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Classical Limit of a non-Abelian Gauge Theory and Asymptotic Freedom*

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ABSTRACT

A simple non-Abelian gauge theory consisting of a charged vector field and the photon is considered in the classical limit. By examining the energy eigenvalues of the charged vector particle in a homogeneous magnetic field of field strength H, and requiring stability of the vacuum (no tachyons), it is found that, for consistency, the effective charge must tend to zero as H tends to infinity. This property is related to the asymptotic freedom of the non-Abelian gauge theory.

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The recent discovery of asymptotic freedom <1> in non-Abelian gauge (massless Yang-Mills) theories has excited considerable interest <2>, but the physical origin of this effect is not clear. Explicit perturbative calculation in these theories shows that the sign of the function $\beta(g)$ in the renormalization group equations is negative for small values of the coupling constant g <3>, whereas this sign is positive in an Abelian gauge theory, such as QED. But why is the sign opposite? The difficulty is exacerbated by the usual explanation of the positive sign in QED <4>, namely that in an appropriate gauge, the positive sign describes the tendency of a bare charge to screen itself by attracting opposite charge ("polarizing the vacuum"). This causes the effective charge as observed at large distances from the charge to be smaller than the bare charge which is observed at small distances, i.e. - at large values of q^2, the spacelike four-momentum transfer to the charge from the scatterer. In the non-Abelian case, a bare charge acts as if it attracts <u>like</u> charge out of the vacuum fluctuations to produce a larger effective charge at small q^2 (large distances). This is certainly contrary to expectation for interactions mediated by vector exchange. A simple SO(3) model is examined here in the classical (single particle quantum mechanics) limit in an attempt to gain understanding of this effect in a more direct fashion than calculating $\beta(g)$.

The model is similar to the SO(3) model of Georgi and Glashow <5>, but consists only of the gauge mesons themselves; there are no Higgs-Kibble scalar fields and no fermions in the theory. The Lagrange function is

$$\mathcal{L} = -\frac{1}{4}\sum_{j=1}^{3} F_j^{\mu\nu} F_{j\mu\nu} \tag{1}$$,

where

$F_j^{\mu\nu} = \partial^\mu A_j^\nu - \partial^\nu A_j^\mu - e\epsilon_{jk\ell} A_k^\mu A_\ell^\nu$ and the electromagnetic coupling, e, corresponds to g in the discussion above. As shown in Ref.5, a gauge-coupled SO(3)-triplet Higgs-Kibble scalar field with a non-vanishing vacuum expectation value (VEV) in the j = 3 component generates via spontaneous symmetry breakdown, a mass for the charged spin-1 field

$$\varphi^\mu = \frac{1}{\sqrt{2}}(\varphi_1^\mu + i\varphi_2^\mu) \tag{2}$$,

and

its complex conjugate φ_μ^*, but A_3^μ remains massless and may be identified as the photon A^μ,

$$A_3^\mu = A^\mu \tag{3}$$.

Despite the lack of scalar fields here, it is assumed that the photon may still be identified as in Eq.(3) by an appropriate choice of coordinates in the group space. The equations of motion for the (c-number) fields are

$$\begin{aligned} &-(\pi^{\nu}\pi_{\nu})\varphi^{\mu} + \pi^{\mu}(\pi^{\nu}\varphi_{\nu}) + 2ieF^{\mu\nu}\varphi_{\nu} \\ &\quad + e^{2}[\varphi^{\nu}\varphi_{\nu}\varphi^{*\mu} - \varphi^{\nu}\varphi^{*}_{\nu}\varphi^{\mu}] = 0 \end{aligned} \qquad (4a)$$

$$\begin{aligned} &\partial_{\nu}F^{\nu\mu} - e^{2}\varphi^{*\sigma}\varphi^{\lambda}(2g_{\lambda\sigma}A^{\mu} - g^{\mu}{}_{\sigma}A_{\lambda} - g^{\mu}{}_{\lambda}A_{\sigma}) \\ &\quad + ie\,\partial_{\nu}(\varphi^{\mu}\varphi^{*\nu} - \varphi^{*\mu}\varphi^{\nu}) = 0 \\ &\quad - ie\,(\varphi^{*\nu}\overleftrightarrow{\partial}^{\mu}\varphi_{\nu} - \varphi^{*\nu}\partial_{\nu}\varphi^{\mu} + \varphi^{\nu}\partial_{\nu}\varphi^{*\mu}) \end{aligned} \qquad (4b)$$

where $\pi^{\mu} = \partial^{\mu} - ieqA^{\mu}$, $q = +1$, and $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$.

Except for the e^2 term, Eq.(4a) is a special case of the equation of motion of a charged spin-1 field of mass m and anomalous magnetic moment K, moving in an electromagnetic field

$$(m^{2} - \pi^{\nu}\pi_{\nu})\varphi^{\mu} + \pi^{\mu}(\pi^{\nu}\varphi_{\nu}) + ieq(1+K)F^{\mu\nu}\varphi_{\nu} = 0 \qquad (5)$$

with $m^2 = 0$, $K = 1$, $q = +1$. The exact energy eigenvalues, E, of a more general equation (including an arbitrary electric quadrupole moment) have been obtained by Goldman and Tsai <6> for the case of a homogeneous static magnetic field of strength H. The eigenvalues may be expressed as (in the case of vanishing quadrupole moment)

$$E^2 = m^2 \Big\{ 1 + (2n+1-2qS_3)\xi + \frac{1-K}{2}\xi^2$$
$$+ \frac{1-K}{2} qS_3 \left[2 + (2n+1-2qS_3)\xi\right]\xi \left(1 - \frac{\left[(2n+1-2qS_3)^2 - 1\right]\xi^2}{\left[2+(2n+1-2qS_3)\xi\right]^2}\right)^{1/2}$$
$$\simeq m^2 \left[1 + 2qS_3(1-K)\left(1 - \frac{1}{(2n+1-2qS_3)^2}\right) \operatorname{sgn}(2n+1-2qS_3)\right]$$
$$+ (2n+1-2qS_3)\, eH \left[1 + qS_3(1-K)\operatorname{sgn}(2n+1-2qS_3)\right]$$
$$+ \frac{1-K}{2} eH\, \xi \left[1 + qS_3 \operatorname{sgn}(2n+1-2qS_3)\right] \qquad (6)$$

where: the approximate form is valid for $\xi \gg 1$; $\xi = eH/m^2$; $n = 0,1,2,\ldots$ is a radial excitation quantum number; and $S_3 = \pm 1$ is the spin projection in the direction of the magnetic field axis. (The case $S_3 = 0$ is of no interest here.)

Consider now, in the SO(3) model, a single charged spin-1 particle placed in a classical, uniform, static, homogeneous magnetic field of large spatial extent. The current source terms in Eq.(4b) that arise from the presence of the charged particle are then negligible. These terms are bilinear in φ and φ^* and so such bilinear terms may consistently be dropped in Eq.(4a) as well. To phrase it another way, the single charged spin-1 particle (wave packet) is not strongly localized. Hence the configuration space probability amplitude for the particle is small everywhere and terms quadratic and cubic in the amplitude are negligibly small. Thus Eq.(4a) can be approximated as a special case of Eq.(5), and the energy eigenvalues of a charged vector SO(3) gauge particle in a homogeneous magnetic field are approximately those given by Eq.(6) in the limit $m^2 \longrightarrow 0$, $K \longrightarrow 1$, (and $q = +1$).

Note that as $m^2 \longrightarrow 0$, $\xi \rightarrow \infty$. It is only the condition $K = 1$, which is a result of the non-Abelian couplings (as opposed to minimal coupling in QED which results in $K = 0$), that keeps the energies finite;

$$E^2 \simeq (2n+1-2qS_3)\,eH \tag{7}$$

Eq.(7) implies that there is exactly one (tachyonic) state with $E^2 < 0$. Such a state violates unitarity and/or causality <7>. This is not entirely satisfactory, but it has been widely speculated that massless Yang-Mills theories undergo a dynamical spontaneous symmetry breaking which provides mass terms for (at least some of) the vector fields <8>.

Eq.(7) indicates that a finite symmetry breaking will be sufficient to raise the E^2 of the tachyonic state above zero <9>.

Some caution is needed here, for the mass terms also provide a scale for eH. In particular, ξ becomes finite and it is necessary to return to the detailed form of Eq(6). For $\xi \ll 1$, the m^2 terms in Eq.(6) guarantee that $E^2 > 0$ for all states. However, $\xi \longrightarrow \infty$ for a strong magnetic field ($H \longrightarrow \infty$) so the effect of the mass terms disappears and the eigenvalues return to the form given by Eq.(7). There is no need to be concerned about K becoming ξ-dependent as were Goldman and Tsai in Ref.6. The congruence between the limits $m^2 \longrightarrow 0$ and $\xi \longrightarrow \infty$ requires

$$K(\xi) \xrightarrow[\xi \to \infty]{} 1 + O(\xi^{-1}) \tag{8}$$

so that in the limit $\xi \longrightarrow \infty$, Eq.(7) is only slightly modified:

$$E^2 \simeq \{2n+1-2qS_3+b[1+qS_3 \,\mathrm{sgn}\,(2n+1-2qS_3)]\}eH \tag{9}$$

where b is some constant (possibly zero). This will not affect the argument below since it refers only to a state with n = 0 for which the term proportional to b vanishes. If $b > 0$, this state is sufficient; if $b < 0$, the argument applies to additional states.

The argument now depends upon the observation that as $H \to \infty$, Eq.(7) (or Eq.(9)) implies that (at least) one state, that with quantum numbers $n = 0$, $qS_3 = +1$, tends to regain a tachyonic character (despite any symmetry-breaking derived mass terms). This effect is exactly due to the non-zero value of the anomalous magnetic moment <6>. However, since K has been fixed <10>, and the mass introduced cannot be H-dependent <11>, the only remaining way to keep $E^2 > 0$ within this framework, is for the coupling e to become H-dependent. This is almost the same point as was made in Ref.6: Effects non-linear in the electromagnetic field, whatever their origin, can be described in Eq.(5) by introducing an H-dependence of e as e(H) in the last term <12>, so that

$$ieq(1+K)F^{\mu\nu}\varphi_\nu \to ie(H)q(1+K)F^{\mu\nu}\varphi_\nu \qquad (10).$$

From Eq.(7)(or Eq.(9)), unitarity and causality are retained only if,

$$e(H) \to 0 \quad \text{as} \quad H \to \infty \qquad (11),$$

that is, in a strong magnetic field, the effective coupling strength (effective charge) of the charged non-Abelian vector particle must be smaller than in a weak field.

Finally, it is necessary to explicitly relate the limit $H \longrightarrow \infty$ to q^2 (spacelike four-momentum transfer) $\rightarrow \infty$. To do this, one need only recall that a charged particle is elastically deflected in a homogeneous magnetic field so that the momentum transfer to it must be non-zero. Since the deflection increases with H and its magnitude reflects the momentum transfer, it follows that the limit $H \longrightarrow \infty$ is directly related to the limit $q^2 \longrightarrow \infty$ in this case.

Thus it has been shown that, in the classical limit of a non-Abelian gauge theory, the preservation of unitarity and causality requires the effective charge to vanish as q becomes large, i.e. - asymptotic freedom.

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References

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<2> S.Weinberg, "Non-Abelian Gauge Theories of the Strong Interactions", MIT preprint MIT-CTP-366 (1973); T.Appelquist and H.Georgi, "e+e- Annihilation in Gauge Theories of the Strong Interactions", Harvard preprint (1973); A.Zee, "Electron-Positron Annihilation in Stagnant Field Theories", Rockefeller Report No.COO-2232B-28 (1973); H.Georgi and H.D.Politzer, "Electroproduction Scaling in an Asymptotically Free Theory of Strong Interactions", Harvard preprint (1973).

<3> This is true provided there are not too many fermion fields or Higgs-Kibble scalar fields; see Ref.1 and T.P.Cheng, E.Eichten, and Ling-Fong Li, "Higgs Phenomena in Asymptotically Free Gauge Theories", SLAC-PUB-1340(T) (1973).

<4> As described on pp.70 and 158-9 of Relativistic Quantum Mechanics by J.D.Bjorken and S.D.Drell, (McGraw-Hill, New York, 1964).

<5> H.Georgi and S.L.Glashow, Phys.Rev.Letters 28, 1494 (1972).

<6> T.Goldman and W.Tsai, Phys.Letters 36B, 467 (1971); T.Goldman, Wu-Yang Tsai and Asim Yildiz, Phys.Rev. D5, 1926 (1972).

<7> At this stage, the model may be reminiscent of a dual string model. Such models are known to be related to gauge theories; see for example L.N.Chang and F.Mansouri, Phys.Rev. D5, 2535 (1972).

<8> R.Jackiw and K.Johnson, "Dynamical Model of Spontaneously Broken Gauge Symmetries", MIT preprint MIT-CTP-348 (1973); J.Cornwall and R.Norton, "Spontaneous Symmetry Breaking Without Scalar Mesons", UCLA preprint UCLA/73/TEP/78 (1973). The symmetry breaking should not depend on elementary Higgs-Kibble scalar fields as this would change Eqs.(4) and also seems to destroy asymptotic freedom; see Ref.3.

<9> A possible origin for such an effect can be seen in the Lagrangian of Eq.(1), interpreted as an operator in a quantum field theory. Then a VEV

$$\langle A_3^\sigma A_3^\lambda \rangle = \eta^2 g^{\sigma\lambda}$$

where η has dimensions of mass, generates a mass term for φ_μ in Eq.(4a)

$$m^2 = 3e^2\eta^2 .$$

<10> K has been fixed by the choice of gauge symmetry and/or by the requirement of finite energy eigenvalues in the $m^2 \rightarrow 0$ limit.

<11> This is true despite the suggestion in Ref.9. The

non-linear (in H) effects referred to below cannot affect the mass term of Eq.(5) because of the antisymmetry of $F^{\mu\nu}$. See Ref.6.

<12> This was expressed in Ref.6 as $1 + K \rightarrow 1 + K(H^2)$. Here, the limit on K as $H \rightarrow \infty$ has already been fixed.