M}}{\mathrm{~s}}\right|^{\alpha+1} \mathrm{f}_{5, \alpha}
\end{align*}
$$

where the $\mathrm{f}_{\mathrm{i}, \alpha}$ are constants analogous to $f_{1, \alpha}$. If one uses this asymptotic behavior in Eq. (3.9) one can isolate the non-vanishing portions of this equation as $\eta, p_{0} \rightarrow \infty$ with $\eta / p_{0}-0$. There will only be a denumerable number of independent nonvanishing functions of $\eta, p_{0}$ as $\eta, p_{0} \rightarrow \infty$ with $\eta / p_{0} \rightarrow 0$. In the appendix, we argue that Eq. (3.9) is to be interpreted as the statement that the coefficients of these functions are to be set equal to zero. This is nothing but a natural generalization of the truncation ideas of CCN. These authors only isolated the constant
and logarithmic terms in $\eta, p_{0}$ in (3.9) as $\eta, p_{0} \rightarrow \infty$ with $\eta / p_{0} \rightarrow 0$ when $\alpha \neq 0$ in (3.10). Here, we may in principle isolate the coefficient of each independent non-vanishing function of $\left(\eta, p_{0}\right)$ in the neighborhood of $(\infty, \infty)$ with $\eta / p_{0} \rightarrow 0$. To illustrate, we presume $\{\alpha\}=\{0,1 / 2,1\}$ in Eqs. (3.10) (the value $\alpha=1 / 2$ is to be representative of the $\mathrm{P}^{\dagger}$ with $\alpha \cong 0.54$ ) and introduce

$$
\begin{align*}
& \widetilde{\mathrm{W}}_{2}^{\bar{\nu}+\nu}=\mathrm{W}_{2}^{\bar{\nu}+\nu}-\sum_{\alpha=\left\{0, \frac{1}{2}, 1\right\}} \mathrm{C}_{2, \alpha}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right) \nu^{\alpha-2} \\
& \widetilde{\mathrm{~W}}_{5}^{\bar{\nu}+\nu}=\mathrm{W}_{5}^{\bar{\nu}+\nu}-\sum_{\alpha=\left\{0, \frac{1}{2}, 1\right\}} \mathrm{C}_{5, \alpha}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right) \nu^{\alpha-1}  \tag{3.21}\\
& \widetilde{\mathrm{~F}}_{1}^{\bar{\nu}+\nu}=\theta(1-\mathrm{x}) \mathrm{F}_{1}^{\bar{\nu}+\nu}-\theta\left(\mathrm{x}_{0}-\mathrm{x}\right) \mathrm{M} \sum_{\alpha=\left\{0, \frac{1}{2}, 1\right\}^{1, \alpha}} \mathrm{f}_{1, \alpha}^{\bar{\nu}+\nu} \mathrm{x}^{-\alpha} \\
& \widetilde{\mathrm{C}}_{1, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})=\mathrm{C}_{1, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})+\theta\left(\mathrm{s}-\mathrm{s}_{0}\right)\left(\frac{2 \mathrm{M}}{\mathrm{~s}}\right)^{\alpha} \mathrm{f}_{1, \alpha}^{\bar{\nu}+\nu}
\end{align*}
$$

where

$$
x_{0}>0 \quad \text { and } \quad s_{0}>0
$$

The usefulness of the $\widetilde{W}_{i}$ and $\widetilde{\mathrm{C}}_{1, \alpha}$ is immediate: they have nice integrability properties in the appropriate regions of their arguments so that they will facilitate the isolation of the various independent functions of ( $\eta, \mathrm{p}_{0}$ ) in Eq. (3.9) in the vicinity of $(\infty, \infty)$ with $\eta / p_{0} \rightarrow 0$. Thus, we rewrite Eq. (3.9) in terms of the $\widetilde{W}_{i}$ and $\widetilde{\mathrm{C}}_{1, \alpha}$ and obtain for $q^{2}<0$

$$
\begin{align*}
& \sigma_{1}+\int_{0}^{\infty} \mathrm{d} \nu\left[\frac{\mathrm{M} \nu}{-\mathrm{q}^{2}} \widetilde{\mathrm{~W}}_{2}^{\bar{\nu}+\nu}+\left.\widetilde{\mathrm{W}}_{5}^{\bar{\nu}+\nu}{\underset{\mathrm{q}}{ }}^{2}\right|_{\text {fixed }}-\int_{0}^{\tau} \mathrm{dx} \frac{\widetilde{\mathrm{~F}}_{1}^{\bar{\nu}+\nu}(\mathrm{x})}{\mathrm{x}}\right. \\
& +\lim _{\eta \rightarrow \infty} \lim _{\mathrm{p}_{0} \rightarrow \infty}\left\{\frac { 1 } { 2 } \sum _ { \alpha = \{ 0 , \frac { 1 } { 2 } , 1 \} } \mathrm { p } _ { 0 } ^ { \alpha } \int _ { \mathrm { q } ^ { 2 } ( 1 + \eta ^ { 2 } \mathrm { M } ^ { 2 } / \mathrm { p } _ { 0 } ^ { 2 } \mathrm { q } ^ { 2 } ) } ^ { \infty } \mathrm { ds } \left[\left(\frac{\mathrm{M}}{\mathrm{~s}}{\left.\widetilde{\mathrm{C}}_{1}^{\bar{\nu}+\nu}+\frac{1}{\mathrm{M}} \mathrm{C}_{4, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})\right)\left(\frac{\left(\frac{\mathrm{s}^{2}}{\mathrm{M}}\right.}{}\right)^{\alpha}}^{\infty}\right.\right.\right. \\
& \left.+\left(\frac{-q^{2}}{M s^{2}} C_{2, \alpha}^{\bar{\nu}+\nu}(s)+\frac{1}{M^{2}} C_{5, \alpha}^{\bar{\nu}+\nu}(s)\right)\left(\frac{\sqrt{S-q^{2}}}{M}\right)^{\alpha-2}\right] \\
& +2 \eta^{\frac{1}{2}}\left(\frac{\mathrm{M}}{-\mathrm{q}^{2}} \mathrm{C}_{2, \frac{1}{2}}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right)+\mathrm{C}_{5, \frac{1}{2}}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right)\right)+\eta\left(\frac{\mathrm{M}}{-\mathrm{q}^{2}} \mathrm{C}_{2,1}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right)+\mathrm{C}_{5,1}^{\bar{\nu}+\nu}\left(\mathrm{q}^{2}\right)\right) \\
& -\mathrm{Mf}_{1,0}^{\bar{\nu}+\nu} \log \frac{2 \mathrm{x}_{0} \mathrm{M} \eta}{-\mathrm{q}^{2}}+2 \mathrm{Mf}_{1, \frac{1}{2}}^{\bar{\nu}+\nu}\left(\frac{1}{\sqrt{\mathrm{x}_{0}}}-\sqrt{\frac{2 \mathrm{M} \eta}{-\mathrm{q}^{2}}}\right) \\
& +\mathrm{Mf}_{1,1}^{\bar{\nu}+\nu}\left(\frac{1}{\mathrm{x}_{0}}+\frac{2 \eta \mathrm{M}}{\mathrm{q}^{2}}\right)-\frac{1}{2} \int_{\mathrm{s}_{0}}^{q^{4} \mathrm{p}_{0}^{2} / \mathrm{M}^{2} \eta^{2}} \mathrm{ds}\left[\frac{\mathrm{Mf} \mathrm{f}_{1,0}^{\bar{\nu}+\nu}}{\mathrm{s}}\right. \\
& \left.+\frac{2 \mathrm{Mf}_{1, \frac{1}{2}}^{\bar{\nu}+\nu}\left({\left.\mathrm{s}-\mathrm{q}^{2}\right)^{\frac{1}{4}}}_{\mathrm{s}^{3 / 2}}\right.}{\sin } \mathrm{p}_{0}^{\frac{1}{2}}+\frac{2 \mathrm{Mf}_{1,1}^{\bar{\nu}+\nu}\left(\mathrm{s}-\mathrm{q}^{2}\right)^{\frac{1}{2}}}{\mathrm{~s}^{2}} \mathrm{p}_{0}\right] \\
& \left.+\frac{1}{2 \mathrm{M}} \int_{\mu_{0}^{2}}^{-2 \mathrm{p}_{0}^{2} \mathrm{q}^{2} / \eta \mathrm{M}} \mathrm{~d} \mu^{2} \frac{\mathrm{H}\left(\mathrm{q}^{2}, \mu^{2}, \mathrm{p}_{0}\right)}{\sqrt{1+\frac{\mu^{2}-\mathrm{q}^{2}-\mathrm{M}^{2}}{p_{0}^{2}}}}\right\}=0 \tag{3.22}
\end{align*}
$$

where $\tau=\max \left\{\mathrm{x}_{0}, 1\right\}, \mathrm{H}$ is given by (3.18), and we are using the fact that $J=0$ is a nonsense point for the amplitudes under discussion.

We may now easily isolate the coefficients of the various independent functions of $\left(\eta, \mathrm{p}_{0}\right.$ ) near ( $\infty, \infty$ ) in this last equation, with the restriction $\eta / p_{0} \rightarrow 0$. In this way we obtain from the coefficient of $\log p_{0}$,

$$
\begin{equation*}
\mathrm{f}_{1,0}^{\bar{\nu}+\nu}=0 \tag{3.23}
\end{equation*}
$$

from the coefficient of $\mathrm{p}_{0}^{\alpha}, \alpha=1 / 2,1$,

$$
\begin{gather*}
\int_{\mathrm{q}}^{2} \mathrm{ds}\left[( \frac { \mathrm { M } } { \mathrm { s } } \mathrm { C } _ { 1 , \alpha } ^ { \overline { \nu } + \nu } ( \mathrm { s } ) + \frac { 1 } { \mathrm { M } } \mathrm { C } _ { 4 , \alpha } ^ { \overline { \nu } + \nu } ( \mathrm { s } ) ) \left({\left.\left.\mathrm{~s}-\mathrm{q}^{2}\right)^{\alpha / 2}+\left(\frac{-\mathrm{Mq}^{2}}{2} \mathrm{C}_{2, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})+\mathrm{C}_{5, \alpha}^{\bar{\nu}+\nu}(\mathrm{s})\right) /\left(\mathrm{s}-\mathrm{q}^{2}\right)^{1-\alpha / 2}\right]}^{\quad=\frac{1}{\mathrm{M}} \int_{\mu_{0}^{2}}^{2} \mathrm{~d} \mu^{2}\left[\hat{\mathrm{~g}}_{1, \alpha}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)+\mathrm{q}^{2} \hat{\mathrm{~g}}_{2, \alpha}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)+\hat{\mathrm{g}}_{4, \alpha}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)-\hat{\mathrm{g}}_{5, \alpha}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)\right]},\right.\right.
\end{gather*}
$$

from the coefficient of $\mathrm{p}_{0}^{\lambda}, 0<\lambda \neq 1 / 2,1$,

$$
\begin{equation*}
\left.\int_{\mu_{0}^{2}}^{\infty} \mathrm{d} \mu^{2}\left[\hat{\mathrm{~g}}_{1, \lambda}^{\bar{\nu}+\nu}\left(\mu^{2}, q^{2}\right)+q^{2} \hat{\mathrm{~g}}_{2, \lambda}^{\bar{\nu}+\lambda^{2}} \mu^{2}, q^{2}\right)+\hat{\mathrm{g}}_{4, \lambda}^{\bar{\nu}+\nu}\left(\mu^{2}, q^{2}\right)-\hat{\mathrm{g}}_{5, \lambda}^{\bar{\nu}+\nu}\left(\mu^{2}, q^{2}\right)\right]=0 \tag{3.25}
\end{equation*}
$$

and, finally, from the constant part of (3.22),

$$
\begin{gather*}
\sigma_{1}+\left.\int_{0}^{\infty} \mathrm{d} \nu\left[\frac{\mathrm{M} \nu}{-\mathrm{q}^{2}} \widetilde{\mathrm{w}}_{2}^{\bar{\nu}+\nu}+\mathrm{W}_{5}^{\bar{\nu}+\nu}\right]\right|_{\mathrm{q}^{2} \text {-fixed }}-\int_{0}^{\tau} \mathrm{dx} \frac{\widetilde{\mathrm{~F}}_{1}^{\bar{\nu}+\nu}(\mathrm{x})}{\mathrm{x}}+\frac{2 \mathrm{M}}{\sqrt{\mathrm{x}_{0}}} \mathrm{f}_{1,1 / 2}^{\bar{\nu}+\nu}+\frac{\mathrm{M}^{\mathrm{f}} \mathrm{\nu}+\nu}{\mathrm{x}_{0}, 1} \\
\\
+\frac{1}{2} \int_{0}^{\infty} \mathrm{ds}\left[\frac{\mathrm{M}}{\mathrm{~s}} \mathrm{C}_{1,0}^{\bar{\nu}+\nu}(\mathrm{s})+\frac{1}{\mathrm{M}} \mathrm{C}_{4,0}^{\bar{\nu}+\nu}(\mathrm{s})+\left(\frac{-\mathrm{Mq}^{2}}{\mathrm{~s}^{2}} \mathrm{C}_{2,0}^{\bar{\nu}+\nu}(\mathrm{s})+\mathrm{C}_{5,0}^{\bar{\nu}+\nu}(\mathrm{s})\right) /\left(\mathrm{s}-\mathrm{q}^{2}\right)\right] \\
 \tag{3.26}\\
-\frac{1}{2 \mathrm{M}} \int_{\mu_{0}^{2}}^{\infty} \mathrm{d} \mu^{2}\left[\hat{\mathrm{~g}}_{1,0}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)+\mathrm{q}^{2} \hat{\mathrm{~g}}_{2,0}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)+\hat{\mathrm{g}}_{4,0}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)-\hat{\mathrm{g}}_{5,0}^{\bar{\nu}+\nu}\left(\mu^{2}, \mathrm{q}^{2}\right)\right] \\
=0
\end{gather*}
$$

where we have again used the fact that $J=0$ is a nonsense point for the amplitudes under discussion. ${ }^{10}$

The constraint (3.23) is well known to follow from the scaling behavior (3.3). The results (3.24) and (3.25) are new. Equation (3.26) is to be compared with the result of CCN, namely, Eq. (3.8). Recall that in obtaining (3.8), CCN assume all $C_{i, 0}=0=g_{i, \gamma}$. With this assumption, Eq. (3.26) obviously becomes identical to (3.8) when $\mathrm{x}_{0} \rightarrow \infty$. Thus, our method is in agreement with the dispersive truncation approach of CCN to the extent that the two overlap.

We should remark that the generalization of (3.23), .. (3.26) to an arbitrary behavior of the $W_{i}$ in the limit in (3.16) is immediate. In particular, if, as our theoretical prejudice ${ }^{10}$ would suggest, the $W_{i}$ are trivial in this limit, then the $g_{i, \rho}$ in (3.24), ..., (3.26) are zero. However, quite independent of the triviality or non-triviality of the $\mathrm{g}_{\mathrm{i}, \rho}$, we find the sum rules (3.24) and (3.25) very surprising. They may be taken to follow from
(1) Equal-time current algebra
(2) Scaling in the form (3.3) and (3.4)
(3) Regge and $q^{2} \rightarrow \infty, \mu^{2}$-fixed behavior in the form (3.5) and (3.16), respectively
(4) Continuation in the dynamics (the correspondence principle)
(5) Inter change of $\lim _{b j}$ and $x$-integration in the region $x<-1$ where the scaling limits vanish.

Obviously, verification of these results will serve as evidence for the least substantiated of these ideas, namely, (4) and (5). ${ }^{19}$

## IV. DISCUSSION

We have extended the approach of Refs. $1,2,3$ to current algebraic sum rules to systematically include the contributions of intermediate states at $x=0,-1$. The extended formalism is succinctly represented by the result (2.10). This equation has been shown to embody naturally a general formulation of the truncation ideas of Cornwall, Corrigan, and Norton. ${ }^{10}$

By applying our extended formalism to the spin independent Schwinger term sum rule of these authors, we have shown that the theory presented here agrees with their dispersive truncative theory to the extent that the theories overlap. At the same time, however, we have obtained several new results, namely, Eqs. (3.24) and (3.25). As we have remarked above, the generalization of these results to an arbitrary behavior in (3.16) is immediate. With this understanding, the results in III may be taken to follow from the scaling assumption II(a) in the form (3.3) and (3.4), assumption II(b) in the region $x<-1$, the Regge behavior (3.5), the correspondence principle, and the 0-i component of equal-time current algebra. A discussion of the other components of the equal-time current algebra in this connection will appear elsewhere. ${ }^{11}$

Let us conclude by emphasizing that the most pleasing feature of the approach to sum rules represented by Eq. (2.10) above is that this approach allows a complete, systematic discussion of these current algebraic relations, both at fixed $q^{2}$ and in the $\lim _{b j}$, which considers the contributions of all classes of intermediate states and which, as its algebraic starting point, needs only the familiar, well-founded, quark equal-time current algebra.

## Acknowledgement

I find it most pleasing to acknowledge helpful discussions with Professors J.D. Bjorken, S.J. Brodsky, F.J. Gilman and S. B. Treiman.

## Appendix

In this appendix we shall discuss the validity of the isolation in (3.9) of non-vanishing independent functions of ( $\eta, \mathrm{p}_{0}$ ) near ( $\infty, \infty$ ) with $\eta / \mathrm{p}_{0} \rightarrow 0$ as described in Eection III. We shall start this discussion with the origin of Eq。(3.9), namely, Eq。(2.4):

$$
\int_{0}^{\infty} \mathrm{d} \nu \mathrm{I}(\mathrm{q}, \mathrm{p})=\mathrm{B}
$$

where in Section III, B $=-\sigma_{1}$ and

$$
\begin{equation*}
\mathrm{I}(\mathrm{q}, \mathrm{p})=\frac{\mathrm{M}^{2} \mathrm{q}_{0}}{\mathrm{p}_{0} \mathrm{q}^{2}} \mathrm{~W}_{1}^{\bar{\nu}+\nu}+\frac{\left(\mathrm{p}_{0}+\mathrm{q}_{0} / 2 \mathrm{x}\right)}{2 \mathrm{xp}_{0}} \mathrm{~W}_{2}^{\bar{\nu}+\nu}+\frac{\mathrm{q}_{0}}{\mathrm{p}_{0}} \mathrm{~W}_{4}^{\bar{\nu}+\nu}+\mathrm{W}_{5}^{\bar{\nu}+\nu} \tag{A.1}
\end{equation*}
$$

What we shall argue is that if we presume that we can interchange the Bjorken limit with integration over $x$ in the region $x<-1$ where the scaling limits $F_{i}(x)$ in (3.3) and (3.4) vanish, then, for the asymptotic behavior (3.5) and (3.16), Eq. (2.4) implies that the LHS of (3.9) is indeed equal to $-\sigma_{1}$, thereby supporting our prescription in III for isolating in this latter equation the non-vanishing independent functions of $\left(\eta, p_{0}\right)$ near $(\infty, \infty)$ with $\eta / p_{0} \rightarrow 0$. To this end, we notice from (A.1) that $I(q, p)$ converges pointwise in $\nu$ to a function of $\nu,-\overrightarrow{\mathrm{q}}^{2}$ in the limit $\mathrm{p}_{0} \rightarrow \infty$ with $\nu=$ fixed. This will also be the case for the $I(q, p)$ which occur in (2.4) for the other components of the equal-time current algebra. As a result, for $\infty>\eta>0$, we have

$$
\mathrm{B}=\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{0}^{\infty} \mathrm{d} \nu \mathrm{I}=\lim _{\mathrm{p}_{0} \rightarrow \infty}\left(\int_{0}^{\eta} \mathrm{d} \nu \mathrm{I}+\int_{\eta}^{\infty} \mathrm{d} \nu \mathrm{I}\right)=\int_{0}^{\eta} \mathrm{d} \nu \lim _{\mathrm{p}_{0} \rightarrow \infty} \mathrm{I}+\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{\eta}^{\infty} \mathrm{d} \nu \mathrm{I}
$$

where the last step follows by the pointwise convergence of I in the limit $p_{0} \rightarrow \infty, \nu$-fixed. Obviously, most of the delicacies are in the term on the RHS of (A.2) involving integration from $\eta$ to $\infty$ and we shall now discuss this term in some detail.

The integral over $\nu$ from $\eta$ to $\infty$ in (A.2) may be rewritten as the sum of two integrals, one over $\mathrm{q}^{2}$, involving the Regge region, and one over x , about which we shall have more to say momentarily. We find

$$
\begin{align*}
\lim _{0 \rightarrow \infty} \int_{\eta}^{\infty} \mathrm{d} \nu \mathrm{I}= & \lim _{\mathrm{p}_{0} \rightarrow \infty}\left[\begin{array}{l}
\frac{1}{2} \int_{-\vec{q}^{2}\left(1-\eta^{2} \mathrm{M}^{2} / \mathrm{p}_{0}^{2} \vec{q}^{-2}\right)}^{\mathrm{q}^{4} \mathrm{p}_{0}^{2} / \eta^{2} \mathrm{M}^{2}} \frac{\mathrm{p}_{0} \mathrm{I}\left(\sqrt{\mathrm{q}^{2}+\vec{q}^{2}}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)}{\mathrm{M} \sqrt{\mathrm{q}^{2}+\vec{q}^{2}}} \\
\\
\end{array} \int_{-\infty}^{-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \mathrm{dx} \frac{\nu^{-} \mathrm{I}\left(\mathrm{q}_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{\frac{1}{2}}}\right]
\end{align*}
$$

where $\epsilon, \nu^{-}$, and $\mathrm{q}_{0}^{-}$are defined in (2.6)。From (A.1), (3.3), and (3.4) it follows that as $p_{0} \rightarrow \infty$, the function $\nu^{-} \mathrm{I}\left(\mathrm{q}_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)$ converges pointwise to a function only of $x$ for $x \leq-\vec{q}^{2} / 2 \eta M$. This will also be the case for the $I(q, p)$ which occur in $(2,4)$ for other components of the equal-time current algebra if we assume Bjorken scaling. As we stated above, in region $x<-1$, where the scaling limits of the structure functions vanish, we assume we can interchange
$\lim _{b j}$ and $x$ integration. But, then, by the pointwise convergence of $\nu^{-} I\left(q_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)$ for $-1 \leq x \leq-\vec{q}^{2} / 2 \eta$ M we have, for $0<\lambda \ll 1$,

Thus, all delicacies in (3.9) reside near the regions $x=0,-1$. Before discussing them, let us summarize what we have accomplished by changing variables:

For $0<\eta<\infty, 0<\lambda \ll 1$,

$$
B=\lim _{p_{0} \rightarrow \infty} \int_{0}^{\infty} \mathrm{d} \nu \mathrm{I}
$$

$$
=\int_{0}^{\eta} \mathrm{d} \nu \lim _{p_{0} \rightarrow \infty} I-\int_{-1}^{-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \mathrm{dx} \frac{\lim _{0 \rightarrow \infty} \nu^{-} \mathrm{I}\left(\mathrm{q}_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)}{\mathrm{x}}
$$

$$
\begin{equation*}
+\lim _{p_{0} \rightarrow \infty}\left\{\int_{-1-\lambda}^{-1} d x \frac{\nu^{-} I\left(q_{0}^{-}, \vec{q}, p\right)}{\left(x^{2}+\epsilon^{2}\right)^{1 / 2}}+\frac{1}{2} \int_{-\vec{q}^{2}\left(1-\eta^{2} M^{2} / \vec{q}^{2} p_{0}^{2}\right)}^{\vec{q}^{4} p_{0}^{2} / \eta^{2} M^{2}} \frac{p_{0} I\left(\sqrt{q^{2}+\vec{q}^{2}}, \vec{q}, p\right)}{M \sqrt{q^{2}+\vec{q}^{2}}}\right\} \tag{A.5}
\end{equation*}
$$

We first turn to the $q^{2}$ integral in this last equation.
It is only necessary to consider the case $\eta, \mathrm{p}_{0} \rightarrow \infty$ with $\eta / \mathrm{p}_{0} \rightarrow 0$, since this is what is involved in (3.9). Evidently, the asymptotic behavior of $p_{0} I$ will be central to our purposes. In this connection, let us note that from (3.5), (A.1), and the correspondence principle it follows that as $\eta, \mathrm{p}_{0} \rightarrow \infty$ with $\eta / p_{0} \rightarrow 0, p_{0} I$ approaches its Regge asymptotic form in the entire region of

$$
\begin{align*}
& \lim _{p_{0} \rightarrow \infty} \int_{\eta}^{\infty} \mathrm{d} \nu \mathrm{I}=-\int_{-1}^{-\vec{q}^{2} / 2 \eta \mathrm{M}} \mathrm{dx} \frac{\mathrm{p}_{0^{\rightarrow \infty}}^{\lim ^{-} \mathrm{I}\left(\mathrm{q}_{0}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)}}{\mathrm{x}}+\lim _{p_{0} \rightarrow \infty}\left\{\int_{-1-\lambda}^{-1} \mathrm{dx} \frac{\nu^{-} \mathrm{I}\left(\mathrm{q}_{0}^{-}, \overrightarrow{\mathrm{q}}, \mathrm{p}\right)}{\left(\mathrm{x}^{2}+\epsilon^{2}\right)^{1 / 2}}\right. \\
& \left.+\frac{1}{2} \int_{-\vec{q}^{2}\left(1-\eta^{2} M^{2} / p_{0}^{2} \vec{q}^{2}\right)}^{\vec{q}^{4} p_{0}^{2} / \eta^{2} M^{2}} \frac{p_{0} I\left(\sqrt{q^{2}+\vec{q}^{2}}, \vec{q}, p\right)}{M \sqrt{q^{2}+\vec{q}^{2}}}\right\} \tag{A.4}
\end{align*}
$$

of integration over $q^{2}$ in (A.5). This will also be the case for the $I(q, p)$ which occur in (2.4) for the other components of the equal-time current algebra. As in the text above, we let $R$ denote the Regge asymptotic form of $p_{0} I$. Then, the convergence of $p_{0} I$ to $R$ in the limit $p_{0} \rightarrow \infty, \eta \rightarrow \infty, \eta / p_{0} \rightarrow 0$ in the region of the $q^{2}$ integration implies that

$$
\begin{equation*}
\lim _{\eta \rightarrow \infty} \lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{-\overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M}} \overrightarrow{\mathrm{q}}^{2} / 2 \eta \mathrm{M} \mathrm{dx} \mathrm{D}=0 \tag{A.6}
\end{equation*}
$$

where $D=\nu^{-}\left(I-R / p_{0}\right) /\left(x^{2}+\epsilon^{2}\right)^{1 / 2}$. Hence we have

$$
\begin{gather*}
\lim _{\eta \rightarrow \infty} \lim _{p_{0} \rightarrow \infty} \int_{-\vec{q}^{2}\left(1-\eta^{2} \mathrm{M}^{2} / \overrightarrow{\mathrm{q}}^{2} \mathrm{p}_{0}^{2}\right)}^{\overrightarrow{\mathrm{q}}^{2} \mathrm{p}_{0}^{2} / \eta^{2} \mathrm{M}^{2}} \mathrm{dq}^{2} \frac{\mathrm{p}_{0} \mathrm{I}}{\mathrm{M}{\sqrt{q^{2}+\vec{q}^{2}}}^{2}}= \\
\quad=\lim _{\eta \rightarrow \infty} \lim _{0 \rightarrow \infty} \int_{-\vec{q}^{2}\left(1-\eta^{2} \mathrm{M}^{2} / \mathrm{p}_{0}^{2} \overrightarrow{\mathrm{q}}^{2}\right)}^{\overrightarrow{\mathrm{q}}^{4} \mathrm{p}_{0}^{2} / \eta^{2} \mathrm{M}^{2}} \mathrm{M}{\sqrt{q^{2}+\vec{q}^{2}}}^{2} \frac{\mathrm{R}}{\sqrt{2}^{2}} \tag{A.7}
\end{gather*}
$$

We consider next the region near $x=-1$ in (A.5).
We note that a possible physical limit near $x=1_{+}$is the fixed hadronic mass limit, $\nu \rightarrow-\infty$ with $\mu^{2}$ fixed. Thus, we change variables from $x$ on $q_{0}^{-}$ near $\mathrm{x}=-1_{-}$to $\mu^{2}$ on $\mathrm{q}_{0}^{+}$near $\mathrm{x}=1_{+}$: we find as $\mathrm{p}_{0} \rightarrow \infty$,

$$
\begin{equation*}
\int_{-1-\lambda}^{-1} d x \frac{\nu^{-} I\left(q_{0}^{-}, \vec{q}, p\right)}{\left(x^{2}+\epsilon^{2}\right)^{1 / 2}}=\int_{M^{2}}^{4 \lambda^{2} p_{0}^{2}} d \mu^{2} \frac{I\left(q_{0}^{+},-\vec{q}, p\right)}{2 M \sqrt{1+\epsilon^{2}+\frac{\mu^{2}-M^{2}}{2}}} \tag{A.8}
\end{equation*}
$$

Here, $q_{0}^{ \pm}$are the functions defined in (2.6) above. As $p_{0} \rightarrow \infty$, the upper limit $4 \lambda^{2} p_{0}^{2}$ in (A.8) corresponds to the $\lim _{b j}$ with $x=-1-\lambda$, a value of $x$ at which

$$
\begin{equation*}
\lim _{\mathrm{p}_{0} \rightarrow \infty} \nu^{-} \mathrm{I}=0 \tag{A.9}
\end{equation*}
$$

If we define H by

$$
\begin{equation*}
\mathrm{I}\left(\mathrm{q}_{0}^{+},-\overrightarrow{\mathrm{q}}, \mathrm{p}\right) \rightarrow \mathrm{H}\left(\mu^{2}, \overrightarrow{\mathrm{q}}^{2}, \mathrm{p}_{0}\right) \tag{A.10}
\end{equation*}
$$

as $p_{0} \rightarrow \infty$ with $\mu^{2}$-fixed, then, by correspondence $I(q, p)$ converges to $H$ in the entire region of integration in (A.8). Thus, presuming $H$ to have the form (3.18) and reasoning analogously to the argument leading to (A.6), we evidently have

$$
\begin{equation*}
\lim _{p_{0} \rightarrow \infty} \int_{0}^{4 \lambda^{2} \mathrm{p}_{0}^{2}} \mathrm{~d} \mu^{2} \frac{\mathrm{I}\left(\mathrm{q}_{0}^{+},-\overrightarrow{\mathrm{q}}, \mathrm{p}\right)}{\sqrt{1+\epsilon^{2}+\frac{\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}}=\lim _{\mathrm{p}_{0} \rightarrow \infty} \int_{0}^{4 \lambda^{2} \mathrm{p}_{0}^{2}} \mathrm{~d} \mu^{2} \frac{\mathrm{H}}{\sqrt{1+\epsilon^{2}+\frac{\mu^{2}-\mathrm{M}^{2}}{\mathrm{p}_{0}^{2}}}} \tag{A.11}
\end{equation*}
$$

Introducing (A.6) and (A.11) into (A.5) yields the desired result (3.9) and, thereby, the isolation prescription given in III. This completes our argument.

Our discussion is seen to depend crucially on the physical notions of correspondence, scaling, and the asymptotic behaviors (3.5) and (3.16). This is as it should be, since, in general, without physical notions there should be no way a priori to argue in favor of the calculations in III. We conclude by remarking that, as we have attempted to indicate throughout our discussion, the arguments in this appendix will pertain to all components of the equal-time current algebra in connection with (2.4).

## References

1. M. A. Keppel-Jones, Phys. Rev. D6 (1972) 1140.
2. Bo F. L. Ward, Nucl. Phys. B61 (1973) 1.
3. M. O. Taha, ICTP preprint, July 1973.
4. D. Dicus, R. Jackiw, and V. L. Teplitz, Phys. Rev. D4 (1971) 1733.
5. See, for example, H. Fritzsch and M. Gell-Mann, Lectures from the 1971 Coral Gables Conference on Fundamental Interactions at High Energy, Vol. 2, Coral Gables, 1971 (Gordon and Breach, New York, 1971) 1-42 and references therein.
6. D.J. Gross and S. B. Treiman, Phys. Rev. D4 (1971) 1059.
7. S. L. Adler and R. Dashen, Current Algebra (Benjamin, New York, 1968).
8. The validity of Eq。 $(2,7)$ is discussed in the spirit of 'Regge" ideas in Refs. (6) and (7) to some extent,the latter discussion being given along the lines of the dispersive approach of Ref. (10) and H. Leutwyler and P. Otterson, "Theoretical problems in deep inelastic scattering", Bern preprint (1972).
9. This principle has been discussed in general by J.D. Bjorken and J. Kogut, Phys. Rev. D8 (1973) 1341. However, its application in the sense of our discussion has been considered by several other authors: H.D.I. Abarbanel, M. L. Goldberger, and S. B. Treiman, Phys. Rev. Letters $\underline{22}$ (1969) 500; R. A. Brandt,ibid $\underline{22}$ (1969) 1149; H. Harari, ibid $\underline{22}$ (1969) 1078; R.A. Brandt, Phys. Rev. D1 (1970) 2808; Cornwall, Corrigan, and Norton ${ }^{10}$. The commutativity of various asymptotic limits of forward scattering amplitudes has also been discussed by P. Vinciarelli and P.Weisz, Phys. Rev. D 7, 3091 (1973).
10. J. M. Cornwall, D. Corrigan, and R.E. Norton, Phys. Rev. Letters $\underline{24}$ (1970) 1141; Phys. Rev。 D3 (1971) 536.
11. B. F. L. Ward, to be published.
12. S. L. Adler, Phys. Rev. 143 (1966) 1144.
13. J. D. Bjorken, Phys. Rev. 179 (1969) 1547.
14. See for example, Ref. 2 and the references therein.
15. Broadhurst, D.J., Gunion, J. F., and Jaffe, R. L. , Ann. Phys. (N. Y.), to be published, have shown how to modify the approach of CCN to allow for $\alpha=0$ fixed poles as well as $\alpha>0$ cuts. In III we only discuss $\alpha=0, \frac{1}{2}, 1$ but it is clear that the incorporation of cuts and other Regge singularities into our formalism is immediate.
16. The notation

$$
\mathrm{R}_{1}=\left.\lim _{\nu \rightarrow \infty} \mathrm{W}_{1}\right|_{q_{-f i x e d}}
$$

is only symbolic for

$$
\begin{gathered}
\mathrm{W}_{1}-\mathrm{R}_{1} \rightarrow 0 \\
\text { as } \nu \rightarrow \infty \\
\mathrm{q}^{2} \text {-fixed }
\end{gathered}
$$

17. S. Deser, W. Gilbert, and E.C. G. Sudarshan, Phys. Rev. 115, 731
(1959); M. Ida, Progr. Theoret. Phys. (Kyoto) 23, 1151 (1960);
N. Nakanishi, ibid 26, 337 (1961); Suppl. 18, 70 (1961).
18. The fixed $-\mu^{2}$ asymptotic limit for $q^{2}$-spacelike has more recently been discussed by P. Vinciarelli, Phys. Rev. D8, 965 (1973).
19. Of course, by now equal-time current algebra, space-like scaling, and space-like Regge asymptotics are well founded. For a discussion of the correspondence principle, see Ref. 9 .

[^0]:    *Work supported by the U. S. Atomic Energy Commission.

