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SCALING, SHORT DISTANCES AND THE LIGHT CONE*

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Preface

These are notes based on lectures delivered by J. Ellis, R. L. Jaffe and M. Nauenberg at the U. C. Santa Cruz Summer School on Particle Physics, June 25th to July 6th, 1973. Chapters 1,2,5,8 and 9 were prepared by J. Ellis, Chapters 3,4,6,7 and 10 by R.L. Jaffe.

The notes are intended as a simple pedagogical introduction to the main ideas of scale invariance at short distances and near the light-cone, and their application to deep inelastic and other phenomena. The notes are not intended as a critical review of the field. We apologize at the outset, to the initiate for ignoring work of which we may not be aware and to the newcomer for leaving out due to limitations of time some large bodies of material which should perhaps be covered in introductory lectures. Among the latter are: the formal theory of broken scale and conformal invariance and renormalization group techniques. Excellent introductions to these subjects can be found in C. Callan in the proceedings of the 1971 Les Houches Summer School and S. Coleman in the proceedings of the 1971 International Summer School of Physics "Ettore Majorana".

We would like to thank Mike Nauenberg for organizing the Santa Cruz School, Sid Drell and SLAC for hospitality while much of this work was done , and for bearing the publication expenses and the secretaries at SLAC and the Center for Theoretical Physics at M.I.T. where the lectures were typed.

Finally we wish to thank the participants in the summer school for many helpful comments, corrections and discussion.

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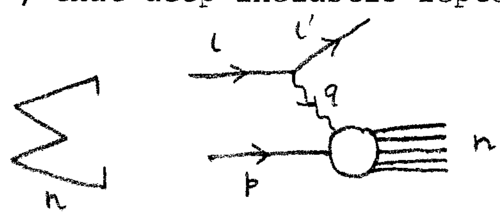
1 - Introduction

1.1 - (Introduction)^2

It is an old idea that in high energy reactions masses and other dimensional parameters might become irrelevant. In the absence of such mass and length scales the reactions would become scale invariant. This hope is clearly not realized in high energy hadronic reactions, which seem to be controlled by Regge singularities, which yield amplitudes of the form

$$(S/S_0)^{\alpha(t)}$$

with dimensional parameters α' and S_0. However it has recently been suggested theoretically[1], and to some extent confirmed experimentally[2], that deep inelastic lepton hadron processes



might be approximately scale invariant in the limit

$$q^2 \to -\infty,\ q\cdot p \to \infty\ ,\ \omega = \frac{-2q\cdot p}{q^2}\ \ \text{fixed}$$

The theoretical basis for this expectation is the observation that the asymptotic behaviours of these processes are related to singularities of current products $J(x)J(0)$ at short distances ($x_\mu \to 0$) and near the light cone[3] ($x^2 \to 0$), and the fact that in many model field theories these singularities are independent of mass parameters (scale invariant).

In these lectures we will discuss these ideas of scale invariance at short distances and near the light-cone, showing how they can be used in the phenomenology of deep inelastic processes, and other effects such as $e^- e^+$ annihilation and current algebra anomalies.[4] Deep inelastic scattering data will be used as crutches to support our theoretical ideas, and are indeed the principal checks on them at the present time. However we hope to make it clear that deep inelastic scattering is just one of several areas where these ideas apply.

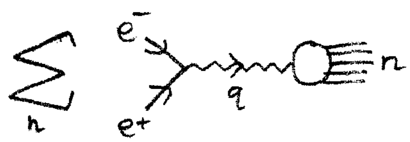
In this lecture we first discuss the relevance of short distance and light-cone behaviour to various phenomena, and then discuss model field theories which are the basis for the subsequent theoretical developments. The next lecture will review the data on deep inelastic reactions, showing the experimental evidence for scale invariance. Then a lecture on the parton model will show how the model field theories can be induced to be scale invariant. The rest of the lectures will discuss scale invariance at short distances more formally. There will be a "technological" lecture on singular functions in field theory, fourier transforms, and so on. Then there will be lectures on operator product expansions at short distances and on the light cone, and the Bjorken-Johnson-Low limit.[5] Finally there will be lectures on anomalies, other processes where light-cone ideas apply, and on sum rules and fixed poles in deep inelastic scattering.

1.2 - The Relevance of Short Distance and Light-Cone Behaviour

In this section we will briefly review the kinematics of a few processes and show how their behaviours are related to short distance and light-cone structure. We also review the Regge limit in x-space, showing that it is not in general related to the light cone.

(a) $e^-e^+ \to$ hadrons

We consider the one-photon exchange contribution to this process:



$$\propto \sum_n |\langle 0 | J_\mu | n \rangle|^2$$

The cross section can be written as

$$\sigma(e^-e^+ \to \gamma \to \text{hadrons}) = -\frac{8\pi^2\alpha^2}{3(q^2)^2} \int d^4x \, e^{iq\cdot x} \langle 0 | J_\mu(x) J^\mu(0) | 0 \rangle \tag{1.1}$$

where q is the sum of the lepton momenta. We will be interested in the behaviour of this cross section as $q^2 \to \infty$. To analyze the expression (1.1) we work in the centre of mass frame so that $q = (Q,0,0,0)$. Also we can replace the product $J_\mu(x)J^\mu(0)$ in Eq. (1.1) by the commutator $[J_\mu(x),J^\mu(0)]$ since in the region $Q > 0$

$$\int d^4x \, e^{iq\cdot x} \langle 0 | J^\mu(0) J_\mu(x) | 0 \rangle = 0$$

because of the spectrum properties. We then have

$$\sigma(e^- e^+ \to \gamma \to \text{hadrons}) = -\frac{8\pi^2\alpha^2}{3(q^2)^2}\int dx_0\, d^3\underline{x}\, e^{iQx_0}\langle 0|[J_\mu(x), J^\mu(0)]|0\rangle \qquad (1.2)$$

The integrand can be written as a function of x_0 and $\underline{x}^2$:

$$\int dx_0\, d^3\underline{x}\, e^{iQx_0} f(x_0, \underline{x}^2)$$

As $Q \to \infty$, contributions to the integral from finite non-zero values of x_0 will vanish because of the rapid phase oscillations of the factor e^{iQx_0}. Hence the asymptotic behaviour of the integral will be controlled by the behaviour of

$$\langle 0|[J_\mu(x), J^\mu(0)]|0\rangle$$

as $x_0 \to 0$. But because it is the matrix element of a commutator, it vanishes for $x^2 = x_0^2 - \underline{x}^2 < 0$. Thus the asymptotic behaviour of $\sigma(e^- e^+ \to \gamma \to \text{hadrons})$ is controlled by the behaviour[4,5] of

$$\lim_{x_\mu \to 0} \langle 0|[J_\mu(x), J^\mu(0)]|0\rangle$$

Suppose this cross section were indeed scale invariant, i.e. did not have any dependence on mass parameters, as $q^2 \to \infty$. Since a cross section has the dimensions $(\text{length})^2$ or equivalently $(\text{mass})^{-2}$, we must have

$$\sigma(e^- e^+ \to \gamma \to \text{hadrons}) \propto \frac{1}{q^2} \qquad \text{as } q^2 \to \infty$$

if mass scales are not to control the limit. Since the kinematic factors outside the integral in (1.2) are mass independent this means that

$$\lim_{x_\mu \to 0} \langle 0 | [J_\mu(x), J^\mu(0)] | 0 \rangle$$

should be independent of any masses. Since currents have dimensions $(\text{mass})^3$ (because charges $\int d^3\underline{x}\, J_0(\underline{x},t)$ are dimensionless), this means the vacuum expectation value should diverge as x^{-6} as $x_\mu \to 0$.

(b) - Commutators and the Bjorken-Johnson-Low Limit

Equal-time commutators are also short-distance properties of the field theory, because they are local. An example is the current algebra commutator

$$\delta(x_0)\left[J_0^a(\underline{x},x_0), J_\mu^b(0)\right] = i f_{abc}\, J_\mu^c(0)\, \delta^4(x) \qquad (1.3)$$

which is transparently independent of masses. These commutators are mass independent (scale invariant) because they are derived from the basic canonical field theory commutator

$$\delta(x_0)\left[\dot\phi(\underline{x},x_0), \phi(0)\right] = -i\,\delta^4(x)$$

for scalar fields and anticommutator

$$\delta(x_0)\left\{\psi(\underline{x},x_0), \psi^+(0)\right\} = \delta^4(x)$$

for spin-1/2 fields, which are themselves mass independent.

Many applications have been made of the current algebra commutators (1.3), and these are now generally accepted as valid. If we want to probe the short distance structure of hadrons further, then we must find ways of looking at other equal-time commutators. One way was proposed by Bjorken, Johnson, and Low[5]. Consider the matrix element

$$\int d^4x \, e^{iq\cdot x} \langle H | R(A(x)B(0)) | H' \rangle$$

and where $R(A(x)B(0)) = \theta(x_0)\ [A(x), B(0)]$ is the retarded commutator, H and H' are two hadronic systems - for convenience we now take them to be identical single particle states of momentum p. Take the limit $q_0 \to i\infty$ with $\underline{q}$ fixed: then by partial integration

$$\int d^4x \, e^{iq\cdot x} \langle p | R(A(x)B(0)) | p \rangle$$
$$= \frac{i}{q_0} \int d^4x \, e^{-i\underline{q}\cdot\underline{x}} \langle p | \delta(x_0) [A(x), B(0)] | p \rangle$$
$$+ \frac{i}{q_0} \int d^4x \, e^{-i\underline{q}\cdot\underline{x}} \langle p | R(\dot{A}(x) B(0)) | p \rangle$$

where the second term is $O\left(\frac{1}{q_0^2}\right)$ and $\dot{A}(x) \equiv \frac{\partial A(x)}{\partial x_0}$.

The procedure can be repeated, the $(1/q_0)^n$ term being related to

$$\langle p | \delta(x_0) \left[\frac{\partial^{n-1} A(x)}{\partial x_0^{n-1}} , B(0) \right] | p \rangle$$

Unfortunately the limit $q_0 \to i\infty$ is rather unphysical. However we will see in a subsequent lecture how

$$\lim_{q_0 \to i\infty} \int d^4x \, e^{iq\cdot x} \langle p | R(J_\mu(x) J_\nu(0)) | p \rangle$$

can be related via dispersion relations to moments of structure functions in deep inelastic lepton-hadron scattering. Using these relations, commutators of the currents and their time derivatives can be probed, giving us ways of exploring short distance behaviour.

(c) - Deep Inelastic Scattering

In electroproduction experiments the lepton scatters off the nucleon target with the exchange of a virtual photon with space-like momentum $q^2 < 0$. In the next lecture the kinematics of this and related processes will be reviewed in detail. For the moment we just consider the simplified case of a spin zero current scattering off a spin-averaged target. The total inelastic cross section

$$\sigma \quad \propto \quad \sum_n |\langle p| J(0) |n\rangle|^2 \tag{1.4}$$

Using the completeness of the intermediate states we can rewrite (1.4) so that

$$\sigma \quad \propto \quad \int d^4x \, e^{iq\cdot x} \langle p| J(x) J(0) |p\rangle \tag{1.5}$$

where q is the momentum of the current.* What we will now argue[3] is that in the limit

$$-q^2, \nu \equiv q\cdot p \to \infty \quad , \quad \omega \equiv -2\nu/q^2 \text{ fixed} \tag{1.6}$$

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* We will always covariantly normalize states

$$\langle p|p'\rangle = (2\pi)^3 \, 2E \, \delta^3(\underline{p} - \underline{p}')$$

then the asymptotic behaviour of the cross section is controlled by the behaviour of

$$\langle p | J(x) J(0) | p \rangle$$

near the light cone $x^2 \sim 0$. The limit(1.6) is called the Bjorken scaling limit. Depending on the strength of the light cone singularity

$$\sigma \sim (q^2)^{-n} f(\omega)$$

in the scaling limit: this is called the scaling behaviour.

To make plausible light-cone dominance we work in the rest frame of the target: $p = (M,0,0,0)$. Then $q = (\frac{\nu}{M},0,0,\sqrt{(\frac{\nu}{M})^2 - q^2}$ $\approx (\frac{\nu}{M},0,0,\frac{\nu}{M}+\frac{M}{\omega})$ in the scaling limit. We can rewrite (1.5) in the form

$$\sigma \propto \int dx_0 d^3\underline{x}\, e^{i\frac{\nu}{M}x_0} e^{-i\underline{q}\cdot\underline{x}} f(x_0^2 - r^2, Mx_0) \tag{1.7}$$

where $|\underline{x}| \equiv r$ and we have expressed the matrix element in (1.5) as

$$f(x^2, x\cdot p) = f(x_0^2 - r^2, Mx_0)$$

Now if we perform the angular integrations in (1.7) we find

$$\sigma \propto \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} dr \frac{r}{\sqrt{(\frac{\nu}{M})^2 - q^2}} e^{i\frac{\nu}{M}x_0} \sin r\sqrt{(\tfrac{\nu}{M})^2 - q^2}\; f(x_0^2 - r^2, Mx_0) \tag{1.8}$$

Now we consider the phase variation of the integrand in (1.8); we can write

$$e^{i\frac{\nu}{M}x_0}\sin r\sqrt{\left(\frac{\nu}{M}\right)^2-q^2} = \frac{e^{i\frac{\nu}{M}x_0}}{2i}\left[e^{ir\sqrt{\left(\frac{\nu}{M}\right)^2-q^2}} - e^{-ir\sqrt{\left(\frac{\nu}{M}\right)^2-q^2}}\right]$$

$$\simeq \frac{e^{i\frac{\nu}{M}(x_0+r)}e^{ir\frac{M}{\omega}}}{2i} - \frac{e^{i\frac{\nu}{M}(x_0-r)}e^{-ir\frac{M}{\omega}}}{2i} \qquad (1.9)$$

in the scaling limit. As before we argue that the integration in (1.8) will only get contributions from values of x_0 and r where the phase variations (1.9) are not large. For the first term in (1.9) this means

$$|x_0+r| = O\left(\frac{M}{\nu}\right), \quad |r| = O\left(\frac{\omega}{M}\right)$$

for the second term

$$|x_0-r| = O\left(\frac{M}{\nu}\right), \quad |r| = O\left(\frac{\omega}{M}\right)$$

In both cases

$$|x^2| = |x_0^2-r^2| = O\left(\frac{1}{q^2}\right)$$

so that in the scaling limit (1.6) we are probing the light-cone behaviour of the matrix element

$$\langle p|\,J(x)\,J(0)\,|p\rangle$$

In electroproduction we measure

$$W_{\mu\nu} \propto \int d^4x\, e^{iq\cdot x}\langle p|J_\mu(x)J_\nu(0)|p\rangle$$

In the next lecture the kinematics will be discussed in detail: here we just note that experiments strongly suggest that $W_{\mu\nu}$ becomes scale invariant (independent of mass parameters) in the scaling limit q^2, $q \cdot p \to \infty$ with ω fixed. This means that

$$\langle p | J_\mu(x) J_\nu(0) | p \rangle$$

has the same type of singularity as found in simple field theory models, which always have mass-independent singularities. (You are not asked to understand these comments now, just asked to accept them - we hope to make things clearer later on!)

It is sometimes convenient to re-express vectors a_μ in terms of light-cone variables

$$\hat{a}_\mu = (a_-, a_1, a_2, a_+)$$

where $a_\pm = 1/\sqrt{2}\,(a_0 \pm a_3)$. Scalar products take the form $a \cdot b = a_+ b_- + a_- b_+ - \vec{a} \cdot \vec{b}$ where $\vec{a} \equiv (a_1, a_2)$. We then see that $q^2 = 2q_+q_-$ and $\nu = q_+p_- + q_-p_+$. In the scaling limit

$$q_+ \to \sqrt{2}\,\frac{\nu}{M} \quad , \quad q_- \to -\frac{M}{\sqrt{2}\,\omega}$$

while

$$\omega = \frac{-2q \cdot p}{q^2} \approx \frac{-p_-}{q_-}$$

remains fixed. Examining the exponential in equation (1.5) we see that $e^{iqx} = e^{i(q_-x_+ + q_+x_-)}$ and the standard phase variation arguments tell us that regions of space with $x_- \to 0$ control the scaling behaviours. This point will be taken up later in discussing the parton model.

(d) The Regge Limit

So far we have given examples of processes which are connected with short distance and light-cone behaviour. Just to reassure you that we are not trying to take over the world, we now give an example of a process clearly unrelated to either $x_\mu \sim 0$ or $x^2 \sim 0$. Consider the same cross section

$$\sigma \propto \int d^4x\, e^{iq\cdot x} \langle p | J(x) J(0) | p \rangle$$

discussed in the previous section, but now consider the limit

$$q^2 \text{ fixed}, \quad q\cdot p \to \infty$$

normally assumed to Regge dominated like hadron-hadron scattering. Going through the phase-variation argument as before we find the dominant contributions come from $|x^2| = 0(1/q^2)$, which does not tend to zero in the Regge limit q^2 fixed, $q\cdot p \to \infty$, so we do not get any closer to the light cone in the limit. The weak constraint $|x^2| = 0(1/q^2)$ means that hadronic mass parameters may well control the asymptotic behaviour, as is true of conventional Regge asymptotic formulae $(S/S_0)^\alpha$.

(e) - Anomalies

All the previous instances where we showed short-distance and light-cone effects were important involved processes at large momenta, where it is intuitively plausible that such effects might turn up. However short-distance effects also occur in processes at small momenta. Examples are current algebra low energy theorems, which tell us a lot about pion interactions at low momenta, and come from equal-time current commutators, which are indeed short-distance effects. There are more interesting and model-dependent effects at low momenta connected with short-distance behavior - the current algebra anomalies.[4] The best-known example is that[6] in the Ward identity relating

$$T_{\mu\nu\rho}(p,k) \equiv \int d^4x\, d^4y\, e^{i(p\cdot x - k\cdot y)} \langle 0| T(J_\mu^{em}(x)\, J_\nu^{em}(0)\, A_\rho^3(0))|0\rangle$$

to

$$T_{\mu\nu}(p,k) \equiv \int d^4x\, d^4y\, e^{i(p\cdot x - k\cdot y)} \langle 0| T(J_\mu^{em}(x)\, J_\nu^{em}(0)\, \partial^\rho A_\rho^3(0))|0\rangle$$

The naive Ward identity is

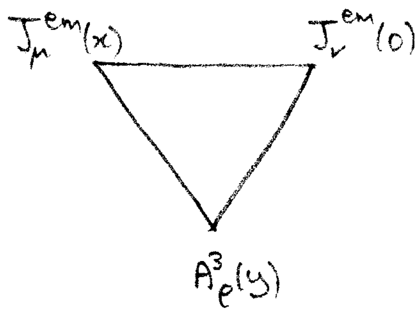
$$i k^\rho T_{\mu\nu\rho}(p,k) = T_{\mu\nu}(p,k) \tag{1.10}$$

which can be used to argue that if PCAC is used to relate $T_{\mu\nu}(p,k)$ to the $\pi^o \to 2\gamma$ decay amplitude, then the latter should vanish. This conclusion is unsound experimentally, and in fact the Ward identity (1.10) is untrue in field theory models. In general an extra term

$$A\, \epsilon_{\mu\nu\rho\sigma}\, p^\rho k^\sigma \tag{1.11}$$

should be added to the right-hand side of (1.10). The $\pi^o \to 2\gamma$ amplitude is then proportional to A in the PCAC approximation.

The anomaly (1.11) arises in perturbation theory because in order to prove (1.10) it is necessary to make illegal changes of variable in the integration for the lowest order fermion loop contributions



to $T_{\mu\nu\rho}$

These operations are illegal because the loop is divergent at large values of the loop momentum. This reason suggests that maybe the anomaly is connected with short-distance behaviour. This suggestion is confirmed by careful studies[4] of (1.10) in configuration space, which show that the partial integration required to derive it may break down if

$$\langle 0 | T(J_\mu^{em}(x) J_\nu^{em}(0) A_\rho^3(y)) | 0 \rangle$$

has an ε^{-9} singularity when $x \sim y \sim \varepsilon \to 0$. Since the currents each have dimension $(\text{mass})^3$, this is indeed the strength of singularity expected if mass parameters become irrelevant at short distances (scale invariance). Therefore the anomaly and the $\pi^0 \to 2\gamma$ decay rate should be taken into account when constructing models for short-distance behaviour.

1.3 - Model Field Theories

After seeing that short-distance and light cone effects are relevant in a number of processes, the next step is to construct models for behaviour in these limits. Because they are the only fully consistent models of currents available, we examine simple field theories, as was done in the proposal and development of current algebra.[7] As remarked earlier, the current algebra commutators

$$\delta(x_0)\left[J_0^a(\underline{x},x_0), J_\mu^b(0)\right] = i\, f_{abc}\, J_\mu^c(0)\, \delta^4(x)$$

$$\delta(x_0)\left[J_0^a(\underline{x},x_0), A_\mu^b(0)\right] = i\, f_{abc}\, A_\mu^c(0)\, \delta^4(x) \qquad (1.12)$$

$$\delta(x_0)\left[A_0^a(\underline{x},x_0), A_\mu^b(0)\right] = i\, f_{abc}\, J_\mu^c(0)\, \delta^4(x)$$

are themselves already statements about short-distance behaviour.

Two models are often used in motivating current algebra: one is the familiar quark model

$$\mathcal{L} = \frac{i}{2}\, \bar\psi_\alpha (\gamma\cdot\overleftrightarrow{\partial})\psi_\alpha - \sum_\alpha m_\alpha \bar\psi_\alpha \psi_\alpha \qquad (1.13)$$

where the α's are SU_3 triplet indices. The vector and axial currents

$$J_\mu^a(x) = \bar\psi(x)\gamma_\mu \frac{\lambda_a}{2}\psi(x) \quad , \quad A_\mu^b(x) = \bar\psi(x)\gamma_\mu\gamma_5 \frac{\lambda_b}{2}\psi(x)$$

obey the $SU_3 \times SU_3$ current algebra. This is true even though the masses m_α break $SU_3 \times SU_3$ symmetry (the currents are not conserved). This is an example of a mass parameter not affecting behaviour at short distances. Interactions can be added to $\mathcal{L}$: the most popular one is to add an SU_3 singlet vector meson ("gluon")

$$\mathcal{L}_{int} = g\, \bar{\psi}_\alpha \gamma^\mu \psi_\alpha B_\mu$$

which also has no effect on the current algebra.

Another possible model has basic octets of spin zero fields:

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu M_a \partial_\mu M_a + \partial^\mu P_a \partial_\mu P_a\right] - \frac{1}{2}\left[\Sigma m_a^2 M_a M_a + \Sigma \mu_a^2 P_a P_a\right]$$

where the M_a and P_a are scalar and pseudoscalar fields respectively. Again $SU_3 \times SU_3$ currents

$$J_\mu(x) \propto M(x) \overleftrightarrow{\partial}_\mu M(x) + P(x) \overleftrightarrow{\partial}_\mu P(x)$$

$$A_\mu(x) \propto P(x) \overleftrightarrow{\partial}_\mu M(x)$$

can be constructed, which obey current algebra despite the scale (and $SU_3 \times SU_3$) symmetry breaking mass terms.

Models like (1.13) and (1.14) are generally mass independent (scale invariant) at short distances and near the light cone. For example the free fermion propagator

$$\sim \frac{\gamma \cdot x}{(x^2)^2} \qquad \text{as} \quad x^2 \to 0$$

and the free boson propagator $\sim 1/x^2$ as $x^2 \to 0$ in a free field theory. We will see later how the mass independence (scale invariance) of these basic singularities suggest the scale invariance of products of currents also.

Connected with the mass independence of the short distance and light-cone singularities is the fact that they generally respect whatever symmetries (SU_3, chirality, etc.) are violated by the mass terms in the theory.

Models like (1.13) and (1.14) can be used to specify commutators other than the current algebra ones: examples include the commutators between space components of currents:

$$\delta(x_0)\left[J_i^a(\underset{\sim}{x},x_0),\ J_j^b(0)\right]$$

and the commutator of a current with its time derivative:

$$\delta(x_0)\left[\dot{J}_\mu^a(\underset{\sim}{x},x_0),\ J_\nu^b(0)\right]$$

The structure of these commutators is model dependent.

A main burden of these lectures will be to argue that a slightly modified version of the model (1.13), with three identical triplets of quark fields is in good agreement with what is presently known from experiment about short distance and light-cone effects, while also making plenty of predictions yet to be tested.[8] We should point out that if model field theories are calculated in perturbation theory, then the scale invariant behaviour found in free field theory, or by manipulations of interacting field theories using canonical commutation relations, gets changed. Scaling gets modified by logarithmic factors, or the power of singularities in x^2 may get altered. There is no evidence from deep inelastic electroproduction experiments that scale invariance is broken, however.

Therefore in these lectures we will generally use canonical field theory results[8] and ignore perturbation theory.

References

1. J.D. Bjorken - Phys. Rev. 179, 1547 (1969).

2. H. Kendall - in Proceedings of the Fifth International Symposium on Electron and Photon Interactions at High Energies, Ithaca, New York, 1971, edited by N.B. Mistry (Cornell Univ. Press, Ithaca, N.Y., 1972): see Lecture 2.

3. B. L. Ioffe - Phys. Lett. 30B, 123 (1969).
   R. A. Brandt and G. Preparata - Nucl. Phys. B27, 541 (1971).
   Y. Frishman - Ann. Phys. (N.Y.) 66, 373 (1971).

4. K. Wilson - Phys. Rev. 179, 1499 (1969).

5. J. D. Bjorken - Phys. Rev. 148, 1467 (1966).
   K. Johnson and F. Low - Progr. Theor. Phys. Suppl. 37-38, 74 (1966).

6. S. L. Adler - Phys. Rev. 177, 2426 (1969).
   J. S. Bell and R. Jackiw - Nuovo Cimento 60A, 47 (1969).

7. M. Gell-Mann - Phys. Rev. 125, 1067 (1962), Physics 1, 63 (1964).

8. H. Fritzsch and M. Gell-Mann, in "Broken Scale Invariance and the Light Cone" edited by M. Dal Cin, G. J. Iverson and A. Perlmutter (Gordon and Breach, New York, 1971).

9. C. G. Callan - Phys. Rev. D2, 1541, (1970).
   K. Symansik - Comm. Math. Phys. 18, 227 (1970).

General References

Useful general references are Refs. 4 and 8. For recent reviews see

C. H. Llewellyn Smith - in Proceedings of the 4th

International Conference on High Energy Collisions, edited by J. R. Smith (Rutherford Laboratory, 1972) Vol. 1, p.87.

Y. Frishman - in Proceedings of the XVI International Conference on High Energy Physics, edited by J.D. Jackson and A. Roberts, (N.A.L., Batavia, 1972) Vol. 4, p.119.

Good references on current algebra are:

S. L. Adler and R. F. Dashen - "Current Algebras and Applications to Particle Physics" (Benjamin, 1968).

B. Renner - "Current Algebras and Their Applications" (Pergamon, 1968).

2 - KINEMATICS, SCALING AND COMPARISON WITH EXPERIMENTS

2.1 - Introduction

In this lecture we review the kinematics of inelastic electroproduction ($e^{\pm} + N \to e^{\pm}$ + hadrons, $\mu^{\pm} + N \to \mu^{\pm}$ + hadrons) and neutrinoproduction ($\nu + N \to \{{\mu^- \atop e^-}\}$ + hadrons, $\bar{\nu} + N \to \{{\mu^+ \atop e^+}\}$ + hadrons*). The crossing properties, analyticity, $q^2 \to 0$ limit, positivity and Regge behaviors of the inclusive structure functions are discussed. Also we review the scaling laws expected[1] for the various structure functions, and some consequences of these laws, particularly in neutrinoproduction. Finally we discuss the qualitative features of the data on scaling in electroproduction[2,3], neutrinoproduction[4,5,6] and electron-positron annihilation[7]. We outline the extent to which scaling ideas have been checked, and give sources where the data may be found.

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*Experiments at CERN[8] and NAL[9] also have events which have not so far been explained away as not being of the type $\nu + N \to \nu$ + hadrons.

and it will be useful to introduce the variables, $\xi = \frac{Q^2}{2\nu}$, $\omega = \frac{1}{\xi} = \frac{2\nu}{Q^2}$, which remain fixed in the scaling limit, and the variable $y = \nu/ME$. See figure 2 for the kinematic range in any experiment: $1 \leq \omega \leq \infty$, $0 \leq \xi$, $y \leq 1$.

B - Electroproduction

In the case of electroproduction the graph in figure 1 is proportional to:

$$A \equiv \bar{u}(k')\gamma^\mu u(k) \langle X|J_\mu(0)|p\rangle \tag{2.1}$$

where u and $\bar{u}'$ are lepton spinors. We then have the differential inelastic cross-section:

$$\propto \quad \sum_X |A|^2 (2\pi)^4 \delta^4(q+p-p_X) \tag{2.2}$$

where the sum is over all final hadronic states X. Substituting (2.1) into (2.2) we find the cross-section is proportional to:

$$t^{\mu\nu} W_{\mu\nu} \tag{2.3}$$

where $$t^{\mu\nu} \equiv \tfrac{1}{2}\mathrm{Tr}(\not{k}'\gamma^\mu \not{k}\gamma^\nu) = 2(k'^\mu k^\nu + k^\mu k'^\nu - g^{\mu\nu} k\cdot k') \tag{2.4}$$

(we assume that the lepton beam is unpolarized) and

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_X \langle p|J_\mu^+(0)|X\rangle\langle X|J_\nu(0)|p\rangle(2\pi)^4\delta^4(q+p-p_X) \tag{2.5}$$

The electromagnetic current is hermitian, but we have written J_μ^+ in (2.5) so as to make a closer connection with neutrinoproduction, where the hadronic weak current is not hermitian. We can rewrite $W_{\mu\nu}$ replacing the δ function by an integral and translating J_μ^+:

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{iqx} \langle p|J_\mu^+(x)J_\nu(0)|p\rangle$$

Further, we can replace the product $J_\mu^+(x)J_\nu(0)$ by the commutator

$[J_\mu^+(x), J_\nu(0)]$ because the product $J_\nu(0)J_\mu^+(x)$ does not contribute for $q_o > 0$. $W_{\mu\nu}$ is simply related to the absorptive part of the forward amplitude for virtual photon-hadron scattering:

$$W_{\mu\nu} = \frac{1}{2\pi} \operatorname{Im} T_{\mu\nu} \quad : \quad T_{\mu\nu} = i\int d^4x\, e^{iq\cdot x} \langle p | T(J_\mu^+(x) J_\nu(0)) | p \rangle \tag{2.6}$$

$T_{\mu\nu}$ has the usual analyticity properties, and in particular dispersion relations for it can be written. We will consider the case where the hadron target is unpolarized: then $T_{\mu\nu}$ depends on just the two momenta q and p, and Lorentz covariance, parity and current conservation decree that it can be written:

$$T_{\mu\nu}(q,p) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T_1(q^2,\nu) + \frac{1}{M^2}\left(p_\mu - \frac{\nu q_\mu}{q^2}\right)\left(p_\nu - \frac{\nu q_\nu}{q^2}\right) T_2(q^2,\nu) \tag{2.7}$$

and $W_{\mu\nu}$ can be similarly expressed in terms of two structure functions:

$$W_{1,2}(q^2,\nu) = \frac{1}{2\pi} \operatorname{Im} T_{1,2}(q^2,\nu) \tag{2.8}$$

In the limit $q^2 \to 0$ the diagonal components give the total photon-nucleon cross-section, and so should be non-singular. This means that $W_2(q^2,\nu) = 0(q^2)$ as $q^2 \to 0$ and that

$$\nu^2 W_2(q^2,\nu) + q^2 W_1(q^2,\nu) = 0((q^2)^2) \text{ as } q^2 \to 0.$$

Knowing these constraints, it is possible to introduce kinematic singularity free amplitudes; this will be done in later lectures. It is often convenient to introduce cross-sections for the absorption of transverse and longitudinal photons:

$$\left.\begin{aligned} \sigma_T(q^2,\nu) &= \frac{4\pi^2\alpha}{MK} W_1(q^2,\nu) \\ \sigma_L(q^2,\nu) &= \frac{4\pi^2\alpha}{MK}\left[-W_1 + W_2\left(1 - \frac{\nu^2}{M^2 q^2}\right)\right] \end{aligned}\right\} \tag{2.9}$$

In the limit $q^2 \to 0$, νW_2 is simply related to the total photoproduction cross-section:

$$\lim_{q^2 \to 0} \frac{-\nu W_2(q^2,\nu)}{M^2 q^2} = \frac{1}{4\pi^2\alpha} \sigma_{tot}(\nu)$$

$\kappa \equiv \nu + q^2/2$ is a conventional flux factor, equal to the equivalent real photon momentum. The diagonal elements of $W_{\mu\nu}$ must be positive semi-definite, and the off-diagonal elements must obey Schwartz inequalities. This requires:

$$0 \leq W_1 \leq \left(1 - \frac{\nu^2}{M^2 q^2}\right) W_2 \tag{2.10}$$

which imply the expected positivity of σ_L and σ_T. From equations (2.3), (2.4), (2.7) and (2.8) it follows that the differential cross-section can be written:

$$\frac{d^2\sigma}{d\Omega d\nu} = \frac{\alpha^2}{4ME^2 \sin^4\frac{\theta}{2}}\left(\cos^2\frac{\theta}{2} W_2 + 2\sin^2\frac{\theta}{2} W_1\right) \tag{2.11}$$

C - Neutrinoproduction

In the case of neutrinoproduction[10] the graph in figure 1 is proportional to:

$$\bar{u}_{\mu\ell}(k')\gamma^\mu(1-\gamma_5)u_\nu(k)\langle X|J_\mu|p\rangle \tag{2.12}$$

where we have assumed the conventional V-A charged current-current theory. An amplitude of the form (2.12) yields a cross-section proportional to:

$$t^{\mu\nu}_{wk} W_{\mu\nu} \tag{2.13}$$

where $\quad t^{\mu\nu}_{wk} = 8\left(k'^\mu k^\nu + k^\mu k'^\nu - g^{\mu\nu} k\cdot k' + i\epsilon^{\mu\nu\alpha\beta} k'_\alpha k_\beta\right)$

and $\quad W_{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|\left[J^+_\mu(x), J_\nu(0)\right]|p\rangle$ as before.

The Lorentz decomposition of $W_{\mu\nu}$ is now more complicated, because the weak currents are not conserved, and have both vector and axial components:

$$\left.\begin{aligned} W_{\mu\nu}(q,p) = &-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\nu) + \frac{1}{M^2}\left(p_\mu - \frac{\nu}{q^2}q_\mu\right)\left(p_\nu - \frac{\nu}{q^2}q_\nu\right)W_2(q^2,\nu) \\ &- i\frac{\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta}{2M^2}W_3(q^2,\nu) + \frac{q_\mu q_\nu}{M^2}W_4(q^2,\nu) \\ &+ \frac{1}{2M^2}\left(p_\mu q_\nu + p_\nu q_\mu\right)W_5(q^2,\nu) \end{aligned}\right\} \tag{2.14}$$

(We have assumed T-invariance to eliminate a possible term $\frac{1}{2M^2}(p_\mu q_\nu - p_\nu q_\mu)\ W_6(q^2,\nu)$.) The forward current amplitude $T_{\mu\nu}$ has a similar structure to (2.14) and as before $W_{\mu\nu} = \frac{1}{2\pi}\,\mathrm{Im}T_{\mu\nu}$.

The cross-section formula for neutrinoproduction is correspondingly more complicated; however, the contributions of W_4 and W_5 can usually be neglected at present. This is because they appear in (2.14) multiplied by q_μ (and/or q_ν) and

$$\bar{u}_\ell\ (k')\cancel{q}(1-\gamma_5)u_\nu(k) = -m_\ell\ \bar{u}_\ell\ (k')\ (1-\gamma_5)\ u_\nu(k)$$

so that the contributions of W_4 and W_5 to the cross-section are proportional to lepton masses.* W_1, W_2 and W_3 obey the positivity constraints:

$$0 \leq \frac{1}{2M^2}\sqrt{\nu^2-M^2q^2}\,|W_3| \leq W_1 \leq \left(1-\frac{\nu^2}{M^2q^2}\right)W_2 \tag{2.15}$$

Retaining just the contributions of W_1, W_2 and W_3 we have (neglecting lepton masses) the following differential cross-section for neutrinoproduction:

$$\frac{d^2\sigma^{\nu,\bar\nu}}{dq^2 d\nu} = \frac{G^2}{8M^2\pi E^2}\left[-2q^2W_1^{\nu,\bar\nu} + \left(4E(E-\nu)+q^2\right)W_2^{\nu,\bar\nu} \pm q^2(2E-\nu)\frac{W_3^{\nu,\bar\nu}}{M}\right] \tag{2.16}$$

On changing from neutrino scattering to anti-neutrinos, the W_3 term changes sign because of the different leptonic coupling. Data on neutrino scattering often are presented in terms of the differential cross-section with respect to $\xi = -q^2/2\nu$ and $y=\nu/E$

$$\frac{d^2\sigma^{\nu,\bar\nu}}{d\xi dy} = \frac{G^2ME}{\pi}\left[y^2\xi W_1^{\nu,\bar\nu} + \left(1-y-\frac{My\xi}{2E}\right)\frac{\nu W_2^{\nu,\bar\nu}}{M^2} \mp y\left(1-\frac{y}{2}\right)\frac{\xi\nu}{M^2}W_3^{\nu,\bar\nu}\right] \tag{2.17}$$

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*In the absence of lepton polarization measurements, only three different combinations of structure functions can be separated.

D - Scaling hypotheses and some consequences

Naive scale invariance demands that the cross-sections (2.11) and (2.17) just be functions of the invariant energies in the Bjorken limit:

$$-q^2 \to \infty, \ \nu \to \infty, \ \omega = \frac{1}{\xi} = \frac{-2\nu}{q^2}$$

This is immediately seen to imply the Bjorken[1] scaling hypotheses:*

$$\left.\begin{array}{lcl} W_1(q^2,\nu) & \to & F_1(\xi) \\ \frac{1}{M^2}\nu W_2(q^2,\nu) & \to & F_2(\xi) \\ \frac{1}{M^2}\nu W_3(q^2,\nu) & \to & F_3(\xi) \end{array}\right\} \tag{2.18}$$

We should note that in many models[12]

$$F_2(\xi) = 2\xi F_1(\xi) \tag{2.19}$$

so that (as can be seen from equation (2.9)) $\frac{\sigma_L}{\sigma_T}$ vanishes in the scaling limit. Evidence on the scaling behavior of the structure functions, and the behavior of σ_L/σ_T, will be discussed in section (2.3).

Data on neutrinoproduction are often presented in integrated forms, because of the low statistics available in experiments up to now. If W_1, νW_2 and νW_3 scale as in (2.18), then in neutrinoproduction:

$$\frac{d\sigma^{\nu,\bar\nu}}{dy} \simeq \frac{G^2 EM}{\pi}\left(ay^2 + by + c\right) \tag{2.20}$$

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*The naive scaling laws for W_4 and W_5 would be νW_4, νW_5 → functions of ξ. These are not generally realized in models: for example in the quark model with interactions $\nu^2 W_4(q^2,\nu) \to F_4(\xi)$, $\nu^2 W_5(q^2,\nu) \to F_5(\xi)$.[11]

where
$$\left.\begin{aligned} a &= \int_0^1 d\xi \left(\xi F_1^{\nu,\bar\nu}(\xi) \pm \tfrac{\xi}{2} F_3^{\nu,\bar\nu}(\xi)\right) \\ b &= \int_0^1 d\xi \left(-F_2^{\nu,\bar\nu}(\xi) \mp \xi F_3^{\nu,\bar\nu}(\xi)\right) \\ c &= \int_0^1 d\xi \; F_2^{\nu,\bar\nu}(\xi) \end{aligned}\right\} \quad (2.21)$$

If we assume the Callan-Gross relation (2.18) and introduce the parameter
$$B = -\int_0^1 d\xi \, \xi \, F_3^{\nu}(\xi) \Big/ \int_0^1 d\xi \, F_2^{\nu}(\xi) \quad (2.22)$$

(constrained by (2.15) to have $|B| \leq 1$) then
$$\frac{d\sigma^{\nu,\bar\nu}}{dy} = \frac{G^2 ME}{\pi} \int_0^1 d\xi F_2^{\nu,\bar\nu}(\xi) \left[1 - (1 \mp B)\left(y - \frac{y^2}{2}\right)\right] \quad (2.23)$$

and
$$\sigma_{\nu,\bar\nu} = \frac{G^2 ME}{\pi} \int_0^1 d\xi F_2^{\nu,\bar\nu}(\xi) \left[\frac{2}{3} \pm \frac{B}{3}\right] \quad (2.24)$$

Note that, as expected from the scale invariance of the interactions apart from the dimensional coupling constant G, the cross-sections rise linearly with E. Note also that since $Q^2 = 2\xi\nu = 2\xi y E M$ that we should expect:
$$\langle Q^2 \rangle \propto E \quad (2.25)$$

in the scaling region.

E - Properties of structure functions

Before looking at the data on scaling, we will just note some more properties of the structure functions $W_1(q^2,\nu)$. First their crossing properties:
$$\left.\begin{aligned} W_{1,2}^{eN}(q^2,\nu) &= -W_{1,2}^{eN}(q^2,-\nu) \\ W_{1,2,3,4}^{\nu N}(q^2,\nu) &= -W_{1,2,3,4}^{\bar\nu N}(q^2,-\nu)\,, \quad W_5^{\nu N}(q^2,\nu) = W_5^{\bar\nu N}(q^2,-\nu) \end{aligned}\right\} \quad (2.26)$$

which may be derived by the substitution $q \to -q$ in the definitions (2.14). We would expect Regge behavior for the structure functions when q^2 is kept fixed, and $\nu \to \infty$. If we assume exchange of a leading factorizable Regge pole with intercept α, then

$$\left.\begin{aligned} &W_1 \sim \nu^{\alpha} \quad , W_2 \sim \nu^{\alpha-2} \quad , W_3 \sim \nu^{\alpha-1} \\ &W_4 \sim \nu^{\alpha} \quad , \quad W_5 \sim \nu^{\alpha-1} \end{aligned}\right\} \quad (2.27)$$

If the leading Regge poles were to contribute in the scaling limit[13], then the residues $\beta_i(q^2)$ would have to behave as:

$$\beta_1 \sim (q^2)^{-\alpha} \;, \beta_2 \sim (q^2)^{1-\alpha} \;, \beta_3 \sim (q^2)^{-\alpha} \;, \text{etc} \qquad (2.28)$$

in the limit $q^2 \to \infty$. They would then dominate the behavior of the structure functions in the limit $\xi = \frac{1}{\omega} \to 0$, yielding:

$$F_1(\xi) \sim \xi^{-\alpha} \;, \; F_2(\xi) \sim \xi^{1-\alpha} \;, \; F_3(\xi) \sim \xi^{-\alpha} \;, \text{etc} \qquad (2.29)$$

The leading Regge poles which might contribute are:

W_1, W_2: Pomeron with $\alpha \simeq 1$

W_3: ω, A_2 with $\alpha \simeq \frac{1}{2}$

Hence it is expected that:

$$F_2(\xi) \to \text{constant} \quad \text{as } \xi \to 0 \qquad (2.30)$$

The difference $F_2^{ep} - F_2^{en}$ should be dominated by the A_2 meson Regge trajectory with $\alpha \simeq \frac{1}{2}$, yielding:

$$F_2^{ep}(\xi) - F_2^{en}(\xi) \sim \xi^{\frac{1}{2}} \quad \text{as } \xi \to 0 \qquad (2.31)$$

Finally we note that the analyticity, crossing and Regge properties enable dispersion relations to be written down for the $T_i(q^2, \nu)$: examples are:

$$\left.\begin{aligned} T_2^{eN}(q^2,\nu) &= 4\int_{-q^2/2}^{\infty} d\nu'\,\nu'\, W_2^{eN}(q^2,\nu')\,\frac{1}{\nu'^2-\nu^2} \\ T_3^{\nu N+\bar{\nu}N}(q^2,\nu) &= 4\int_{-q^2/2}^{\infty} d\nu'\,\nu'\, W_3^{\nu N+\bar{\nu}N}(q^2,\nu)\,\frac{1}{\nu'^2-\nu^2} \end{aligned}\right\} \quad (2.32)$$

There could also be real polynomials on the right hand sides of equations (2.32), which would not contribute to the absorptive parts W_i of the T_i. The equations (2.32) will be essential later on when we write sum rules which test models for Bjorken scaling.

2.3 - Qualitative Features of Data on Scaling

In this section we briefly discuss the experimental evidence on scaling in electroproduction, neutrinoproduction and electron-positron annihilation, referring to sources where the data may be found.

A - Electroproduction

Experiments, principally those by a SLAC-MIT collaboration, seem to have established the following facts about electroproduction.

a) The structure functions W_1 and νW_2 seem to scale[2], for both neutrons and protons, for 1 or $2 \leq Q^2 \leq 10$ GeV^2.

b) For both neutrons and protons, the ratio σ_L/σ_T seems[3] to be small, as expected from the Callan-Gross relation, and a scaling behavior of the type $\nu\, \sigma_L/\sigma_T \to f(\xi)$ is not ruled out by the data.

c) The structure function $\nu W_2^{en} < \nu W_2^{ep}$ in the observed range[2,3], the ratio lying between 0.3 and 0.9 approximately.

d) The data are consistent[2,3] with $\nu W_2^{en} < \nu W_2^{ep}$ approaching the same limit as $\xi \to 0$, as expected if the structure functions are dominated by the Pomeron in this limit.

B - Neutrinoproduction

As remarked earlier, the low statistics of data on neutrinoproduction mean that they are usually plotted in an integrated form, and detailed checks of the scaling behaviors of the structure functions are not yet possible. Also, data are so far only available from nuclear targets which are almost equal mixtures of neutrons and protons. However, various consequences of the scaling hypotheses discussed in the previous section have been tested experimentally.

a) Data from CERN (gargamelle)[4] are consistent with σ_ν and $\sigma_{\bar\nu}$ rising linearly with E for E $\gtrsim 2$ GeV.

b) The ratio $\sigma_{\bar\nu}/\sigma_\nu$ seems to be roughly constant at around 0.4 for $2 \leq E < 50$ GeV[4,5]. As $F_2^{\nu N} + F_2^{\nu P} = F_2^{\bar\nu N} + F_2^{\bar\nu P}$ by charge symmetry, examination of formula (2.24) shows that $\sigma_{\bar\nu}/\sigma_\nu = \frac{1}{3}$ would correspond to B=1, or maximal V-A interference. Thus the data suggest that V-A interference is quite large. This is confirmed by data on the y distributions[4,6] in neutrino and anti-neutrino reactions below 50 GeV which are close to $\frac{d\sigma^{\bar\nu}}{dy} \propto (1-y)^2$, $\frac{d\sigma^{\nu}}{dy}$ = constant, as expected from equation (2.23) if $B \simeq 1$.

c) Data from NAL[6] are consistent with $\langle Q^2 \rangle$ rising linearly with E in neutrino reactions up to 150 GeV.

Thus data on neutrinoproduction are nicely consistent with scaling, although it should be emphasized that many of the CERN data have $Q^2 < 1$ GeV^2, and so are really only shallow inelastic.

C - Electron-positron Annihilation

As discussed in Lecture 1, the simple scaling prediction is that $\sigma(e^-e^+ \to \gamma \to \text{hadrons}) \propto \frac{1}{q^2}$ as $q^2 \to \infty$. Since $\sigma(e^-e^+ \to \gamma \to \mu^-\mu^+)$ should also $\sim \frac{1}{q^2}$ by QED, this means we should observe

$$\frac{\sigma(e^-e^+ \to \gamma \to \text{hadrons})}{\sigma(e^-e^+ \to \gamma \to \text{hadrons})} \to \text{constant as } q^2 \to \infty$$

Data from energies below $q^2=25GeV^2$ so far show little sign of this limiting behaviour setting in[7]. Indeed the CEA data at $q^2=16$ and 25 suggest that the hadron-muon ratio rises in this energy range. However, in view of the large error bars on the data, probably no firm conclusions about scaling should be drawn until more accurate data are available. SPEAR should provide some very soon.

FIGURE CAPTIONS

1. Kinematics of inelastic lepton scattering.
2. Kinematic range of inelastic lepton scattering.

REFERENCES

1. J.D. Bjorken - Phys. Rev. 179, 1547 (1969)
2. J.I. Friedman and H. Kendall - Ann. Rev. of Nuclear Sciences 22, 203 (1972)
3. E.M. Riordan - M.I.T. Laboratory for Nuclear Science Technical Report no. COO-3069-176 (1973)
4. D.H. Perkins - Proceedings of the XVI International Conference on High Energy Physics (N.A.L., Batavia, 1972) p. 189
5. A. Benvenuti et al. - Phys. Rev. Letters 30, 1084 (1973)
6. B.C. Barish et al. - Phys. Rev. Letters 31, 565 (1973)
7. H. Newman - Proceedings of the S.L.A.C. Summer Institute (1973)
8. F.J. Hasert et al - Phys. Letters 46B, 138 (1973)
9. A. Benvenuti et al - N.A.L. preprint (1973)
10. For a comprehensive discussion see:
    C.H. Llewellyn Smith - Physics Reports 3C, 261 (1972)
11. W.-C. Ng and P. Vinciarelli - Phys. Letters 38B, 219 (1972)
12. C.G. Callan and D.J. Gross - Phys. Rev. Letters 22, 156 (1969)
13. H.D.I. Abarbanel, M.L. Goldberger and S.B. Treiman - Phys. Rev. Letters 22, 500 (1969)

3 - Parton Model

3.1 Introduction

The parton model was originally introduced by Feynman[1] to account for the systematics of high energy hadron-hadron collisions. Soon after the SLAC-MIT inelastic electroproduction experiments were performed, it was observed[2] that the parton model provides a natural explanation for the observed Bjorken scaling. Unfortunately, in its early (and simplest) formulations[3] the model suffers from several diseases e.g., it is not Lorentz covariant, and it is sufficiently poorly defined to make it difficult to determine which of the model's predictions are reliable and which are not.

Light-cone and short-distance expansions were developed independently and are not afflicted with these diseases. However, it was apparent almost from the beginning that light-cone expansions and algebra are in some sense equivalent to the more general features of parton models. This equivalence has been worked out in detail[4], and the parton model has been cured of several of its diseases by Landshoff, Polkinghorne, and Short[5] in a formulation which is in many ways a momentum space representation of light-cone expansions.

Despite subsequent developments, the naive parton model remains the simplest approach to the derivation of the fundamental results of short distance and light-cone expansions. Hence the inclusion of this lecture is a series primarily dedicated to light-cone and short distance physics. It should be kept in mind that the rather awkward assumptions made in parton models have analogs in the coordinate space approach. It is important to resist being seduced by the elegance of that approach into the misapprehension that it involves no unconventional assumptions.

The outline of this lecture is as follows: First, we introduce light-cone coordinates which will simplify our subsequent work; second, we give a brief derivation of the parton model in electroproduction; third, we derive a selection of sum rules and other relations in a quark-parton model; fourth, we discuss briefly the parton model of the e^+e^- total annihilation cross section; fifth, we outline a hierarchy for the generality of parton model results; lastly, we enumerate some other processes to which parton models have been applied, both those which can also be treated with light-cone or short distance methods and those which can't.

3.2 - Light-cone variables

It will be convenient to introduce the coordinates[6]

$$\hat{a}_\mu \equiv (a_-, a_1, a_2, a_+)$$
$$a_\pm \equiv \frac{1}{\sqrt{2}}(a_0 \pm a_3)$$

for any four vector a_μ (likewise for any tensor $t^{\alpha\beta\cdots}$). We reserve the notation a_μ for the usual coordinate representation $a_\mu = (a_0, a_1, a_2, a_3)$. In terms of $a_\pm$ we find

$a^2 = 2a_+a_- - \vec{a}_\perp^{\,2}$ ($a_\perp$ is the Euclidean vector $[a_1,a_2]$) which may be summarized in terms of a metric tensor $\hat{g}^{\mu\nu}$ with $\hat{g}^{+-} = \hat{g}^{-+} = 1$, $\hat{g}_{ii} = -1$ for $i = 1,2$ and all other entries zero. Then $a^2 = \hat{a}_\mu \hat{a}_\nu \hat{g}^{\mu\nu}$.*

An entire canonical formalism for field theory may be developed in these coordinates.[7] The usual association of the Hamiltonian, H, with propagation in time is replaced by the identification of $P_- \equiv \frac{1}{\sqrt{2}}(P_0 - P_3)$ with propagation in $x_+ \equiv \tau$. The mass shell equation; $P_- = \frac{(P_\perp^2 + M^2)}{2P_+}$ is reminiscent of nonrelativistic dynamics with $P_- \leftrightarrow$ Energy; $P_+ \leftrightarrow$ mass. This analogy is developed fully by Kogut and Soper.[7]

3.3 - Derivation of the Parton Model

Derivations of the parton model tend to be unsatisfying. Since we intend to use the model only heuristically and as a mnemonic for light-cone calculations, this need not overly concern us. A more satisfactory derivation is possible,[5] though the physics is the same as presented here. Our derivation parallels that of Drell, Levy, and Yan[3,8] except that we prefer light-cone variables to the infinite momentum frame.

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*This notation has the rather unfortunate side effect that $a^{\pm} = a_{\mp}$. However, we will be careful to use only covariant (as opposed to contravariant) vectors and avoid this complexity. Note that $\hat{g}_{+-} = \hat{g}_{-+} = \hat{g}^{+-} = 1$.

We begin with the usual expression for $W_{\mu\nu}$:

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \langle P|\, J_\mu(x)\, J_\nu(0)\, |P\rangle \tag{3.1}$$

where J_μ and $|P\rangle$ are in the Heisenberg picture. Now introduce the operator:

$$\mathcal{U}(\tau) \equiv \mathcal{T}\left(\exp -i\int_{-\infty}^{\tau} d\tau'\, \mathcal{P}_-^{I}(\tau')\right) \tag{3.2}$$

(remember $\tau \equiv 1/\sqrt{2}\,(x_0 + x_3)$) which transforms from the Heisenberg to interaction picture in light-cone coordinates. [The τ development is determined by the "Hamiltonian":

$$\mathcal{P}_-(\tau) = \mathcal{P}_-^{0}(\tau) + \mathcal{P}_-^{I}(\tau)$$

where $\mathcal{P}_-^{\circ}$ is the non-interacting part. $\mathcal{T}$ signifies τ-ordering, analogous to time ordering]. $\mathcal{U}(\tau)$ converts $J_\mu(x)$ to the free current $j_\mu(x)$:

$$J_\mu(x) = \mathcal{U}^{-1}(\tau)\, j_\mu(x)\, \mathcal{U}(\tau) \tag{3.3}$$

So Eq. (3.1) becomes

$$W_{\mu\nu} = \frac{1}{4\pi}\int e^{i(q_-\tau + q_+x_- - \vec{q}_\perp\cdot\vec{x}_\perp)} \langle P|\mathcal{U}^{-1}(\tau)\, j_\mu(x)\, \mathcal{U}(\tau)\, \mathcal{U}^{-1}(0)\, j_\nu(0)\, \mathcal{U}(0)|P\rangle\, d^4x \tag{3.4}$$

As John Ellis described in the introductory lecture, we may take the Bjorken limit by letting $q_-\to\infty$ with all other momenta fixed ($\xi \equiv -q^2/2P\cdot q = -q_+/P_+$, etc.). As Ellis discussed $q_- \to \infty$ implies $\tau \to 0$. Let us suppose then that τ may be set to zero in the matrix element of Eq. (3.4) (we shall return to this question):

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4x \, e^{iq\cdot x} \langle UP | j_\mu(x) j_\nu(0) | UP \rangle \tag{3.5}$$

where $|UP\rangle \equiv U(\tau = 0)|P\rangle$

Eq. (3.5) <u>is</u> the parton model for inelastic electroproduction. To see this consider $|UP\rangle$:

$$U(0)|P\rangle = \sqrt{Z_2}\Big\{ |P\rangle + \sum_n{}' \frac{|n\rangle\langle n|P_-^I(0)|P\rangle}{P_-(P) - P_-(n)} + \sum_{n,m}{}' \frac{|n\rangle\langle n|P_-^I(0)|m\rangle\langle m|P_-^I(0)|P\rangle}{[P_-(P)-P_-(n)][P_-(P)-P_-(m)]} \tag{3.6}$$

Here Σ' indicates a sum over all states except $|P\rangle$. This is "old-fashioned" perturbation theory* in light-cone variables, in particular: $\vec{P}_\perp, P_+$ and mass are conserved at each vertex while P_- is not. Also, the states $|n\rangle$ (and $|m\rangle$) are free (multi-) particle states, i.e., eigenstates of P_-^0. In a

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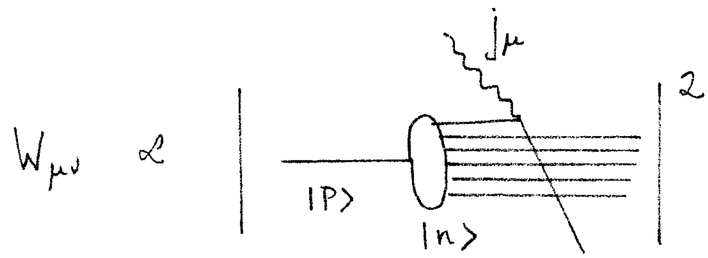
* See Drell and Yan, Refs. 3 and 8 for a reminder of the rules of old fashioned perturbation theory.

simplified notation:

$$U(0)|P\rangle = \sum_n a_n |n\rangle \tag{3.7}$$

with $\sum_n |a_n|^2 = 1$. The free currents in Eq. (3.5) can only scatter off a free particle in the state $|n\rangle$, which may be illustrated as follows:



(Fig. 1)

Note there is no form factor at the photon vertex and no final state interaction. This is the essence of the parton model: The elastic and incoherent scattering off of constituents in the Bjorken limit. Scaling will be seen to follow directly from a reading of Figure 1.

The essential dynamical ingredient of the model is the replacement $U(\tau)U^{-1}(0) \to 1$ as $q_- \to \infty$. It is not clear that this replacement may be interchanged with the implied sum over intermediate states in Eq. (3.4), and in fact, in renormalizable theories it can't. Consider for example, $\langle n|U(\tau)U^{-1}(0)|P\rangle$ with $|P\rangle$ a single particle state of momentum

$$\hat{P} = \left(\frac{M^2}{2P_+}, 0, 0, P_+ \right)$$

Using Eq. (3.2):

$$\langle n|U(\tau)U^{-1}(0)|P\rangle = \langle n|P\rangle + \frac{\left[e^{i[P_-(n)-P_-(P)]\tau}-1\right]}{P_-(P)-P_-(n)}$$

$$\otimes \langle n|\rho_-^I(0)|P\rangle + \ldots\ldots \tag{3.8}$$

Let $|n\rangle$ be a collection of particles with momenta

$$\hat{P}_i = \left(\frac{m_i^2+\vec{P}_{\perp i}^{\,2}}{2x_iP_+}\,,\ \vec{P}_{\perp i}\,,\ x_iP_+\right)$$

[Momentum conservation demands $\sum_i x_i = 1$, $\sum_i \vec{P}_{\perp i} = 0$].
The form of $\hat{P}_i$ guarantees 3-momentum and mass conservation.
As $q_-\to\infty$ $\tau \sim 1/q_-$, but the interaction dependent terms in Eq. (3.8) vanish only if $(P_-(n) - P_-(P))$ is finite for all states $|n\rangle$

$$P_-(n)-P_-(P) = \sum_i \frac{m_i^2+\vec{P}_{\perp i}^{\,2}}{2x_iP_+} - \frac{M^2}{2P_+} \tag{3.9}$$

One way to enforce this (due originally to Drell, Levy, and Yan[3]) is to introduce _ad hoc_ a transverse momentum cutoff into $\langle n|\rho_-^I(0)|P\rangle$. Then $\vec{P}_{\perp i}^{2}$ never becomes large and Eq. (3.8) reduces to $\langle n|P\rangle$ as desired. A covariant formulation of the same (_ad hoc_) dynamical supression was introduced by Landshoff, Polkinghorne, and Short.[5] From here on we shall realize the parton model by the simple expedient of assuming a transverse momentum cutoff, i.e., by assuming all particles in the state $|n\rangle$ in Eq. (3.7) have momentum $|P_{\perp i}| < P_{\perp max}$ relative to the incident proton.

$P_-(n)-P_-(P)$ also grows large as $x_i \to 0$. Disposal of this region and the proof that $1>x_i>0$ in most models may be constructed following Drell, Levy and Yan.[3]

In renormalizable perturbation theory without the introduction of <u>ad hoc</u> cutoffs the interaction dependent terms in Eq. (3.8) remain important in the Bjorken limit and neither a parton model nor scaling is obtained.

Combining Eq's. (3.5) and (3.7) with the transverse momentum cutoff we may demonstrate Bjorken scaling:

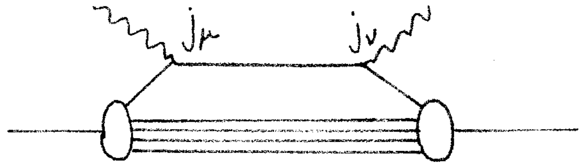
$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \sum_n |a_n|^2 \langle n|\, j_\mu(x) j_\nu(0)\,|n\rangle$$

where the state $|n\rangle$ consists of a collection of particles with limited transverse momentum:

$$\hat{P}_i = \left(\frac{m_i^2 + \vec{P}_{\perp i}^{\,2}}{2x_i P_+},\ \vec{P}_{i\perp},\ x_i P_+ \right) \tag{3.10}$$

$(\sum_i x_i = 1, \sum_i \vec{P}_{i\perp} = 0)$. Terms with $a_m^* a_n$ cannot occur on account of the transverse momentum cutoff and absence of final state interactions. By virtue of this $j_\mu(x) j_\nu(0)$ is a single particle operator:

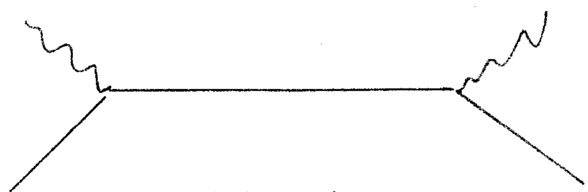
$W_{\mu\nu} \propto$



(Fig. 2)

$$W_{\mu\nu} = \sum_n |a_n|^2 \sum_i w_{\mu\nu}^{\text{Born}}(P_i, q) \tag{3.11}$$

where $w_{\mu\nu}^{\text{Born}}$ is the structure function for the Born graph



(Fig. 3)

for scattering off of parton i. Consider (e.g.) a spin 1/2 parton

$$\begin{aligned} w_{\mu\nu}^{\text{Born}} &= \tfrac{1}{4} Q_i^2 \,\mathrm{Tr}\,\{\not{P}_i \gamma_\mu (\not{P}_i + \not{q}) \gamma_\nu\}\, \delta[(P_i+q)^2 - m_i^2] \\ &= Q_i^2\, \delta(q^2 + 2P_i \cdot q) \{ P_{i\mu}(P_i+q)_\nu \\ &\quad + P_{i\nu}(P_i+q)_\mu - g_{\mu\nu} P_i \cdot (P_i+q) \} \end{aligned} \tag{3.12}$$

The δ-function is the imaginary part of the parton propagator in Figure 3, Q_i is the parton's charge.

From here on we consider the laboratory frame:
$\vec{q}_\perp = \vec{P}_\perp = 0$; $q^2 = 2q_+q_-$, $P^2 = M^2 = 2P_+P_-$, $\nu = P_+q_- + P_-q_+$, and the Bjorken limit: $q_- \to \infty$, $\xi = \dfrac{-q_+}{P_+}$ fixed $(0 \le \xi \le 1)$.

To isolate W_1 consider $W_{yy} \equiv -g_{yy}W_1 = W_1$:

$$\lim_{Bj} W_{yy} = \sum_n |a_n|^2 \sum_i \delta(2q_+q_- + 2x_iP_+q_-)$$
$$\otimes (2P_{iy}^2 + m_i^2 + x_i\nu) Q_i^2$$
$$= \frac{1}{2\nu} \sum_{n,i} |a_n|^2 \delta(\xi - x_i) x_i \nu Q_i^2$$
$$= \frac{\xi}{2} \sum_{n,i} |a_n|^2 \delta(\xi - x_i) Q_i^2$$

$\sum_{ni} |a_n|^2 \delta(\xi - x_i)$ is the probability per unit phase space to find a parton with momentum $P_{+i} = \xi P_+$ in a proton. In light-cone variables:

$$d\Pi = d^4p\, \delta(p^2 - m^2) = \frac{dp_+}{p_+} d^2p_\perp = \frac{d\xi}{\xi} d^2p_\perp$$

so $\xi \sum_{n,i} |a_n|^2 \delta(\xi - x_i)$ is the parton probability density per unit $d\xi$. A more convenient notation is

$$\xi \sum_{n,i} |a_n|^2 \delta(\xi - x_i) = \sum_a u_a(\xi)$$

where $u_a(\xi)$ is the probability of finding a parton of <u>species</u> "a" with momentum ξP_+. Finally then

$$F_1(\xi) \equiv \lim_{Bj} W_1(Q^2, \nu) = \frac{1}{2} \sum_a u_a(\xi) Q_a^2 \qquad (3.13)$$

Analogously

$$F_2(\xi) \equiv \lim_{Bj} \frac{\nu}{M^2} W_2(Q^2, \nu) = \xi \sum_a u_a(\xi) Q_a^2 \qquad (3.14)$$

thereby establishing Bjorken scaling.

3.4 Sum Rules and Spectrum Relations in the Quark Model

To illustrate the convenience of the parton model, we shall quickly derive a series of sum rules and spectrum relations which will be explored in more detail later. For the present we confine ourselves to the quark model and return to the question of generality later. From the electromagnetic:*

$$j_\mu^{e.m.} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \qquad (3.15)$$

and weak

$$j_\mu^{+} = \cos\theta_c\,\bar{u}\gamma_\mu(1-\gamma_5)d + \sin\theta_c\,\bar{u}\gamma_\mu(1-\gamma_5)s \qquad (3.16)$$

currents of the model and Eq. (3.14) and its neutrino analog the following table may be constructed:

$$F_1^{ep}(\xi) = \frac{1}{2}\left\{\frac{4}{9}\left(u(\xi)+\bar{u}(\xi)\right) + \frac{1}{9}\left(d(\xi)+\bar{d}(\xi)+s(\xi)+\bar{s}(\xi)\right)\right\} \qquad (3.17a)$$

$$F_1^{en}(\xi) = \frac{1}{2}\left\{\frac{4}{9}\left(d(\xi)+\bar{d}(\xi)\right) + \frac{1}{9}\left(u(\xi)+\bar{u}(\xi)+s(\xi)+\bar{s}(\xi)\right)\right\} \qquad (3.17b)$$

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* We use Gell-Mann's notation u,d,s for the conventional p,n,λ quark triplet with charges 2/3,-1/3,-1/3 and strangeness 0,0,-1 respectively. $u(\xi)$, $d(\xi)$, $s(\xi)$ denote the probability densities for the quarks as partons.

$$F_1^{\nu p}(\xi) = F_1^{\bar{\nu} n}(\xi) = d(\xi) + \bar{u}(\xi) \tag{3.17c}$$

$$F_1^{\bar{\nu} p}(\xi) = F_1^{\nu n}(\xi) = u(\xi) + \bar{d}(\xi) \tag{3.17d}$$

$$F_3^{\nu p}(\xi) = F_3^{\bar{\nu} n}(\xi) = 2\bar{u}(\xi) - 2d(\xi) \tag{3.17e}$$

$$F_3^{\bar{\nu} p}(\xi) = F_3^{\nu n}(\xi) = 2\bar{d}(\xi) - 2u(\xi) \tag{3.17f}$$

Here θ_c has been set to zero[9] and charge symmetry has been used:

$$|p\rangle = e^{i\pi I_2}|n\rangle, \quad J_\mu^-(x) = e^{-i\pi I_2} J_\mu^+(x)\, e^{i\pi I_2}$$

Expressions for $F_2(\xi)$ are not given since as we have already seen for electroproduction: $F_2(\xi) = 2\xi F_1(\xi)$. The quark density distributions u, d, s, etc. are non-zero only for $0 < \xi < 1$.

A. Callan-Gross Relation:[10]

As already remarked $F_2(\xi) = 2\xi F_1(\xi)$ for ep, en, νp, or νn scattering. This is a consequence of the spin structure of a spin-1/2 current. Had quarks spin-0, $F_1(\xi) = 0$ would have been obtained. As discussed in lecture 2, experiments support the spin-1/2 assignment although the evidence is not at all conclusive.[11,12]

B. Adler-Sum Rule:[13]

This is the simplest but least precise derivation of Adler's Sum Rule: From Eqs. (3.17c) and (3.17d):

$$\int_0^1 d\xi \left[F_1^{\bar{\nu}P}(\xi) - F_1^{\nu P}(\xi)\right] = \int_0^1 d\xi \left[u(\xi) - d(\xi) + \bar{d}(\xi) - \bar{u}(\xi)\right]$$
$$= 1 \qquad (3.18)$$

because the integral just measures twice the proton's third component of isospin. Note we have had to use the fact that all partons have x_i between zero and one.

C. Gross-Llewellyn Smith Sum Rule:[14]

From (3.17e) and (3.17f):

$$\int_0^1 d\xi \left[F_3^{\nu P}(\xi) + F_3^{\bar{\nu}P}(\xi)\right] = -2\int_0^1 d\xi \left[u(\xi) + d(\xi) - \bar{u}(\xi) - \bar{d}(\xi)\right]$$
$$= -6B(P) + 2S(P) = -6 \qquad (3.19)$$

where B(P) and S(P) are the baryon number and strangeness of the proton.

D. "Duality" Sum Rule:[15]

From Eq. (3.17a) and (3.17b):

$$\int d\xi \left[F_1^{eP}(\xi) - F_1^{en}(\xi)\right] = \frac{1}{6}\int_0^1 d\xi \left[u(\xi) + \bar{u}(\xi) - d(\xi) - \bar{d}(\xi)\right]$$

Now _if_ the only $\bar{u}$ and $\bar{d}$ quarks in the proton and neutron reside in some isospin symmetric "sea" of $q\bar{q}$ pairs denoted by c(x) then:

$$u(\xi) = u_0(\xi) + c(x)$$
$$d(\xi) = d_0(\xi) + c(x)$$
$$\bar{u}(\xi) = \bar{d}(\xi) = c(x)$$

The integral is just proportional to the proton's isospin and we obtain:

$$\int_0^1 d\xi \left[F_1^{ep}(\xi) - F_1^{en}(\xi) \right] = 1/6 \tag{3.20}$$

E. Nachtmann relation:[16]

From Eq. (3.17a) and (3.17b) it follows immediately that

$$1/4 \leq F_2^{ep}(\xi)/F_2^{en}(\xi) \equiv \mathcal{R} \leq 4$$

Current data show $\mathcal{R} \simeq 1/3$ near $\xi = 1$.

F. Llewellyn Smith relation:[17]

Eqs. (3.17) involve 6 independent structure functions, but $s(\xi)$ and $\bar{s}(\xi)$ occur only in the combination $s(\xi) + \bar{s}(\xi)$ from which it follows:

$$F_3^{\nu p}(\xi) - F_3^{\bar{\nu} p}(\xi) = 12 \left[F_1^{ep}(\xi) - F_1^{en}(\xi) \right] \tag{3.21}$$

G. Momentum Sum Rules:[18]

Momentum conservation requires $\int d\xi \sum_a \xi u_a(\xi) = 1$ Uncharged "gluons" also contribute to this sum rule although they do not participate in lepton scattering and their distributions cannon be directly measured. We may separate their contribution:

$$1 - \epsilon = \int_0^1 d\xi \sum_a \xi u_a(\xi) \tag{3.22}$$

where the sum now ranges only over charged constituents and ε is the fraction of the proton's momentum carried by the gluons. From Eq. (3.22) we obtain the inequality

$$\int_0^1 d\xi \left[F_2^{ep}(\xi) + F_2^{en}(\xi) \right] \leq 5/9$$

(which follows from the observation $u(\xi), \bar{u}(\xi) \ldots \bar{s}(\xi) \geq 0$, as required by positivity discussed in the previous lecture). ε itself may be measured using neutrino structure functions:

$$\epsilon = 1 + \int_0^1 d\xi \left[\frac{3}{4} F_2^{\nu p + \bar{\nu} p}(\xi) - \frac{9}{2} F_2^{ep+en}(\xi) \right] \tag{3.23}$$

which may be evaluated from present data:[19]

$$\epsilon = 0.46 \pm 0.21$$

Gluons seem to be required, which is not unexpected since they presumably supply the forces which bind quarks together.

Finally, let me list a series of results which may be reqarded as exercises for the impassioned reader:

(1)
$$F_1^{\nu p}(\xi) + F_1^{\nu n}(\xi) \leq \frac{18}{5} \left(F_1^{ep}(\xi) + F_1^{en}(\xi) \right)$$

(2) In the vector gluon model, where $\partial_\mu J^{\mu +} = 2im_u \bar{u}\gamma_5 d$ ($m_u = m_d$ is the mass of the u-quark, and we have set $\theta_c = 0$), calculate divergence-proton "inelastic scattering" and show[20]

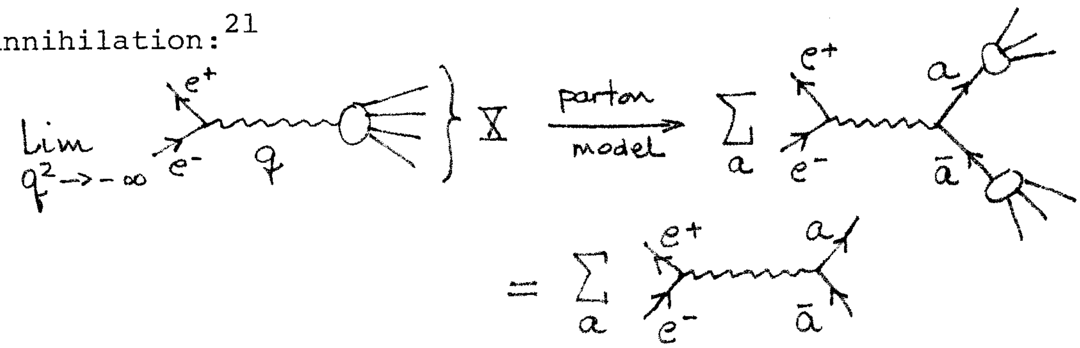
$$F_4^{\nu p}(\xi) - \frac{1}{2\xi} F_5^{\nu p}(\xi) = \frac{m_u^2}{4M^2\xi^3} F_2^{\nu p} \tag{3.25}$$

This practically unobservable result shows that the bare quark mass (m_u) is in principle observable and finite in the parton model.

Before returning to discuss the generality of these results, we discuss briefly application of the parton model to e^+e^- annihilation.

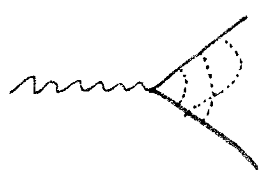
3.5 $e^+e^- \rightarrow \quad \rightarrow$ hadrons

The same analysis which led us to the parton model for electroproduction leads to the following picture of e^+e^- annihilation:[21]



(Fig. 4)

Briefly: the virtual photon decays into a parton-antiparton pair which subsequently develop into hadrons without substantial interaction. The vanishing of interaction terms such as



is in analogy with the vanishing of vertex corrections in the parton graphs for electroproduction. Consequently:

$$\lim_{q^2 \to \infty} \sigma(e^+e^- \to \text{anything}) = \sum_a Q_a^2 \, \sigma(e^+e^- \to a\bar{a})$$
$$= \frac{4\pi\alpha^2}{3q^2} \sum_a Q_a^2 \qquad (3.26)$$

(for spin-1/2 partons)

a result which we will see to be very fundamental from a coordinate space point of view.

3.6 A Hierarchy of Parton Model Results

The time has come to discuss the generality of the various results we have derived in the parton model. Much insight into the reliability of the parton model comes from experience with light-cone and Bjorken-Johnson-Low techniques and will be obvious later on.

The first class of results are scaling laws. These are shared by all approaches in which interactions are negligible near the light-cone. Although they are all violated in lowest order of perturbation theory in renormalizable theories, nevertheless all techniques which predict scaling produce the same scaling laws.*

Sum rules are far more variable. Those which depend solely on the relation between parton distribution functions

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*This is not the case for the structure functions which violate chiral symmetry ($W_{4,5}[q^2,\nu]$). These depend non-trivially on the interactions which violate the symmetry. See Ref. 20 for a discussion.

and conserved quantities such as I_3, S and B may be abstracted from the algebra of current commutators near the light-cone. Examples of these are the Adler and Gross-Llewellyn Smith sum rules. The correct algebraic derivations of these sum rules are quite different. The Adler sum rule depends only on the isospin algebra of the weak currents and is valid in all reasonable models which scale. Moreover, it is actually the Bjorken limit of a fixed-q^2 sum rule and thereby acquires a rather different significance than the G-LS sum rule which doesn't have a fixed-q^2 analog. The G-LS sum rule is a sensitive test of the fractional baryon number which distinguishes the quark model: For example, in the Sakata model the right-hand side is -2 rather than -6.

Sum rules which rely upon "intuition" regarding the parton distribution of hadrons should be kept distinct from the "algebraic" sum rules discussed above. Whether they follow from the absence of exotic exchanges in the t-channel[22] (as does the "duality" sum rule) or from more elaborate hypotheses (as, for example, the "mean-square charge" sum rule of Bjorken and Paschos)[23], these results are a step removed from those which rely solely on the algebra.

Spectrum relations may be classified similarly. All of the ones discussed in the previous section have been shown to follow from the algebraic properties of the currents.[24] (The Callan-Gross relation follows from the Dirac algebra rather than an internal symmetry.) The momentum sum rule is rather complex and will be discussed in a later lecture more carefully.

Finally we note that the parton model predictions for $e^+e^- \to X$ are actually consequences only of scale invariance at short distances. The scaling law and proportionality to $\sum_a Q_a^2$ are shared by all reasonable models which have Bjorken scaling.

3.7 Other Parton Processes

Since its development in the regime of electroproduction, the parton model has been applied a wide variety of processes augmented with various additional assumptions. A few of these (numbers 1,2, and 4 below) will be discussed later in John Ellis' lectures. For the sake of completeness, I list a selection of these determined purely on the basis of personal taste. They are listed more or less in order of increasing numbers of additional assumptions, in all cases the model is applied in a particular scaling limit:

	Process	References
1.	$e^+e^- \to h$ + anything	25
2.	$ep \to eh$ + anything	26
3.	$PP \to \mu^+\mu^-$ + anything	27
	$\gamma P \to \mu^+\mu^-$ + anything	28
4.	$ep \to \gamma e$ + anything	29
5.	Connections between form factors and $ep \to e$ + anything	30

	Process	References
6.	$ab \to cd$ at wide angles	31
7.	$ab \to c$ + anything when $P_{c_\perp}$ is large	32

Many of the predictions of the parton model follow from its operator structure near the light-cone and at short distances. Nevertheless the model is more predictive than is its short-distance and light-cone structure alone. It is important to remember this although we have introduced the model merely as an expedient. Ellis will discuss the distinction between partons and the light-cone in some detail later.

REFERENCES

1. R. P. Feynman, unpublished and Phys. Rev. Lett. 23, 1415 (1969), and in "High Energy Collisions", London; Gordon and Breach, 1969.
2. J. D. Bjorken and E. A. Paschos, Phys. Rev. 158, 1975 (1969), and R. P. Feynman, unpublished.
3. J. D. Bjorken and E. A. Paschos, op. cit.; S. D. Drell, D. J. Levy and T. M. Yan, Phys. Rev. D1, 1035 (1970); S. D. Drell and T. M. Yan, Ann. Physics (N.Y.) 66, 555(1971).
4. R. L. Jaffe, Physics Letters 37B, 517 (1971), Phys. Rev. D5, 2622 (1972). J. C. Polkinghorne, Nuovo Cimento 8A, 572 (1972).
5. P.V. Landshoff, J. C. Polkinghorne and R. D. Short, Nucl. Physics B20, 225 (1971); see P.V. Landshoff and J. C. Polkinghorne, Physics Reports 5C, 1 (1972).
6. L. Susskind, Phys. Rev. 165, 1535 (1968).
7. J. B. Kogut and D. E. Soper, Phys. Rev. D1, 2901 (1970) develop the canonical formulation and Feynman rules for QED.
8. See, for example, S. D. Drell, "Inelastic Electron Scattering, Asymptotic Behavior, and Sum Rules", in Subnuclear Phenomenon, edited by A. Zichichi, (Academic Press, London, 1970).
9. Predictions for the case $\theta_C \neq 0$ are to be found in C. H. Llewellyn Smith, Physics Reports 3C, 261 (1972).
The possibility of large violations of charge symmetry as evidence of charm required in gauge theories of weak and electromagnetic interactions is discussed by A. deRujula

and S. Glashow in a series of recent Harvard preprints.

10. C. Callan and D. J. Gross, Phys. Rev. Lett. 22, 156 (1969).
11. J. F. Gunion and R. L. Jaffe, M.I.T.-preprint MIT-CTP-342 and erratum.
12. See E. Riordan, M.I.T. Ph.D. thesis (unpublished) for recent data.
13. S. L. Adler, Phys. Rev. 143, 1144 (1965).
14. D. J. Gross and C. H. Llewellyn Smith, Nuclear Physics B14, 337 (1969).
15. K. Gottfried, Phys. Rev. Lett. 18, 1174 (1967); P. V. Landshoff and J. C. Polkinghorne, Nucl. Phys. B28, 240 (1971).
16. O. Nachtmann; J. Physics (Paris) 32, 97 (1971); Nucl. Phys. B38, 397 (1972); Phys. Rev. D5, 686 (1972).
17. C. H. Llewellyn Smith, Nucl. Physics B17, 277 (1970).
18. C. H. Llewellyn Smith, Phys. Rev. D4, 2392 (1971).
19. D. H. Perkins, "Neutrino Interactions" in XVI International Conference on High Energy Physics, Chicago-Batavia, Sept., 1972.
20. R. L. Jaffe and C. H. Llewellyn Smith, Phys. Rev. D7, 2506 (1973).
21. N. Cabbibo, G. Parisi, M. Testa, Nuovo Cim. Lett. 4, 35 (1970).
22. See, for example, Landshoff and Polkinghorne, Ref. 15.
23. J. D. Bjorken and E. A. Paschos, Phys. Rev. D1, 1450 (1970).
24. See, for example, C. G. Callan, M. Gronau, A. Pais, E. A. Paschos and S. B. Treiman, Phys. Rev. D6, 387 (1972).

25. S. D. Drell, D. J. Levy and T. M. Yan, Phys. Rev. D1, 1617 (1970).

26. S. D. Drell and T. M. Yan, Phys. Rev. Letters 24, 855 (1970); E. W. Colglazier and F. Ravndal, Phys. Rev. D7, 1537 (1973); J. D. Bjorken, Phys. Rev. D7, 282 (1973). Recent work is referenced in R. N. Cahn, J. W. Cleymans and E. W. Colglazier, Phys. Letters 43B, 323 (1973).

27. S. D. Drell and T. M. Yan, Phys. Rev. Letters 25, 316 (1970) P.V. Landshoff and J. C. Polkinghorne, Nuclear Physics B33, 221 (1971) and Erratum, ibid, B36, 642 (1972).

28. R. L. Jaffe, Phys. Rev. D4, 1507 (1971).

29. S. J. Brodsky, J. F. Gunion and R. L. Jaffe, Phys. Rev. D6, 2488 (1972).

30. S. D. Drell and T. M. Yan, Phys. Rev. Letters 24, 181 (1970). G. West, Phys. Rev. Letters 24, 1206 (1970).

31. J. F. Gunion, S. J. Brodsky, R. Blankenbecler, Phys. Letters, 34B, 649 (1972), SLAC preprint SLAC-PUB-1183.

32. J. F. Gunion, S. J. Brodsky, R. Blankenbecler, Phys. Rev. D6, 2625 (1972); S. Berman, J. D. Bjorken and J. B. Kogut, Phys. Rev. D4, 3388 (1971); P. V. Landshoff and J. C. Polkinghorne, Cambridge Preprints DAMTP 72/43 and 72/48.

4. - Singular Functions and Fourier Transforms

4.1 Introduction

Practical use of short-distance and light-cone expansions requires facility with the Fourier transformation of various singular functions. Derivations of the necessary formulae are usually rather technical and too frequently non-experts are put off by this technology. In this lecture we attempt to introduce the required formulae without referring to the Bateman manuscript or other sources of arcana. First we study the general properties of the singular functions which are expected to occur in operator product expansions. Second we discuss the propagator functions of free scalar field theory in some detail and quote an explicit expression. From this we derive a formula for the Fourier transform (F.T.) of simple (integer power) light-cone singularities. Lastly we extend this, via analytic continuation, in d - the dimension of the singularity - to the general case. Use of analytic continuation allows us to circumvent a lot of algebra but is open to the criticism that the F.T. may not be analytic in d. We may be assured of the correctness of the proceedure by its agreement with the often quoted result of the more tedious analysis.

The casual reader may wish to skip this lecture. He must do so at the price of accepting Eqs. (4.15), (4.16), (4.17) and (4.21) on faith.

4.2 General Properties

The singular functions which occur in operator product expansions are generally of the form $(1/x^2)^d$ plus some boundary conditions which tell us how to treat the singularity at $x^2=0$. The boundary conditions are conveniently included in the singular expression itself by addition of a small imaginary part ($i\varepsilon$) to x^2. For general operator products the determination of the proper $i\varepsilon$-prescription may be a delicate matter.[1] We will frequently encounter three different prescriptions: First that appropriate to a simple product:

$$\frac{1}{(-x^2+i\epsilon x_0)^d} \sim \langle 0|\phi(x)\phi(0)|0\rangle$$

second, that associated with a commutator:

$$\frac{1}{(-x^2+i\epsilon x_0)^d} - \frac{1}{(-x^2-i\epsilon x_0)^d} \sim \langle 0|[\phi(x),\phi(0)]|0\rangle$$

and third, that appropriate to a time-ordered product:

$$\frac{1}{(x^2-i\epsilon)^d} \sim \langle 0|T(\phi(x)\phi(0))|0\rangle$$

The equivalence to the commutator, T-product or ordinary product is to be understood as follows: for a free, massless, scalar field, calculation of the matrix element on the right will yield the singular function of the left with d = 1. For integer values of d the second singular function reduces to either*

$\delta^{(n)}(x^2)\varepsilon(x_0)$ or $(x^2)^m\theta(x^2)\ \varepsilon(x_0)$ according to whether d is positive or negative. The explicit formulae are derived below (see Eq's. (4.19) and (4.20)).

4.3 Free Field Propagator Functions

The Feynman propagator for a free scalar field is familiar in momentum space:

$$\Delta_F(k) = (k^2 - m^2 + i\epsilon)^{-1} \tag{4.1}$$

The $i\varepsilon$-prescription is determined by the boundary conditions, in this case that Δ_F propagate particles forward and anti-particles backward in time. The coordinate space propagator is defined as follows:

$$\Delta_F(x,m^2) \equiv (2\pi)^{-4}\int d^4k\, e^{-ik\cdot x}\,(k^2 - m^2 + i\epsilon)^{-1} \tag{4.2}$$

$$= -i\int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[\theta(x_0)e^{-ik\cdot x} + \theta(-x_0)e^{ik\cdot x}\right] \tag{4.3}$$

where $\omega_k \equiv (\vec{k}^2 + m^2)^{1/2}$. $\Delta_F(x,m^2)$ is the usual two point function:

$$\Delta_F(x,m^2) = \frac{1}{i}\langle 0|T(\phi(x)\phi(0))|0\rangle \tag{4.4}$$

---

* $\delta^{(n)}(t) \equiv d^n/dt^n\delta(t)$ and is defined by $\int_{-\infty}^{\infty}dt\ \delta^{(n)}(t)f(t) = (-1)^n f^{(n)}(0)$ for a suitable test function $f(t)$. Also $\varepsilon(x) = \begin{cases} 1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{cases}$.

The equivalence of Eq's. (4.2) and (4.4) may be verified by substituting the plane-wave decomposition of $\phi(x)$ and using the canonical commutation relations.[2,3]

Often we will require the free field commutator function:

$$\Delta(x,m^2) \equiv i \langle 0|[\phi(x),\phi(0)]|0\rangle \tag{4.5}$$

$$= -2\epsilon(x_0)\,\mathrm{Im}\, i\,\Delta_F(x,m^2) \tag{4.6}$$

$$= \frac{1}{(2\pi)^3 i}\int d^4k\, e^{ik\cdot x}\,\delta(k^2-m^2)\,\epsilon(k_0) \tag{4.7}$$

all of which may be verified by taking the imaginary parts of Eq's. (4.2), (4.3) and (4.4).

Other propagator functions appropriate to retarded or advanced propagation, free field anticommutators and the like are defined by suitable $i\varepsilon$-prescriptions and may be found in appendices to any reputable field theory text.[2,3] Generally the T-product, simple product and commutator suffice for our purposes. For future reference note that the singular function of Eq. (4.7) may be rewritten as follows:

$$\pi\,\delta(k^2-m^2)\,\epsilon(k_0) = \frac{i}{2}\left[\frac{1}{-k^2+m^2+i\epsilon k_0} - \frac{1}{-k^2+m^2-i\epsilon k_0}\right] \tag{4.8}$$

We enumerate some properties of $\Delta_{\mp}(x,m^2)$ and $\Delta(x,m^2)$:

I. $\Delta(x,m^2)$ vanishes for $x^2 < 0$. This is required by locality (see Eq. (4.5)): $\phi(x)$ should commute with itself at spacelike separation. We shall not verify this explicitly.†

II. $\Delta(x,m^2) = -\Delta(-x,m^2)$, directly from Eq. (4.6) or (4.7).
$\Delta(x,m^2)|_{x_0=0} = 0$.

III. $\Delta(x,m^2) = \Delta^+(x,m^2) + \Delta^-(x,m^2)$, and $\Delta_F(x,m^2) = \theta(x_0)\,\Delta^+(x,m^2) - \theta(-x_0)\Delta^-(x,m^2)$

where

$$\Delta^{\pm}(x,m^2) \equiv \pm\frac{1}{(2\pi)^3 i}\int d^4k\, e^{ik\cdot x}\,\theta(\pm k_0)\,\delta(k^2-m^2)$$
$$= \left[\Delta^{\mp}(x,m^2)\right]^* \tag{4.9}$$

To learn more we must construct an explicit representation in terms of familiar functions. It is sufficient to consider only $\Delta^{\pm}(x,m^2)$. First[2] perform the energy and angular integrals of Eq. (4.9):

$$\Delta^+(x,m^2) = \frac{1}{4\pi r}\,\partial/\partial r\, f(x) \tag{4.10}$$

with

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k}\,e^{i(\omega_k x_0 + kr)} \tag{4.11}$$

---

†From property II and Lorentz invariance it is not hard to see $\Delta(x,m^2) = 0$ for all $x^2 < 0$.

where $k = |\vec{k}|$ and $\omega_k = (\vec{k}^2 + m^2)^{1/2}$. Already from Eq's. (4.10) and (4.11) we see that $\Delta^+(x,m^2)$ is singular when $x_0=r$: the derivative $(\partial/\partial r)$ of the integral in Eq. (4.11) diverges at its limits.

For a complete evaluation of the integral of Eq. (4.11) we refer to Bogoliwbov and Shirkov.[2] Before quoting their result we can at least motivate the appearance of Bessel functions. Define

$$\begin{aligned} k &= m \sinh \theta \\ \omega_k &= m \cosh \theta \\ x_0 &= \sqrt{x^2} \cosh \theta_0 \\ r &= \sqrt{x^2} \sinh \theta_0 \end{aligned}$$

and consider (eg.) $x_0 < r$. Substitution into Eq. (4.11) yields:

$$f(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\theta \, \exp\left\{ i m \sqrt{x^2} \cosh(\theta + \theta_0) \right\}$$

which is a familiar representation of a Bessel function[4]:

$$f(x) = -\frac{1}{2} J_0\left(m\sqrt{x^2}\right) - \frac{i}{2} Y_0\left(m\sqrt{x^2}\right)$$

Other regions of coordinate space are treated analagously, and one finds discontinuities across the light cone, $x_0 = \pm r$. When the required r-differentiation is performed the discontinuities become light-cone singularities.

Without further ado we quote the result of this calculation[2]:

$$\Delta(x,m^2) = \frac{1}{2\pi} \epsilon(x_0) \delta(x^2) - \frac{m^2}{4\pi} \epsilon(x_0) \theta(x^2) \frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}} \tag{4.12}$$

Note that the most singular term is independent of mass: the leading light-cone or short-distance singularity of free field theory is just what would be expected only the basis of scale invariance.

4.4 Integer Power Singularities

From the definition (Eq. (4.7)) and the explicit form (Eq. (4.12)) of $\Delta(x,m^2)$ we may recover the Fourier transform of all singularities of the form:

$$\delta^{(n)}(x^2)\epsilon(x_0)$$

or

$$(x^2)^m\,\theta(x^2)\epsilon(x_0)$$

As mentioned earlier these are the singularities appropriate to commutator functions. The treatment of other iε-prescriptions is entirely analagous.

From Eq. (4.6) note that

$$\left(\frac{d}{dm^2}\right)^n \Delta(x,m^2)\bigg|_{m^2=0} = \frac{(-1)^n}{(2\pi)^3 i}\int d^4k\, e^{ik\cdot x}\,\delta^{(n)}(k^2)\epsilon(k_0) \tag{4.13}$$

We are finished as soon as we can learn how to differentiate Eq. (4.12) with respect to mass. For this we employ the Taylor expansion to be found in any treatice on Bessel functions:[4]

$$J_1(z) = \frac{1}{2}z\sum_{k=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^k}{k!(k+1)!}$$

It is then a matter of algebra to verify:

$$n \geq 1 \quad \left(\frac{d}{dm^2}\right)^n \Delta(x,m^2)\Big|_{m^2=0} = \left(-\frac{1}{4}\right)^n \frac{(x^2)^{n-1}}{2\pi\,(n-1)!}\,\epsilon(x_0)\,\theta(x^2)$$

$$n = 0 \qquad \Delta(x,m^2)\Big|_{m^2=0} = \frac{1}{2\pi}\,\epsilon(x_0)\,\delta(x^2) \tag{4.14}$$

Comparing Eq's. (4.13) and (4.14) we conclude:

$$\int d^4x\, e^{ik\cdot x}\,\delta(x^2)\,\epsilon(x_0) = 4\pi^2 i\,\delta(k^2)\,\epsilon(k_0) \tag{4.15}$$

$$\int d^4x\, e^{ik\cdot x}\,\delta^{(n)}(x^2)\,\epsilon(x_0) = \frac{2^{2-2n}}{(n-1)!}\,\pi^2 i\,(k^2)^{n-1}\,\theta(k^2)\,\epsilon(k_0) \tag{4.16}$$

and by inversion:

$$\int d^4x\, e^{ik\cdot x}\,(x^2)^n\,\theta(x^2)\,\epsilon(x_0) = 2^{2n+4}\,\pi^2 i\, n!\,\delta^{(n+1)}(k^2)\,\epsilon(k_0) \tag{4.17}$$

Aside from the numerical factors these results could have been anticipated on the basis of dimensional analysis (to guess the powers), symmetry (to guess the $\varepsilon(k_0)$) and locality (to guess the $\theta(k^2)$). These three equations are the fundamental tools for light-cone and short-distance calculations. It is instructive and useful to generalize them to the case of non-integer singularities.

4.5 General Power Singularities

We wish to find the Fourier transforms of generalized functions of the form:

$$E_d(x) = \left[(-x^2+i\epsilon x_0)^d - (-x^2-i\epsilon x_0)^d \right] \tag{4.18}$$

Generalization to time ordered, retarded or other products is a matter of i-epsilonics and is left to the reader.

Our strategy is to relate $E_d(x)$ to δ-functions or θ-functions for integer d, rewrite Eqs. (4.16) and (4.17) in terms of $E_d(x)$ and then analytically continue the result to non-integer d. We will not justify this analytic continuation but note that it agrees with the result of direct and more tedious integration.

Consider first Eq. (4.8), replace the dummy variable k by x, differentiate n-times with respect to m^2 and set $m^2=0$:

$$\left[(-x^2+i\epsilon x_0)^{-n} - (-x^2-i\epsilon x_0)^{-n} \right] = \frac{-2\pi i}{(n-1)!}\,\delta^{(n-1)}(x^2)\,\epsilon(x_0) \tag{4.19}$$

Second, consider Eq. (4.18) for positive d:

$$(-x^2 \pm i\epsilon x_0)^d = \exp\{d \log(-x^2 \pm i\epsilon x_0)\}$$

Define the logarithm with its branch cut running from 0 to $-\infty$ along the negative real axis, so

$$\log(-x^2 \pm i\epsilon x_0) = \log|x^2| \qquad x^2<0$$
$$= \log|x^2| \pm i\pi\epsilon(x_0) \qquad x^2>0$$

Then

$$E_d(x) = (x^2)^d\, 2i \sin\pi d\; \theta(x^2)\epsilon(x_0)$$

For positive integer d, $E_d(x) = 0$ as expected. For any $d>0$ we obtain:

$$(x^2)^d \theta(x^2)\epsilon(x_0) = \frac{1}{2i\sin\pi d} E_d(x) \tag{4.20}$$

Let us now substitute Eq's. (4.19) and (4.20) into Eq. (4.16) (and also use $(n-1)! = \Gamma(n)$ to facilitate the continuation):

$$\int d^4x\, e^{ik\cdot x}\left[(-x^2+i\epsilon x_0)^{-n-1} - (-x^2-i\epsilon x_0)^{-n-1}\right]$$
$$= \frac{2^{2-2n}\pi^3 i}{\Gamma(n)\Gamma(n+1)\sin\pi n}\left[(-k^2+i\epsilon k_0)^{n-1} - (-k^2-i\epsilon k_0)^{n-1}\right]$$

Finally use the identity:

$$\Gamma(n)\Gamma(1-n) = \pi\csc\pi n$$

and let $n = -d-1$

$$\int d^4x\, e^{ik\cdot x} E_d(x) = \frac{\Gamma(d+2)}{\Gamma(-d)}\, 2^{2d+4}\pi^2 i\, E_{-d-2}(k) \tag{4.21}$$

Now we forget that d is an integer and let Eq. (4.21) define the F.T. of E_d for any d. The assumption is that the F.T. of E_d is an analytic function, which I will not attempt to prove here. Eq. (4.21) agrees with the result of the conventional calculation quoted, for example, by Frishman.[5]

REFERENCES

1. See, for example, J. Ellis and Y. Frishman, Phys. Rev. Letters 31, 135 (1973).
2. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, (Interscience, New York, 1959).
3. See, for example, J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill, New York, 1964).
4. See, for example, M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington, 1964).
5. Y. Frishman, Annals of Physics 66, 373 (1971).

5 - OPERATOR PRODUCT EXPANSIONS

5.1 - Introduction

We saw in the first lecture how various processes probe the short distance and light-cone singularity structure of hadrons. Also we have seen in the parton model how model field theories might be used in describing the data. In this lecture we extend the formal discussion of the previous lecture to discuss the general structure in field theories of operator products at short distances and near the light cone, which is the basis for the rest of the lectures.

It turns out that in simple model field theories[1] operator products at short distances (or near the light cone[2]) can be expanded as a series of other operators multiplied by successively less singular c-number functions

$$A(x)\,B(0) \underset{x_\mu \to 0}{\sim} \sum_i C_i(x)\,O_i(0) \tag{5.1}$$

$$A(x)\,B(0) \underset{x^2 \to 0}{\sim} \sum_i C_i'(x)\,O_i'(x|0) \tag{5.2}$$

Some significant features of Eqs. (5.1) and (5.2) include:

(a) The singularities are c-numbers, not operators. It should be emphasized that this property has not been checked experimentally. It would require for example seeing whether the Bjorken scaling behaviour of virtual Compton scattering away from the forward direction was the same as that of the imaginary part in the forward direction, as measured in deep inelastic scattering.

(b) The operators appearing in the short distance expansion (5.1) are local and generally include familiar objects measurable directly, such as currents. The expansions therefore give connections between two current processes (in certain limits) with one current processes, which are the bases for deriving sum rules in deep inelastic scattering.

(c) The bilocal operators $O_i(x|o)$ appearing in the light - cone expansion (5.2) are analytic as $x_\mu \to o$ and the first terms in their expansion are given by the leading terms in the short distance expansion (5.1).

In this lecture we will seek to make plausible the postulation of such operator production expansions for hadrons, discussing how they occur in the canonical manipulations of free and interacting field theories, and how one might try to investigate their existence in renormalizable field theories. We discuss the connections between short distance and light-cone expansions. Also we examine the form expected for the short distance expansions of pairs of hadronic currents, and finish by relating the c-number term in the product of two electromagnetic currents to the asymptotic cross section $\sigma(e^-e^+ \to \gamma \to \text{hadrons})$.

5.2 - OPERATOR PRODUCT EXPANSIONS IN FREE-FIELD THEORY

As a first example of how operator product expansions like (5.1) and (5.2) occur in field theory[1], we consider the simplest case of a free scalar field. We are interested in composite operators made out of products of the constituent fields: objects like $\theta(x) \equiv \phi(x)\phi(x)$. Actually, as written $\theta(x)$ is not a sensible definition because it has a non-zero (and in fact infinite) vacuum expectation value. This is because the negative frequency parts of $\phi(x)$ contain creation operators $a^{+}(k)$ which do not annihilate the vacuum. To remove this problem we must redefine $\theta(x)$ (and indeed all other composite field operators) so that the positive frequency parts of $\phi(x)$

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E}\, a(k)\, e^{-ik\cdot x}$$

always stand to the right of the negative frequency parts. Thus we redefine

$$\Theta(x) = :\phi(x)\phi(x): \equiv \phi^{(-)}(x)\phi^{(-)}(x) + 2\phi^{(-)}(x)\phi^{(+)}(x) + \phi^{(+)}(x)\phi^{(+)}(x)$$

This procedure is called normal ordering. Consider now the product

$$\Theta(x)\Theta(0) = :\phi(x)\phi(x): \; :\phi(0)\phi(0):$$

Clearly the complete four field product is not normal ordered, and so will become infinite when $x \sim 0$. To see what the singularity is, we rewrite $\theta(x)\theta(0)$, using the canonical commutation relations of free field theory to interchange positive and negative frequency parts of $\phi(x)$ and $\phi(0)$:

$$[\phi^{(+)}(x), \phi^{(-)}(0)] = \frac{1}{(2\pi)^3} \int d^4q\, e^{-iq\cdot x}\, \Theta(q_0)\, \delta(q^2 - M^2) = \frac{1}{i} \Delta^{-}(x, M^2)$$

in the notation of lecture 4. We get:

$$\Theta(x)\Theta(0) = -2\left[\Delta^{-}(x, m^2)\right]^2 - 4i\Delta^{-}(x, M^2) :\phi(x)\phi(0): + :\phi(x)\phi(x)\phi(0)\phi(0): \qquad (5.3)$$

The normal ordered products on the right-hand side of (5.3) are non-singular as $x_\mu \to 0$: all the singularities have been absorbed into the quantities $\Delta^-(x,M^2)$ which

$$\sim \frac{-i}{4\pi^2} \frac{1}{x^2 - i\epsilon x_0} \qquad \text{as} \qquad x^2 \to 0$$

We can expand $:\phi(x)\phi(0):$ about the limit $x = 0$:

$$:\phi(x)\phi(0): \; = \; :\phi(0)\phi(0): \; + \; x_\mu :\partial^\mu\phi(0)\,\phi(0): \; + \cdots$$

so that introducing

$$J_\mu(x) = \; :\partial^\mu\phi(x)\,\phi(x):$$

we have in the limit $x_\mu \to 0$ an expansion:

$$\Theta(x)\,\Theta(0) \approx$$

$$\frac{1}{8\pi^4} \frac{1}{(x^2 - i\epsilon x_0)^2} - \frac{\Theta(0)}{\pi^2 (x^2 - i\epsilon x_0)} - \frac{x_\mu J^\mu(0)}{\pi^2 (x^2 - i\epsilon x_0)} + \cdots \tag{5.4}$$

Note that the expansions (5.3) and (5.4) are of the form promised in the introduction: they contain c-number singular functions multiplying operators. The short distance expansion (5.4) contains local operators, and the bilocal operator $:\phi(x)\phi(0):$ appearing in (5.3) (the prototype of a light-cone expansion) is analytic as $x_\mu \to 0$, with the first terms in its expansion given by the leading terms in the short distance expansion.

Two comments are in order. The first is that expansions analogous to (5.3) and (5.4) will also hold for other current products, such as time-ordered products and commutators. They are readily obtained by just taking the relevant combinations of expansions of simple products. The singular functions then get changed to $1/x^2 - i\varepsilon$ for time ordered products, and $\varepsilon(x_0)\delta(x^2)$ for commutators. The second comment is that operator product expansions similar to (5.3) and (5.4) can easily be derived in free fermion field theories. $\Delta^-(x, M^2)$ is now replaced by the fermion singular function

$$S^-(x, M^2) = -(i\gamma\cdot\partial + M)\,\Delta^-(x, M^2)$$

and there are complications of spinology, but the idea is the same.

Consider the physically interesting case of two SU_3 vector currents made up of fundamental free fermion fields $\psi(x)$ belonging to a triplet representation[3] (quarks!)

$$J_\mu^a(x) = :\bar{\psi}(x)\gamma_\mu \frac{\lambda^a}{2}\psi(x):$$

Manipulations similar to those carried out earlier in the section give an expansion resembling (5.3). The analogue of the $:\phi(x)\phi(0):$ term in (5.3) has pieces proportional to

$$:\bar{\psi}(x)\gamma_\mu\lambda^a\left[\gamma\cdot\frac{\partial}{\partial x}\Delta^-(x, M^2)\right]\gamma_\nu\lambda^b\psi(0): \tag{5.5}$$

For the moment we will only be interested in this object at short distances, where it is proportional to

$$\frac{x^\rho}{(x^2 - i\epsilon x_0)^2} : \bar{\psi}(0)\gamma_\mu\gamma_\rho\gamma_\nu\lambda^a\lambda^b\psi(0):$$

Using

$$\gamma_\mu\gamma_\rho\gamma_\nu = g_{\mu\rho}\gamma_\nu + g_{\rho\nu}\gamma_\mu - g_{\mu\nu}\gamma_\rho - i\,\epsilon_{\mu\rho\nu\sigma}\gamma^\sigma\gamma_5$$

and

$$\lambda^a\lambda^b = (d_{abc} + i\,f_{abc})\lambda^c$$

this can be re-expressed in terms of the vector current and the axial current

$$A^a_\mu(x) = :\bar{\psi}(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi(x):$$

Adding all the pieces together one finishes up with

$$J^a_\mu(x)J^b_\nu(0) \underset{x_\mu\to 0}{\sim} \frac{\delta_{ab}}{2\pi^4}\frac{(g_{\mu\nu}x^2 - 2x_\mu x_\nu)}{(x^2 - i\epsilon x_0)^4}$$

$$+ \frac{d_{abc}}{2\pi^2}\frac{\epsilon_{\mu\nu\alpha\beta}\,x^\alpha A^\beta_c(0)}{(x^2 - i\epsilon x_0)^2}$$

$$+ \frac{f_{abc}}{2\pi^2}\frac{x^\rho(g_{\mu\rho}g_{\nu\sigma} + g_{\nu\rho}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma})J^\sigma_c(0)}{(x^2 - i\epsilon x_0)^2}$$

$$+ \quad \cdots \qquad (5.6)$$

where the dots indicate less singular terms. The short distance expansion for the product of two axial currents is exactly the same as for the product of vector currents. The expansion for the product of an axial current with a vector current has the form

$$J_\mu^a(x)\, A_\nu^b(0) \underset{x_\mu \to 0}{\sim} \frac{d_{abc}}{2\pi^2} \frac{\epsilon_{\mu\nu\alpha\beta}\, x^\alpha J_c^\beta(0)}{(x^2 - i\epsilon x_0)^2}$$

$$+ \frac{f_{abc}}{2\pi^2} x^\rho \frac{\left[g_{\mu\rho} g_{\nu\sigma} + g_{\nu\rho} g_{\mu\sigma} - g_{\mu\nu} g_{\rho\sigma}\right] A_c^\sigma(0)}{(x^2 - i\epsilon x_0)^2}$$

$$+ \quad \cdots \tag{5.7}$$

For time ordered products we just replace $(x^2 - i\varepsilon x_0)$ in (5.6) and (5.7) by $(x^2 - i\varepsilon)$.

We could go on to consider more complicated products of operators. For example the short distance singularities of the triple product

$$T\left(J_\mu(x)\, J_\nu(0)\, A_\rho(y)\right)$$

could be studied. In a subsequent lecture we will be concerned with the c-number part in an expansion of this product, as it is connected with anomalies in current algebra low energy theorems. In a free fermion field theory it will be proportional to

$$Tr\left[\gamma_\mu S_F(x,M^2)\gamma_\nu S_F(y,M^2)\gamma_\rho\gamma_5 S_F(x-y,M^2)\right]$$

$$\propto Tr\left\{\frac{\gamma_\mu\gamma\cdot x\ \gamma_\nu\ \gamma\cdot y\ \gamma_\rho\gamma_5\ \gamma\cdot(x-y)}{\left[(x^2-i\epsilon)(y^2-i\epsilon)((x-y)^2-i\epsilon)\right]^2}\right\}$$

at short distances.

5.3 - OPERATOR PRODUCT EXPANSIONS FOR TWO CURRENTS

In the previous section we discussed the form taken by the operator product expansions of pairs of currents in the canonical quark model. We should ask ourselves what happens in other models and how to parametrize the operator product expansion (5.6), (5.7) so as to accommodate more general possibilities.

First we examine what happens in theories with more triplets of quarks[4], where the current takes the form

$$J^a_\mu(x) = \sum_c :\bar\psi_c(x)\gamma_\mu\frac{\lambda_a}{2}\psi_c(x):$$

and the sum over c indicates a sum over a new internal index "colour". The operator parts of the expansion will be unaffected: replacing eq.(5.5), where one pair of fermions has been reduced, will be something proportional to

$$\sum_c \sum_{c'} \; : \bar{\psi}_c(x)\gamma_\mu\lambda^a\left[\delta_{cc'}\,\gamma\cdot\frac{\partial}{\partial x}\Delta^-(x,M^2)\right]\gamma_\nu\lambda^b\psi_{c'}(0):$$

yielding

$$-\frac{i}{2\pi^2(x^2-i\epsilon x_0)^2}\; x^\rho \sum_c \; : \bar{\psi}_c(0)\gamma_\mu\gamma_\rho\gamma_\nu\lambda^a\lambda^b\psi_c(0):$$

at short distances, which gives the same currents in the same combinations as in the single triplet model. However, when the second pair of fermions is reduced in to get the c-number part, there will be a sum over the colour index, and the c-number will be multiplied by the number of colours.* Since the c-number is SU_3 symmetric, if we assume the usual SU_3 assignment for the electromagnetic current, then the coefficient of the c-number term in (5.6) is proportional to the sum of the squares of the charges of the fundamental spin 1/2 fields

$$J^a_\mu(x)\,J^b_\nu(0) \underset{x_\mu\to 0}{\sim} \frac{S_{JJ}\,\delta_{ab}}{2\pi^4}\,\frac{(g_{\mu\nu}x^2-2x_\mu x_\nu)}{(x^2-i\epsilon x_0)^4}+\dots$$

where

$$S_{JJ} = \frac{3}{2}\sum Q^2_{\frac{1}{2}}$$

(S_{AA} is defined analogously, and in general equals S_{JJ}.[5])

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*The same will be true of the c-number parts of other expansions.

In a theory made up from fundamental spin zero fields, both the first and second terms in (5.6) are altered: the first term now has a coefficient

$$\frac{3}{8} \sum Q_0^2$$

and the second term vanishes (why?). Accordingly, we allow the axial current term in (5.6) to have an arbitrary coefficient K_{JJ} in general, and define K_{AA} and K_{JA} analogously. (In fact there are arguments why the K's should be equal.[5])

The coefficient of the term

$$f_{abc}\, x_\mu\, J_\nu^c(0)$$

in (5.6) is determined by current algebra. It contributes to $[J_0^a(x), J_\nu^b(0)]$ a term

$$\frac{f_{abc}}{2\pi^2}\, x_0\, J_\nu^c(0) \left[\frac{1}{(x^2 - i\epsilon x_0)^2} - \frac{1}{(x^2 + i\epsilon x_0)^2}\right]$$

The square bracket is just* $-2\pi i \varepsilon(x_0)\delta'(x^2)$ which at $x_0 \simeq 0$ gives $\frac{2\pi^2 i}{x_0}\,\delta^3(\underline{x})$, and hence the current algebra equal time commutator. The

---

*Recall from Lecture 4 that

$$(x^2 - i\epsilon x_0)^{-1} - (x^2 + i\epsilon x_0)^{-1} = 2\pi i\, \epsilon(x_0)\, \delta(x^2)$$

and that for $n = 1, 2, \cdots$

$$(x^2 - i\epsilon x_0)^{-n-1} - (x^2 + i\epsilon x_0)^{-n-1} = \frac{(-1)^n 2\pi i}{n!}\, \epsilon(x_0)\, \delta^{(n)}(x^2)$$

$$f_{abc}\, x^{\rho} \left(g_{\nu\rho}\, g_{\mu\sigma} - g_{\mu\nu}\, g_{\rho\sigma} \right) J_c^{\sigma}(0)$$

terms do not concern us, and we will not discuss them further.

Notice that the second term in (5.6) appears in the commutator of two space components of the currents*:

$$\left[J_i^a(x),\, J_j^b(0) \right] \sim \frac{-i\, K_{JJ}\, d_{abc}\, \epsilon_{ijk0}}{\pi} \left(x^k A_c^0(0) - x^0 A_c^k(0) \right) \epsilon(x_0)\, \delta'(x^2)$$

(where the letters i,j,k etc. are 3-space indices)

Near $x_0 \approx 0$

$$\sim\ i\, K_{JJ}\, d_{abc}\, \epsilon_{ijk0} \left(x^k A_c^0(0) - x^0 A_c^k(0) \right) \frac{\delta^3(\underline{x})}{x^0}$$

so that the space-space commutator contains a term

$$\left[J_i^a(\underline{x},0),\, J_j^b(0) \right] = -i\, \epsilon_{ijk}\, K_{JJ}\, d_{abc}\, A_c^k(0)\, \delta^3(\underline{x}) \tag{5.8}$$

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*See footnote on preceding page

For reference we also quote the c-number part of the commutator of two SU_3 currents:

$$\left[J^a_\mu(x), J^b_\nu(0)\right] \underset{x_\mu \to 0}{\sim} -i \frac{S_{JJ}\,\delta_{ab}(g_{\mu\nu}x^2 - 2x_\mu x_\nu)}{6\pi^3}\,\epsilon(x_0)\,\delta'''(x^2)$$

so that

$$\left[J^{em}_\mu(x), J^{em}_\nu(0)\right] \underset{x_\mu \to 0}{\sim} \frac{-iR}{3\pi^3}(g_{\mu\nu}x^2 - 2x_\mu x_\nu)\,\epsilon(x_0)\,\delta'''(x^2)$$

where

$$R = \frac{2}{3} S_{JJ} = \sum Q_i^2 + \frac{1}{4}\sum Q_0^2$$

5.4 - OPERATOR PRODUCT EXPANSIONS IN THEORIES WITH INTERACTIONS

In this section we briefly discuss what happens to operator product expansions when interactions are switched on. One way of studying this question is to use the canonical commutation relations of a theory with interactions to study its short distance and light-cone singularities.

Consider for example a quark theory with the vector gluon interaction

$$\mathcal{L}_{int}(x) = g\,\bar{\psi}(x)\gamma_\mu\psi(x)B^\mu(x)$$

In their studies of this model Gross and Treiman[6] found that in order to extract the leading singularities it was sufficient to treat the gluon as an external c-number field. (We refer to their paper[6] for details.) They then showed that the only effect of a c-number gluon field on the singular functions on the light cone was to multiply them by a phase. For $x^2 \sim 0$

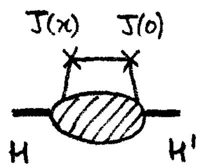
$$S^-(x,M^2) \rightarrow \exp\left(-ig\int_0^x dy_\mu B^\mu(y)\right) S^-(x,M^2)$$

where the integral is along a straight light-like path from 0 to x.

The short distance expansions are therefore unaffected because they come from the tip of the light cone where the phase is zero. Away from short distances along the light cone, the phase does not affect the Lorentz or internal symmetry structure of the bilocal operator, but it does means that the bilocal operator is not simply expressible in terms of quark fields alone.

Another method of studying operator product expansions in an interacting field theory is to use perturbation theory.[7] It has been shown that in general operator product expansions of the type (5.1) or (5.2) remain valid, but that their form changes from that in free field theory. Let us sketch what happens. Consider an object $T(J(x)J(0))$ where J is a product of two constituent fields, which we take for convenience to be scalar $J(x) = :\phi(x)\phi(0):$. Then the matrix element of $T(J(x)J(0))$ between any pair of hadronic states H and H' can be written as a sum of graphs with interactions at the points $y_1 \ldots y_n$, and particles propagating between the interactions, the points x and 0, and the external particles, each integrated over $y_1 \ldots y_n$. This sum of graphs can be divided into two classes, in one of which a particle propagates freely between x and 0, and in the other not. Figuratively speaking:

$$\langle H|T(J(x)J(0))|H'\rangle =$$



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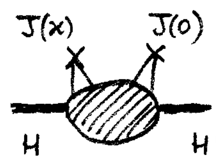
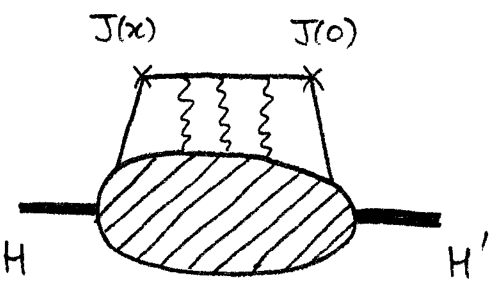


where the shaded areas indicate interaction regions. The first class of diagrams looks like

$$\Delta_F(x)\,\langle H|T(\phi(x)\phi(0))|H'\rangle$$

and so gives to the operator product expansion a term like that in the free field case except that $\langle H|T(\phi(x)\phi(0))|H'\rangle$ may, and in general does, become divergent[8] as $x^2 \to 0$. What about diagrams in the second class? It is apparent that before the integrations over the interactions $y_1 \ldots y_n$ the diagrams will not have any $1/x^2$ singularities. However, the integrations over $y_1 \ldots y_n$ can in general yield $1/x^2$ singularities, or even stronger ones. If all the integrations were convergent, there would be no problem, and the second class of diagrams would not be singular on the light cone; this is what happens in super-renormalizable field theories, like ϕ^3. In renormalizable field theories, including for example ϕ^4 and the quark-vector gluon theory, the integrals are in general divergent and must be renormalized. The singularities at short distances and near the light cone are then modified by logarithmic factors in each order of perturbation theory. If all orders of perturbation theory could be summed, these logarithms might well exponentiate to powers. In a vector gluon theory problems come from graphs looking like



where the 's are vector gluons. The canonical manipulations discussed earlier, which just modified the light-cone singularity by a phase, are completely false in perturbation theory.

Happily, the data on deep inelastic scattering are consistent with canonical light-cone singularities, and give no evidence for logarithms, though no one really understands how this is possible. In all that follows we will assume that the perturbation theory calculations are inapplicable, and that operator product expansions for hadronic currents in fact have a canonical structure*. This and time are the reasons we have not discussed expansions in interacting theories in more detail.

5.5 - APPLICATION TO $e^-e^+\to\gamma\to$HADRONS

In Section 5.3 we saw how in canonical manipulations of field theories, the c-number part of the commutator of two electromagnetic currents at short distances is

$$[J_\mu(x), J_\nu(0)] \underset{x_\mu\to 0}{\sim} \frac{-iR}{3\pi^3}\left(g_{\mu\nu}x^2 - 2x_\mu x_\nu\right)\epsilon(x_0)\,\delta'''(x^2) + \cdots \tag{5.8}$$

where R is a model-dependent parameter (2/3 for fractionally

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*Recently[9] it has been discovered that non-Abelian gauge theories have canonical short distance singularities, while violating Bjorken scaling in electroproduction by inverse powers of logarithms.[10]

charged uncoloured quarks, 2 for coloured quarks, etc.).

We have

$$\sigma(e^-e^+ \to \gamma \to \text{hadrons}) = -\frac{8\pi^2\alpha^2}{3(q^2)^2}\int d^4x\, e^{iq\cdot x}\langle 0|[J_\mu(x), J^\mu(0)]|0\rangle$$

so that the short distance contribution is

$$\frac{i\,16R\,\alpha^2}{9\pi(q^2)^2}\int d^4x\, e^{iq\cdot x}\, x^2\, \epsilon(x_0)\,\delta'''(x^2)$$

$$= \frac{-i\,16R\,\alpha^2}{9\pi(q^2)^2}\,\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q}\int d^4x\, e^{iq\cdot x}\,\epsilon(x_0)\,\delta'''(x^2)$$

Now we use an identity given in Lecture 4:

$$\int d^4x\, e^{iq\cdot x}\,\epsilon(x_0)\,\delta^{(n)}(x^2)$$

$$= \frac{2^{2-2n}}{(n-1)!}\,\pi^2 i\,(q^2)^{n-1}\,\theta(q^2)\,\epsilon(q_0) \tag{5.9}$$

which in the case n=3 implies

$$\int d^4x\, e^{iq\cdot x}\,\epsilon(x_0)\,\delta'''(x^2) = \frac{i\pi^2}{2^5}(q^2)^2\,\theta(q^2)\,\epsilon(q_0) \tag{5.10}$$

For $q^2 \to \infty$ with $q_0 > 0$, this gives

$$\sigma = \left(\frac{-i\,16 R \alpha^2}{9\pi (q^2)^2}\right)\left(\frac{i\pi^2}{2^5}\right)\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q}(q^2)^2$$

$$= \frac{4\pi R \alpha^2}{3}\frac{1}{q^2} \tag{5.11}$$

It is apparent from (5.9) and (5.10) that if we considered a weaker short distance singularity than (5.8) then its contribution to the cross section would fall off more rapidly than $1/q^2$.

Hence, as asserted earlier, the asymptotic behaviour of the cross section is determined by the leading short distance singularity. The parton model result[11] for the cross section is identical with (5.10) because the model diagram

has a canonical singularity of the form (5.8). At large q^2, the cross section for $e^-e^+ \to \gamma \to \mu^-\mu^+$ is also determined by a leading short distance singularity, which is canonical when we work to leading order in α. It is of the form (5.8) with R=1. Hence the expectation is that

$$\frac{\sigma(e^- e^+ \to \gamma \to \text{hadrons})}{\sigma(e^- e^+ \to \gamma \to \mu^- \mu^+)} \to R \tag{5.12}$$

as $q^2 \to \infty$.

As was emphasized in the second lecture, there is as yet no overwhelming experimental evidence in favour of this behaviour. We regard it as a fundamental test of the ideas of broken scale invariance, perhaps even more basic than Bjorken scaling in electroproduction. In all field theory models studied, (5.12) holds if Bjorken scaling works. Also renormalizable gauge theories[9] have the property (5.12), but Bjorken scaling fails by inverse powers of logarithms[10]. We will keep our fingers crossed.

References

1. K. G. Wilson - Phys. Rev. 179, 1499 (1969).

2. R. A. Brandt and G. Preparata - Nucl. Phys. B27, 541 (1971).
   Y. Frishman - Ann. Phys. (N.Y.) 66, 373 (1971).

3. H. Fritzsch and M. Gell-Mann - in "Broken Scale Invariance and the Light Cone" edited by M. Dal Cin, G. J. Iverson and A. Perlmutter (Gordon and Breach, New York, 1971).

4. H. Fritzsch and M. Gell-Mann in Proceedings of the International Conference on Duality and Symmetry in Hadron Physics, edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

5. J. Ellis - unpublished (1972). See Lecture 8.

6. D. J. Gross and S. B. Treiman - Phys. Rev. D4, 1059 (1971).

7. See for example W. Zimmermann in Lectures on Elementary Particles and Quantum Field Theory, edited by S. Deser et.al. (M.I.T. Press, Cambridge, Mass., 1971).

8. R. Jackiw and R. E. Waltz - Phys. Rev. D6, 702 (1972).

9. D. J. Gross and F. Wilczek - Phys. Rev. Lett. 30, 1343 (1973).
   H. D. Politzer - Phys. Rev. Lett. 30, 1346 (1973).
   K. Syranzik informs people that this result was also known to G. 't Hooft.

10. D. J. Gross and F. Wilczek - NAL Preprint, NAL-PUB-73/49-THY- "Asymptotically Free Gauge Theories-I" (1973).

11. N. Cabibbo, G. Parisi and M. Testa - Lett. al Nuovo Cimento 4, 35 (1970).
    S. D. Drell, D. J. Levy and T.-M. Yan - Phys. Rev. D1, 1617 (1970).

6. Light-Cone Expansions and the Quark Light-Cone Algebra

6.1 Introduction

We have already argued that the light cone dominates the Bjorken limit of inelastic lepton scattering. Also we have shown that in free field theory and in interacting theories treated canonically (but not in lowest order perturbation theory) operator products have a rather simple expansion about the light cone. Applied to inelastic lepton scattering these ideas have an immediate, striking consequence: the experimentally observed (Bjorken) scaling requires that the product of electromagnetic currents behave like a product of free currents near the light cone. Here we will show how this comes about, explore its consequences regarding bilocal operators, discuss light-cone current algebra and relate all this to the parton model.

6.2 Measuring Light-Cone Singularities and Bilocal Operators

We suppose that in the vicinity of the light cone the commutator of two currents may be expressed as a sum of c-number functions singular on the light cone multiplying bilocal operators, regular at $(x-y)^2 = 0.$[1]

$$[J(x), J(y)] = \sum_{[\alpha]} C_{[\alpha]}(x-y)\, O^{[\alpha]}(x,y) \tag{6.1}$$

For simplicity we will consider scalar currents until further notice. The sum on $[\alpha]$ covers Lorentz indices (if any), internal symmetry labels and strength of singularity. From Eq. (6.1) we construct the current correlation function

$$\mathcal{F}(x^2, x\cdot p) \equiv \langle P|[J(x), J(0)]|P\rangle = \sum_{[\alpha]} C_{[\alpha]}(x) \langle P|O^{[\alpha]}(x|0)|P\rangle \tag{6.2}$$

$$= \sum_{[\beta]} E_{[\beta]}(x^2)\, F^{[\beta]}(x^2, x\cdot p) \tag{6.3}$$

By assumption $F(0,x\cdot p)$ is finite*.

We can now show a) that the scaling laws determine the light-cone singularity of $E_{[\beta]}$ and b) that scaling structure functions measure the Fourier transform of $F^{[\beta]}(0,x\cdot p)$. Although this may be done for an arbitrary singularity using the Fourier transform relation derived at the end of Lecture 4, we prefer to use the more physical example of singularities which are integer powers of x^2. Locality and crossing require the following of Eq. (6.3):

1. $\mathcal{F}(x^2,x\cdot p) = 0$ for $x^2 < 0$ Locality

2. $\mathcal{F}(x^2,x\cdot p) = -\mathcal{F}(x^2,-x\cdot p)$ Crossing

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* Since we will only consider a single matrix element of this expansion, it is impossible to conclude that the singularity is actually a c-number. In the models we discuss this will be the case.

Suppose now the most singular term in Eq. (6.3) were

$$\frac{1}{i\pi}\,\delta(x^2)\epsilon(x\cdot p)\,F(x\cdot p) \qquad\qquad \big(F(\lambda) = F(-\lambda)\big)$$

The contribution of such a term to the structure function determined by $\mathcal{F}(x^2, x\cdot p)$,

$$\left(V(Q^2,\nu) \equiv \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\,\mathcal{F}(x^2, x\cdot p)\right)$$

is then:

$$V_1(Q^2,\nu) = \frac{1}{4\pi^2 i}\int d^4x\, e^{iq\cdot x}\,\delta(x^2)\epsilon(x\cdot p)\,F(x\cdot p) \tag{6.4}$$

Substituting for $F(x\cdot p)$ its Fourier transform:

$$F(x\cdot p) \equiv \int_{-\infty}^{\infty} d\alpha\; e^{i\alpha x\cdot p}\, f(\alpha) \tag{6.5}$$

we obtain

$$\begin{aligned} V_1(Q^2,\nu) &= 2\pi\int_{-\infty}^{\infty} d\alpha\, f(\alpha)\;\Delta(\alpha p+q,0) \\ &= \int_{-\infty}^{\infty} d\alpha\, f(\alpha)\,\delta(\alpha^2M^2 + 2\alpha\nu - Q^2)\,\epsilon(\alpha M+\nu) \end{aligned} \tag{6.6}$$

having used the definition of the free causal propagator and

the fact that $\Delta(x,0) = (2\pi)^{-1}\,\delta(x^2)\varepsilon(x_0)$.
There are two solutions to the δ-function:

$$\alpha_\pm = -\frac{\nu}{M^2} \pm \sqrt{\frac{\nu^2}{M^4} + \frac{Q^2}{M^2}}$$

which in the Bjorken limit reduce to:

$$\alpha_+ \cong \xi \qquad\qquad \alpha_- \cong -\frac{2\nu}{M^2}$$

and yield:

$$\lim_{Bj} V_1(Q^2,\nu) = \frac{1}{2\nu}\left(f(\xi) - f\left(-\frac{2\nu}{M^2}\right)\right) \tag{6.7}$$

As discussed in Lecture 2, $V(Q^2,\nu)$ should vanish below threshold, i.e. for $|2\nu| < Q^2$ Eq. (6.7) fulfills this requirement if $f(\alpha) = 0$ for $|\alpha| > 1$. Whatever dynamics generates the dominant light cone singularity must also respect this spectral condition[2], and we therefore assume $f(\alpha) = 0$ for $\alpha > 1$ henceforth. Eq. (6.7) reduces to $\frac{1}{2\nu}\, f\ (\xi)$.

The result of this brief calculation could have been guessed on dimensional grounds alone: $V_1(Q^2,\nu)$ has dimension $[\text{mass}]^{-2}$ (see Eq. (6.4)) which we expect to be supplied by an inverse factor of ν or Q^2 in the Bjorken limit.

Suppose, now, that we consider a term in $\mathcal{F}(x^2, x\cdot p)$ with one power weaker a light-cone singularity:

$$\frac{1}{\pi i} M^2 \theta(x^2)\epsilon(x\cdot p)G(x\cdot p)$$

where the constant, M^2 (dimension $[\text{mass}]^2$), is necessary to preserve the dimension of $\mathcal{J}(x^2,x\cdot p)$. This contributes to $V(Q^2,\nu)$ a term of the form:

$$V_2(Q^2,\nu) = \frac{M^2}{4\pi^2 i}\int d^4x\, e^{iq\cdot x}\theta(x^2)\epsilon(x\cdot p)G(x\cdot p) \tag{6.8}$$

This may be evaluated by inserting the Fourier transform of $G(x\cdot p)$ and using the observation from an earlier lecture that

$$\frac{d}{dm^2}\Delta(x,m^2)\bigg|_{m^2=0} = -\frac{1}{8\pi}\theta(x^2)\epsilon(x_0) \tag{6.9}$$

From which we obtain

$$\lim_{Bj} V_2(Q^2,\nu) \propto \frac{M^2}{\nu^2}g'(\xi) \tag{6.10}$$

where $g'(\xi)$ is the derivative of the Fourier transform of $G(x\cdot p)$ [Eq. (6.10) could also have been guessed from dimensional analysis].

The relation, apparent in Eqs. (6.7) and (6.10) between the light-cone singularity of a term in $\mathcal{J}(x^2,x\cdot p)$ and the power of ν or q^2 in the ensuing scaling law is completely general. [The reader is invited to derive the result for an arbitrary singularity from the technology of Lecture 4.[1]] Moreover, the scaling structure function measures, as promised, the Fourier transform of the matrix element of the bilocal operator along the light cone.

What is remarkable about the SLAC-MIT experiments is that scaling for W_1 and νW_2 is what would be expected if the electromagnetic currents were constructed from free fields: i.e., the singular functions obtained from commuting two currents constructed of free fields determine that W_1 and νW_2 scale, provided one assumes the bilocal operators to be smooth near the light-cone. Jackiw, van Royen and West[3] constructed an explicit form for the leading light-cone singularity of the electromagnetic current correlation function consistant with the scaling of W_1 and νW_2:

$$\langle P|[J_\mu(x), J_\nu(0)]|P\rangle = \frac{2i}{\pi}\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\Big[\delta(x^2)\epsilon(x\cdot p) \otimes \int_0^1 d\xi/\xi \cos\xi x\cdot p\, F_L(\xi) + \ldots\Big] + \frac{2i}{\pi}\Big[S_{\mu\rho\nu\sigma} \otimes P_\rho\,\partial_\sigma\left(\delta(x^2)\epsilon(x\cdot p)\right)\int_0^1 d\xi\, \frac{\sin\xi x\cdot p}{\xi} F_2(\xi) + \ldots\Big] \tag{6.11}$$

where omitted terms are less singular on the light-cone.*

$S_{\mu\rho\nu\sigma}$ is defined as follows:

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*Actually Eq. (6.11) differs from Ref. 3 in the second term. The reason we have chosen the form shown will become clear when the quark model light-cone algebra is explored. The second term of Eq. (6.11) is current conserving only to leading order on the light-cone (which is all that need concern us here) in contrast to Ref. 3.

$$S_{\mu\rho\nu\sigma} \equiv \frac{1}{4} \mathrm{Tr}\{\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma\}$$

$$= g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma} \tag{6.12}$$

and $F_L(\xi) \equiv F_1(\xi) - \frac{1}{2\xi} F_2(\xi)$.

The first integral in Eq. (6.11) would appear linearly divergent if $F_L(\xi)$ has the Regge behavior expected of it (see Lecture 2). The understanding and removal of this sort of divergence will be discussed at length later. First we introduce the Fritzsch-Gell-Mann light-cone algebra.

6.3 The Quark Model Light-Cone Algebra

The observation that Bjorken scaling is equivalent to free-field light-cone singularities suggests that symmetry relations known to be violated by interactions might nevertheless be valid to leading order on the light cone. The prime candidate for such a symmetry is Gell-Mann's SU(3) and indeed, Fritzsch and Gell-Mann have proposed[4] that the leading light-cone structure of current commutators be abstracted from the free quark model. As something of a bonus, they found that the bilocal operators which are generated by commuting two currents form an algebra among themselves. Here we develop the quark model light-cone algebra and apply it to the derivation of sum rules.

The strategy is to suppose that the electromagnetic and weak currents may be constructed from currents of the form:

$$j_{\mu a}(x) = \frac{1}{2}\bar{\psi}(x)\lambda_a\gamma_\mu\psi(x)$$

$$j^5_{\mu a}(x) = \frac{1}{2}\bar{\psi}(x)\lambda_a\gamma_\mu\gamma_5\psi(x) \tag{6.13}$$

where $\psi(x)$ is a free, three component quark spinor field and λ_a are the usual U(3) matrices* normalized to $\mathrm{Tr}\,\lambda_a^2 = 2(a=0\ldots8)$. Using the free field anti-commutator

$$\{\psi(x),\bar{\psi}(0)\} \equiv S(x) = (-\not{\partial}+im)\Delta(x,m^2) = -\frac{\not{\partial}}{2\pi}\delta(x^2)\epsilon(x_0) + \ldots$$

where only the leading singularity is of interest, we find:†

$$[j_{\mu a}(x), j_{\nu b}(0)] = -\frac{1}{4\pi}\partial^\rho\left(\delta(x^2)\epsilon(x_0)\right)\{S_{\mu\rho\nu\sigma}\,d_{abc}\,A^\sigma_c(x|0) + iS_{\mu\rho\nu\sigma}\,f_{abc}\,S^\sigma_c(x|0) - i\epsilon_{\mu\rho\nu\sigma}\,d_{abc}\,S^{5\sigma}_c(x|0) + \epsilon_{\mu\rho\nu\sigma}\,f_{abc}\,A^{5\sigma}_c(x|0)\} \tag{6.14}$$

where

$$A_{\sigma c}(x|0) \equiv \frac{1}{2}[\bar{\psi}(x)\gamma_\sigma\lambda_c\psi(0) - \bar{\psi}(0)\gamma_\sigma\lambda_c\psi(x)]$$
$$S_{\sigma c}(x|0) \equiv \frac{1}{2}[\bar{\psi}(x)\gamma_\sigma\lambda_c\psi(0) + \bar{\psi}(0)\gamma_\sigma\lambda_c\psi(x)]$$
$$A^5_{\sigma c}(x|0) \equiv \frac{1}{2}[\bar{\psi}(x)\gamma_\sigma\gamma_5\lambda_c\psi(0) - \bar{\psi}(0)\gamma_\sigma\gamma_5\lambda_c\psi(x)]$$
$$S^5_{\sigma c}(x|0) \equiv \frac{1}{2}[\bar{\psi}(x)\gamma_\sigma\gamma_5\lambda_c\psi(0) + \bar{\psi}(0)\gamma_\sigma\gamma_5\lambda_c\psi(x)] \tag{6.15}$$

and

$$\{\lambda_a,\lambda_b\} = 2d_{abc}\lambda_c$$
$$[\lambda_a,\lambda_b] = 2if_{abc}\lambda_c$$

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*The SU(3) algebra is not closed though the U(3) algebra is.

†The identity $\gamma_\mu\gamma_\rho\gamma_\nu = S_{\mu\rho\nu\sigma}\gamma_\sigma - i\epsilon_{\mu\rho\nu\sigma}\gamma_\sigma\gamma_5$ has been used.

For completeness we note that the vector-axial vector commutator is obtained from Eq. (6.14) by the substitutions:

$$A_{c\sigma}(x|o) \longleftrightarrow A^5_{c\sigma}\ (x|o)$$
$$S_{c\sigma}(x|o) \longleftrightarrow S^5_{c\sigma}\ (x|o)$$

and that the axial-axial commutator is identical to the vector-vector.

Note that the bilocal operators in these expansions are the bilocal generalizations of the local currents themselves. Fritzsch and Gell-Mann[4] showed that the bilocals generate a closed algebra. For example:

$$\begin{aligned}[j_{\mu a}(x|y), j_{\nu b}(u|v)] \propto\ & i\partial^\rho_y [\delta[(y-u)^2]\epsilon(y_0-u_0)] \\ & \otimes \{S_{\mu\rho\nu\sigma}(d_{abc}+if_{abc})j^\sigma_c(x|v) - i\epsilon_{\mu\rho\nu\sigma}(d_{abc}+if_{abc})j^{5\sigma}_c(x|v)\} \\ & + i\partial^\rho_v[\delta[(v-x)^2]\epsilon(v_0-x_0)] \\ & \otimes\{S_{\mu\rho\nu\sigma}(d_{abc}-if_{abc})j^\sigma_c(u|y) + i\epsilon_{\mu\rho\nu\sigma}(d_{abc}-if_{abc})j^{5\sigma}_c(u|y)\}\end{aligned}$$

where $j^\sigma_c(x|y) = \bar{\psi}(x)\lambda_c\gamma^\sigma\psi(y)$ and $j^{5\sigma}_c(x|y) = \bar{\psi}(x)\lambda_c\gamma^\sigma\gamma^5\psi(y)$

Testing the bilocal algebra is a difficult undertaking since it requires a four current process in which all spatial separations may be forced to be light-like. The processes $ep\to e(\mu^+\mu^-) + X$; $e^+e^-\to\mu^+\mu^- + X$ and $e^+e^-\to e\ e^- + X$ have been proposed as candidates[5]. However, the likelihood that these experiments will be performed in the required kinematic regime is negligible for the forseeable future. Unless we have overlooked some application, it seems we must content ourselves with the algebra of local currents for the time being.

Before proceeding to the derivation of sum rules, we note that Eq. (6.14) motivated the choice of tensor structure in Eq. (6.11). The reader should verify for himself that Eq. (6.14) generates the correct scaling behavior for $F_2(\xi)$ in electroproduction.

6.4 Sum Rules From the Quark Light-Cone Algebra

Here we derive several of the sum rules previously derived from the parton model and also hopefully learn why others cannot be derived.

1. Callan-Gross Relation

Eq. (6.14) contains no term proportional to $(g_{\mu\nu}\Box-\partial_\mu\partial_\nu)$ and by comparison with Eq. (6.11) necessitates $F_L(\xi) = 0$ which is the Callan-Gross relation. Note, no statement about the detailed nature of the proton has been made.

2. Adler Sum Rule

Consider the term in $W_{\mu\nu}^{\nu p}$ symmetric in $\mu \leftrightarrow \nu$

$$\frac{1}{2}\left(W_{\mu\nu}^{\nu p}+W_{\nu\mu}^{\nu p}\right) = -\frac{S_{\mu\rho\nu\sigma}}{4\pi^2}\int d^4x\, e^{iq\cdot x}\,\partial^\rho\left[\delta(x^2)\epsilon(x_0)\right]\Big\{\langle P|S_3^\sigma(x|0)|P\rangle + \frac{1}{\sqrt{3}}\langle P|A_8^\sigma(x|0)+\sqrt{2}\,A_0^\sigma(x|0)|P\rangle\Big\} \tag{6.16}$$

and define *

---

*The terms involving A_k^σ will drop out of the sum rule since they vanish at $x_\mu = 0$. For expediency we drop them henceforth.

$$\langle P|S_3^\sigma(x|0)|P\rangle = P^\sigma f_3^S(x\cdot p, x^2) + x^\sigma g_3^S(x\cdot p, x^2)$$

(6.17)

In the Bjorken limit only $f_3{}^s(x\cdot p,0)$ will contribute to W_2.

Define

$$f_3^S(x\cdot p,0) = \int_{-1}^{1} d\alpha\, e^{i\alpha x\cdot p}\, \hat{f}_3^S(\alpha)$$

(6.18)

the limits being set by spectral requirements. Combining Eqs. (6.16-6.18) and isolating the coefficient of $P_\mu P_\nu$ we obtain:*

$$\frac{1}{M^2} W_2^{\nu p}(q^2,\nu) = -\frac{1}{\nu}\, \xi\, \hat{f}_3^S(\xi)$$

or

$$F_2^{\nu p}(\xi) = -\xi\, \hat{f}_3^S(\xi)$$

Now refer to Eq. (6.18) and note:

$$\int_{-1}^{1} \frac{d\xi}{\xi} F_2^{\nu p}(\xi) = -f_3^S(0,0) = -2$$

$f_3{}^s(0,0)$ is two since $S_3^\sigma\ (0|0) = 2j_3^\sigma(0)$ whose proton matrix element is

$$\langle P|j_3^\sigma(0)|P\rangle = 2P^\sigma I_3(\text{proton}) = P^\sigma$$

Finally the usual form of the sum rule follow from the observation that crossing requires $F_2^{\nu p}(\xi) = F_2^{\bar{\nu} p}(-\xi)$.

---

*Factors of two and π are formidable in this calculation. Among the pitfalls are
1) 2 from the vector-vector and axial-axial commutators
2) $j_{+\sigma}(x) = j_{1\sigma}^3(x) - j_{1\sigma}^{5\,3}(x) + i j_{2\sigma}(x) - i j_{2\sigma}^5(x)$
3) $\langle P|P'\rangle = (2\pi)^3\ 2E\ \delta^3(\vec{P}-\vec{P}')$ which implies $\langle P|j_\mu^{em}(0)|P\rangle = 2P_\mu$

Several remarks are in order. First, the parton model derivation is easier. Second, it is clear we have assumed nothing specific to the proton except its isospin. Third, the derivation still has flaws, e.g., how do we know that $F_2(\xi)/\xi$ does not contain a term proportional to $\delta(\xi)$, necessary to validate the sum rule, but unobservable in neutrino scattering, or that the strong assumption of the existence of the bilocal operators is really necessary? We will return to these later.

3. Bjorken's Spin Dependent Sum Rule[6]:

The matrix element $\langle P|S^5_{a\sigma}(0|0)|P\rangle$ vanishes between spin averaged proton states. If we do not spin average (and choose for example a = 3):

$$\langle PS|\, S^5_{3\sigma}(0|0)\,|PS\rangle = 2g_A S_\sigma \tag{6.19}$$

where $g_A \cong 1.2$ is the axial weak charge. Equation (6.19) may be exploited to derive a sum rule analogous to Adler's Consider the spin dependent terms in electron scattering:*

$$\frac{1}{2}(W_{\mu\nu}-W_{\nu\mu}) = \frac{i\epsilon_{\mu\nu\alpha\beta}}{M^2}\, q^\alpha S^\beta \left[G_1(q^2,\nu) + \frac{\nu}{M^2}G_2(q^2,\nu)\right] + \frac{i\epsilon_{\mu\nu\alpha\beta}}{M^4}\, p^\alpha q^\beta\, q\cdot S\; G_2(q^2,\nu) \tag{6.20}$$

---

*That is, we define $W_{\mu\nu}$ as before except that the spin of the proton is not averaged. In that case two additional structure functions appear, both antisymmetric in $\mu \leftrightarrow \nu$. For an introduction to spin dependent effects and a parton model derivation of Bjorken's sum rule see Kuti and Weisskopf.[7]

From Eq. (6.14) we obtain:

$$\frac{1}{2}(W_{\mu\nu}-W_{\nu\mu}) = \frac{i\epsilon_{\mu\nu\alpha\beta}}{4\pi^2}\int d^4x\, e^{iq\cdot x}\,\partial^\alpha \delta(x^2)\epsilon(x_0)$$
$$\otimes \langle P|\sqrt{\tfrac{2}{27}}\, S^5_{\beta 0}(x|0) + \tfrac{1}{6} S^5_{\beta 3}(x|0) + \frac{1}{6\sqrt{3}} S^5_{\beta 8}(x|0)|P\rangle$$

If we take the proton neutron difference only the $S^5_{\beta 3}(x|0)$ term survives [λ_8 and λ_0 are invariant under isospin rotations]. Fixing our attention on the coefficient of $\epsilon_{\mu\nu\alpha\beta}q^\alpha s^\beta$ we obtain in the Bjorken limit:[8]

$$2\int_0^1 d\xi\,[g_1(\xi)^{ep} - g_1(\xi)^{en}] = g_A/6$$

where

$$\lim_{Bj} \frac{\nu}{M^2} G_1(q^2,\nu) \equiv g_1(\xi)$$

This sum rule may also be derived in the parton model[7,10] although the treatment of spin is probably simpler in the light-cone approach.

4. Other Sum Rules and Spectral Relations

Most of the sum rules and spectral relations discussed in Lecture 3 may be derived in much the same manner as the Callan-Gross relation and Adler sum rule were derived. The obvious exception is the duality sum rule. If we imitate the derivation of the Adler sum rule for $F_2^{ep} - F_2^{en}$, we find:

$$F_2^{ep-en}(\xi) = \frac{1}{3}\xi \hat{f}_3^A(\xi)$$

where

$$\langle P|A_3^\sigma(x|0)|P\rangle = p^\sigma f_3^A(x\cdot p, x^2) + x^\sigma g_3^A(x\cdot p, x^2)$$

and $$f_3^A(x\cdot p,0) = \int_{-1}^{1} d\alpha\, e^{i\alpha x\cdot p} \hat{f}_3^A(\alpha)$$

If we attempt a sum rule $\int_{-1}^{1} d\xi/\xi\, F_2^{ep-en}(\xi)$ we get zero by crossing ($f^A{}_3(0,0) = 0$). On the other hand the integral:

$$\int_0^1 d\xi/\xi\, \left[F_2^{ep}(\xi) - F_2^{en}(\xi)\right]$$

is some integral along the light-cone whose value is unknown. The parton model result which depended upon the proton's composition cannot be derived.

6.5 Regge Behavior and Bilocal Operators[11, 12]

We return now to the question of how to interpret

$$F_L(x\cdot p) \equiv \int_0^1 \frac{d\xi}{\xi} \cos \xi x\cdot p\; F_L(\xi) \tag{6.19}$$

in the event $F_L(\xi)$ has the Regge behavior ($\sim 1/\xi$ as $\xi \to 0$) expected of it. Such questions are important in understanding BJL techniques and the sum rules they generate. Clearly if Eq. (6.19) diverges, the Fourier transform does not exist in the usual sense but may be understood as a generalized function. To display $F_L(x\cdot p)$ in a finite form proceed as follows: Let $f_R(\xi) = \sum_{\alpha>0} \gamma(\alpha)|\xi|^{-\alpha}\varepsilon(\xi)$ be the sum of all Regge terms which would cause a divergence in Eq. (6.19). Define

$$\tilde{F}_L(\xi) \equiv F_L(\xi) - f_R(\xi)$$

and add and subtract from Eq. (6.19) the quantity

$$\int_0^\infty \frac{d\xi}{\xi} \cos \xi x.p \, f_R(\xi) \, ,$$

$$F_L(x.p) = \int_0^1 \frac{d\xi}{\xi} \cos \xi x.p \, \tilde{F}_L(\xi) - \int_1^\infty \frac{d\xi}{\xi} \cos \xi x.p \, f_R(\xi)$$
$$+ \int_0^\infty d\xi \cos \xi x.p \sum_{\alpha>0} \gamma(\alpha) \, |\xi|^{-\alpha-1}$$

Note now*

$$|\xi|^{-\alpha-1} = \frac{1}{2i\sin\pi\alpha} \left[\frac{1}{(-\xi+i\epsilon)^{\alpha+1}} - \frac{1}{(-\xi-i\epsilon)^{\alpha+1}} \right]$$

for $\xi>0$. For $\xi<0$, the quantity in the brackets vanishes. Then

$$\int_{-\infty}^{\infty} d\xi \cos\xi x.p |\xi|^{-\alpha-1} = \frac{x.p}{2i\alpha\sin\pi\alpha} \int_{-\infty}^{\infty} d\xi \sin\xi x.p \left[\frac{1}{(-\xi+i\epsilon)^{\alpha}} - \frac{1}{(-\xi-i\epsilon)^{\alpha}} \right]$$

$$= -\frac{x.p}{\alpha} \int_0^\infty d\xi/\xi^\alpha \sin \xi x.p$$

$$= -\frac{|x.p|^\alpha}{\alpha} \Gamma(1-\alpha) \cos\frac{\pi\alpha}{2}$$

Finally, we obtain:

$$F_L(x.p) = \int_0^1 d\xi/\xi \cos \xi x.p \, \tilde{F}_L(\xi) - \int_1^\infty d\xi/\xi \cos \xi x.p \, f_R(\xi)$$
$$- \sum_{\alpha>0} \gamma(\alpha) \frac{|x.p|^\alpha}{\alpha} \Gamma(1-\alpha) \cos \pi\alpha/2$$

which is finite. In particular $F_L(0) = \int_0^1 \frac{d\xi}{\xi} \tilde{F}_L(\xi) - \sum_{\alpha>0} \frac{\gamma(\alpha)}{\alpha}$

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*Proof is direct from: $(-\xi \pm i\epsilon)^{-\alpha-1} = \exp\{(\alpha-1)[\log|\xi| \pm i\pi]\}$

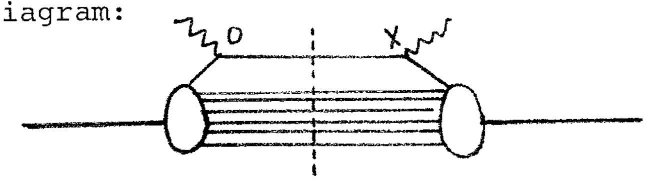
The seemingly exceptional values $\alpha=0$ and 1 may be easily disposed of.[12,13]

It may seem that the generalized function prescription is somewhat arbitrary. However, momentum space techniques[13] make it abundantly clear that this is the proper way to interpret these integrals: That the bilocal operators' matrix elements are finite and well-defined when the relevant structure functions scale. Actually it is clear from Eq. 6.11 that they are analytic in $x\cdot p$.[12]

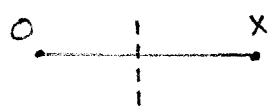
6.6 Partons and the Light Cone

It is instructive to return briefly to the parton model and search out the elements which it shares with the light cone:[14]

(1) The free-field singularity which yields Bjorken scaling is generated by the free propagation of the scattered parton into the final state. Remember the diagram:



In coordinate space, the propagator



is $\Delta(x) \to \frac{1}{2\pi}\,\delta(x^2)\,\varepsilon(x_0)$ near the light cone.

(2) An algebra of currents corresponds to a symmetry among partons.

If these are kept in mind, it is always possible to determine which parton model results are consequences of the model's light-cone structure alone.

6.7 Abstraction from Free-Field Theory

Finally, I would like to comment briefly on the abstraction of singularity structure from free field theory. Gell-Mann[4] speculated that abstraction must stop when the abstracted results would be invalid in interacting theory treated canonically (by which I mean ignoring the divergences of perturbation theory). All the results of this lecture satisfy this criterion.[15] Indeed we have not gone as far as we could: Mandula[16] showed $\frac{\nu}{M^2} W_L(q^2,\nu)$ scales in the canonical quark gluon model (as it would in free quark theory). Recently, however, Broadhurst, Gunion, and I[17] showed that in the quark, vector gluon model the leading chiral symmetry violating structure, $F_5(x)$, is explicitly proportional to the quark-gluon coupling, g. Abstraction from free field theory implies $F_5(x) = 0$ and this by a series of arguments implies a rather trivial theory. This supports Gell-Mann's suggestion on the limits of free field theory.

REFERENCES

1. Y. Frishman, Phys. Rev. Lett. 25, 966 (1970); Annals of Physics 66, 373 (1971); R.A. Brandt and G. Preparata, Nucl. Physics B27, 541 (1971).
2. R. Jackiw and R. Waltz, Phys. Rev. D6, 702 (1972), have shown explicitly that the current correlation function obeys these spectral conditions in model field theories.
3. R. Jackiw, R. van Royen, and G. B. West, Phys. Rev. D2, 2473 (1970).
4. H. Fritzsch and M. Gell-Mann, in Broken Scale Invariance and the Light-Cone, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson and A. Perlmutter (Gordon and Breach, New York, 1971) Vol. 2, p. 1.
5. D. J. Gross and S. B. Treiman, Phys. Rev. D4, 2105 (1971); J. Iliopoulos and E. A. Paschos, Phys. Rev. D6, 1340 (1972); M. Chaichian, DESY 72/15 (1972); G. Murtaya and M. S. K. Rayni, University of Islamabad preprint (1971); H. Terazawa, Phys. Rev. D5, 2259 (1972) and T. F. Walsh and P. Zerwas, Nuclear Physics B41, 551 (1972).
6. J. D. Bjorken, Phys. Rev. 148, 1467 (1966).
7. J. Kuti and V. F. Weisskopf, Phys. Rev. D4, 3418 (1972).
8. As we have derived it, this sum rule should involve both G_1 and G_2. G_2 may be isolated as the coefficient of $\varepsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta q\cdot S$ and is found to obey the sum rule[9] $\int_0^1 d\xi\, g_2(\xi) = 0$ where $g_2(\xi) = \lim_{Bj} \frac{\nu^2}{M^4} G_2(q^2,\nu)$. The form of Bjorken's sum rule

in the text makes use of Burkhardt and Cottingham's result.

9. H. Bürkhardt and W. N. Cottingham, Ann. Physics (N.Y.) 56, 453 (1970). Actually this form is not quite correct (it is Regge divergent) the correct form is given in R.L. Jaffe and C. H. Llewellyn Smith, Phys. Rev. D7, 2506 (1973). The changes have no effect on Bjorken's Sum Rule.

10. See, for example, R. P. Feynman, Photon-Hadron Interactions (Benjamin, Reading, Mass. 1972), pp. 156-159.

11. Y. Frishman, Acta Physica Austriaca, 34, 351 (1971).

12. Y. Frishman, "Light Cone and Short Distance Singularities" in Proceedings of the XVI International Conference on High Energy Physics, edited by J. D. Jackson and A. Roberts (NAL, Batavia, 1973) Vol. 4, p. 119.

13. D. J. Broadhurst, J. F. Gunion, and R. L. Jaffe, SLAC-PUB-1197, to be published in Annals of Physics.

14. R. L. Jaffe, Physics Letters 37B, 517 (1971), Phys. Rev. D5, 2622 (1972). J. C. Polkinghorne, Nuovo Cimento 8A, 572 (1972).

15. C. H. Llewellyn Smith, Phys. Rev. D4, 2392 (1971); D. J. Gross and S. B. Treiman, Phys. Rev. D4, 1059 (1971).

16. J. Mandula, Phys. Rev. D8, 328 (1973).

17. D. J. Broadhurst, J. F. Gunion, and R. L. Jaffe, MIT preprint MIT-CTP-339, to be published in Physical Review.

7. - BJL Limit

7.1 Introduction

Bjorken's original investigation[1] of deep inelastic electroproduction employed neither light-cone nor parton model techniques. Rather it was based on a momentum space realization of the short distance expansion developed some years earlier by Bjorken[2] and independently by Johnson and Low[3], and known as the BJL limit. This momentum space approach turns out to be a straightforward way to expose the assumptions which underly the sum rules we have been discussing and consequently to be a useful framework to explore possible violations of canonical scaling results. We proceed as follows:

First we discuss the BJL limit for the simple case of scalar currents. Then following Wilson[4] we relate the commutators which occur in the BJL limit to the short distance expansion. We will then discuss BJL limit sum rules and their validity in perturbation theory. Finally we discuss the "light cone" BJL limit and the derivation of fixed-q^2 sum rules including Adler's.

7.2 Derivation of the Limit

Consider the amplitude for scalar-current, proton forward scattering

$$\begin{aligned} T(q^2,\nu) &= i \int d^4x\, e^{iq\cdot x} \langle P|T^*(J(x)\,J(0))|P\rangle \\ &= i \int d^4x\, e^{iq\cdot x} \langle P|T(J(x)\,J(0))|P\rangle + \text{Polynomials in } \nu, q^2 \\ &= i \int d^4x\, e^{iq\cdot x} \langle P|\theta(x_0)[J(x),J(0)]|P\rangle + \text{Polynomials} \end{aligned} \qquad (7.1)$$

where T denotes time ordering. We have allowed for the possibility that $J(x)\ J(0)$ is so singular near $x = 0$ that the time ordered product is not covariant.[5] The covariantizing terms are, however, polynomials in q_μ and p_μ. Also we have converted the time ordered product to a retarded commutator - allowable provided the state $|p\rangle$ is stable. Consider now the limit $q_0 \to i\infty$ with $\vec{q} = 0$ and (for the moment) p_μ fixed, and repeatedly partially integrate Eq. (7.1):

$$\lim_{BJL} T(q^2,\nu) = i \sum_{n=1}^{\infty} \left(\frac{i}{q_0}\right)^{2n} \int d^3x \, \langle p | [\partial_0^{2n-1} J(x), J(0)] | p \rangle \Big|_{x_0=0} \tag{7.2}$$

Crossing $[T(q^2,\nu) = T(q^2, -\nu)]$ eliminated the terms in Eq. (7.2) proportional to an odd power of q_0.

$q_0 \to i\infty$ is a rather unphysical limit but may be related to observables via fixed-q^2 dispersion relations for $T(q^2,\nu)$. Immediately the question of subtractions arises: $T(q^2,\nu)$ is determined by its imaginary part along the real axis only up to a real polynomial in ν:

$$T(q^2,\nu) = 4 \int_{-q^2/2}^{\infty} \frac{d\nu' \, \nu' \, W(q^2,\nu')}{\nu'^2 - \nu^2} + \sum_{n=0}^{\infty} \nu^{2n} B_n(q^2) \tag{7.3}$$

(we have used the fact that $T(q^2,\nu)$ is even in ν and have defined

$W(q^2,\nu) = \frac{1}{2\pi}$ Im $T(q^2,\nu)$). If the integral is not convergent the polynomial may be used to provide subtractions in the usual fashion.* The dispersion integral may or may not need subtraction depending upon the (measureable) asymptotic behavior of $W(q^2,\nu)$ for large ν. Nevertheless one cannot rule out a priori the presence of a real polynomial regardless of the behavior of W.

I belabor this point because results obtained from BJL techniques depend sensitively upon assumptions about subtraction. For the moment we will assume no subtraction is necessary and no polynomial is present. Later the effect of a polynomial will be made apparent in a specific example. In any practical application the reader is forwarned to explore the sensitivity of his analysis to subtraction hypotheses.

To proceed, rewrite Eq. (7.3) (without the polynomial) in terms of the variable $\omega = 2\nu/-q^2 = 2p_0/q_0$:

$$T(q^2,\omega) = 4\int_1^\infty \frac{d\omega'\,\omega'\,W(q^2,\omega')}{\omega'^2-\omega^2}$$

and take $q_0 \to i\infty$, $\vec{q} = 0$, p_μ fixed (whence $\omega \to 0$). As $\omega \to 0$ the

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*Suppose, for example, the integral is barely divergent, then formally write:

$$T(q^2,0) = 4\int_{-q^2/2}^\infty \frac{d\nu'}{\nu'} W(q^2,\nu') + B_0(q^2)$$

and subtract from Eq. (7.3):

$$T(q^2,\nu) = T(q^2,0) + 4\nu^2\int_{-q^2/2}^\infty \frac{d\nu'}{\nu'(\nu'^2-\nu^2)} W(q^2,\nu') + \sum_{n=1}^\infty \nu^{2n} B_n(q^2)$$

denominator may be expanded:

$$\lim_{BJL} T(q^2,\omega) = \lim_{q_0 \to i\infty} 4 \sum_{n=0}^{\infty} \left(\frac{2P_0}{q_0}\right)^{2n} \int_1^{\infty} d\omega' \, \omega'^{-2n-1} W(q^2,\omega')$$

(7.4)

We would like to derive sum rules for moments of $W(q^2,\omega)$ by equating powers of $1/q_0$ in Eq's. (7.2) and (7.4). To do so we must assume something about the behavior of $W(q^2,\omega)$ as $q^2 \to -\infty$. We study here only the simple case when $W(q^2,\omega)$ scales:*

$$\lim_{q^2 \to -\infty} W(q^2,\omega) = F(\omega)$$

More complex situations in which the limit $q^2 \to -\infty$ cannot be taken underneath the integral are of considerable interest[6] but are beyond this simple presentation. Comparing Eq's. (7.2) and (7.4) we obtain (n=0):

$$4 \int_1^{\infty} \frac{d\omega}{\omega} F(\omega) = \lim_{q_0 \to i\infty} (\text{Polynomials})$$

and for $n \geqslant 1$:

$$\int_1^{\infty} d\omega \, \omega^{-2n-1} F(\omega) = \frac{i}{4} \left(\frac{i}{2P_0}\right)^{2n} \int d^3x \, \langle P | [\partial_0^{2n-1} J(x), J(0)] | P \rangle \Big|_{x_0=0}$$

These equations are the fundamental results of the BJL

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*If, for example, $\nu W(q^2,\omega)$ scaled (as is the case for W_2) the only modification is a realignment of which moments are identified with which commutators.

analysis. However the observant reader will realize he has been swindled. Although $W(q^2,\omega)$ may scale to $F(\omega)$ there are surely corrections to the scaling law which would enter all lower order sum rules. For example suppose

$$\lim_{q^2\to-\infty} W(q^2,\omega) = F(\omega) + \frac{M^2}{q^2}G(\omega)$$

then the above equation for n=1 should be corrected to read:

$$\int_1^\infty d\omega \left[\frac{F(\omega)}{\omega^3} + \frac{M^2 G(\omega)}{4P_0^2\omega}\right] = \frac{-i}{16P_0^2}\int d^3x \langle P|[\dot{J}(x), J(0)]|P\rangle\Big|_{x_0=0}$$

where the new term arises from inserting the correction in the n=0 relation. To rid ourselves of these unwanted terms we now take $p_0 \to \infty$:

$$4\int_1^\infty \frac{d\omega}{\omega} F(\omega) = \lim_{P_0\to\infty}\left(\lim_{q_0\to i\infty}(\text{Polynomials})\right) \tag{7.5}$$

and for $n \geq 1$:

$$\int_1^\infty d\omega\, \omega^{-2n-1} F(\omega) = \lim_{P_0\to\infty} \frac{i}{4}\left(\frac{i}{2P_0}\right)^{2n}\int d^3x \langle P|[\partial_0^{2n-1} J(x), J(0)]|P\rangle\Big|_{x_0=0} \tag{7.6}$$

Our assumptions (no subtractions, W scales) require the right hand side of Eq. (7.5) to exist. Scaling also requires that for every n, the commutator of Eq. (7.6) exist and contain a finite term[7] proportional to p_0^{2n}. * In this language scaling seems quite miraculous - requiring the existance of an infinite tower of high spin (to obtain increasing powers of p_0) operators.

A simple example is in order. Consider a free massless spinor theory:

$$J(x) = :\bar{\psi}(x)\psi(x):$$

Dimensional analyses leads us to expect $W(q^2, \omega)$ to scale. To calculate the $n = 1$ term we need

$$\{\psi(x), \psi^{+}(0)\} = \delta^3(\vec{x})$$

and

$$\partial_0 \psi(x) = -\gamma^0 \vec{\gamma} \cdot \vec{\nabla} \psi(x)$$

A brief calculation yields

$$\langle P | [\dot{J}(x), J(0)] | P \rangle \Big|_{x_0 = 0} = -\frac{4}{i} \langle P | \bar{\psi}(x) i \gamma^0 \overleftrightarrow{\partial^0} \psi(x) | P \rangle \delta^3(\vec{x})$$

This is the spin-2 part of the stress-energy tensor: $\theta_{\mu\nu} \equiv i\bar{\psi}\gamma_\mu \partial_\nu \psi - g_{\mu\nu} L$ where L is the free spinor Lagrangian. $\theta_{\mu\nu}$ is conserved and normalized:

$$\langle P | \theta_{\mu\nu} | P \rangle = 2 P_\mu P_\nu$$

Eq. (7.6) with $n = 1$ reads:

$$\int_1^\infty \frac{d\omega}{\omega^3} F(\omega) = \lim_{P_0 \to \infty} \frac{1}{4 P_0^2} \left[2 P_0^2 + \langle P | L | P \rangle \right] = 1/2$$

---

*The commutator of Eq. (7.6) grows no faster than p_0^{2n} — 2n-1 powers come from ∂_0^{2n-1} and one from $\int d^3x = \int d^4x\ \delta(x_0)$.

This is correct: since only the Born graph exists in this theory:

$$F(\omega) = \lim_{-q^2,\nu \to \infty} \frac{1}{4} \mathrm{Tr}\, \not{p}(\not{p}+\not{q})\, \delta(2\nu+q^2) = \frac{1}{2}\delta(\omega-1)$$

On the other hand, consider a free massless scalar theory:

$$J(x) = :\phi(x)\phi(x):$$

Dimensional analysis predicts $\frac{\nu}{M^2} W(q^2,\omega)$ to scale so $F(\omega) = 0$ and we expect to get zero by computing the right hand side of Eq. (7.6) for n = 1. Using

$$[\phi(x),\phi(0)]\Big|_{x_0=0} = 0 \qquad [\dot{\phi}(x),\phi(0)]\Big|_{x_0=0} = -i\delta^3(\vec{x})$$

we obtain:

$$[\dot{J}(x), J(0)]\Big|_{x_0=0} = -2i\delta^3(\vec{x})\phi(0)\phi(0)$$

Since the matrix element $\langle p|\phi(0)\phi(0)|p\rangle$ has no term proportional to $p_0^{\,2}$ we find:

$$\int_1^\infty \frac{d\omega}{\omega^3} F(\omega) = 0$$

as expected. To obtain non-trivial results in a scalar theory assume $\nu W(q^2,\nu)$ to scale and repeat the analysis.

In the realistic case of vector and/or axial vector currents, this sort of calculation yields sum rules like that of Gross and Llewellyn Smith if the commutators are calculated canonically* (as we just did) and if it is assumed that dispersion relations have no more subtractions than they need.**

In models where scaling is violated (e.g. order by order in renormalizeable perturbation theory) commutators become anomalous[8] and gratuitous subtraction constants appear.[9] Although the naive (and useful) form of the sum rules is lost, nevertheless the BJL theorem - which is actually little more than a definition of an equal time commutator as the coefficient of $1/(q^0)^n$ in the expansion of an amplitude - remains valid and useful.[10] We will encounter a specific example later. For the rest of this lecture we assume, where required, that the commutators which appear in the BJL expansion are canonical.

7.3 Relation to Short Distance Expansion

Wilson[4] showed the relation between his short distance expansion and equal time commutators. Consider the short distance expansion:

$$[J(x), J(0)] = \sum_n E_n(x)\, O_n(0) \tag{7.7}$$

with

$$E_n(x) = e_n\left[(-x^2+i\epsilon x_0)^{-d_n} - (-x^2-i\epsilon x_0)^{-d_n}\right] \tag{7.8}$$

valid near $x_\mu = 0$.

*That is: using canonical commutation relations of the fields and the naive (unrenormalized) equations of motion

**This may be weakened, see lecture 10.

At small x_0, $E_n(x)$ is equivalent to a sum of δ-functions. Consider

$$A_n([v], x_0) \equiv \int E_n(x_0, \vec{x})\, v(\vec{x})\, d^3x \tag{7.9}$$

for some smooth function $v(\vec{x})$. Since $E_n=0$ if $|\vec{x}| > x_0$ it makes sense to Taylor expand $v(\vec{x})$.

$$v(x) = v(0) + \vec{x}\cdot\vec{\nabla} v(0) + \ldots$$

$$A_n([v], x_0) = F_n^0(x_0)\, v(0) + \vec{F}_n^1(x_0)\cdot\vec{\nabla} v(0) + \ldots \tag{7.10}$$

where
$$F_n^0(x_0) \equiv \int E_n(x_0, \vec{x})\, d^3\vec{x}$$
$$\vec{F}_n^1(x_0) \equiv \int E_n(x_0, \vec{x})\, \vec{x}\, d^3\vec{x}, \text{ etc.}$$

Since by definition:

$$\int v(\vec{x})\, \delta^3(\vec{x})\, d^3x = v(0)$$
$$\int v(\vec{x})\, \vec{\nabla}\delta^3(\vec{x})\, d^3x = -\vec{\nabla} v(0), \text{ etc.}$$

Eq. (7.10) allows us to rewrite Eq. (7.9) as:

$$A_n([v], x_0) = \int d^3x \left[F_n^0(x_0)\, \delta^3(x) - \vec{F}_n^1(x_0)\cdot\vec{\nabla}\delta^3(x) \ldots \right] v(x)$$

So the quantity in brackets is to be identified with $E_n(x_0, \vec{x})$. Finally note from dimensional analysis and from the observation that Eq's. (7.7) and (7.8) are odd in x_0 that:

$$F_n^0(x_0) = A_n^0\, \epsilon(x_0)\, |x_0|^{-2d_n+3}$$
$$\vec{F}_n^1(x_0) = \vec{A}_n\, \epsilon(x_0)\, |x_0|^{-2d_n+4}$$

where the constants A_n^0, $\vec{A}_n^1, \ldots$ require further analysis to determine.* So we conclude:

$$
\begin{aligned}
E_n(x) = {} & \epsilon(x_0)|x_0|^{-2d_n+3} A_n^0 \, \delta^3(\vec{x}) \\
& + \epsilon(x_0)|x_0|^{-2d_n+4} \vec{A}_n^1 \cdot \vec{\nabla} \delta^3(\vec{x}) \\
& + \epsilon(x_0)|x_0|^{-2d_n+5} (A_n^2)_{ij} \nabla^i \nabla^j \delta^3(\vec{x}) \\
& + \ldots
\end{aligned} \tag{7.11}
$$

Since Eq. (7.11) is odd in x_0 the equal time commutator of Eq. (7.7) vanishes as it should. The first time derivative is non-vanishing at equal times. From Eq's. (7.8) and (7.11) it is clear that the spatial integral of the equal time commutator of $\dot{J}(x)$ with $J(0)$ measures the most singular term in the expansion and that it is

$$
\begin{aligned}
0 &\text{ if } 2d_n - 2 < 0 \\
\text{finite} &\text{ if } 2d_n - 2 = 0 \\
\infty &\text{ if } 2d_n - 2 > 0
\end{aligned}
$$

If $d_n < 1$ for the most singular term, then after some number of time derivatives it will become finite or infinite at $x_0=0$. We conclude that the first non-vanishing moment integral in the BJL expansion is given by the leading short distance

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*Actually a rather pleasant exercize in complex analysis yields:

$$A_n^0 = \frac{2\pi^{3/2}}{i} e^{i\pi d_n} \frac{\Gamma(n-3/2)}{\Gamma(n)}$$

while $\vec{A}_n^1 = 0$ by rotational symmetry.

singularity. Higher moment integrals determine lower singularities in a way the reader may work out for himself.

7.4 BJL Limit Sum Rules

1. Gross-Llewellyn Smith Sum-Rule:

To exemplify BJL techniques we return to this (by now familiar) sum rule and expose more clearly its structure. Consider the amplitude $T_{\mu\nu}$ in neutrino production with $\mu=x$, $\nu=y$ and both $\vec{q}$ and $\vec{p}$ in the z-direction

$$T_{xy} = \frac{-i\epsilon_{xyoz}}{2M^2}\left(p^0q^z - p^zq^0\right) T_3(q^2,\nu)$$

Take the sum of νp and $\bar{\nu}p$ amplitudes and write a dispersion relation for T_3:

$$T_3^{\nu+\bar{\nu}}(q^2,\nu) = 4\int_{-q^2/2}^{\infty} \frac{\nu' d\nu'}{\nu'^2-\nu^2} W_3^{\nu+\bar{\nu}}(q^2,\nu) + \sum_{n=0}^{\infty} f_n(q^2)\,\nu^{2n}$$

We have explicitly allowed for a polynomial in ν^2 even though the dispersion integral (without subtraction) converges in Regge theory. Taking the limit $q_0 \to i\infty$ (with $\vec{q} = 0$) and defining $\frac{\nu}{M^2} W_3(q^2,\nu) \equiv F_3(q^2,\omega)$ we obtain:

$$-\frac{4iP_z}{q_0} \lim_{q^2\to-\infty} \int_1^{\infty} \frac{d\omega}{\omega^2} F_3^{\nu+\bar{\nu}}(q^2,\omega) + \frac{iP_z}{2M^2q_0}\bar{f}_0 + \cdots$$

$$= -\frac{1}{q_0}\int d^3x \left\langle P\middle|\left[J_x^+(x), J_y^-(0)\right] + \left[J_x^-(x), J_y^+(0)\right]\middle|P\right\rangle\Big|_{x_0=0} + \cdots \tag{7.12}$$

On the left hand side only the term proportional to p_z/q_0 has been written, while on the right only the $1/q_0$ term is written. $\bar{f}_0$ is a possible $1/q^2$ term in $f_0(q^2)$ as $q^2 \to -\infty$. All the ignored terms must be separately equal unless there are the sorts of anomalies mentioned earlier. The commutator in Eq. (7.12) may be calculated in a canonical quark field theory:*

$$J_\mu^+(x) = \bar{\psi}(x)\gamma_\mu(1-\gamma_5)\lambda^+\psi(x) \qquad \lambda^\pm = \frac{1}{2}(\lambda^1 \pm i\lambda^2)$$

using

$$\left\{\psi^+(0), \psi(x)\right\}\Big|_{x_0=0} = \delta^3(\vec{x})$$

with the result:

$$[J_x^+(x), J_y^-(0)] + [J_x^-(x), J_y^+(0)] = -4i\bar{\psi}(0)\{\lambda_+,\lambda_-\}\gamma_z\psi(0)\,\delta^3(\vec{x})$$

$$= -[12i\,B_z(x) - 4i\,S_z(x)]\,\delta^3(\vec{x})$$

where $B_\mu(x)$ and $S_\mu(x)$ are the baryon and strangeness currents in the model. Since

$$\langle P|S_\mu(x)|P\rangle = 0 \qquad \text{and} \qquad \langle P|B_\mu(x)|P\rangle = 2P_\mu \,,$$

we obtain finally:

$$\lim_{q^2 \to -\infty} \int_0^1 d\xi\, F_3^{\nu p+\bar{\nu}p}(\xi,q^2) - \bar{f}_0/4M^2 = -6 \tag{7.14}$$

---

*This commutator may be extracted directly from the short distance expansion (cf. Eq. (5.8)).

Eq. (7.14) reduces to the Gross-Llewellyn Smith sum rule provided:

a. The constant $\bar{f}_0$ is absent. Like the $J = 0$ fixed poles to be discussed in Lecture 10, $f_0(q^2)$ should be a polynomial in q^2 and therefore not contain a term ($\bar{f}_0$) proportional to $1/q^2$.

b. Our canonical evaluation of the commutator is correct. Perturbation theory is plagued by anomalous commutators.[8] For this sum rule 6 is replaced by $6 - 3g^2/2\pi^2$ in second order in the vector gluon model.[11]

c. $F_3(\xi,q^2)$ scales so that the limit may be taken underneath the integral sign. If not, the sum rule may make sense as the limit of the integral but the implied non-uniformity makes its experimental significance dubious.

It is appropriate to emphasize that neither a,b, or c is obtained in perturbation theory. But, for that matter, neither is Bjorken scaling for F_1 and F_2 found in perturbation theory.

2. Cornwall-Norton Sum Rules:[12]

These are the moment sum rules for W_1 and W_2 generated by exactly the method we applied to the commutator of scalar currents. In canonical quark theories these sum rules will contain the Callan-Gross relation (moment by moment) so we need only write the sum rules for (say) W_2. To isolate W_2 consider $T_{zz} - T_{xx}$ in a frame where $\vec{q}$ is in the $\hat{y}$ direction and $\vec{p}$ is in the $\hat{z}$ direction:

Now take $q_0 \to i\infty$, set $\vec{q}$ to zero and subsequently let $p_0 \to \infty$:*

$$2 \lim_{q^2 \to -\infty} \int_1^\infty \frac{d\omega}{\omega^{2n+2}} F_2(q^2,\omega) = \lim_{P_0 \to \infty} (2P_0)^{-2n-2}$$

$$\otimes \int d^3x \left\{ \langle P | [(i\partial_0)^{2n+1} J_z(x), J_z(0)] | P \rangle \Big|_{x_0=0} - \langle P | [(i\partial_0)^{2n+1} J_x(x), J_x(0)] | P \rangle \Big|_{x_0=0} \right\} \tag{7.15}$$

Since the equal time commutators of Eq. (7.15) are determined by the short distance expansion, these sum rules are useful if the short distance behavior of a theory is understood. Recently it has been discovered that this is the case for some non-Abelian Yang-Mills theories.[13] Although the structure functions cannot themselves be computed in these theories, their moment integrals in principle can.[14] For a discussion of other applications of these sum rules see for example Frishman's NAL Rapporteur's Talk.[6] As an example of an application we follow Llewellyn Smith[15] to derive the parton model momentum sum rule from the n = 0 Cornwall-Norton Sum Rule.

3. Momentum Sum Rule:[15]

For n=0, Eq. (7.15) reduces to (assuming scaling):

$$\int_0^1 d\xi\, F_2^{ep}(\xi) = \lim_{P_0 \to \infty} \frac{i}{8P_0^2} \int d^3x \langle P | [\dot{J}_z(x), J_z(0)] - [\dot{J}_x(x), J_x(0)] | P \rangle \Big|_{x_0=0} \tag{7.16}$$

To proceed one must make a model for the currents (eg. canonical gluon model, perturbation theory, etc.). We will consider only

---

*As $p_0 \to \infty$, $p_3 \to p_0$

the free quark model except to quote results which may be found in Llewellyn Smith's Hamburg Summer School Lecture.[16] Explicit calculation of the commutator yields:

$$\langle P|[\dot{J}_i(x), J_i(0)]|P\rangle\Big|_{x_0=0} = -4i\,\delta^3(\vec{x})\langle P|\bar{\psi}\, i(\gamma_0\partial_0 - \gamma_i\partial_i)Q^2\psi|P\rangle \tag{7.17}$$

Were it not for the factor Q^2 this operator would be the spin 2 part of the stress-energy tensor. Suppose we had begun with currents proportional to baryon number rather than charge. Since $B^2 = \frac{1}{9}I$ in the quark model we would obtain:

$$\langle P|[\dot{J}_i^B(x), J_i^B(0)]|P\rangle\Big|_{x_0=0} = -\frac{4i}{9}\,\delta^3(\vec{x})\langle P|\bar{\psi}\, i(\gamma_0\partial_0 - \gamma_i\partial_i)\psi|P\rangle$$

Now use $\theta_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi - g_{\mu\nu}L$ (L is the Lagrangian) and the normalization $\langle p|\theta_{\mu\nu}|p\rangle = 2p_\mu p_\nu$ to reduce the right hand side to $-\,4/9\; i\;\delta^3(\vec{x})\;(2p_0{}^2 - 2p_i{}^2)$ + terms lower order in $\vec{p}$. It is easy to show that:

$$\int_0^1 d\xi\,\left[\frac{3}{4}F_2^{ep+en}(\xi) - \frac{9}{2}F_2^{\nu p+\nu n}(\xi)\right] \tag{7.18}$$

reproduces the baryon number and therefore the stress energy tensor. Combining Eq. (7.16) with (7.17) and taking the appropriate combination of structure functions:

$$\int_0^1 d\xi\,\left[\frac{3}{4}F_2^{ep+en}(\xi) - \frac{9}{2}F_2^{\nu p+\nu n}(\xi)\right] = 1$$

In an interacting theory uncharged gluons also contribute to $\theta_{\mu\nu}$ but not to the structure functions. Let ε be the fraction of proton's momentum residing in the glue:

$$\langle P|\theta_{\mu\nu}^{\text{gluons}}|P\rangle = 2P_\mu P_\nu \epsilon$$

The sum rule now reads:

$$\int_0^1 d\xi \left[\frac{3}{4} F_2^{ep+en}(\xi) - \frac{9}{2} F_2^{\nu p+\nu n}(\xi)\right] = 1-\epsilon$$

Llewellyn Smith[15] showed $1-\varepsilon \geq 0$ from the positivity of baryon-current, nucleon deep inelastic scattering. It has not so far been possible to prove $\varepsilon \geq 0$ in general[15] so the full parton model result is not obtained.

7.5 Light Cone BJL Limit and Fixed Mass Sum Rules

Jackiw and Cornwall[17] modified the BJL limit to make use of the light-cone formulation of field theories developed in recent years. Specifically they considered the limit $q_- \to i\infty$ with $q_+ = 0$ and $\vec{q}_\perp$ fixed ($\vec{p}\cdot\vec{q} = 0$). In terms of invariants we find:

$$q^2 = 2q_+q_- - q_\perp^2 = -q_\perp^2 \quad (\text{fixed})$$
$$\nu = 2P_+q_- \longrightarrow \infty$$

To exemplify the applications of this limit we derive the Adler sum rule. Consider T_{++} for neutrino scattering:

$$T_{++}^{\nu P} = \frac{P_+P_+}{M^2} T_2^{\nu P}(q^2,\nu) = i\int d^4x\, e^{iq\cdot x}\langle P|\delta(x_+)[J_+^+(x), J_+^-(0)]|P\rangle \tag{7.19}$$
$$+ \text{polynomials}$$

We have used the light-cone formulation of field theory (assumed equivalent to the equal time formulation) to write the amplitude as an x_+-retarded commutator. Writing a dispersion relation for T_2 allowing for Regge behavior and a real polynomial:

$$T_2^{\nu p}(q^2,\nu) = 2\int_{-q^2/2}^{\infty} \frac{\nu' d\nu' W_2^{\nu p+\bar{\nu} p}(q^2,\nu')}{\nu'^2-\nu^2} + 2\nu\int_{-q^2/2}^{\infty} \frac{d\nu' W_2^{\nu p-\bar{\nu} p}(q^2,\nu')}{\nu'^2-\nu^2}$$
$$+ \sum_n f_n(q^2)\,\nu^n \qquad (7.20)$$

Combining Eq. (7.19) and (7.20) and taking q_- to $i\infty$:

$$-\frac{2}{p_+q_-}\int_{-q^2/2}^{\infty} d\nu' W_2^{\nu p-\bar{\nu} p}(q^2,\nu') + O\left(\frac{1}{q_-^2}\right) + \sum_n f_n(q^2)\,(p_+q_-)^n$$
$$= -1/q_- \int dx_- \, d^2x_\perp \langle P|[J_+^+(x), J_+^-(0)]|P\rangle\Big|_{x_+=0}$$
$$+ \text{Polynomials} + O\left(\frac{1}{q_-^2}\right)$$

If we identify coefficients of $1/q_-$

$$\int_{-q^2/2}^{\infty} d\nu' W_2^{\nu p-\bar{\nu} p}(q^2,\nu') = \frac{p_+}{2}\int dx_- \, d^2x_\perp \langle P|[J_+^+(x), J_+^-(0)]|P\rangle\Big|_{x_+=0}$$

this commutator may be calculated using the canonical <u>light-cone</u> anti-commutator of $\psi(x)$ with its adjoint.[17] Suffice it to say that <u>if</u> the commutator is canonical then the Adler sum rule at fixed-q^2 is obtained:

$$\frac{1}{M^2}\int_{-q^2/2}^{\infty} d\nu \, W_2^{\bar{\nu} p-\nu p}(q^2,\nu) = 2$$

Note, unlike Gross-Llewellyn Smith Sum Rule, Adler's sum rule exists at every $q^2 < 0$. If one tried to derive a fixed-q^2 analog of the G-LS sum rule one encounters a light-cone commutator which involves interactions in a non-trivial way. Adler's is unique among the sum rules we have discussed in possessing a fixed-q^2 form. Note also that the "canonicity" of the light-cone commutator above is sufficient to derive the sum rule, there are no real constants to be contended with. Indeed one can show that even in second order perturbation theory in the vector gluon model - in which νW_2 doesn't scale - the sum rule is nevertheless valid.[7] We refer to Dicus, Jackiw, and Teplitz[18] for the application of this technique to the derivation of several fixed-mass sum rules.

REFERENCES

1. J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
2. J. D. Bjorken, Phys. Rev. 148, 1467 (1966).
3. K. Johnson and F. E. Low, Prog. Theor. Phys. Supp. 37-38, 74 (1966).
4. K. Wilson, Phys. Rev. 179, 1499 (1969).
5. K. Johnson, Nucl. Phys. 25, 431 (1961).
6. Y. Frishman, "Light Cone and Short Distance Singularities", in Proceedings of the XVI International Conference on High Energy Physics, Edited by J. D. Jackson and A. Roberts (NAL, Batavia, 1973) Vol. 4, pg. 119.
7. See the discussion in Y. Frishman and S. Yankielowicz, Phys. Rev. D7 (1973).
8. S. L. Adler and W.-K. Tung, Phys. Rev. Letters 22, 478 (1969). R. Jackiw and G. Preparata, Phys. Rev. Letters 22, 975 (1969); 22, 1162 (E) (1969).
9. D. Corrigan, Phys. Rev. D4, 1053 (1971).
10. D. J. Broadhurst, J. F. Gunion and R. L. Jaffe, SLAC-PUB-1197. to be published in Annals of Physics (N.Y.)
11. R. L. Jaffe and H. R. Quinn, unpublished.
12. J. M. Cornwall and R. E. Norton, Phys. Rev. 177, 2584 (1969).
13. H. D. Politzer, Phys. Rev. Letters 30, 1346 (1973). D. Gross and F. Wilczek, Phys. Rev. Letters 30, 1343 (1973).
14. D. J. Gross and F. Wilczek, "Asymptotically Free Gauge Theories - I", NAL-PUB 73/49 - THY (1973).
15. C. H. Llewellyn Smith, Phys. Rev. D4, 2392 (1971). See also

H. Fritzsch and M. Gell-Mann in Center for Theoretical Studies, University Miami, Tracts in Mathematics and Natural Science Vol. 2, Gordon and Breach 1971.

16. C. H. Llewellyn Smith, Springer Tracks in Modern Physics 62, 51 (1972).

17. J. M. Cornwall and R. Jackiw, Phys. Rev. D4, 367 (1971).

18. D. Dicus, R. Jackiw and V. Teplitz, Phys. Rev. D4, 1733 (1971).

8 - ANOMALIES

8.1 - Introduction

The experimental value of the $\pi^0 \to 2\gamma$ life-time is $(0.84 \pm 0.10) \times 10^{-16}$ sec. Current algebra and PCAC can be used to calculate the decay amplitude; they imply that it vanishes.[1] To accommodate the experimental decay rate, one must believe either that the PCAC extrapolation is badly wrong, or that the current algebra low energy theorem is wrong, or both.

It has been known since 1949 that if the $\pi^0 \to 2\gamma$ decay rate is calculated in lowest order perturbation theory using a fermion loop, then the rate is non-zero.[2] A calculation using the usual fractionally charged quarks comes out about a factor of three wrong in the amplitude (an order of magnitude in the decay rate).* It was Wilson[3] who first clearly pointed out how the result of perturbation theory[4] could be understood as a short distance effect, caused by the breakdown of naive current algebra calculations due to the strong singularity of the Green's function $\langle 0|T(J_\mu(x)J_\nu(0)A_\rho(y))|0\rangle$ when $x_\mu \sim y_\mu \sim 0$. This anomaly arises not because the current algebra commutators are incorrect, but because the delicate manipulations required to drive the Ward identity from the commutators break down if current Green's functions are as singular as suggested by scale invariance arguments.

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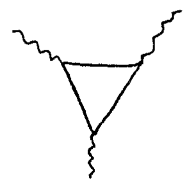
*A calculation using an elementary nucleon loop would give about the right decay rate.

That the anomaly is indeed a short distance effect is made plausible by the following argument. As we have seen in previous lectures, graphs which have no strong interaction vertices, and hence are zeroth order in the strong interactions, give an accurate representation of the conjectured canonical short distance behavior of hadrons. In perturbation theory the anomaly arises because the triangle graph



which is zeroth order in the strong interactions, is highly divergent at large momenta (i.e. short distances) which can be shown to invalidate naive manipulations. Hence this anomaly arises in theories with canonical short distance behavior, and we may call it a canonical anomaly, to distinguish it from the Callan-Symanzik anomalies[5] which result from the logarithmic modifications of short distance behavior found in higher orders of perturbation theory.

In this lecture we first review the simple current algebra low energy theorem for $\pi^0 \rightarrow 2\gamma$ decay which suggests that it should vanish.[1] We then discuss the perturbation theory calculation,[2,4] and then the Wilson analysis[3] showing how short distance effects can cause an anomaly. Finally we use the analysis of Crewther[6] to show how the singularities of two current products $\langle 0|T(J_\mu(x)J_\nu(0))|0\rangle$ and $\langle 0|T(A_\beta(y)A_\rho(0))|0\rangle$ are related to the singularity of $\langle 0|T(J_\mu(x)J_\nu(0)A_\rho(y))|0\rangle$ which is responsible for the anomaly. This gives model-independent relations between the $\pi^0 \to 2\gamma$ decay rate, $\sigma(e^-e^+ \to \gamma \to \text{hadrons})$ and the Bjorken[7] sum rule for polarization effects in deep inelastic electroproduction.

8.2 - The Current Algebra Low Energy Theorem

The $\pi^0 \to 2\gamma$ decay amplitude is proportional to:

$$\int d^4x e^{ipx} \langle 0|T(J_\mu(x)J_\nu(0))|\pi(k)\rangle$$

and hence, using the axial divergence as an interpolating field for the pion,[8] to:

$$\frac{1}{f_\pi^2 m_\pi^2}(m_\pi^2 - k^2)\int d^4x\, d^4y \langle 0|T(J_\mu(x)J_\nu(0)\partial^\lambda A_\lambda(y))|0\rangle e^{ip\cdot x}e^{-ik\cdot y}\Big|_{k^2=m_\pi^2} \tag{8.1}$$

where f_π is defined by $\langle 0|A^\mu(0)|\pi(k)\rangle = ik^\mu f_\pi$. The straightforward way of estimating the $\pi^0 \to 2\gamma$ decay amplitude would be to use a current algebra Ward identity and PCAC. Defining:

$$T_{\mu\nu\lambda}(p,k) \equiv \int d^4x d^4y e^{ipx} e^{-iky} \langle 0|T(J_\mu(x)J_\nu(0)A_\lambda(y))|0\rangle$$

and $T_{\mu\nu}(p,k) \equiv \int d^4x d^4y e^{ipx} e^{-iky} \langle 0|T(J_\mu(x)J_\nu(0)\partial^\lambda A_\lambda(y))|0\rangle$

we first observe that:

$$\frac{\partial}{\partial y_\lambda}\langle 0|T(J_\mu(x)J_\nu(0)A_\lambda(y))|0\rangle = \langle 0|T(J_\mu(x)J_\nu(0)\partial^\lambda A_\lambda(y))|0\rangle$$
$$+\ \delta(y_0 - x_0)\langle 0|T([A_0(y),J_\mu(x)]J_\nu(0))|0\rangle + \delta(y_0)\langle 0|T(J_\mu(x)[A_0(y),J_\nu(0)])|0\rangle \tag{8.2}$$

According to the equal-time current algebra, the last two terms in (8.2) should vanish. Fourier transforming (8.2), partially integrating with respect to y, and discarding the integral of a total derivative, we get the naive Ward identity:

$$ik^\lambda T_{\mu\nu\lambda}(p,k) = T_{\mu\nu}(p,k) \tag{8.3}$$

If we now define the form factor $\bar{T}(k^2,p^2,(k-p)^2)$:

$$T_{\mu\nu}(p,k) = \frac{1}{-2\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}p^\alpha k^\beta \bar{T}(k^2,p^2,(k-p)^2)$$

then the physical $\pi^0 \to 2\gamma$ decay rate is given by:

$$\lim_{k^2 \to m_\pi^2} \frac{(m_\pi^2 - k^2)\bar{T}(k^2,0,0)}{f_\pi m_\pi^2}$$

The usual PCAC assumption is that this $\simeq \bar{T}(0,0,0)$. Then the Ward identity (8.3) requires that $\bar{T}(0,0,0)$ should vanish.[1] One way to see this is to expand (8.3) in powers of the momenta, noting that the invariant form factors in $T_{\mu\nu\lambda}$ may have no poles, as there are no massless hadrons. Another way[3] to see this is to observe that:

$$\bar{T}(0,0,0) = -\frac{\pi^2}{12}\epsilon_{\mu\nu\alpha\beta}\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial k_\beta}T^{\mu\nu}(p,k)\Big|_{p=k=0}$$
$$= -\frac{\pi^2}{12}\epsilon_{\mu\nu\alpha\beta}\int d^4x\, d^4y\, x^\alpha y^\beta \langle 0|T(J^\mu(x)J^\nu(0)\partial^\lambda A_\lambda(y))|0\rangle \qquad (8.4)$$

and proceed as follows.

If we define $F_{\mu\nu\lambda}(x,y) \equiv \langle 0|T(J_\mu(x)J_\nu(0)A_\lambda(y))|0\rangle$ there is the identity:

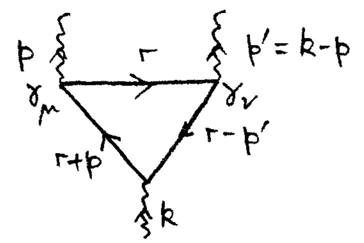
$$x_\alpha y_\beta \frac{\partial}{\partial y_\lambda}F_{\mu\nu\lambda}(x,y) = \frac{\partial}{\partial y_\lambda}(x_\alpha y_\beta F_{\mu\nu\lambda}) - \left(\frac{\partial}{\partial x_\lambda}+\frac{\partial}{\partial y_\lambda}\right)(x_\alpha y_\nu F_{\mu\lambda\beta})$$
$$+ \frac{\partial}{\partial x_\lambda}(x_\mu y_\nu F_{\lambda\alpha\beta}) + x_\alpha y_\nu\left(\frac{\partial}{\partial x_\lambda}+\frac{\partial}{\partial y_\lambda}\right)F_{\mu\lambda\beta} - x_\mu y_\nu \frac{\partial}{\partial x_\lambda}F_{\lambda\alpha\beta} \qquad (8.5)$$

The first three terms in (8.5) are total derivatives, and so should not contribute to (8.4), and the last two terms should vanish identically because of the conservation of J_μ and J_ν. Hence $\bar{T}(0,0,0)$ should vanish. The experimental $\pi^0 \to 2\gamma$ decay rate corresponds to $\bar{T}(m_\pi^2,0,0) \simeq 0.5$. We will see in the following sections that this apparent conflict with PCAC and the current algebra low energy theorem can be resolved, the reason being that in general the Ward identity (8.3) is expected to be false, and should have an extra anomalous term proportional to: $\epsilon_{\mu\nu\alpha\beta}p^\alpha k^\beta$.

8.3 - The Anomaly in Perturbation Theory

In this section we discuss the Ward identity (8.3) and see how it breaks down in the lowest order of perturbation theory.[2] We do not attempt to discuss the anomaly in perturbation theory thoroughly,[4] merely to give a flavor of it and motivate the idea that short distance singularities may be responsible for the anomaly.

The lowest order nucleon loop contributions to $T_{\mu\nu\lambda}$ in perturbation theory come from the graphs*:



+ (diagram with electromagnetic currents interchanged)

The lowest order graphs for $T_{\mu\nu}$ are identical, except that the vertex $\frac{1}{2}\gamma_\lambda\gamma_5$ is replaced by $im\gamma_5$, where m is the mass of the nucleon. Applying the Feynman rules we find:

$$T^0_{\mu\nu\lambda} = i\int\frac{d^4r}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{(\not r-\not p'+m)\gamma_\nu(\not r+m)\gamma_\mu(\not r-\not p+m)\gamma_\lambda\gamma_5}{[(r-p')^2-m^2+i\epsilon][r^2-m^2+i\epsilon][(r-p)^2-m^2+i\epsilon]}\right\} \tag{8.6a}$$

$$T^0_{\mu\nu} = 2m\int\frac{d^4r}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{(\not r-\not p'+m)\gamma_\nu(\not r+m)\gamma_\mu(\not r-\not p+m)\gamma_5}{[(r-p')^2-m^2+i\epsilon][r^2-m^2+i\epsilon][(r-p)^2-m^2+i\epsilon]}\right\} \tag{8.6b}$$

where the superscript ^{0} signifies lowest order contributions. Before we try to verify the Ward identity (8.3) we must first make sense of the two integrals in (8.6), both of which are linearly divergent for large values of the loop momentum r. Fortunately both $T^0_{\mu\nu\lambda}$ and $T^0_{\mu\nu}$ can be expressed in terms of convergent

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*Here we use elementary nucleons in the loops - calculations with any other fermions are similar.

integrals, as follows. By parity and Lorentz invariance:

$$T^{o}_{\mu\nu\lambda} = T_1^{o} p^{\tau}\epsilon_{\tau\mu\nu\lambda} + T_2^{o} p'^{\tau}\epsilon_{\tau\mu\nu\lambda} + T_3^{o} p_\nu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\mu\lambda} + T_4^{o} p'_\nu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\mu\lambda} + T_5^{o} p_\mu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\nu\lambda} + T_6^{o} p'_\mu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\nu\lambda} \quad (8.7)$$

with the T^{o}'s Lorentz scalar functions of p and p'. The quantities $T_1{}^{o}$ and $T_2{}^{o}$ are formally divergent, whereas the other T's are finite. However, current conservation

$$p^{\mu} T^{o}_{\mu\nu\lambda} = 0 = p'_\nu T^{o}_{\mu\nu\lambda} \quad (8.8)$$

relates $T^{o}_{1,2}$ to $T^{o}_{3,4,5,6}$. Using also the symmetry property $T^{o}_{\mu\nu\lambda}(p,p') = T^{o}_{\nu\mu\lambda}(p',p)$ it can be shown[8] that

$$T^{o}_{\mu\nu\lambda} = \left[(p\cdot p')p^{\tau}\epsilon_{\tau\mu\nu\lambda} + p_\nu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\mu\lambda}\right] T_3^{o} + \left[p'^2 p^{\tau}\epsilon_{\tau\mu\nu\lambda} + p'_\nu p^{\xi} p'^{\tau}\epsilon_{\xi\tau\mu\nu}\right] T_4^{o} + (p \longleftrightarrow p', \mu \longleftrightarrow \nu) \quad (8.9)$$

where T_3^{o} and T_4^{o} are finite:

$$T_3^{o} = \frac{i}{2\pi^2} I_{11}(p,p')$$
$$T_4^{o} = \frac{-i}{2\pi^2}\left[I_{20}(p,p') - I_{10}(p,p')\right]$$

where $I_{st}(p,p') = \int_0^1 dx \int_0^{1-x} dy\, x^s y^t / [y(1-y)p^2 + x(1-x)p'^2 + 2xy\, p\cdot p' - m_o^2]$

A similar analysis shows

$$T^{o}_{\mu\nu} = -\frac{m^2}{2\pi^2} I_{oo}(p,p')\, p^{\xi} p'^{\tau}\epsilon_{\xi\tau\mu\nu}$$

Comparison of these expression for $T^{o}_{\mu\nu\lambda}$ and $T^{o}_{\mu\nu}$ yields

$$i k^{\lambda} T^{o}_{\lambda\mu\nu}(p,k) = T^{o}_{\mu\nu}(p,k) + \frac{1}{2\pi^2} A p^{\xi} k^{\tau}\epsilon_{\xi\tau\mu\nu} \quad (8.10)$$

The precise value of A is model-dependent; for a simple nucleon loop it is 1/2, for a simple quark loop 1/6, and so on.

Why is there the extra anomalous term in (8.10)? The perturbation theory explanation runs as follows: because the integrals for $T^{0}_{\mu\nu\lambda}$ and $T^{0}_{\mu\nu}$ are linearly divergent at large values of r, they are ambiguous. To specify them, extra inputs such as in this case current conservation, are necessary. Naive manipulations of the integrands in (8.6) using the Dirac algebra and freely translating linearly divergent r integrals would have led to a "derivation" of the false non-anomalous Ward identity (8.3). Translations of linearly divergent integrals are not generally valid, and our calculation using current conservation to specify the linearly divergent integrals shows that in this case the translations are invalid.

There are many technical questions about the perturbation theory calculation, and we refer to the review of Adler[4] for a detailed discussion. Let us assume for now that there is an anomaly of the form (8.10) and discuss its implications.

(1) In contrast with the argument of section (8.2) we now find $\bar{T}(0,0,0) = A$, so that we have a smooth extrapolation from the zero momentum point to the $\pi^0 \to 2\gamma$ decay point $k^2 = m_\pi^2$ if we choose a model with $A = 1/2$.

(2) Can we understand the anomaly from a short distance point of view?[3] The lowest order graphs calculated in this section have the canonical free field singularity structure at short distances. For large momentum the integrand in (8.5a) is just the fourier transform of the canonical c-number ninth

order singularity in the expansion of $T(J_\mu(x)J_\nu(0)A_\lambda(y))$ discussed in section (5.2). As discussed above, the anomaly arises in perturbation theory because of the divergence of the integrand in (8.6a) at large momenta. Hence, in general a theory of hadrons which has the canonical ninth order singularity structure for $\langle 0|T(J_\mu(x)J_\nu(0)A_\lambda(y))|0\rangle$ will have an anomaly like (8.10). For this reason we may call it a canonical anomaly.

We should perhaps reemphasize that we have not abandoned our previous disbelief in perturbation theory as a reliable guide to the short distance behavior of hadrons. The triangle graph has no hadronic insertions; it is zeroth order in the strong interactions. From our point of view, considerations of higher order graphs involving hadronic corrections to the simple loop graphs are irrelevant.

8.4 - The Wilson Argument

We have seen that in perturbation theory the simple argument of section (8.2) leading to the Ward identity (8.3) breaks down, and that this may well be a result of short distance effects. Accordingly we now go through the analysis of section (8.2) again,[3] this time taking more care with short distance singularities. We will examine the representation (8.4) for $\bar{T}(0,0,0)$. First we make a Wick rotation of the x_0 and y_0 integrals and go over to an $0(4)$ metric. Then we exclude the regions $|x_0|<\varepsilon$, $|y_0|<\varepsilon'$ and $|x_0-y_0|<\varepsilon''$ from the integrations in (8.4), thereby excluding the short distance singularities. We will subsequently take the limits ε, ε' and $\varepsilon''\to 0$; the integral is absolutely convergent, so that it is independent of the order in which these limits are taken. Comparing (8.4) and (8.5) we see that:

$$\bar{T}(0,0,0) = -\frac{\pi^2}{12}(I+J+K)$$

where

$$I = \epsilon^{\mu\nu\alpha\beta}\int d^4x\, d^4y\, \partial/\partial y_\lambda \left(x_\alpha y_\beta F_{\mu\nu\lambda}(x,y)\right)\Big|_{|x_0|>\epsilon,\,|y_0|>\epsilon',\,|x_0-y_0|>\epsilon''} \quad (8.11)$$

and J and K are similarly related to the second and third terms in equation (8.5). The integral over y in I can be expressed in terms of surface integrals at $y_0=\pm\varepsilon'$, $x_0\pm\varepsilon''$. Consider the contribution of the surface integral:

$$I_1 = \epsilon^{\mu\nu\alpha\beta}\int d^4x\, d^4y\, [\delta(y_0+\epsilon') - \delta(y_0-\epsilon')]\, x_\alpha y_\beta F_{\mu\nu 0}\Big|_{|x_0|>\epsilon,\,|x_0-y_0|>\epsilon''}$$

There are singularities in x and y-x in $F_{\mu\nu 0}(x,y)$, and these can cause a non-cancellation of the two surface terms in I_1 unless $|x_0|>>\varepsilon'$. This we can assume if we take the limit $\varepsilon\to 0$ after the limit $\varepsilon'\to 0$. Similarly, we can assume that the other surface integrals in I, namely

$$I_2 = \epsilon^{\mu\nu\alpha\beta}\int d^4x\, d^4y\, [\delta(y_0-x_0+\epsilon'') - \delta(y_0-x_0-\epsilon'')]\, x_\alpha y_\beta F_{\mu\nu 0}(x,y)\Big|_{\substack{|x_0|>\epsilon \\ |x_0-y_0|>\epsilon''}}$$

cancel if we take the limit $\varepsilon\to 0$ after the limit $\varepsilon''\to 0$. Hence we find that I vanishes if we take the limit $\varepsilon\to 0$ last. Similar analyses of J and K show that they will vanish if we take ε'' or $\varepsilon'\to 0$ last, respectively.

Unfortunately, in equation (8.11) only one of the limits ε, ε', $\varepsilon''\to 0$ can be taken last, hence only one of I, J and K can be guaranteed to vanish. Suppose we take $\varepsilon'\to 0$ last, so that K vanishes. Then it easy to see that in a world which is scale invariant at short distance I cannot be assumed to vanish. Consider for example I_1: when $x\sim y\sim\varepsilon'$ $F_{\mu\nu\lambda}(x,y)\sim(\varepsilon')^{-9}$ if it is scale invariant (mass-independent) at short distances,

Consider for example I_1 which in the limit ε, $\varepsilon''\to 0$ can be written

$$I_1 = \epsilon^{\mu\nu\alpha\beta} \int d^4x\, d^3y \left[x_\alpha y_\beta F_{\mu\nu o}(x,y) \right]_{y_o=\epsilon'}^{y_o=-\epsilon'}$$

Scale invariance at short distance say that as the currents have dimensions $(\text{mass})^3$, $F_{\mu\nu\lambda}(x,y)\sim(\varepsilon')^{-9}$ for $x\sim y\sim\varepsilon'\sim 0$, so that $x_\alpha y_\beta\, F_{\mu\nu o}(x,y)\sim(\varepsilon')^{-7}$. As the region of integration over this singularity has a volume $(\varepsilon')^7$, I_1 could very well be non-zero in the limit $\varepsilon'\to 0$. Similar arguments apply to I_2, and to the contributions to J.

We conclude that in a world where $F_{\mu\nu\lambda}(x,y)$ has a scale invariant ninth order singularity as $x\sim y\sim 0$, $\bar{T}(0,0,0)$ may be non-zero, or equivalently there may be an extra anomalous term in the Ward identity (8.3), as in equation (8.10).

8.5 - The Crewther Relation

The Wilson argument only indicates that an anomaly is possible; precise evaluation of the anomaly requires a model for short distance behavior. If the chosen model is the canonical behavior of a model field theory, one method of evaluation is to calculate the lowest order of perturbation theory loop. As the anomaly only depends on a multiple short distance behavior, this would give the same result as an analysis in configuration space.[9] However, there is one relation involving the anomaly which is independent of the specific model for short distance behavior, which was derived by Crewther.[6] It relates the anomaly to singularities in products of pairs of currents, which can be measured in other processes.

The idea is as follows: consider some triple current product with operator product expansion:

$$T(\,A(x)\,B(0)\,C(y)) \underset{x\sim y\sim 0}{\sim} \sum_n F_n(x,y)\,O_n(0) \tag{8.12}$$

where the $O_n(0)$ are local operators, and the $F_n(x,y)$ singular c-number functions. We also have the simpler expansion:

$$T(A(x)\,B(0)) \underset{x\to 0}{\sim} \sum_m E_m(x)\,O_m(0) \tag{8.13}$$

(where the $E_m(x)$ are c-number singular functions) which means that:

$$T(A(x)\,B(0)\,C(y)) \sim \sum_m E_m(x)\,T(O_m(0)\,C(y)) \tag{8.14}$$

when $x\sim 0$ and y, $y-x \gg x$. There are also expansions:

$$T(O_m(0)\,C(y)) \underset{y\sim 0}{\sim} \sum_n E_{mn}(y)\,O_n(0) \tag{8.15}$$

Substituting (8.15) into (8.14) we find that

$$T(A(x)\,B(0)\,C(y)) \sim \sum_{m,n} E_m(x)\,E_{mn}(y)\,O_n(0) \tag{8.16}$$

when both x and y are small, but $x \ll y$, $y-x$. Comparing (8.12) and (8.16) we see that there is a consistency condition on $F_n(x.y)$ in terms of the E_m and E_{mn}:

$$F_n(x,y) \sim \sum_m E_m(x)\,E_{mn}(y) \tag{8.17}$$

when $x \ll y$, $y-x$.

We now apply this relation between the singularities in two and three current products to the expansion

$$T(J_\mu(x)\,J_\nu(0)\,A_\lambda(y)) \underset{x,y\sim 0}{\sim} \Delta_{\mu\nu\lambda}(x,y)\,I + \cdots \tag{8.18}$$

The c-number function $\Delta_{\mu\nu\lambda}(x,y)$ has a ninth order singularity as x, $y\sim 0$, and is responsible for the anomaly, as argued in the previous section. First we take $x \ll y$, $y-x$ in equation (8.18), using:

$$T(J_\mu(x)\,J_\nu(0)) \sim \frac{K_{JJ}}{3\pi^2}\,\epsilon_{\mu\nu\alpha\beta}\,\frac{x^\alpha A^\beta(0)}{(x^2-i\epsilon)^2} + \cdots \tag{8.19}$$

where we have used equation (5.5) with the $i\varepsilon$-prescription for time-ordered products, taken the value of the d_{abc} appropriate for our combination of currents, allowed for the model-dependent factor K_{JJ} discussed in section (5.3), and the dots in (8.19) stand for terms which are less singular or have different quantum numbers. Hence:

$$T(J_\mu(x)J_\nu(0)A_\lambda(y)) \underset{x\sim 0}{\sim} \frac{K_{JJ}}{3\pi^2}\epsilon_{\mu\nu\alpha\beta}\frac{x^\alpha T(A^\beta(0)A_\lambda(y))}{(x^2-i\epsilon)^2}+\cdots \quad (8.20)$$

Now we use

$$T(A_\beta(0)A_\lambda(y)) \underset{y\sim 0}{\sim} \frac{S_{AA}}{2\pi^4}\frac{(g_{\beta\lambda}y^2-2y_\beta y_\lambda)}{(y^2-i\epsilon)^4}+\cdots \quad (8.21)$$

which comes from the axial current analogue of equation (5.6), with the model-dependent factor S_{AA} discussed in section (5.3). Substituting (8.21) into (8.20) and comparing with (8.18) we get an equation analogous to (8.17):

$$\Delta_{\mu\nu\lambda}(x,y) \sim \frac{K_{JJ}S_{AA}}{6\pi^6}\frac{\epsilon_{\mu\nu\alpha\beta}x^\alpha(g^\beta{}_\lambda y^2-2y^\beta y_\lambda)}{(x^2-i\epsilon)^2\,(y^2-i\epsilon)^4} \quad (8.22)$$

valid in the limit $x \ll y$, $y-x$.

Hence $\Delta_{\mu\nu\lambda}(x,y)$ is determined by two-current singularities in the region $x \ll y$. To determine the anomaly it is necessary to know $\Delta_{\mu\nu\lambda}(x,y)$ for all small x and y. Fortunately it can be argued that $\Delta_{\mu\nu\lambda}(x,y)$ has a unique form in the short distance region. In spin zero model field theories $\Delta_{\mu\nu\lambda}$ is zero; in spin 1/2 theories its SU_3 generalization is (as argued in section (5.2)):

$$\Delta^{abc}_{\mu\nu\lambda}(x,y) \underset{x,y\sim 0}{\sim} \frac{N_{JJA}d^{abc}}{16\pi^6}\,T_r\left\{\frac{\gamma_\mu\gamma\cdot x\gamma_\nu\gamma\cdot y\gamma_\lambda\gamma_5\gamma\cdot(x-y)}{(x^2-i\epsilon)^2(y^2-i\epsilon)^2((x-y)^2-i\epsilon)^2}\right\} \quad (8.23)$$

where N_{JJA} is a model-dependent constant ($N_{JJA} = 1$ in the quark model, 3 in the three-colored quark model, etc.) Furthermore, it can be proved using conformal invariance,[10] which is expected to be a good symmetry at short distances, that the form (8.23) is unique.

Taking the limit x<<y, y-x of (8.23) and comparing it with (8.21) we deduce:

$$N_{JJA} = K_{JJ} S_{AA} \tag{8.24}$$

To get from the relation (8.24) to experimental quantities, certain steps are necessary. In all models, $S_{AA} = S_{JJ}$. Also, in all models with the usual SU_3 assignment for the electromagnetic current $J_\mu^{em} = J_\mu^3 + \frac{1}{\sqrt{3}} J_\mu^8$ we have $R = 2/3\ S_{JJ}$ Explicit integration of $\Delta_{\mu\nu\lambda}$ to get the anomaly (or equivalently a lowest order perturbation theory calculation) yields: $N_{JJA} = 6\bar{T}(0,0,0)$. Hence we eventually conclude that:

$$\bar{T}(0,0,0) = \frac{1}{4} K_{JJ} R \tag{8.25}$$

The Crewther technique can be applied to deduce relations involving other parameters in the short distance expansions of section (5.3). For example, by considering the limit y<<x, x-y∿0 one obtains:

$$N_{JJA} = K_{AA} S_{JJ} \tag{8.26}$$

By considering the Crewther argument applied to $\langle 0|T(A_\mu(x)A_\nu(0)A_\lambda(y))|0\rangle$ one deduces (in an obvious notation):

$$N_{AAA} = K_{AA}\, S_{AA} \tag{8.27}$$

N_{AAA} is related to the anomaly in the Ward identity for the three axial- current Green's function. By current algebra this is related[10] to the $\pi \to 2\gamma$ anomaly, such that:

$$N_{AAA} = N_{JJA} \tag{8.28}$$

Combining (8.24), (8.26), (8.27) and (8.28) we see that:

$$K_{JJ} = K_{JA} = K_{AA} \tag{8.29}$$

which is a useful restriction on the multiplicity of parameters in the expansions of Chapter 5.

8.6 - Implications for Models of Short Distance Behavior

Let us now consider the implications of the Wilson analysis[3] and the Crewther relation[6] for building models of short distance behavior. As mentioned earlier, the experimental value of $\bar{T}(m_\pi^2,0,0)$, deduced from the $\pi^0 \to 2\gamma$ decay rate, is about three times larger than its quark model value. In the quark model, $\bar{T}(0,0,0) = \frac{1}{6}$ and $K_{JJ} = 1$, $R = \frac{2}{3}$. If we believe PCAC so that $\bar{T}(m_\pi^2,0,0) \simeq \bar{T}(0,0,0)$, then equation (8.24) tells us that the quark model for either or both K_{JJ} or R should be modified. As mentioned in Lecture 5, R is measurable in principle:

$$\frac{\sigma(e^-e^+\to\gamma\to\text{hadrons})}{\sigma(e^-e^+\to\gamma\to\mu^-\mu^+)} \xrightarrow[q^2\to\infty]{} R$$

The other parameter K_{JJ} is the coefficient of the axial current in the product of two vector currents at short distances, and as the coefficient in the Bjorken[7] sum rule for spin-dependence in deep inelastic electroproduction:

$$2\int_0^1 d\xi\, g_1(\xi)^{ep-en} = \frac{g_A}{6} K_{JJ} \tag{8.30}$$

This sum rule was derived and notation defined in Lecture 6, using the quark light-cone algebra in which $K_{JJ} = 1$. It could also be derived using the BJL techniques of Lecture 7. Unfortunately, polarized deuterium data are necessary for testing (8.30), but will not be available for some time. However, using reasonable additional parton model assumptions, a sum rule can be written down for scattering off polarized protons alone, which should enable the quark model prediction $K_{JJ} = 1$ to be checked.[12] In all models with fundamental spin zero fields, $K_{JJ} = 0$ and there is no anomaly. In models with spin $\frac{1}{2}$ fields, K_{JJ} is always 1. This suggests it is likely that the quark model prediction $R = \frac{2}{3}$ should be modified by a factor of 3. This is what happens in the quark model of many colors. In this case $\bar{T}(0,0,0) = \frac{C}{6} K_{JJ} = 1$ and $R = \frac{2}{3}C$, where C is the number of colors. Because of the $\pi^0 \to 2\gamma$ decay rate, $C = 3$ is fashionable,[9,13] yielding $R = 2$, but as discussed in Lecture 2, there is as yet no overwhelming experimental evidence in favor of this value.

REFERENCES

1. D. G. Sutherland - Nucl. Phys. B2, 433 (1967).

2. J. Steinberger - Phys. Rev. 76, 1180 (1949).

3. K. G. Wilson - Phys. Rev. 179, 1499 (1969).

4. For a comprehensive review of anomalies in perturbation theory, see for example - S. L. Adler - "Lectures on Elementary Particles and Quantum Field Theory, Brandeis Summer Institute, Massachusetts Institute of Technology Press, Cambridge, Massachusetts, 1970.

5. C. G. Callan, Phys. Rev. D2, 1541 (1972). K. Symanzik, Comm. Math. Physics, 18, 227 (1970); Springer Tracts in Modern Physics 57, 222 (1971).

6. R. J. Crewther - Phys. Rev. Lett. 28, 1421 (1972).

7. J. D. Bjorken - Phys. Rev. 148, 1467 (1966).

8. L. Rosenberg - Phys. Rev. 129, 2786 (1963) - see also S. L. Adler, reference 4.

9. M. S. Chanowitz and J. Ellis - Phys. Rev. D7, 2490 (1973).

10. E. J. Schreier - Phys. Rev. D3, 980 (1971).

11. See for example - J. Wess and B. Zumino - Phys. Lett. 37B 95 (1971).

12. J. Ellis and R. L. Jaffe - preprint SLAC - PUB - 1288 (1973).

13. H. Fritzsch and M. Gell-Mann - "Proceedings of the International Conference on Duality and Symmetry in Hadron Physics", edited by E. Gotsman (Weismann Science Press, Jerusalem, 1971).

9 - FURTHER APPLICATIONS OF LIGHT-CONE ANALYSIS

9.1 - Introduction

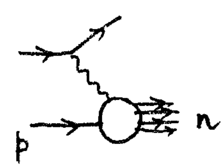
So far our applications of light-cone ideas have just been to the total deep inelastic cross sections. In this lecture we will discuss how these ideas can be applied to other processes. We will discuss $e^- + e^+ \to$ hadron + anything first, then $L + p \to L$ + hadron + anything, then $p + p \to (\mu^- + \mu^+)$ + anything, and finally list some other processes and make some philosophical comments.

Even in the most favourable case of $e^- + e^+ \to$ hadron + anything the light-cone analysis is not nearly as predictive as in deep inelastic scattering. There are no sum rules, so that the fundamental structures of the field theory underlying hadrons is not being probed as deeply. Also there are many cases where the parton model can be applied but not the light cone. Thus the limitations of the light cone become apparent, and it becomes important to understand better what properties of the parton model give it its power and can be safely abstracted.

9.2 - $e^-e^+ \to$ Hadron + Anything

Many people have pointed out the kinematic similarities[1,2,3] between deep inelastic scattering

$$L + p \to L + \text{anything} :$$



and one particle inclusive annihilation

$e^- + e^+ \Rightarrow$ Hadron + anything :

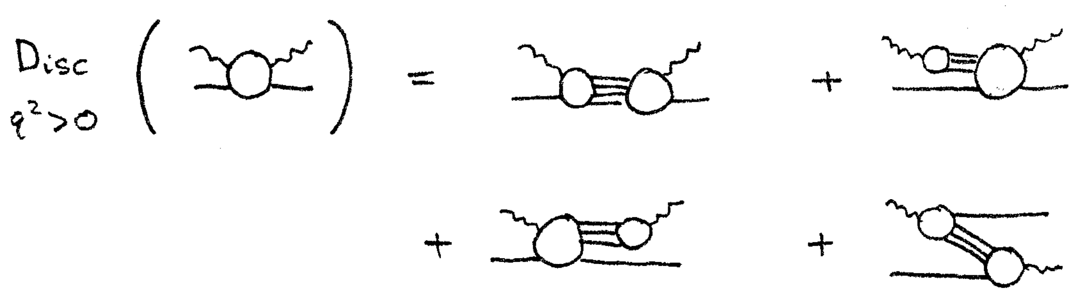
The former process is (apart from lepton kinematics)

$$\sum_n \langle p | J_\mu(0) | n\rangle\langle n | J_\nu(0) | p\rangle \, \delta^4(q+p-p_n)$$

and the latter

$$\sum_n \langle 0 | J_\mu(0) | p,n\rangle\langle p,n | J_\nu(0) | 0\rangle \, \delta^4(q-p-p_n) \qquad (9.2)$$

and one sees that what is different is that the hadron line has been crossed from the initial to the final state. Another way of looking at the similarity is in the style of Mueller[4]. Electroproduction is the total discontinuity of the forward off-shell Compton amplitude for $q^2 < 0$. However, the annihilation process is just one of the terms in the discontinuity of the forward off-shell Compton amplitude for $q^2 > 0$. This can be represented graphically:



The first term is the discontinuity in $(q + p)^2$, the second and third are discontinuities in q^2, and the last term is the inclusive annihilation cross section, a discontinuity in $(q - p)^2$, very analogous to the Mueller diagram for inclusive hadronic processes.

The annihilation process has been treated many times in the parton model[5], can we treat it using light-cone dominance? First let us do some kinematics. We can rewrite (9.2), replacing the δ function by an x integration:

$$\overline{W}_{\mu\nu} \equiv \int d^4x\, e^{iq\cdot x} \sum_n \langle 0|J_\mu(x)|p,n\rangle\langle p,n|J_\nu(0)|0\rangle \tag{9.3}$$

which we can express in terms of two gauge invariant tensors (p is the momentum of the observed hadron H, and $\nu \equiv p\cdot q$)

$$\overline{W}_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\overline{W}_1(\nu,q^2) + \left(p_\mu - \frac{\nu}{q^2}q_\mu\right)\left(p_\nu - \frac{\nu}{q^2}q_\nu\right)\overline{W}_2(\nu,q^2)$$

We note that $\overline{W}_1$ and $\overline{W}_2$ are very analogous to the deep inelastic structure functions. Including the lepton kinematics, the cross section for $e^-e^+ \to \gamma \to$ hadron + anything (neglecting terms of order $1/(q^2)^2$ is

$$\frac{d^2\sigma}{d\xi\, d(\cos\Theta)} = \frac{\pi\alpha^2}{\xi^3}\left\{\overline{\sigma}_T(1+\cos^2\Theta) + \overline{\sigma}_L(1-\cos^2\Theta)\right\} \tag{9.5}$$

where we have introduced the longitudinal and transverse "cross sections"

$$\bar{\sigma}_T(\nu, q^2) \equiv \frac{1}{q^2}\bar{W}_1(\nu, q^2)$$

and

$$\bar{\sigma}_L(\nu, q^2) \equiv \frac{1}{q^2}\left(\bar{W}_1(\nu, q^2) + \left(\frac{\nu}{2m^2\xi} - 1\right)\bar{W}_2(\nu, q^2)\right)$$

where M is the mass of the observed hadron and the variable $\xi \equiv q^2/2q\cdot p$ which has the kinematic range (for large q^2) $1 < \xi \leq \sqrt{q^2}/2M$. We see immediately that the naive scaling hypothesis is that

$$\bar{W}_1 \to \bar{f}_1(\xi) \quad , \quad \frac{\nu \bar{W}_2}{M^2} \to \bar{f}_2(\xi) \tag{9.6}$$

in the limit q^2, $q\cdot p \to \infty$, ξ fixed,
so that the right-hand side would feature no dimensional constants. Integrating (9.5) over θ, getting $\frac{d\sigma}{d\xi} = \frac{4\pi\alpha^2}{3\xi^3}(2\bar{\sigma}_T + \bar{\sigma}_L)$ in the limit q^2, $q\cdot p\to\infty$, ξ fixed, and then over ξ we get an expression for the multiplicity

$$\bar{n}(q^2)\sigma_{tot}(q^2) = \frac{4\pi\alpha^2}{3}\int_1^{\sqrt{q^2}/2M} d\xi \frac{1}{\xi^3}(2\bar{\sigma}_T + \bar{\sigma}_L) \tag{9.7}$$

If we weight $d\sigma/d\xi$ with the energy of the observed particle, and then integrate we get (assuming there is just one type of observed particle) the total energy. We can write the final sum rule in the form

$$q_0\,\sigma_{tot}(q^2) = \int_1^{\sqrt{q^2}/2M} d\xi\, p_0 \frac{d\sigma}{d\xi}$$

In the lepton centre of mass frame, $p_0 = q_0/2\xi$, so

$$\sigma_{tot}(q^2) = \int_1^{\sqrt{q^2}/2M} d\xi \frac{1}{2\xi}\frac{d\sigma}{d\xi} = \frac{2\pi\alpha^2}{3}\int_1^{\sqrt{q^2}/2M} d\xi \frac{1}{\xi^4}(2\bar{\sigma}_T+\bar{\sigma}_L) \qquad (9.8)$$

From Eq. (9.8) we see again that if $\sigma_{tot} \sim 1/q^2$, then it is natural that $\bar{W}_1$, $\nu\bar{W}_2$ also scale as in (9.6). Also it is apparent from (9.7) that in order for $\bar{n} \sim \log q^2$, analogously to the present fashion for hadronic multiplicities, we should have

$$q^2(2\bar{\sigma}_T+\bar{\sigma}_L) \sim \xi^2 \text{ as } \xi \to \infty$$

To see if the light-cone is relevant[2] in the limit q^2, $q\cdot p\to\infty$, ξ fixed we use the standard phase variation arguments.

Neglecting Lorentz indices for the moment as they are irrelevant to this argument, we can write (9.3) as

$$\int d^4x \; e^{iq\cdot x} \bar{f}(x^2, p\cdot x) \qquad (9.9)$$

working in the rest frame of the observed hadron, so that

$$p = (M,0,0,0) \quad , \quad q = \left(\frac{\nu}{M}, 0, 0, \sqrt{\left(\frac{\nu}{M}\right)^2 - q^2}\right)$$

(9.9) becomes

$$\int_{-\infty}^{\infty} dx_0 \int d^3\underline{x} \; e^{i\frac{\nu}{M}x_0} e^{-i\underline{q}\cdot\underline{x}} f(x_0^2-\underline{x}^2, Mx_0)$$

Performing the angular integration we get (where $|\underline{x}| \equiv r$)

$$\int_{-\infty}^{\infty} dx_0 \int_0^{\infty} dr \frac{r}{\sqrt{(\frac{\nu}{M})^2-q^2}}\, e^{i\frac{\nu}{M}x_0} \sin\sqrt{(\tfrac{\nu}{M})^2-q^2}\; r\, f(x_0^2-r^2, Mx_0)$$

The phases of the terms coming from the two exponentials in $r\sqrt{(\frac{\nu}{m})^2-q^2}$ can be written in leading order as $\nu, q^2 \to \infty$ with fixed: $e^{i\frac{\nu}{m}(x_0+r)}$ $e^{\mp i\xi rm}$ and as usual the dominant contributions come from

$$|x_0 \pm r| = O\left(\frac{M}{\nu}\right) \quad , \quad |r| = O\left(\frac{1}{M\xi}\right)$$

$$\Rightarrow \quad |x^2| = |x_0^2 - r^2| = O\left(\frac{1}{q^2}\right)$$

so that the light cone between the two currents in (9.3) is indeed being probed.

As suggested by our analysis of the discontinuity, this is not the light-cone singularity of a simple operator product. We can reduce in the hadron H appearing in (9.3), getting[2]

$$\sum_n \int d^4x\, d^4y\, d^4z\, e^{iq\cdot x} e^{ip\cdot(z-y)} \langle 0|T(J_\mu(x)\, S^+(y))|n\rangle \times \langle n|\bar{T}(J_\nu(0)\, S(z))|0\rangle$$

$$= \int d^4x\, d^4y\, d^4z\, e^{iq\cdot x} e^{ip\cdot(z-y)} \langle 0|T(J_\mu(x) S^+(y))\bar{T}(J_\nu(0) S(z))|0\rangle \tag{9.10}$$

where T and $\bar{T}$ stand for time-(anti-time-) ordered products, and the hadronic sources are represented by $S^+(y)$, $S(z)$.

What light-cone singularity for (9.10) is required to give scaling of the form (9.6), and what form is expected in various different models? It is convenient[5] to introduce the structure functions $\bar{V}_1$ and $\bar{V}_2$:

$$\left(-g_{\mu\nu}q^2+q_\mu q_\nu\right)\bar{V}_1(\nu,q^2)+\left(\nu(q_\mu p_\nu+q_\nu p_\mu)-p_\mu p_\nu q^2-g_{\mu\nu}\nu^2\right)\bar{V}_2(\nu,q^2)$$

Quantities V_1 and V_2 analogous to $\bar{V}_1$ and $\bar{V}_2$ can be defined for electroproduction. It is suggested by the data that as expected in models using fundamental fermion fields, σ_L and hence V_1 vanish in the scaling limit. We will make the analogous assumption for $\bar{V}_1$ in the annihilation process. Comparing the above equation and (9.5) we see that the canonical scaling expectation for $\bar{V}_2$ is

$$\nu^2\bar{V}_2(\nu,q^2) \rightarrow \text{function of } \xi$$

We can represent $\bar{V}_2$ in the form

$$\bar{V}_2 \sim \int d^4x\, e^{iq\cdot x}\sum_i C_i(x)\bar{f}_i(p\cdot x)$$

where the $C_i(x)$ are the leading c-number singular functions, and $\bar{f}_i(p\cdot x)$ are analogous to (but more complicated than) the matrix elements of the bilocal operator in electroproduction. Then dimensional arguments suggest that the $C_i(x)$ be singular functions of order zero.

Because $J_\mu(x)$ is always to the left of $J_\nu(0)$ in the expression (9.3), it, and hence the singular function, should be analytic in the lower complex x_0 plane. This suggests that the leading singularity is of the form

$$\ln \mu^2(-x^2 + i\epsilon x_0) \qquad (9.11)$$

where μ is some mass parameter. Singularities of the form (9.11) do indeed occur in free field theory* and super-renormalizable field theory models[6]. Thus we have

$$\bar{V}_2 \sim \int d^4x\, e^{iq\cdot x} \ln \mu^2(-x^2+i\epsilon x_0)\bar{f}(p\cdot x)$$

Introducing the fourier transform

$$\bar{g}(\alpha) \equiv \frac{1}{2\pi}\int d(p\cdot x)\, e^{i\alpha(p\cdot x)}\bar{f}(p\cdot x)$$

we get

$$\bar{V}_2 \sim \int d\alpha \int d^4x\, e^{i(q-\alpha p)\cdot x} \ln \mu^2(-x^2+i\epsilon x_0)\bar{g}(\alpha)$$

$$\propto \int d\alpha\, \bar{g}(\alpha)\, \Theta(q_0-\alpha p_0)\, \delta'\left((q-\alpha p)^2\right)$$

$$\propto \nu^{-2}\, \bar{g}'(\xi) \qquad (9.12)$$

Thus we have recovered the desired scaling law for this process: notice that at this level $\bar{g}'(\xi)$ is completely arbitrary (except that it must vanish for $\xi < 1$ in order to respect the spectrum condition). In particular we can choose $\bar{g}'(\xi) \sim \xi^2$ as

---

*It is clear that carrying through the free field theory analysis of Lecture 5 will always yield a singular function of $(-x^2+i\epsilon x_0)$ in current products where $J(x)$ appears to the left of $J(0)$.

$\xi\to\infty$, so that the multiplicity integral (9.7) diverges as $\xi\to\infty$, giving a logarithmic increase in hadronic multiplicity.

If $\bar{g}'(\xi)\sim\xi^2$, then $\bar{g}(\xi)\sim\xi^3$ as $\xi\to\infty$ and $\bar{f}(p\cdot x)^{-4}$ as $(p\cdot x)\to 0$. At first sight this seems rather strange[7]. The matrix element of the bilocal operator in deep inelastic electroproduction is analytic as $(p\cdot x)\to 0$, and one might have expected $\bar{f}(p\cdot x)$ to be similar. In fact there is no reason known why $\bar{f}(p\cdot x)$ should not be singular as $(p\cdot x)\to 0$, and in general it is singular in models. For example the analogous quantity in a super-renormalizable ϕ^3-type theory which scales has a $\log(p\cdot x)$ singularity in lowest order perturbation theory[6]. A singularity is possible because there is no simple representation for the operator product (9.10) controlling the annihilation cross section, and in models this complexity is reflected in the expression for $\bar{f}(p\cdot x)$ which has a structure like

$$\int d^4y\, d^4z\, e^{ip(z-y)} \langle 0|T(\phi(x)S^{+}(y))\bar{T}(\phi^{+}(0)S(z))|0\rangle \tag{9.13}$$

where ϕ is a fundamental constituent field. The integrations over y and z in the expression (9.13) can give singularities in $(p\cdot x)$, because they involve integrating over singularities in $(x-y)$, $(x-z)$, y and z. Brandt and Ng[8] have demonstrated explicitly how these integrations over singularities can give rise to singularities in $(p\cdot x)$.

We have shown that the light cone is relevant to $e^-e^+\to\gamma\to$ hadron + anything, and that a canonical singularity gives rise to canonical scaling analogous to that in deep inelastic scattering. Immediately the question arises whether any interesting predictions can be made from the light cone. In deep inelastic we obtained two classes of results: sum rules (Adler, Gross-Llewellyn Smith etc.) and spectrum relations

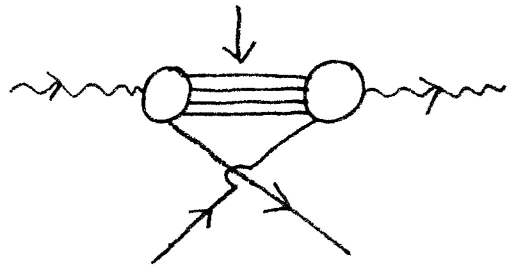
$$\frac{1}{4} \leq \frac{F_2^{eN}(\xi)}{F_2^{ep}(\xi)} \leq 4 \; , \quad 12\left(F_1^{ep} - F_1^{eN}\right) = F_3^{\nu p} - F_3^{\nu N} \quad , \text{ etc}$$

In annihilation it is not possible to derive sum rules and this can be seen in two ways:

a) In deriving sum rules we used dispersion relations to connect the real part (given by a canonical commutator) with the absorptive part (proportional to the structure function). When $q^2>0$, the annihilation cross section is only a portion of the full absorptive part of the amplitude, so that the connection breaks down.

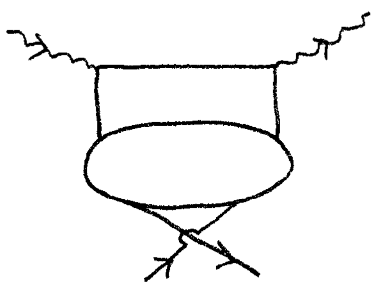
b) The sum rules related moments of deep inelastic structure functions to matrix elements of local operators which were terms in the expansion of the bilocal operator around $(x \cdot p)=0$. In annihilation the analogue of the bilocal operator is not in general analytic at $(x \cdot p)=0$, and is not closely related to local operators.

However it is possible to derive spectrum relations in the annihilation region, since these just depend on the internal symmetry properties of the leading light-cone singularity and do not require any analyticity properties. Suppose we write cross section for $e^-e^+ \to \gamma \to$hadron + anything in the form:



(9.14)

Then the SU_3 representations in the current-current channel indicated by the arrow will be restricted by the choice of the fundamental fields which the current is formed. This is because the diagram (9.14) can be written (literally in the parton model, figuratively in a more general light-cone analysis) as:



(9.15)

If the fundamental fields just belong to isospin singlets and doublets (or SU_3 triplets) then only $I=0,1$ (or SU_3 singlets and octets) will appear in the current-current channel. This gives a number of predictions: consider for example $e^-e^+\to\pi^{\pm}_{0}+$ anything. Then

$$\frac{d\sigma}{d\xi}(\pi^+) - 2\frac{d\sigma}{d\xi}(\pi^0) + \frac{d\sigma}{d\xi}(\pi^-) \tag{9.16}$$

corresponds to $I=2$ exchange in the cross channel

$$\frac{d\sigma}{d\xi}(\pi^+) - \frac{d\sigma}{d\xi}(\pi^-) \qquad \text{to } I=1 \tag{9.17}$$

$$\frac{d\sigma}{d\xi}(\pi^+) + \frac{d\sigma}{d\xi}(\pi^0) + \frac{d\sigma}{d\xi}(\pi^-) \quad \text{to } I=0 \tag{9.18}$$

Our assumption on the quantum numbers of the constituents determines that the combination (9.16) is zero. Charge conjugation requires that (9.17) be zero. Hence

$$\frac{d\sigma}{d\xi}(\pi^+) = \frac{d\sigma}{d\xi}(\pi^0) = \frac{d\sigma}{d\xi}(\pi^-) \tag{9.19}$$

This prediction gives a precise check of our ideas about the quantum numbers of constituent fields, and should be relatively easy to test at e^-e^+ colliding rings, pions being the most copiously produced particles. Predictions analogous to (9.19) can be made for other multiplets.[10]

9.3 - L + p → L + Hadron + Anything

A similar analysis to the previous one can be made for this process[2,10] (for simplicity we will restrict ourselves to spin zero currents composed of products of spin zero fields). The cross section for $L + p \to L + H$ + anything is then

$$\propto \sum_n \int d^4x\, e^{iq\cdot x} \langle p|J(x)|k,n\rangle\langle k,n|J(0)|p\rangle \tag{9.20}$$

where the target hadron has momentum $p = (M,0,0,0)$, and the observed hadron has momentum k. As before this could be rewritten

$$\propto \int d^4x\, d^4y\, d^4z\, e^{iq\cdot x} e^{ik\cdot(z-y)} \langle p|T(J(x)S^+(y))\bar{T}(J(0)S(z))|p\rangle \qquad (9.21)$$

The light-cone singularity again has the $(-x^2 + i\varepsilon x_0)$ structure: in this case it will be $(-x^2 + i\varepsilon x_0)^{-1}$ because of our choice of scalar currents and spin zero fundamental fields.

The expression (9.20) or (9.21) can then be written

$$\propto \int d^4x \frac{e^{iq\cdot x}}{-x^2+i\epsilon x_0} f(x\cdot p, x\cdot k, p\cdot k) + \left(\begin{array}{l}\text{terms non-leading}\\ \text{on the light-cone}\end{array}\right)$$

Introducing

$$g(\alpha,\beta,p\cdot k) = \frac{1}{(2\pi)^2}\int d(x\cdot p)\,d(x\cdot k)\,e^{i\alpha(x\cdot p)}e^{i\beta(x\cdot k)} f(x\cdot p, x\cdot k, p\cdot k)$$

we get

$$\int d\alpha\, d\beta\; g(\alpha,\beta,p\cdot k)\int d^4x \frac{e^{i(q-\alpha p-\beta k)\cdot x}}{-x^2+i\epsilon x_0}$$

$$\propto \int d\alpha\, d\beta\; g(\alpha,\beta,p\cdot k)\,\Theta(\nu-\alpha M-\beta k_0)\,\delta\left((q-\alpha p-\beta k)^2\right) \qquad (9.22)$$

Introducing $\omega' = 2q\cdot k/-q^2$ and $\kappa = 2k\cdot p/-q^2$ we find the argument of the δ function becomes

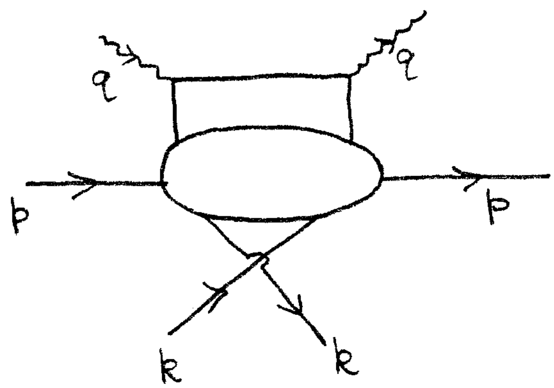
$$\delta\left(q^2(1-\alpha\omega-\beta\omega'-\alpha\beta\kappa)+\alpha^2M^2+\beta^2M_H^2\right)$$

In the limit $q^2\to-\infty$ with ω,ω' and κ fixed Eq. (9.22) becomes

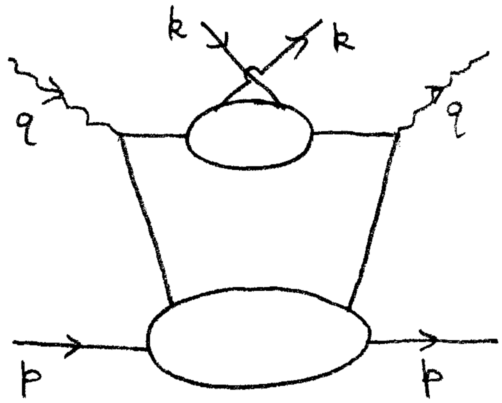
$$\propto \frac{1}{q^2} \int d\alpha\, d\beta\; g(\alpha,\beta,p\cdot k)\, \Theta(\omega - \beta\varkappa)\, \delta\left(1-\alpha\omega-\beta\omega'-\alpha\beta\varkappa\right)$$

It is evident that if $p \cdot k$ is kept fixed as q^2, $q \cdot p$, $q \cdot k \to \infty$ $(\kappa \to 0)$ then there is a fair possibility that the $(-x^2 + i\varepsilon x_0)^{-1}$ light-cone singularity will dominate. However, if we also take $p \cdot k \to \infty$ the leading light-cone singularity will not in general dominate. Non-leading singularities will in general have extra powers of $1/q^2$ associated with them, but the kernels analogous to $g(\alpha,\beta,p \cdot k)$ might well blow up as $(p \cdot k) \to \infty$ in such a way as to cancel out the powers of $1/q^2$.

This situation is precisely what is found in parton models of this process[5,11]: for $(p \cdot \kappa)$ finite in the scaling limit, parton model diagrams like



dominate, and scaling is given by the leading light-cone singularity However when $(p \cdot k) \to \infty$ with κ fixed $\neq 0$, then graphs like



dominate. This graph clearly has no singularity on the light cone. In terms of the five regions introduced by Bjorken[12] in discussing final states in electroproduction, the light cone dominates in the target fragmentation region and plateau, the hole fragmentation region and any finite part of the current plateau[12]. But it does not dominate in the central region of the current plateau, or in the parton fragmentation region near the high momentum boundary of phase space.

Consequences of applying light cone ideas to the final particle distribution in electroproduction have been pursued by several authors.[2,10,13] The scaling laws obtained agree with the parton model and a Mueller-Regge analysis, and interesting testable predictions of spectrum relations can be made. As in the case of annihilation, no sum rules can be obtained.

9.4 - $p + p \to (\mu^- + \mu^+)$ + Anything

The differential cross section for producing a μ-pair of mass = $\sqrt{q^2}$ from two incident protons p,p' with total centre of mass energy $\sqrt{s}$ is (for large q^2,s)

$$\frac{d\sigma}{dq^2} \propto \frac{1}{q^2 s} W(q^2,s) \tag{9.23}$$

where*

$$W(q^2,s) \propto \int d^4y\, \Delta_+(y,q^2) \langle pp'_{in}|J_\mu(y)J^\mu(0)|pp'_{in}\rangle$$

We are interested in the scaling limit q^2, $s\to\infty$ with $\tau\equiv q^2/s$ fixed.

It is difficult[14] to see how dominance of this process by the leading light cone singularity of $\langle pp'_{in}|J_\mu(y)J^\mu(0)|pp'_{in}\rangle$ can be made plausible except under extra assumptions. Consider a term in $\langle pp'_{in}|J_\mu(y)J^\mu(0)|pp'_{in}\rangle$ which behaves like $(y^2-i\varepsilon y_0)^n$ near the light cone, being of the form

$$(y^2-i\epsilon y_0)^n f\left(p\cdot y,\, p'\cdot y\,,\, s=(p+p')^2\right) \tag{9.24}$$

Suppose for example that $f(p\cdot y,\, p'\cdot y,\, s)$ were of the form

$$\bar{f}(p\cdot y,\, p'\cdot y)\, s^m$$

for some exponent m. Then in the scaling limit q^2, $s\to\infty$ with $\tau=q^2/s$ fixed:

$$\int d^4y\, \Delta_+(y,q^2)(y^2-i\epsilon y_0)^n \bar{f}(p\cdot y, p'\cdot y) \propto (q^2)^{n'} g(\tau)$$

for some exponent n' which depends on n. Hence (9.24) would give to (9.23) a contribution

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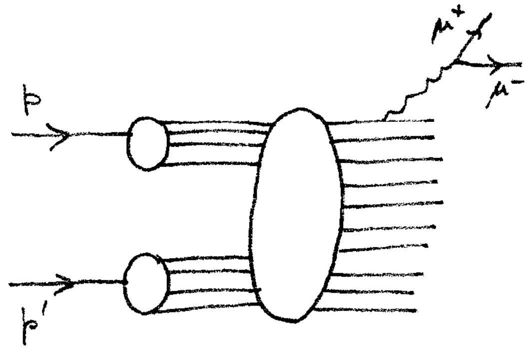
*The factor of $\Delta_+(y,q^2)$ in (9.23) comes from the lepton kinematics.

$$\propto \quad s^m (q^2)^{n'} g(\tau)$$

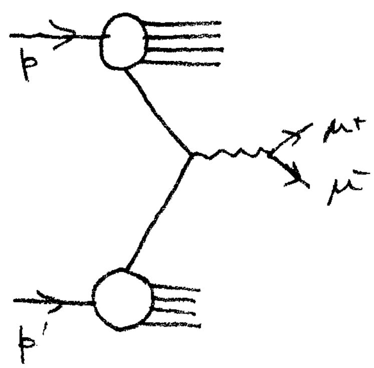
Clearly the possible dependence on s in the matrix element allows non-leading light-cone singularities to contribute or even dominate in the scaling limit.

This is exactly what happens in the parton model[15], where a term nonsingular on the light cone controls the scaling behavior. In the parton model we must consider two classes of diagrams

Bremsstrahlung:



and Parton-anti-parton Annihilation



It can be argued that the bremsstrahlung diagrams do not contribute in the scaling limit. Let us work in the centre of mass frame using light-cone variables and consider a parton from proton p with momentum

$$\left(\frac{m^2 + p_\perp^2}{2xp_+} , \underline{p}_\perp , x p_+ \right)$$

which emits a photon of momentum

$$q = (q_- , \underline{q}_\perp , q_+)$$

leaving behind a positive energy on mass-shell parton with momentum

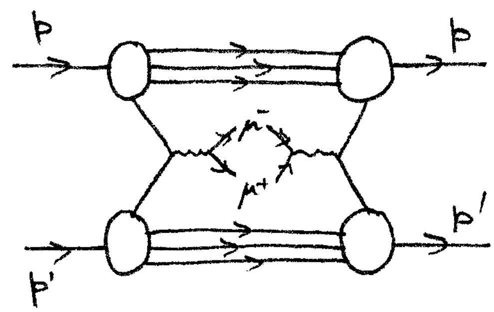
$$k = \left(\frac{m^2 + k_\perp^2}{2k_+} , \underline{k}_\perp , k_+ \right)$$

Since $q^2 = 0(s)$, both $q_\pm$ are $0(\sqrt{s})$. But $k_+ > 0$ because the parton is physical (!) so by energy momentum conservation

$$\frac{m^2 + p_\perp^2}{2xp_+} = q_- + \frac{m^2 + k_\perp^2}{2k_+} = O(\sqrt{s})$$

This condition cannot be satisfied because p_+ is $0(\sqrt{s})$, $|p_\perp|$ is cut-off and wee partons ($x \to 0$) can be argued away. Hence the bremsstrahlung diagram is suppressed.

The surviving annihilation diagrams have no light-cone singularity, because they have no parton propagating between y and 0, in contrast with electroproduction

$$W(q^2,s)_{ann} =$$ 

Working in the centre of mass frame, the off-shell photon is produced by a parton from proton p of type a with momentum xp + (finite part) annihilating with an antiparton from proton p' with momentum x'p' + (finite part). The probability of finding a parton of type a with momentum $\sim$ xp is (from Lecture 3) $u_a(x)$ (and similarly for the antiparton $u_{\bar{a}}(x')$). We have the restriction $(xp + x'p')^2 \gtrsim Q^2 \Rightarrow xx' = \tau$. Hence

$$W(q^2,s)_{ann} = \int_0^1 dx \int_0^1 dx' \, \delta(xx'-\tau) \sum_a u_a(x) u_{\bar{a}}(x') Q_a^2 \qquad (9.25)$$

To obtain the coordinate space structure of (9.25) we use[15]

$$\int d^4y \, \Delta_+(y,q^2) e^{i(p_1+p_2)\cdot y} = \frac{2\pi}{s} \delta(x_1x_2-\tau) \qquad (9.26)$$

where p_1 and p_2 are momenta such that

$$(p_1+p_2)^2 = xx's + \text{(terms vanishing as } s,q^2 \to \infty)$$

Possible choices are (in conventional coordinates)

$$p_1 = \left(\sqrt{x^2\underline{p}^2+M^2}\,,\ x\underline{p}\right)$$

$$p_2 = \left(\sqrt{x'^2\underline{p}^2+M^2}\,,\ -x'\underline{p}\right)$$

where $p = (\sqrt{\underline{p}^2+m^2}, \underline{p})$. Substituting (9.26) into (9.25) we find

$$\langle p p'_{in} | J_\mu(y) J^\mu(0) | p p'_{in} \rangle$$

$$= \frac{s}{2\pi} \int_0^1 dx \int_0^1 dx' e^{i(p_1+p_2)\cdot y} \sum_a u_a(x) u_{\bar{a}}(x') Q_a^2 \qquad (9.27)$$

$$+ \quad \left(\text{terms not contributing to } W(q^2,s)\right)$$

which is non-singular on the light cone unless u_a and $u_{\bar{a}}$ are very bizarre. From (9.23) and (9.25) we see that

$$\frac{d\sigma}{dq^2} \quad \propto \quad \left(\frac{1}{q^2}\right)^2 F(q^2/s)$$

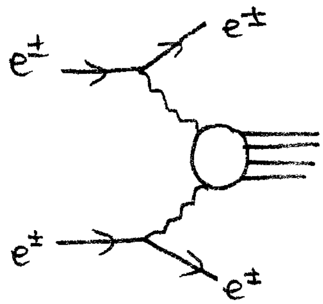
in the parton model in the scaling limit. From (9.27) we see that this scaling law is not given by the leading light-cone singularity.

9.5 - Comments

There are other processes where light-cone ideas can be applied, for example to two photon processes in colliding rings

$$e^\pm + e^\pm \rightarrow e^\pm + e^\pm + \text{Hadrons}$$

diagrammatically



where the virtual photons are a long way off their mass shell. The hadronic state produced may have finite mass -- for example a π^0.

Also there is a class of processes[16] which probe the commutators of pairs of light-cone commutators. Examples are

(a) $$e^- e^+ \to \gamma \to \mu^- \mu^+ + \text{Hadrons}$$

where the $(e^- e^+)$, $(\mu^- \mu^+)$ and hadronic systems all have large masses,

(b) $$L + p \to L + (\mu^- \mu^+) + \text{Hadrons}$$

in the deep inelastic region with the $(\mu^- \mu^+)$ pair having a large mass

(c) $$e^\pm + e^\pm \to e^\pm + e^\pm + \text{Hadrons}$$

where both the virtual photons are a long way off mass shell and the hadrons have large mass.

At the moment we feel that the most interesting problem is to understand the parton model better, particularly in configuration space. Just what are the basic assumptions and properties of the model which make it applicable outside the regions of light-dominance? If we had such a formulation, then we might understand better how seriously to take these other predictions of the parton model. After equal-time commutators, short distance behaviour and the light cone, where next in x-space?

REFERENCES

1. J. Pestieau and P. Roy - Phys. Letters 30B, 483(1969).
2. J. Ellis - Phys. Letters 35B, 537(1971).
3. J. Stack - Phys. Rev. Letters 28, 57(1972).
4. A.H. Mueller - Phys. Rev. D2, 2963(1971).
5. S.D. Drell, D.J. Levy and T.-M. Yan - Phys. Rev. D1, 1617(1971). P.V. Landshoff and J.C. Polkinghorne - Nucl. Phys. B33, 221 (1971).
6. J. Ellis and Y. Frishman - Phys. Rev. Letters 31, 135(1973).
7. C.G. Callan and D.H. Gross - IAS Princeton preprint (1972).
8. R.A. Brandt and W.-C. Ng - Stony Brook preprint (1973).
9. J. Ellis and C.H. Llewellyn Smith - unpublished - see footnote 36 of C.H. Llewellyn Smith in Proceedings of the 4th International Conference on High Energy Collisions, Edited by J.R. Smith (Rutherford Laboratory, 1972) vol. 1, p. 87.
10. H. Fritzsch and P. Minkowski - Nucl. Phys. B55, 363(1973).
11. S.D. Drell and T.-M. Yan - Phys. Rev. Letters 24, 855(1970).
12. J.D. Bjorken - Phys. Rev. D7, 282(1973).
13. A. Schwimmer - Caltech preprint CALT-68-376(1973).
14. S. D. Drell and T.-M. Yan - Phys. Rev. Letters 25, 316 (1970).
15. R. L. Jaffe - Phys. Letters 37b, 517 (1971).
16. D.J. Gross and S.B. Treiman - Phys. Rev. D4, 2105(1971).

10. J = 0 Fixed Poles

10.1 Introduction

Up to now we have applied operator product expansions to lepton scattering only at asymptotically large momentum transfer. At finite q^2 all terms in the expansions will in general contribute and there is no particular advantage in this approach. $J = 0$ fixed poles are an exception: There is good reason to believe that their residues are (simple) polynomials in q^2. This may be used to establish enticing though difficult to test correspondences between large and small (even zero) q^2 phenomena.

I wish to avoid detailed Regge theory (for lack of both time and competence) and will use it only to provide a parametrization of the large ν behavior of $T_{1,2}(q^2,\nu)$. Identical results are obtained whatever parametrization is used (provided it possesses the usual analyticity and crossing symmetry). Since Regge ideas are not a priori valid at asymptotic q^2,[1] it is comforting that this analysis is not tied to them.

I will proceed as follows: first to say what $J = 0$ fixed poles are; next to show why they should not occur in purely hadronic processes; third to argue that their residues might be expected to be polynomials in q^2; finally to apply these ideas to the study of electroproduction, specifically in deriving the Cornwall, Corrigan, Norton, Rajaraman, Rajasakharan[2] sum rule and the Schwinger term sum rule of Jackiw, van Royen and West[3]. The latter provides a rich example of the unity of the techniques developed in these lectures.

10.2 Definitions

For our purposes a J = 0 fixed pole (hereafter F.P.) is any purely real term in a scattering amplitude at infinite energy with an energy dependence corresponding to an $\alpha = 0$ Regge pole. We will discuss only forward (t=0) scattering and therefore have no experimental reassurance that J = 0 F.P. are indeed fixed (independent of t), although theories which generate them inevitably generate no t dependence.

To be specific, a real term at infinite ν, of the form $C_1(q^2)$ in $T_1(q^2,\nu)$ (which Reggeizes like ν^α), or of the form $C_2(q^2)\frac{\mu^4}{\nu^2}$ in $T_2(q^2,\nu)$ (which Reggeizes like $\nu^{\alpha-2}$) is a J = 0 fixed pole.*

For completeness we write out the Regge parametrization of (e.g.) T_1:

$$T_1^R(q^2,\nu) = 2\pi \sum_\alpha \gamma(\alpha,q^2) \left(\frac{1+e^{i\pi\alpha}}{\sin\pi\alpha} \right) \nu^\alpha \tag{10.1}$$

If, as we are assuming, the Regge terms scale, then[1]

$$\lim_{q^2 \to -\infty} \gamma^1(\alpha,q^2) \propto (q^2)^{-\alpha}$$

Note the behavior of Eq. (10.1) for $\alpha = 0$: If Re $T_1^R(q^2,\nu)$ is to remain finite for $\alpha = 0$ (which it must) then $\gamma(0,q^2) = 0$ and Im $T_1^R(q^2,\nu) = 0$ at $\alpha = 0$. A J = 0 pole in Eq.(1) is purely

*For $t \neq 0$ C_1 and C_2 become functions of t.

real. A more complex J-plane structure (e.g. a Regge cut or dipole) can generate an $\alpha = 0$ term in Im T_1^R and logarithms in Re T_1^R. We will stick to Eq. (10.1). More complex J-plane singularities present no more than technical problems.[4]

10.3 Exclusion from Hadronic Processes

Fixed poles are forbidden in hadronic processes by unitarity and hermitian analyticity. A simple proof is as follows[5]: consider the t-channel partial wave unitarity equation below inelastic threshold

$$b(\ell, t+i\epsilon) - b^*(\ell^*, t+i\epsilon) = \frac{2ik}{k^2+m^2}\, b(\ell, t+i\epsilon)\, b^*(\ell^*, t+i\epsilon)$$

$$t_{inel} > t > 4m^2 \tag{10.2}$$

By hermitian analyticity ($b^*(\ell^*, t+i\epsilon) = b(\ell, t-i\epsilon)$) Eq. (10.2) becomes:

$$b(\ell, t+i\epsilon) - b(\ell, t-i\epsilon) = \frac{2ik}{k^2+m^2}\, b(\ell, t+i\epsilon)\, b(\ell, t-i\epsilon) \tag{10.3}$$

$b(\ell, t)$ cannot have a pole at $\ell = \ell_0$ independent of t, for then the left-hand side of Eq. (10.3) would have a single pole while the right-hand side would have a double pole, which is, of course, a contradiction in the absence of essential singularities and the like.

Current scattering processes such as electro- and neutrino-production are immune to such arguments. They are calculated to lowest order in electromagnetism or weak interactions only and therefore do not obey non-linear unitarity equations like

Eq. (10.2). Soon after Regge poles were introduced into particle physics it was shown that a $J = 1$ fixed pole was necessary to compensate the decoupling of the Pomeron in Compton scattering.[6] Subsequently Creutz, Drell and Paschos[7] called attention to the possibility of a $J = 0$ fixed pole in Compton scattering.

10.4 q^2 - Dependence of $J = 0$ Fixed Pole Residues

Light-cone expansions with leading free-field singularities lead to the expectation that residues of $J = 0$ F.P. in kinematic constraint free (KCF) amplitudes are (simple) polynomials in q^2. The restriction to KCF amplitudes is to prevent the introduction of inverse powers of q^2 into the F.P. residue by arbitrary redefinition of the invariant amplitudes.

The best "physical" motivation for polynomial residues comes from parton models.[8] There the $J = 0$ fixed poles arise from the real part of the diagrams whose imaginary part yields scaling (we consider spin-1/2 partons for the moment):

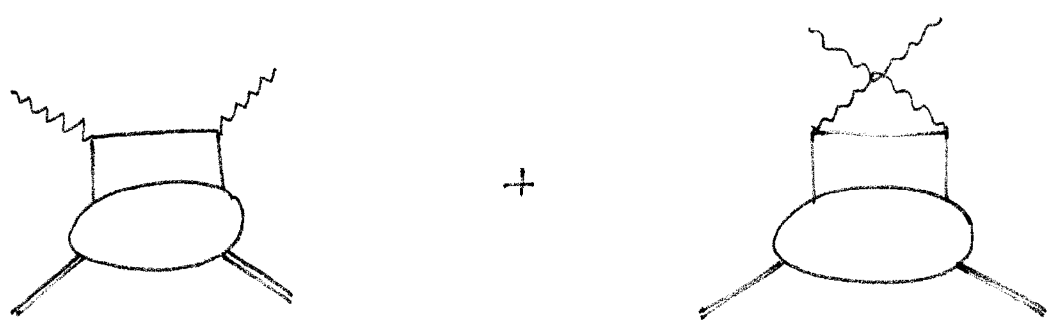


Figure 1

The fixed pole originates in the Born graphs:

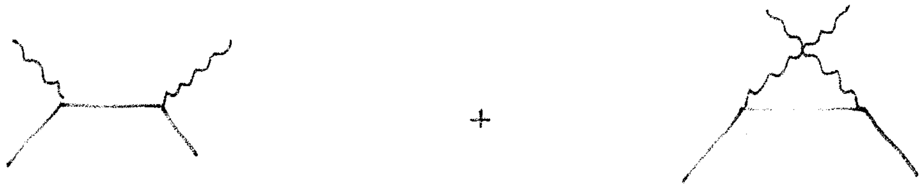
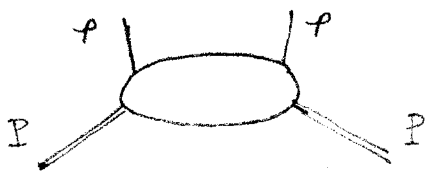


Figure 2

which are "glued" on to the parton-proton amplitude:



The born graphs (averaged over parton spin) yield

$$T_{\mu\nu}^{\text{Born}} \propto \frac{\mathrm{Tr}\{\not p \gamma_\mu (\not p + \not q)\gamma_\nu\}}{2p\cdot q + q^2} + \frac{\mathrm{Tr}\{\not p \gamma_\nu (\not p + \not q)\gamma_\mu\}}{q^2 - 2p\cdot q}$$

with $p \cdot q = xP \cdot q = x\nu$. As $\nu \to \infty$:

$$T_{\mu\nu}^{\text{Born}} \longrightarrow \text{const } g_{\mu\nu} + \text{other terms}$$

This constant is a $J = 0$ term in T_1 with polynomial (in fact constant) in q^2 residue. It remains to be verified that this term persists when the Born graph is grafted on to the rest of the amplitude. To do this would be to duplicate the coordinate space analysis which follows. Reference 8 treats the problem in detail. Finally it is necessary to argue that no other diagrams have $J = 0$ F.P. This is possible because all other diagrams are of the form:

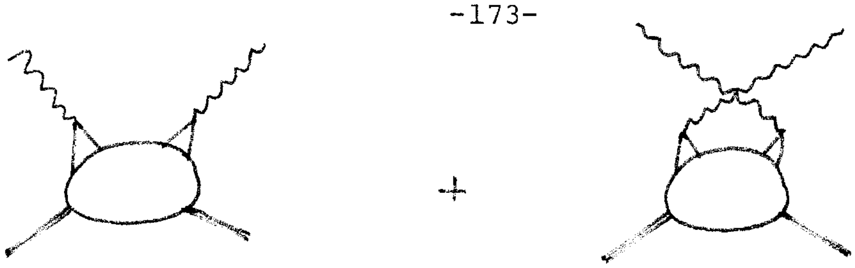
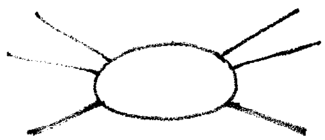


Figure 3

It is argued that these are no more than particular contractions of parton-proton 6-point functions:



which are <u>assumed</u> to be strong interaction amplitudes and by virtue of this to possess no fixed poles. In the coordinate space approach an analogous assumption must be made. In any case the consequence that fixed poles at $J = 0$ have polynomial residues is testable (via the sum rule of Section 10.5). The reader may either accept polynomial residues as an assumption (with the above motivation) and proceed to Section 10.5 or read on to discover the origins of $J = 0$ F.P. in coordinate space.

To proceed in coordinate space[9] it is necessary first to review the origins of Regge behavior[11] in the light-cone formalism. A treatment of the physical Compton amplitude, $T_{\mu\nu}$, involves too much unnecessary tensor algebra. We shall study instead a hypothetical amplitude $T(q^2,\nu)$ with the following properties:

1. Up to polynomials, $T(q^2,\nu)$ is the Fourier transform of a scalar function of x^2 and $x\cdot p$ with the boundary conditions appropriate to a time ordered product:

$$T(q^2,\nu) \equiv i\int d^4x\, e^{iq\cdot x}\, C(x^2-i\epsilon,\, x\cdot p)$$
$$+ \text{ polynomials} \qquad (10.4)$$

2. $T(q^2,\nu)$ Reggeizes like $\nu^{\alpha-1}$ at fixed q^2.
3. $W(q^2,\nu) = \frac{1}{2\pi}$ Im $T(q^2,\nu)$ is crossing odd:
   $W(q^2,\nu) = -W(q^2,-\nu)$.
4. $T(q^2,\nu)$ is analytic in the cut ν-plane with cuts from $-\infty$ to $q^2/2$ and from $-q^2/2$ to ∞.

The strategy will be to show that a light-cone singularity in $C(x^2-i\varepsilon,\, x\cdot p)$ generates a fixed pole at given J-value dependent on the strength of singularity. The way we have concocted $C(x^2-i\varepsilon,\, x\cdot p)$, a free-field singularity $(\frac{1}{x^2-i\varepsilon})$ will generate a J = 0 fixed pole. While other singularities may generate moving poles at any J-value or fixed poles at other $J \neq 0$, only the $\frac{1}{x^2-i\varepsilon}$ term generates a J = 0 fixed pole and it has polynomial (in fact, constant) residue. It so happens that this is also the case for (eg.) T_2 in electroproduction (or neutrinoproduction): a free field singularity in $\langle P|T(J_\mu(x)\; J_\nu(0)|P\rangle$ generates a J = 0 F.P. with polynomial residue, while all the other terms generate only moving poles or F.P. at $J < 0$.

The analysis is not without assumptions. Most important is the assumption that as t changes from zero any quantity which can be t-dependent, is. This is crucial and is violated if there are J = 0 fixed poles in photoproduction (or electroproduction) of specific final states (eg. $\gamma p \to \rho p$).

To begin, consider the Regge behavior which might arise from a leading free field singularity in $C(x^2-i\varepsilon,\ x\cdot p)$:

$$C_1(x^2-i\epsilon,\ x\cdot p) = -\frac{1}{\pi^2}\,\frac{f(x\cdot p)}{x^2-i\epsilon}$$

which contributes to $T(q^2,\nu)$ as follows:*

$$T_1(q^2,\nu) = 4\int_{-1}^{1} d\lambda\ \frac{f(\lambda)}{(2\nu\lambda - Q^2 + \lambda^2 M^2 - i\epsilon)} \tag{10.5}$$

where

$$f(x\cdot p) = \int_{-1}^{1} d\lambda\, e^{i\lambda x\cdot p} f(\lambda)$$

Eq. (10.5) establishes that $\nu W(q^2,\nu)$ scales. As $\xi = -q^2/2\nu$ goes to zero Regge behavior in $W(q^2,\nu)$ is obtained if $f(\lambda)$ is proportional to $\lambda^{-\alpha}$ for small λ:

$$\lim_{\nu\to\infty} W(q^2,\nu) = \frac{1}{\nu}\,\xi^{-\alpha} = \frac{1}{2}(Q^2)^{-\alpha}(2\nu)^{\alpha-1}$$

We will therefore parameterize Regge behavior in $T_1(q^2,\nu)$ by:

$$f^R(\lambda) = \sum_\alpha \beta(\alpha)\,|\lambda|^{-\alpha}\epsilon(\lambda) \tag{10.6}$$

The subscript on T indicates this particular contribution where the absolute value and $\varepsilon(\lambda)$ are required by crossing. Eqs. (10.5) and (10.6) are not entirely correct as they stand. $T_1^R(q^2,\nu)$ must be analytic in ν. Im $T_1^R(q^2,\nu)$ generated by Eqs. (10.5) and (10.6) is zero for $|\xi| > 1$ and equals to $2\pi \sum_\alpha \beta(\alpha)|\xi|^{-\alpha}\varepsilon(\xi)$ for $|\xi| < 1$ - which is clearly not analytic. To remedy this we define:

$$T_1{}^R(q^2,\nu) \equiv 4\int_{-\infty}^{\infty} \frac{d\lambda \sum_\alpha \beta(\alpha)|\lambda|^{-\alpha}\epsilon(\lambda)}{2\nu\lambda - Q^2 + \lambda^2 M^2 - i\epsilon} \tag{10.7}$$

We will need only to consider trajectories with non-negative intercept ($\alpha > 0$). These may be isolated from Eq. (10.7) regardless of q^2:

$$\lim_{\substack{\nu\to\infty \\ q^2 \text{ fixed}}} T_1{}^R(q^2,\nu) = \frac{4\pi}{Q^2}\sum_{\alpha>0} \beta(\alpha)\left(\frac{1+e^{i\pi\alpha}}{\sin\pi\alpha}\right)\xi^{-\alpha+1} \tag{10.8}$$

Note the following:

1) An $\alpha = 0$ term in Eq. (10.7) would contribute a logarithm

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* This is the familiar problem that Regge parametrization of the form ν^α does not incorporate the physical threshold. Since only asymptotic ($\nu \to \infty$) behavior is of interest, this is unimportant. Alternatively the parametrization $(\nu-\nu_0)^\alpha$ (or in our case $(\omega-1)^\alpha$) may be used. This form has the correct threshold and is analytic. Use of this Regge parametrization is developed in an appendix to Ref. 4.

to Eq. (10.8), <u>not a real constant</u>. For convenience we ignore such a term. It presents only technical problems which are treated in Ref. 4.

2) Eq. (10.7) generates <u>no purely real constant term</u>.

To proceed we subtract from Eq. (10.5) its leading Regge terms (all $\alpha > 0$)

$$T_1(q^2,\nu) - T_1^R(q^2,\nu) = 4\int_{-\infty}^{\infty} \frac{d\lambda\,\left[f(\lambda) - \sum_{\alpha>0} \beta(\alpha)|\lambda|^{-\alpha}\epsilon(\lambda)\right]}{2\nu\lambda - Q^2 + \lambda^2 M^2 - i\epsilon} \tag{10.9}$$

It is understood that $f(\lambda)$ vanishes outside $[-1,1]$. All terms which vanish less rapidly than $1/\nu$ have been explicitly subtracted. Therefore the $\nu \to \infty$ limit of Eq. (10.9) exposes the $J = 0$ F.P.:*

$$\lim_{\substack{\nu\to\infty \\ q^2 \text{ fixed}}} \left(T_1(q^2,\nu) - T_1^R(q^2,\nu)\right) = \frac{2}{\nu}\int_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\left[f(\lambda) - \sum_{\alpha>0}\beta(\alpha)|\lambda|^{-\alpha}\epsilon(\lambda)\right]$$

$$\equiv \frac{1}{\nu} T^{FP}(q^2) = 2\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\tilde{f}(\lambda) - 4\sum_{\alpha>0}\frac{\beta(\alpha)}{\alpha} \tag{10.10}$$

where

$$\tilde{f}(\lambda) = f(\lambda) - \sum_{\alpha>0}\beta(\alpha)|\lambda|^{-\alpha}\epsilon(\lambda)$$

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* Remember our assumption that $T(q^2,\nu)$ Reggeizes like $\nu^{\alpha-1}$, so a $J = 0$ F.P. is a term proportional to $1/\nu$ at large ν.

is explicitly finite since all terms which would have produced divergences in the integral have been subtracted off. Also it is manifestly a (constant) polynomial in q^2.

As for t-dependence, we shall assume that as t departs from zero everything which may develop a t-dependence does:

$$f(\lambda) \to f(\lambda,t)$$

$$\alpha \to \alpha(t)$$

$$\beta(\alpha) \to \beta(\alpha,t)$$

However the light-cone singularity is a c-number so its strength is independent of t. The location of the fixed pole at $J = 0$ was determined by the strength of the singularity (we shall make that clearer momentarily) and is consequently independent of t.*

So far we have discussed only the contribution of a leading free-field singularity. It remains to show that <u>non-leading singularities generate no $J = 0$ F.P.</u> To do this consider a specific example: a singularity one power weaker:

$$[C(x^2-i\varepsilon,\ x\cdot p)]_{\text{non-leading}} = \log\ (x^2-i\varepsilon)\ h(x\cdot p)$$

whose contribution to $T(q^2,\nu)$ we label T_2:

$$T_2\,(q^2,\nu) \propto \int_{-1}^{1} \frac{d\lambda\ h(\lambda)}{(2\nu\lambda - Q^2 + \lambda^2 M^2 - i\epsilon)^2}$$

Consider a term of the form $|\lambda|^{-\alpha}\varepsilon(\lambda)$ in $h(\lambda)$ [Again the limits of the λ-integral must be extended to $\pm\infty$]. Such a term contributes

*Note, of course, the residue $T^{FP}(q^2)$ may develop a t-dependence although the location is fixed at $J = 0$.

to $T_2(q^2,\nu)$ as follows in the Regge limit:

$$\lim_{\substack{\nu \to \infty \\ q^2 \text{ fixed}}} T_2^R(q^2,\nu) \propto \frac{1}{Q^2\nu}\,\xi^{-\alpha}$$

Non-leading singularities are damped only by factors of Q^2 not ν in the Regge limit.*

Now a term of the form $\beta(0)|\lambda|^0\epsilon(\lambda)$ in $h(\lambda)$ will contribute a real constant to $T(q^2,\nu)$ at large ν. According to our assumption (everything develops a t-dependence if possible) this is a moving pole:

$$|\lambda|^0 = |\lambda|^{\alpha(0)=0} \longrightarrow |\lambda|^{\alpha(t)}$$

and is of no concern to us. If we subtract off all Regge contributions with $\alpha > -1$ then a $J = -1$ F.P. emerges:

$$\lim_{\substack{\nu \to \infty \\ q^2 \text{ fixed}}} \left(T_2(q^2,\nu) - T_2^R(q^2,\nu)\right) \propto \frac{1}{\nu^2}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2}\left(h(\lambda) - \sum_{\alpha>-1}\beta(\alpha)|\lambda|^{-\alpha}\epsilon(\lambda)\right)$$

but no $J = 0$ F.P. is found. (Actually the residue of this $J = -1$ F.P. is zero by crossing $h(\lambda) = -h(-\lambda)$ though this is not the case for other singularities).

In fact: The location of the fixed pole generated by a given light-cone singularity is determined by the strength of

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* Parenthetically this is explicit evidence that the Regge limit is not light cone dominated.

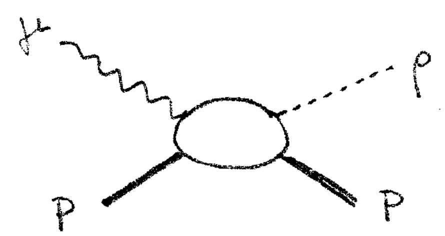
the singularity. A free field singularity generates a $J = 0$ F.P. The pole is fixed because the singularity is a c-number.

This completes the argument that $J = 0$ F.P. residues are simple polynomials in q^2. Generalizations to vector currents and non-integer dimensional singularities are found in Refs. 10 and 11. Several warnings are in order:

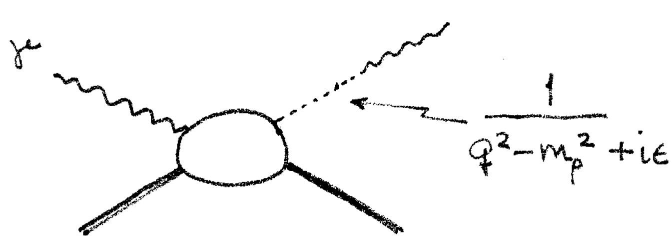
1. Such poles are very difficult to isolate experimentally since moving poles, cuts, etc. may have intercept arbitrarily close to or even fortuitously at $J = 0$.

2. We have assumed that it makes sense to expand the current correlation function about the light cone to all orders. Perhaps some non-uniformity in this expansion generates an $\alpha = 0$ fixed pole in lower order and foils the proof.

3. $J = 0$ fixed poles are not excluded from photoproduction, e.g.:



If we now append a photon to the rho, a non-polynomial residue F.P. would seem to occur in Compton scattering:



It is instructive to see what goes wrong with our argument if there are fixed poles in photoproduction. The fixed pole term is of the form*

$$T^{F.P.}_{\substack{\text{NON}\\ \text{POLYNOMIAL}}}(q^2,\nu) = \frac{A}{\nu}\,\frac{1}{q^2 - m_\rho^2 + i\epsilon}$$

In coordinate space this yields a string of light-cone singularities multiplying functions of $x\cdot p$ which are t-independent. We had assumed such terms (which are matrix elements of bilocal operators) to be t-dependent. The presence of fixed poles in photoproduction will generally destroy the proof that F.P. in Compton scattering have polynomial residues.

10.5 Application I. CCNRR Sum Rule

This sum rule is simply the precise statement of the fact that the residues of fixed poles in virtual Compton scattering are polynomials in q^2. The derivation followed here was developed with Llewellyn Smith.[12] Parton model[8] and light-cone[9] derivations are also instructive. To begin we must write a kinematic constraint free decomposition to $T_{\mu\nu}$. The usual decomposition is not satisfactory: as noted in Lecture 2, if $T_{\mu\nu}$ is to remain finite at $q^2 = 0$ T_1 and T_2 are constrained to obey

$$\lim_{q^2\to 0} \frac{1}{q^2} T_2(q^2,\nu) \sim \text{finite}$$

$$\lim_{q^2\to 0} \frac{1}{q^2}\left(T_1(q^2,\nu) + \frac{\nu^2}{M^2 q^2} T_2(q^2,\nu)\right) \sim \text{finite}$$

*We continue to pretend that the amplitude in question Reggeizes like $\nu^{\alpha-1}$.

A constraint free decomposition is:

$$T_{\mu\nu} = -\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) t_L(q^2,\nu) + \frac{1}{M^2}\left(P_\mu P_\nu q^2 - \nu(P_\mu q_\nu + P_\nu q_\mu) + \nu^2 g_{\mu\nu}\right) t_2(q^2,\nu) \qquad (10.11)$$

$$T_2(q^2,\nu) = q^2 t_2(q^2,\nu)$$
$$T_1(q^2,\nu) + \frac{\nu^2}{M^2 q^2} T_2(q^2,\nu) = q^2 t_L(q^2,\nu) \qquad (10.12)$$

A rerun of the derivation of the last section with appropriate free-field light-cone singularities establishes that t_L and t_2 have polynomial residue J = 0 fixed poles.

Consider $t_2(q^2,\nu)$:

$$t_2(q^2,\nu) = \frac{4}{q^2}\int_{-q^2/2}^{\infty} \frac{d\nu'\,\nu' W_2(q^2,\nu')}{\nu'^2-\nu^2} \qquad (10.13)$$

The J = 0 fixed pole is a term proportional to $1/\nu^2$ as $\nu \to \infty$. $t_2(q^2,\nu)$ contains all sorts of leading ($\alpha > 0$) Regge terms which mask the fixed pole. We will assume these to be only simple poles with $0 < \alpha \leq 1$ - cuts and a possible $\alpha = 0$ term in W_2 are discussed in detail in Ref. 4. To remove leading Regge terms:

$$t_2^R(q^2,\nu) = \frac{2\pi}{q^2} \sum_{\alpha>0}^{1} \gamma(\alpha,q^2) \frac{\nu^{\alpha-2} + (-\nu)^{\alpha-2}}{\sin \pi\alpha} ,$$

write the dispersion relation

$$t_2^R(q^2,\nu) = \frac{4}{q^2} \int_0^\infty \frac{\nu' d\nu'}{\nu'^2-\nu^2} \sum_{\alpha>0}^{1} \gamma(\alpha,q^2)\, \nu'^{\,\alpha-2}$$

(10.14)

Note, the threshold is $\nu = 0$ rather than the physical threshold: $-q^2/2$. Now subtract Eqs. (10.13) and (10.14):

$$\begin{aligned} t_2(q^2,\nu) - t_2^R(q^2,\nu) &\equiv \tilde{t}_2(q^2,\nu) \\ &= \frac{4}{q^2} \int_0^\infty \frac{d\nu'}{\nu'^2-\nu^2} \left[W_2(q^2,\nu') - \sum_{\alpha>0}^{1} \gamma(\alpha,q^2)\, \nu'^{\,\alpha-2} \right] \\ &= \frac{4}{q^2} \int_0^\infty \frac{\nu' d\nu'}{\nu'^2-\nu^2} \tilde{W}_2(q^2,\nu') \end{aligned}$$

The leading term in $\tilde{t}_2(q^2,\nu)$ as $\nu \to \infty$ is the fixed pole.

$$\begin{aligned} \lim_{\nu\to\infty} \tilde{t}_2(q^2,\nu) &\equiv \frac{M^2}{\nu^2} C_2(q^2) \\ &= -\frac{4}{\nu^2 q^2} \int_0^\infty \nu' d\nu'\, \tilde{W}_2(q^2,\nu') \end{aligned}$$

(10.15)

The integral over $\tilde{W}_2$ is convergent since all terms which vanish slower than $1/\nu'^2$ have been subtracted out. Eq. (10.15) may be evaluated in the Bjorken limit:

$$\lim_{q^2 \to -\infty} C_2(q^2) = 2\int_0^\infty d\omega\, \tilde{F}_2(\omega) \tag{10.16}$$

or at $q^2 = 0$

$$C_2(0) = 2\left[1 + \frac{1}{2\pi^2\alpha}\int_0^\infty d\nu\, \tilde{\sigma}_{TOT}(\nu)\right] \tag{10.17}$$

To obtain Eq. (10.17) from Eq. (10.15) note from Lecture 2 that:

$$\lim_{q^2 \to 0} \frac{-\nu W_2(q^2,\nu)}{M^2 q^2} = \frac{1}{4\pi^2\alpha}\sigma_{TOT}(\nu)$$

where σ_{TOT} is the total photoabsorption cross section for real photons. The "1" in Eq. (10.17) is from the Born term (Thompson limit) which has been separated out from under the integral. $\tilde{\sigma}$ is defined in analogy to $\tilde{W}_2$:

$$\tilde{\sigma}_{TOT}(\nu) \equiv \sigma_{TOT}(\nu) - \sum_{\alpha>0}^{1} \beta(\alpha)\,\nu^{\alpha-1}$$

$F_2(\omega)$ is defined from $W_2(q^2,\nu)$

$$\tilde{F}_2(\omega) \equiv \lim_{\nu, q^2 \to \infty} \frac{\nu}{M^2}\tilde{W}_2(q^2,\nu)$$

The final step is the trivial observation that the only polynomial in q^2 which is constant at $q^2 = 0$ and at $q^2 = \infty$ is a constant, hence $C_2(0) = C_2(\infty)$ or:

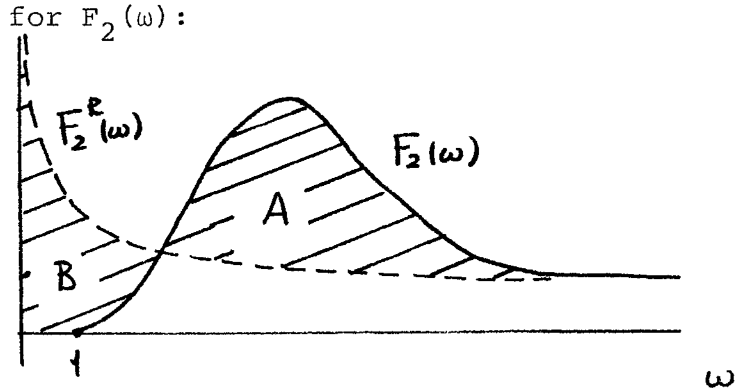
$$\int_0^\infty d\omega\, \tilde{F}_2(\omega) = 1 + \frac{1}{2\pi^2\alpha}\int_0^\infty d\nu\, \tilde{\sigma}_{TOT}(\nu) \tag{10.18}$$

which is the CCNRR sum rule.

Damashek and Gilman[13] and Dominguez, Suaya and Ferro-Fontan[14] have evaluated the right-hand side of Eq. (10.18) from photo-production data.* They find within rather wide limits, $C_2(0)$ to be consistent with 1, i.e.

$$\int_0^\infty d\nu\, \tilde{\sigma}_{TOT}(\nu) = 0$$

From the sum rule we are led to expect the behavior pictured below for $F_2(\omega)$:



*These phenomenological analyses are open to various criticisms. Most important we have shown that moving poles and/or cuts may have contributions arbitrarily close to or even at $J = 0$. One must assume that the operation of subtracting off known trajectories with $\alpha > 0$ isolates the $J = 0$ fixed pole. Additional ambiguities regarding the intercept of Regge trajectories and the onset of Regge behavior are discussed in Ref's. 13 and 14.

$$\text{Area A} - \text{Area B} \cong 1$$

In particular $F_2(\omega)$ must fall substantially below its maximum before the onset of Regge behavior. Close and Gunion[15] have verified that current data are not in contradition to this - neither however is there any substantial support for the value 1. A significant test of the sum rule awaits NAL data at high ω.

10.6 Application II Schwinger Term Sum Rule

Combining BJL techniques with the assumption of polynomial fixed pole residues, a sum rule may be derived for the operator Schwinger term.[16] This sum rule is rather important in several theoretical applications including electromagnetic mass differences.[17] It also provides a superb laboratory for studying the validity of canonical manipulations in perturbation theory.[18,19,4] Our analysis again follows Ref. 4.

Schwinger terms are non-canonical derivative terms in current commutators. The operator Schwinger term in the commutator of electromagnetic currents is defined by:

$$\langle P | [J_0(x), J_i(0)] | P \rangle \Big|_{x_0=0} = i\partial_i \delta^3(\vec{x})\, S$$

We may isolate it in electroproduction by taking the BJL limit of T_{0i} with q_i fixed but not zero:

$$\lim_{BJL} T_{0i} = -1/q_0 \int d^3x\, e^{iq\cdot x} \langle P|[J_0(x), J_i(0)]|P\rangle\Big|_{x_0=0} + O(1/q_0^2) + \text{Polynomials}$$
$$= -q_i/q_0\, S + O(1/q_0^2) + \text{Polynomials}$$

Using our kinematic constraint free decomposition of $T_{\mu\nu}$ we obtain (taking $\vec{P} = 0$):

$$\lim_{BJL} q_0 q_i \left(t_L(q^2,\nu) - t_2(q^2,\nu)\right) = -\frac{q_i}{q_0} S + O\left(\frac{1}{q_0^2}\right) + q_i P(q^2,\nu)$$

where $P(q^2,\nu)$ is some polynomial in q^2 and ν which is to be identified with terms on the left-hand side which do not vanish as $q_0 \to i\infty$. Identifying coefficients of $1/q_0$:

$$S = -\lim_{BJL} q^2 \left(\bar{t}_L(q^2,\nu) - \bar{t}_2(q^2,\nu)\right) \tag{10.19}$$

where $\bar{t}_{L,2}$ are defined to have all terms subtracted off which in the limit vanish like $1/\nu$ or more slowly.

We write dispersion relations for t_L and t_2:

$$t_L(q^2,\nu) = t_L(q^2,0) - \frac{4\nu^2}{q^2}\int_{-q^2/2}^{\infty} \frac{d\nu'}{\nu'(\nu'^2-\nu^2)} W_L(q^2,\nu') + \sum_{n=1}^{\infty} f_n(q^2)\nu^{2n}$$

$$t_2(q^2,\nu) = \frac{4}{q^2}\int_{-q^2/2}^{\infty} \frac{\nu' d\nu'}{\nu'^2-\nu^2} W_2(q^2,\nu') + \sum_{n=0}^{\infty} g_n(q^2)\nu^{2n} \tag{10.20}$$

Where $W_L(q^2,\nu) = -q^2/2\pi \; \mathrm{Im}\; t_L(q^2,\nu)$.

In both dispersion relations we allow a real polynomial; in that for t_L we explicitly use the first term in the polynomial to subtract the dispersion relation. Changing variables from ν to ω

$$t_L(q^2,\omega) = t_L(q^2,0) - \frac{4\omega^2}{q^2}\int_1^\infty \frac{d\omega'}{\omega'(\omega'^2-\omega^2)} W_L(q^2,\omega') + \sum_{n=1}^{\infty} f_n(q^2)\left(\frac{q^2\omega}{2}\right)^{2n}$$

$$t_2(q^2,\omega) = -\frac{8M^2}{q^4}\int_1^\infty \frac{d\omega'}{\omega'^2-\omega^2} F_2(q^2,\omega') + \sum_{n=0}^{\infty} g_n(q^2)\left(\frac{q^2\omega}{2}\right)^{2n} \tag{10.21}$$

In the BJL limit $\nu, -q^2 \to \infty$ but $\omega \to 0$. From Eqs. (10.21) we isolate the finite terms for Eq. (10.19):

$$\lim_{BJL} q^2\left(\bar{t}_L(q^2,\omega) - \bar{t}_2(q^2,\omega)\right) = \lim_{q^2 \to -\infty} q^2\, \bar{t}_L(q^2,0) = -S \tag{10.22}$$

The Schwinger term is the $q^2 \to -\infty$ limit of a subtraction constant. <u>Any other result implicitly assumes the absence of the subtraction constant or otherwise disposes of it</u>.

We shall dispose of it and obtain a useful sum rule from the assumption of polynomial residues. We perform the same analysis on $t_L(q^2,\nu)$ as we did on $t_2(q^2,\nu)$ when obtaining the CCNRR sum rule. That is we isolate the fixed pole by taking

ν to infinity in a finite energy sum rule:

$$C_L(q^2) = t_L(q^2,0) + \frac{4}{q^2}\int_0^\infty \frac{d\nu}{\nu}\,\tilde{W}_L(q^2,\nu)$$

(10.23)

and $C_L(q^2)$ is to be a polynomial in q^2. We require $\lim_{q^2\to-\infty} q^2\bar{t}_L(q^2,0)$ (remember the bar indicates the removal of all parts which fall slower than $1/\nu$). Since $C_L(q^2)$ is a polynomial in q^2 it makes no contribution to $\bar{t}_L(q^2,\nu)$ and we find

$$\lim_{q^2\to-\infty} q^2\bar{t}_L(q^2,0) = -4\lim_{q^2\to-\infty}\int_0^\infty \frac{d\nu}{\nu}\,\tilde{W}_L(q^2,\nu)$$

$$= -S$$

(10.24)

which is the sum rule. Now if the polynomial residue assumption were inoperative[20], a $1/q^2$ term in $C_L(q^2)$ would contribute to the sum rule. We call such a term $a_L(q^2)$. In its most general form the sum rule reads:

$$S = \lim_{q^2\to-\infty}\left[-a_L(q^2) + 4\int_0^\infty \frac{d\nu}{\nu}\,\tilde{W}_L(q^2,\nu)\right]$$

(10.25)

Note that S is never infinite unless the $q^2 \to -\infty$ limit does not exist, despite frequent assertions to the contrary.

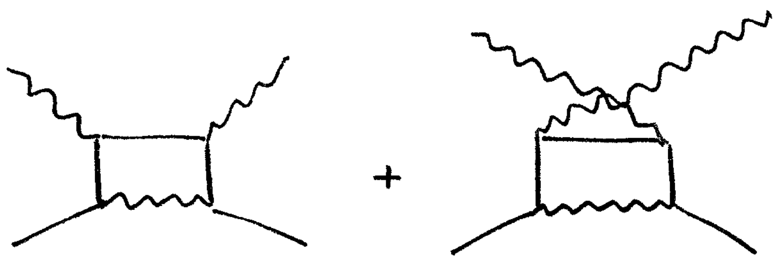
10.7 Breakdown of Canonical Sum Rules in Perturbation Theory

Canonically we expect Eq. (10.25) to be fulfilled with $a_L(q^2) = 0$ and $\lim_{q^2 \to -\infty} W_L(q^2,\omega) = F_L(\omega)$:

$$S = \int_0^\infty \frac{d\omega}{\omega} 4 \tilde{F}_L(\omega)$$

so that $S = 0$ in spin-1/2 theories where $F_L(\omega) = 0$ by the Callan-Gross relation.

In second order perturbation theory this does not happen. Consider first a spin-1/2 quark vector-gluon model.[18,19] Since Eq. (10.25) should be valid Feynman diagram by Feynman diagram, we may confine our attention to the box graph:



Direct calculation yields $S = 0$. However $F_L(\omega) \neq 0$ instead

$$F_L(\omega) = \frac{g^2}{8\pi^2 \omega} \Theta(\omega^2 - 1)$$

and moreover

$$\lim_{\nu \to \infty} t_L(q^2,\nu) = \frac{g^2}{\pi^2} \int_0^\infty \frac{dx\, x^2 (1-x)}{q^2 x(1-x) - M^2}$$

This is a non-polynomial fixed pole residue:

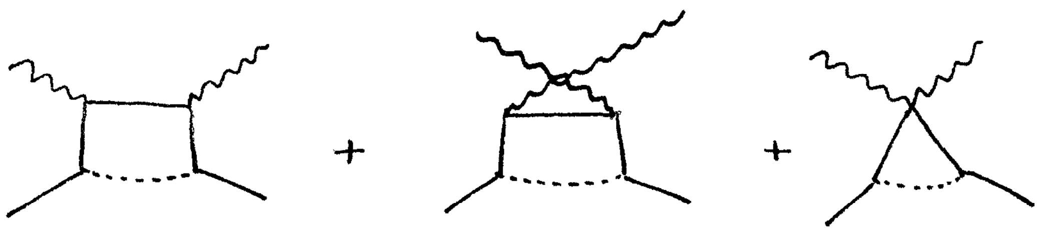
$$\lim_{q^2 \to -\infty} a(q^2) = \frac{1}{2} g^2/\pi^2$$

So the sum rule reads:

$$0 = -g^2/2\pi^2 + g^2/2\pi^2 \int_1^\infty \frac{d\omega}{\omega^2}$$

and is valid. The breakdown of the Callan-Gross relation is accompanied by the appearance of a compensating non-polynomial fixed pole.

In a spin-0 theory[4] the following graphs



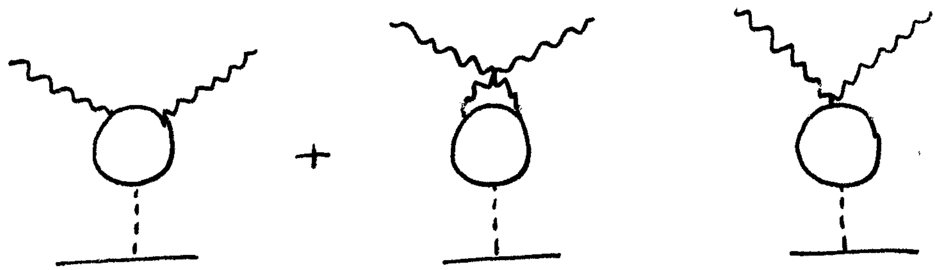
contribute to $t_L(q^2,\nu)$; the first two contribute to $F_L(\omega)$:

$$F_L(\omega) = g^2/32\pi^2 \; \frac{\omega - 1}{\omega\mu^2 + (1-\omega)^2 M^2}$$

and only the seagull graph contributes to S:

$$S = \frac{g^2}{8\pi^2} \int_0^1 dx \; \frac{1-x}{x\mu^2 + (1-x)^2 M^2}$$

Again the sum rule is verified, this time conventionally. There are however three more graphs in a spin zero theory:



These have no imaginary part and therefore do not contribute to $F_L(\omega)$. The third contributes a logarithmic divergence to the Schwinger term

$$S = \lim_{\Lambda^2 \to \infty} g^2/8\pi^2\mu^2 \log \Lambda^2/M^2$$

where Λ is some cutoff. And the three graphs contribute a non-polynomial fixed pole:

$$\lim_{q^2 \to -\infty} a(q^2) = \lim_{q^2 \to -\infty} \frac{-g^2}{8\pi^2\mu^2} \left(\beta \log \frac{\beta+1}{\beta-1} - 2\right)$$

$$\beta \equiv (1 - 4M^2/q^2)^{1/2}$$

which validates the sum rule

$$S = -\lim_{q^2 \to -\infty} a(q^2)$$

In summary, the breakdown of canonical scaling laws is accompanied by the appearance of non-polynomial residual fixed poles which conspire to satisfy the sum rule. Of course the sum rule is only experimentally useful if the fixed pole has polynomial residue since $a(q^2)$ is not directly measurable.

Finally we return to the question, raised several days ago, regarding δ-functions at $\xi = 0$ in structure functions. The problem was that light-cone derivations of sum rules involve the integration of the Fourier transform of a bilocal operator over the range $-1 \leq \xi \leq 1$. For $\xi \neq 0$ the F.T. is proportional to an (observable) structure function. A useful sum rule is obtained only if distributions at $\xi = 0$ are excluded by fiat. Now we can identify these distributions with the asymptotic limit of non-polynomial residue fixed poles. Again the Schwinger term sum rule provides the test case. Jackiw, van Royen and West[3] derived the S.T. sum rule by purely coordinate space techniques and found (assuming no Regge terms with $\alpha \geq 0$):

$$S = 4\int_0^1 \frac{d\xi}{\xi} F_L(\xi) \tag{10.26}$$

This seems to be violated in (e.g.) second order perturbation theory of spin-1/2 particles: $S = 0$; $\int_0^1 \frac{d\xi}{\xi} F_L(\xi) = g^2/8\pi^2$ However, Zee[18] showed that a careful calculation of $F_L(\xi)$ in perturbation theory uncovers an additional term:

$$F_L(\xi) = \frac{g^2}{8\pi^2}\left(\xi - 2\xi\,\delta(\xi)\right) \tag{10.27}$$

The δ-function is not present at finite q^2 and ν and its rather arcane origin in the non-uniformity of the Bjorken limit in perturbation theory need not concern us. Clearly Eq. (10.27) now satisfies the sum rule, Eq. (10.26).

This treatment is completely equivalent to our earlier (BJL + finite energy sum rule) formulation. In Eq. (10.25) the integral is to be evaluated <u>before</u> letting $q^2 \to -\infty$ so that any δ-function which materializes in the Bjorken limit will not be encountered. The non-polynomial fixed pole then cancels the integral. If, on the other hand, one <u>insists</u> on formulating the sum rule as Eq. (10.26) it is necessary to treat $\frac{1}{\xi}F_L(\xi)$ (or its regulated analog) as a distribution with singularities at $\xi = 0$ which reflect the existence of non-polynomial fixed poles.

Finally let me note that there are many intriguing and perhaps useful consequences of the techniques developed in this lecture. The reader is directed to References 4, 12 and especially 21 for further applications.

REFERENCES

1. H.D.I. Abarbanel, M. L. Goldberger and S. B. Treiman, Phys. Rev. Letters 22, 500 (1969).
2. J. M. Cornwall, D. Corrigan and R. Norton, Phys. Rev. Letters 24, 1141 (1970); R. Rajaraman and G. Rajesakhran, Phys. Rev. D3, 266 (1971) and Erratum D4, 2940 (1971).
3. R. Jackiw, R. van Royen and G. B. West, Phys. Rev. D2, 2473 (1970).
4. D. J. Broadhurst, J. F. Gunion and R. L. Jaffe, SLAC-PUB-1197 (to be published in Annals of Physics).
5. See, for example, E. J. Squires, Complex Angular Momenta and Particle Physics (W.A. Benjamin, N.Y., 1966).
6. V. D. Mur, Zh. Eksperim. i Teor. Fiz. 44, 2173 (1963); 45, 1051 (1963) (English transl.:Sov. Phys. - JETP 17, 1458 (1963); 18, 727 (1964)); H. K. Shepard, Phys. Rev. 159, 1331 (1967); H.D.I. Abarbanel and S. Nussinov, ibid. 158, 1462 (1967); A. H. Mueller and T. L. Trueman, ibid., 160, 1296, 1306 (1967); H.D.I. Abarbanel, F. E. Low, I.J. Muzinich, S. Nussinov, and J. H. Schwarz, ibid., 160, 1329 (1967).
7. M. Creutz, S.D. Drell and E. A. Paschos, Phys. Rev. 178, 2300 (1969).
8. S. J. Brodsky, F. E. Close and J. F. Gunion, Phys. Rev. D5. 1384 (1972); P. V. Landshoff and J. C. Polkinghorne, Phys. Rev. D5, 2050 (1972). The first arguments for polynomial residues were made by T.P. Cheng and W.-K. Tung, Phys.

Rev. Letters 24, 851 (1970).

9. M. Bander, Phys. Rev. D5, 3274 (1972); G. Mack, Phys. Lett. 35B, 234 (1971). Our analysis is developed from Ref. 10. See also Ref. 4.

10. Y. Frishman, "Light Cone and Short Distance Singularities" in Proceedings of the XVI International Conference on High Energy Physics, edited by J.D. Jackson and A. Roberts (NAL, Batavia, 1973) Vol. 4, pg. 119.

11. Y. Frishman, Annals of Physics 66, 373 (1971). See also R. L. Jaffe, Annals of Physics 75, 545 (1973).

12. R. L. Jaffe and C. H. Llewellyn Smith, Phys. Rev. D7, 2506 (1973).

13. M. Damashek and F. J. Gilman, Phys. Rev. D1, 1319 (1970).

14. C. A. Dominguez, C. Ferro-Fontan and R. Suaya, Phys. Letters 31B, 365 (1970).

15. F. E. Close and J. F. Gunion, Phys. Rev. D4, 742, 1576 (1971).

16. See for example Ref. 4.

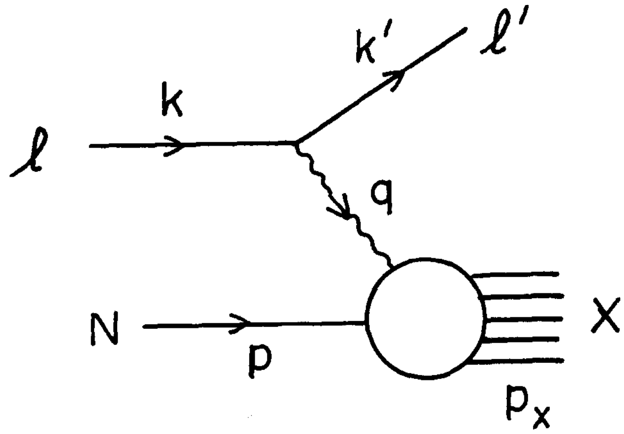
17. R. Jackiw and H. J. Schnitzer, Phys. Rev. D5, 2008 (1972); J. F. Gunion MIT preprint MIT-CTP-333 (to be published in Phys. Rev.).

18. A. Zee, Phys. Rev. D3, 2432 (1971).

19. D. Corrigan, Phys. Rev. D4, 1053 (1971).

20. R. Ziegler, quoted in New York Times, March 30, 1973.

21. D. J. Broadhurst, J. F. Gunion and R. L. Jaffe, MIT preprint, MIT-CTP-339 (to be published in Physical Review).



k′
ℓ′
k
ℓ
q
N
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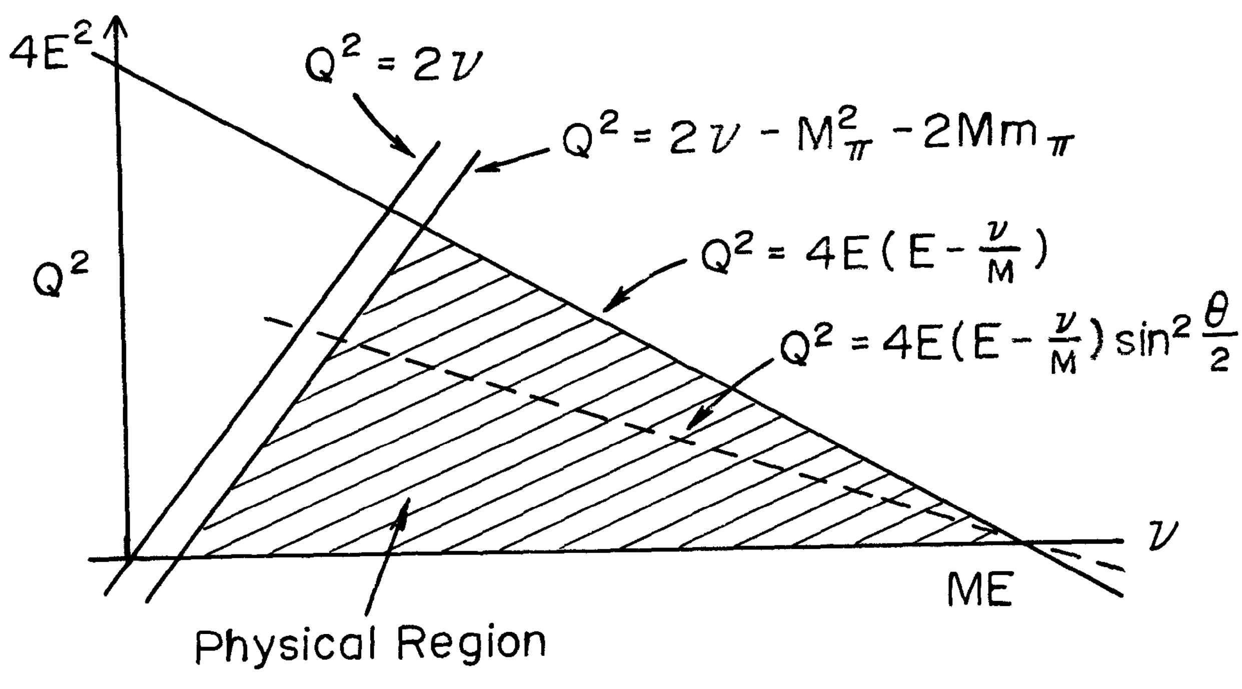
X
p_x

Fig. 1



$4E^2$
$Q^2 = 2\nu$
$Q^2 = 2\nu - M_\pi^2 - 2Mm_\pi$
$Q^2 = 4E(E - \frac{\nu}{M})$
Q^2
$Q^2 = 4E(E - \frac{\nu}{M}) \sin^2 \frac{\theta}{2}$
ν
ME
Physical Region

Fig. 2