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SPIN MOTION IN e^+e^- STORAGE RINGS

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ABSTRACT

General properties of the spin motion in e^+e^- storage rings are studied in order to determine the conditions where radiative beam polarization is likely to occur. A general first-order theory of depolarization is developed and applied to specific examples of nonresonant depolarization in storage rings. It is found that under many practical conditions radiative beam polarization should occur and therefore the beam polarization will be a significant parameter in high energy e^+e^- colliding beam experiments.

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In view of the interest in the possibilities of significant beam polarization occurring in modern high-energy electron-positron storage rings, it is useful to review some facts concerning spin motion in storage rings, with emphasis on depolarization effects. A general method for solving equations of spin motion is developed and applied to the approximate treatment of depolarization in several specific storage ring examples. A convenient method for handling quantum fluctuations necessary for understanding depolarization away from spin resonances is also developed. Some of the results that follow have appeared in the published literature.[1,2,3,4,8]

As is well known,[1,5] the process of synchrotron radiation in an e^+e^- storage ring leads to transversely polarized beams where the electrons (positrons) are polarized antiparallel (parallel) to the guide magnetic field. The magnitude of the polarization builds up in time, according to:

$$P(t) = P_0 (1 - e^{-t/\tau})$$

$$P_0 = \frac{8\sqrt{3}}{15} \left(\frac{\tau_{dep}}{\tau_{pol} + \tau_{dep}}\right) \approx 0.924 \left(\frac{\tau_{dep}}{\tau_{pol} + \tau_{dep}}\right) \tag{1}$$

$$\frac{1}{\tau} = \frac{1}{\tau_{pol}} + \frac{1}{\tau_{dep}} \quad .$$

τ_{dep} is the "depolarization" time constant and is the primary subject of this note. τ_{pol} is the polarization buildup time and is given by:

$$\tau_{pol} \simeq 98 \text{ seconds} \times \frac{R_0^3(\text{meters})}{E^5(\text{GeV})} \times \frac{R}{R_0} \quad . \tag{2}$$

R_0 is the magnetic bending radius of the storage ring, R is the average radius, and E is the single-beam energy. For example, in the case of the Stanford

3-GeV e^+e^- storage ring SPEAR, τ_{pol} is numerically:

$$\tau_{SPEAR} \approx \frac{165 \text{ hours}}{E^5 \text{ (GeV)}} \quad .$$

The general form of the equation of spin motion of a particle in an electromagnetic field, neglecting damping terms,[6] is

$$\bar{S}' = \bar{\Omega} \times \bar{S} \ , \tag{3}$$

where ' denotes differentiation with respect to the azimuthal coordinate $\theta = \omega_0 t$; ω_0 is a constant equal to the orbital rotation frequency in the case of closed-orbital motion, as in a storage ring; $\bar{\Omega}$ is a function of the laboratory electric and magnetic fields and the state of motion ($\bar{v}$, $\dot{\bar{v}}$) of the particle. In the case of a storage ring, $\bar{\Omega}$ can be split into a periodic part $\bar{\Omega}_0$ and a part due to motion away from the equilibrium orbit $\bar{\omega}$. In this note, $\bar{\omega}$ is treated as a small perturbation to $\bar{\Omega}_0$ and only first-order effects are calculated. The important point, however, is that $\bar{\Omega}_0$ may describe a very general guide field composed of, say, uniform field-bending magnets, quadrupoles, electrostatic deflection plates, solenoids, etc. Yet, due to the fact that there exists a closed-equilibrium orbit, $\bar{\Omega}_0$ is periodic, i.e.,

$$\bar{\Omega}_0(\theta + 2\pi) = \bar{\Omega}_0(\theta) \quad . \tag{4}$$

An important rule[2] which follows directly from the general equation of spin motion [Eq. (3)] is that the scalar product of any two solutions to Eq. (3) is a constant of the motion:

$$\frac{d}{dt}\left[\bar{S}_a(\theta) \cdot \bar{S}_b(\theta)\right] = \omega_0(\bar{S}_a \cdot \bar{S}_b)' = 0 \quad . \tag{5}$$

The general solution for the spin motion along the equilibrium orbit can be written as a linear combination of three orthonormal-basis vectors $\hat{x}_\alpha$, satisfying

$$\hat{x}'_\alpha = \bar{\Omega}_0 \times \hat{x}_\alpha \quad . \tag{6}$$

The orthonormality of the basis vectors,

$$\hat{x}_\alpha \cdot \hat{x}_\beta = \delta_{\alpha\beta} \tag{7}$$

is preserved in time by virtue of Eq. (5). The general solution to Eq. (6) is therefore written:

$$\bar{S}(\theta) = \sum_\alpha S_\alpha \hat{x}_\alpha(\theta) \quad . \tag{8}$$

Now, consider the relationship between $\bar{S}$ at some azimuth θ and its evolved spin $\tilde{S}$, one orbital period later.

$$\tilde{S}(\theta) = \sum_\beta S_\beta \hat{x}_\beta(\theta+2\pi) = \sum_\alpha S'_\alpha \hat{x}_\alpha(\theta) \equiv \Lambda \bar{S}(\theta) \quad , \tag{9}$$

$$S'_\alpha = \sum_\beta \Lambda_{\alpha\beta} S_\beta \quad .$$

It is easily shown that the matrix connecting spin vectors on successive orbital revolutions $\Lambda_{\alpha\beta}$ is given by

$$\Lambda_{\alpha\beta} = \hat{x}_\alpha(\theta) \cdot \hat{x}_\beta(\theta + 2\pi) \quad . \tag{10}$$

From the periodicity of $\bar{\Omega}_0$, it follows that $\hat{x}_\beta(\theta + 2\pi)$ is also a solution of Eq. (6) and therefore Eq. (5) applies, indicating that the matrix elements of Λ are constants of the motion. Since the transformation Λ preserves the scalar product and its matrix elements are constants in time, we arrive at the very important conclusion that at any given position on the equilibrium orbit of a storage ring the spin vector of a particle is related to the spin vector in the previous revolution by a simple rotation of constant magnitude about a constant direction in space. Thus, once established, a net polarization along this direction will not change in time. Depolarization occurs when an aperiodic term $\bar{\omega}$ is added to $\bar{\Omega}_0$ in the equation of motion. An example of such a term would be one due to the betatron motion of a particle in a storage ring, where the electric and magnetic fields on

the trajectory of the particle are not periodic because the trajectory does not close after one orbital period. Depolarization also occurs with or without the term $\bar{\omega}$ when the spin-precession frequency is equal to an integer, for in that case the matrix Λ is the identity matrix and no polarization direction can be defined.

Because Λ is a rotation operator, it can be expressed in the following exponential form:

$$\Lambda = e^{-2\pi i \nu \hat{n} \cdot \bar{J}} , \tag{11}$$

where the precession frequency ν, polarization direction $\hat{n}$, and the usual angular momentum operators $\bar{J}$ ($\bar{J}$ is a set of 3×3 matrices satisfying $\bar{J} \times \bar{J} = i\bar{J}$ and $J^2 = 2$) have been introduced. The requirement that $\tilde{S}$ and $\bar{S}$ each satisfy the same equations of motion Eq. (3) with $\bar{\Omega} = \bar{\Omega}_0$ forces $\hat{n}$ to obey these equations and to be a periodic function of θ. Therefore, $\hat{n}$ is a particular solution of the spin motion on the equilibrium orbit. This fact is used to write the general solution to the spin motion on the equilibrium orbit in a form suggested by Eq. (11):

$$\bar{S}(\theta) = e^{-i\Phi(\theta)\hat{n} \cdot \bar{J}} \bar{x}(\theta) , \tag{12}$$

where

$$\Phi(\theta) = \int_0^\theta d\theta' \hat{n} \cdot \bar{\Omega}_0$$

and $\bar{x}(\theta)$ is a periodic function of θ. Application of the transformation Λ shows that the spin-precession frequency is given by:

$$\nu = \frac{1}{2\pi} \int_\theta^{\theta+2\pi} d\theta' \hat{n} \cdot \bar{\Omega}_0 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \hat{n} \cdot \bar{\Omega}_0 . \tag{13}$$

Equation (13) explicitly shows that ν is independent of θ for arbitrary periodic guide fields.

It is useful to express $\hat{x}(\theta)$ as a sum of the eigenvectors of the operator $\hat{n} \cdot \bar{J}$ which are defined by:

$$\hat{n} \cdot \bar{J} \begin{pmatrix} \hat{x}_+ \\ \hat{x}_0 \\ \hat{x}_- \end{pmatrix} = \begin{pmatrix} +\hat{x}_+ \\ 0 \\ -\hat{x}_- \end{pmatrix} \quad . \tag{14}$$

$\hat{x}_0$ is obviously equal to $\hat{n}$ and the other eigenvectors are most easily constructed from a knowledge of $\hat{n}$, the properties of $\bar{J}$, and the eigenvalue properties of Eq. (14). All three eigenvectors are periodic functions of θ and are related to each other in the following manner:

$$\begin{aligned} &\hat{n} \times \hat{x}_\pm = \pm i\hat{x}_\pm && \hat{x}_+ \times \hat{x}_- = i\hat{n} \\ &\hat{n} \cdot \hat{x}_\pm = \hat{x}_+ \cdot \hat{x}_+ = \hat{x}_- \cdot \hat{x}_- = 0 && \hat{x}_+ \cdot \hat{x}_- = \hat{n} \cdot \hat{n} = 1 \ . \end{aligned} \tag{15}$$

The eigenvectors of the transformation Λ can now be written down by inspection and form a very useful set of basis vectors for the description of spin motion on the equilibrium orbit and for perturbation calculations of spin motion away from the equilibrium orbit. They are:

$$\begin{aligned} \bar{S}_\pm(\theta) &= e^{\mp i\Phi(\theta)}\, \hat{x}_\pm \\ \bar{S}_0(\theta) &= \hat{n} \quad , \end{aligned} \tag{16}$$

with corresponding eigenvalues $e^{\mp 2\pi i\nu}$, 0.

When aperiodic perturbing fields are present, the projection of the spin along $\hat{n}$ is no longer conserved and depolarization results. Instead, the projection of the spin along a new direction $\hat{n} + \delta\hat{n}$ is a constant of the motion and the equation of motion for the difference vector $\delta\hat{n}$ to first order in the perturbing field $\bar{\omega}$ is:

$$\delta\hat{n}' = \bar{\Omega}_0 \times \delta\hat{n} + \bar{\omega} \times \hat{n} \quad . \tag{17}$$

The procedure for solving Eq. (17) is straightforward; $\delta\hat{n}$ is expressed as a sum of the eigenvectors of the unperturbed motion with coefficients that depend on θ. Equation (17) yields a set of differential equations for the coefficients and they are easily integrated to give:

$$\delta\hat{n} = 2\ \mathrm{Im}\left(\bar{S}_+(\theta)\int_0^{\theta} d\theta'\ \bar{S}_-(\theta')\cdot\bar{\omega}\right) \quad . \tag{18}$$

Up to an overall phase, $\delta\hat{n}$ can be written as a sum over contributions from successive orbital revolutions $\delta_j\hat{n}$, the index j referring to the jth orbit. The $\delta_j\hat{n}$ are given by:

$$\delta_j\hat{n} = 2\ \mathrm{Im}\left\{\bar{S}_+(\theta)\int_{\theta+2\pi(j-1)}^{\theta+2\pi j} d\theta'\ \bar{S}_-(\theta')\cdot\bar{\omega}\right\}. \tag{19}$$

In Eq. (19), the angle θ refers to the position along the orbit and is bounded by 0 and 2π. The reason we choose to study $\delta_j\hat{n}$ is that quantum fluctuations often dominate the depolarization when the spin-precession frequency is sufficiently far away from certain resonance values and in many practical cases the effect of these fluctuations is negligible during a single orbital period, but comes into play over many revolutions.

The derivations leading to Eq. (18) and Eq. (19) are completely general and depend only on certain basic properties of the spin motion, such as the periodicity of $\bar{\Omega}_0$. Therefore, these equations are valid first-order expressions for arbitrary storage rings, although there may be rather formidable technical problems in the computation of the eigenvectors $\hat{n}$, $\hat{x}_\pm$ in cases where $\bar{\Omega}_0$ does not always point in the same direction.

Let us now turn to the specific case of spin motion in a storage ring where the particles move primarily in the horizontal plane in a nearly uniform vertical magnetic field. Radial and longitudinal magnetic fields are treated as perturbations

and only first-order terms in these fields are retained. The main guide field is taken to point vertically down in the $\hat{y}$ direction. The $\hat{z}$ unit vector points along the direction of motion of the positively charged particle and x points radially outward. In this approximation, $\hat{n} = -\hat{y}$ and $\hat{x}_{\pm} = (\hat{x} \mp i\hat{z})/\sqrt{2}$. Expanding the BMT[6] equation to first order in magnetic fields away from the $\hat{y}$ direction and motion away from the $\hat{z}$ direction and ignoring electric fields, an expression for $\bar{\omega}$ is found:

$$\bar{\omega}(\theta) = -\frac{1}{\langle B \rangle}\left[(1+\gamma a)\, B_r \hat{x} + \frac{g}{2} B_z \hat{z}\right] + a(\gamma - 1) y'(\theta)\, G(\theta)\hat{z} \quad . \tag{20}$$

$\langle B \rangle$ is the average value of the main magnetic field taken over an entire orbit; B_R, B_z are the perturbing radial and longitudinal fields, respectively; γ is the electron energy in units of the electron rest mass; a is the anomalous part of the electron gyromagnetic ratio $a = (g-2)/2$; $G(\theta)$ is the "bending" function of the storage ring, defined by Sands[7] and is related to the guide magnetic field. $y(\theta)$ is the vertical (downward) excursion of the particle from the equilibrium orbit. Again, the ' denotes differentiation with respect to θ. $\bar{\Omega}_0$ is given by:

$$\bar{\Omega}_0 = \gamma a R G(\theta)\hat{n} \quad , \tag{21}$$

where R is the mean radius of the storage ring and is related to $G(\theta)$ by:

$$\frac{1}{R} = \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\, d\theta \quad .$$

The spin-precession frequency ν is equal to γa.

Since radial magnetic fields give rise to vertical focusing, the equations of particle motion can be used to simplify Eq. (2) to the following form:

$$\bar{\omega}(\theta) = \frac{1}{R}(1+\gamma a)\, y''(\theta)\hat{x} + a(\gamma - 1)\, y'(\theta)\, G(\theta) z - \frac{gB_z}{2\langle B \rangle}\hat{z} \quad . \tag{22}$$

We see from Eq. (22) that the contributions to $\bar{\omega}$ fall into two categories:

(1) those terms which are due to the normal focusing properties of the storage ring, and

(2) terms due to the inclusion on the guide field of the storage ring, of special elements, such as a solenoid magnet, which can give rise to longitudinal magnetic fields.

Let us consider terms of the first category and ignore, for the time being, longitudinal fields. The expression for $\delta_j \hat{n}$ requires calculating the following integral:

$$I = \int_{\alpha}^{\alpha+2\pi} d\theta' \quad \bar{\omega} \cdot \hat{x}_{-} \, e^{i\Phi(\theta')}$$

$$= \frac{1}{\sqrt{2}} \int_{\alpha}^{\alpha+2\pi} d\theta' \left\{ \frac{1}{R} (1 + \gamma a) \, y''(\theta') + i \, a(\gamma - 1) \, y'(\theta) G(\theta') \right\} e^{i\Phi(\theta')} \tag{23}$$

where

$$\Phi(\theta') = \gamma a R \int_0^{\theta'} d\theta'' \, G(\theta'') \quad .$$

Note that Eq. (23) is particularly simple to calculate in the case of a separated-function storage ring because of the following property of such machines:

$$G(\theta) \, y''(\theta) = 0 \quad .$$

Rather than compute I for some particular machine, it is instructive to make the sinusoidal approximation to the betatron motion and assume that $G(\theta)$ is a constant. With these approximations and assuming γ is large compared to 1, I can be written:

$$I \simeq \frac{1}{\sqrt{2}} \left(\frac{a_y}{\sqrt{\beta_y}} \right) \int_{\alpha}^{\alpha+2\pi} d\theta' \left\{ \nu_y (1+\nu) \cos(\nu_y \theta' + \phi_0) - i\nu \sin(\nu_y \theta' + \phi_0) \right\} e^{+i\nu\theta'} \tag{24}$$

where a_y is the vertical-betatron amplitude parameter, ν_y is the vertical "tune" of the storage ring, $\nu = \gamma a$ is the spin-precession frequency, β_y is the mean value of the vertical-betatron function ($= R/\nu_y$ in the sinusoidal approximation), and ϕ_0 is a constant. Evaluating Eq. (24) gives the following expression for the $\hat{z}$ projection of $\delta_j\hat{n}$ at $\theta = 0^o$ (the $\hat{x}$ projection has the same form, merely differing in phase by 90°):

$$\delta_j\hat{n}(0)\cdot\hat{z} = \frac{a_y}{\sqrt{\beta_y}} \; f_1 \cos\left[t\left(j+\frac{1}{2}\right) - \phi_0\right] + f_2 \cos\left[s\left(j+\frac{1}{2}\right) - \phi_0\right]$$

$$f_1 = \frac{\sin(t/2)}{(\nu - \nu_y)}\left[1 + \nu(\nu_y + 1)\right] \qquad f_2 = \frac{\sin(s/2)}{(\nu + \nu_y)}\left[1 + \nu(\nu_y - 1)\right] \quad .$$

$$s = 2\pi(\nu + \nu_y) \qquad t = (\nu - \nu_y) \tag{25}$$

s and t are to be evaluated modulo 2π.

In the absence of quantum fluctuations, Eq. (25) shows that there will be no net depolarization after many revolutions through the perturbing fields, provided that the spin-precession frequency is not equal to a betatron frequency sideband of some integer. That is to say, the betatron motion in a storage ring introduces a new set of spin resonances whose frequencies are given by integral multiples, sums, and differences of the betatron tune of the storage ring. Since, in general, there is coupling between the vertical and horizontal betatron motions, the general conditions on the spin-precession frequency ν for a depolarization resonance to occur can be written:

$$\nu = i \pm j\nu_x \pm k\nu_y \quad , \tag{26}$$

where i, j, k are integers; ν_x, ν_y are the horizontal, vertical storage ring tunes, respectively.

The so-called stochastic depolarization comes about from quantum fluctuations to the particle motion during the synchrotron radiation process and will

lead to depolarization even when Eq. (26) is not satisfied. In our model of depolarization, quantum fluctuations are introduced through the amplitude parameter a_y by noting that on the jth revolution, a_y can be written as the sum of the radiation-damped amplitude from some previous orbital revolution plus an uncorrelated part due to quantum excitations following that revolution:

$$a_{y_j} = a_{y_k} e^{-(j-k)/N_0} + a_u \qquad (27)$$

$$N_0 = \omega_0 \tau_y, \quad k\ j \quad .$$

τ_y is the vertical damping time of the storage ring and a typical value for N_0 would be ~ 1000. These quantum fluctuations lead to a distribution of the $\delta\hat{n}$, where $\delta\hat{n}$ is the change in the polarization direction after N orbital periods

$$\left(\delta\hat{n} = \sum_{j=1}^{N} \delta_j \hat{n}\right) \quad .$$

Since $\delta\hat{n}$ has the same form in both the x and z directions, the distribution function of the x and z projections of $\delta\hat{n}$ are equal. It is easily seen that the relative depolarization after N revolutions, $-\Delta P/P$ is given by one minus the cosine of the angle between the original polarization direction and the new direction, which is simply equal to $1 - \frac{1}{2}(\delta\hat{n})^2$. Since δn has a distribution of values, we have to perform an ensemble average to find the beam depolarization:

$$-\left\langle \frac{\Delta P}{P} \right\rangle = \frac{1}{2} \langle (\delta\hat{n})^2 \rangle = \frac{1}{2} \langle n_x^2 + n_y^2 \rangle$$

$$\left\langle \frac{\Delta P}{P} \right\rangle = - \langle n^2 \rangle \quad . \qquad (28)$$

The symbol $\langle n^2 \rangle$ stands for the ensemble average of the second moment of the distribution function of the x or z projection of δn and includes an average over the parameter ϕ_0 introduced in Eq. (24). We calculate this quantity

using Eq. (25).

$$\langle n^2 \rangle = \frac{\nu_y \langle a_y^2 \rangle}{2R} \sum_{j=1}^{N} \sum_{k=1}^{N} e^{-|j-k|/N_0} \Big\{ f_1^2 \cos t(j-k) + f_2^2 \cos s(j-k)$$

$$+ 2f_1 f_2 \cos \left[\frac{s+t}{2}(j-k)\right] \cos \left[\frac{s-t}{2}(j+k+1)\right] \Big\} \tag{29}$$

The last term in Eq. (29) oscillates in the sum of j and k, while all other terms are even in the difference. Since the double summation can be expressed as a summation over the sums and differences of j and k, this last term will be negligible compared with the others, when N is large (except when ν is equal to an integer, in which case this term will give rise to an integer resonance). Finally, since we are interested in depolarization times long compared to the radiation damping time, the summation over the difference in j and k is replaced by an integral. That is to say,

$$\sum_{j=1}^{N} \sum_{k=1}^{N} f(j-k) \rightarrow 2N \int_0^{\infty} dv\, f(v) \quad .$$

With this approximation, we obtain

$$\langle n^2 \rangle = \frac{\nu_y \langle a_y^2 \rangle}{R} \frac{t}{\tau_y} \left[\frac{f_1^2}{(1/N_0)^2 + t^2} + \frac{f_2^2}{(1/N_0)^2 + s^2} \right] , \tag{30}$$

where the running time t ($= N/\omega_0$) has been introduced.

We see that the relative depolarization is proportional to the running time and, therefore, the beam polarization decays exponentially in time with a time constant τ_{dep}, given by:

$$\frac{1}{\tau_{dep}} = \frac{\nu_y \langle a_x^2 \rangle}{\tau_y R} \left[\frac{f_1^2}{(1/N_0^2) + t^2} + \frac{f_2^2}{(1/N_0^2) + s^2} \right] . \tag{31}$$

From the theory[7] of quantum fluctuations on the orbital motion of particles in a storage ring, we know

$$\langle a_y^2 \rangle = \frac{2\sigma_y^{*2}}{\beta_y^*} \quad , \tag{32}$$

where σ_y^*, β_y^* is the standard deviation of the vertical beam size, vertical betatron function, evaluated at the interaction point.

In order to maintain the periodicity of Eq. (29) in the variables s and t, Eqs. (30) and (31) should be averaged with a similar expression in the variables $2\pi - t$ and $2\pi - s$. The full expression for τ_{dep} is therefore:

$$\frac{1}{\tau_{dep}} = \frac{\nu_y}{4\tau_y}\left(\frac{\sigma_y^{*2}}{R\beta_y^*}\right)\left\{\left[\frac{1+\nu(\nu_y-1)}{\nu-\nu_y}\right]^2 S(t) + \left[\frac{1+\nu(\nu_y-1)}{\nu+\nu_y}\right]^2 S(s)\right\} \tag{33}$$

$$S(x) = 4\sin^2(x/2)\left[\frac{1}{1/N_0^2+x^2} + \frac{1}{1/N_0^2+(2\pi-x)^2}\right] \quad .$$

To a good approximation, $S(x) \simeq 1$, and in many applications $\nu + \nu_y >> \nu - \nu_y$, so the second term in Eq. (33) can be neglected. For order-of-magnitude estimates, we can neglect 1 compared to ν, ν_y, and we arrive at the simple result found by other authors,[1,4,8]

$$\frac{1}{\tau_{dep}} \approx \frac{\nu_y}{4\tau_y}\left(\frac{\sigma_y^{*2}}{R\beta_y^*}\right)\left(\frac{\nu\nu_y}{\nu-\nu_y}\right)^2 \quad . \tag{34}$$

From Eq. (1), we see that the critical parameter determining whether or not there will be a significant buildup of beam polarization is the ratio τ_{pol}/τ_{dep}. Since τ_{pol}, the normalized horizontal beam size $\sigma_x^*/\sqrt{\beta_x^*}$, and the horizontal damping time τ_x are all related to the emission of synchrotron radiation, we might expect some simple relation to hold among these parameters.

Using simplified assumptions discussed by Sands,[7] there is, in fact, such a relationship and we can use it to write the ratio τ_{pol}/τ_{dep} in the following simple form:

$$\frac{\tau_{pol}}{\tau_{dep}} = \frac{11}{144} \left(\frac{\tau_x}{\tau_y}\right)\left(\frac{\sigma_y^*}{\sigma_x^*}\right)^2 \left(\frac{\beta_x^*}{\beta_y^*}\right)\left(\frac{\nu_y}{\nu_x}\right)^3 \left(\frac{\nu}{\nu - \nu_y}\right)^2 \quad . \tag{35}$$

In most practical cases, $\tau_x \approx \tau_y$, and the vertical beam size is dominated by coupling, so Eq. (35) can be simplified to:

$$\frac{\tau_{pol}}{\tau_{dep}} = \frac{11}{144} K^2 \left(\frac{\nu_y}{\nu_x}\right)^3 \left(\frac{\nu}{\nu - \nu_y}\right)^2 \quad , \tag{36}$$

where K is the coupling constant of the storage ring Note that, strictly speaking, the preceding derivation is inconsistent with respect to coupling because we have used K to determine the vertical beam size, but have not included the x motion in the various integrals leading up to Eq. (34) and therefore there are no horizontal-betatron-frequency resonance sidebands in Eqs. (34) and (36); yet we expect them to occur. For this reason, Eq. (36) should be relied on for only those ν values that are well away from the resonance condition of Eq. (26).

Since K is usually on the order of a few percent, Eq. (36) indicates that the depolarization time is long compared with the polarization buildup time for typical storage-ring parameters. On the other hand, Eq. (36) also indicates certain conditions when depolarization effects are not negligible and therefore careful studies of the depolarization effects must be performed for specific machine conditions if the beam polarization is an important experimental parameter.

Let us now examine the effect of longitudinal magnetic fields by calculating the depolarization to be expected from a nearly compensated solenoid placed

in a drift region of the equilibrium orbit. Fringing fields at the ends of the solenoid will give rise to radial magnetic fields for particles that are displaced horizontally from the equilibrium orbit. The expression for $\bar{\omega}$ in this example is:

$$\bar{\omega}(\theta) \approx \frac{1}{2<B>}\left[(1+\gamma a)\frac{dB_s}{dz}x(\theta)\,\hat{x} - g\,B_s(\theta)\hat{z}\right] \quad . \tag{37}$$

We assume that the x (horizontal) motion which is linear in z in a drift section, is not affected to first order by the solenoid, and therefore Eq. (19) is easily integrated by parts to give:

$$\delta_j\hat{n}(\theta)\cdot\hat{z} = \frac{\int B_s dz}{2<B>R}\left\{\left(\frac{a_x}{\sqrt{\beta_x^*}}\right)(1+\gamma a)\cos\left[2\pi\nu(j-1)\right] \times \right.$$

$$\left. \times \sin\left[2\pi\nu_x(j-1)+\phi_0\right] - g\,\sin\left[2\pi\nu(j-1)\right]\right\} \quad , \tag{38}$$

where the horizontal betatron parameters a_x, β_x, ν_x, similar to those defined in Eq. (24) have been introduced. Note that $\int B_s dz$ is generally designed to be as close to zero as possible by the addition of compensating antisolenoids in order that the storage ring performance is not degraded by the solenoid. The second term in Eq. (38) only contributes to the integer resonances and is therefore neglected in what follows. Using the same procedure for evaluating the ensemble average $<n^2>$ as was used previously, the depolarization rate due to the solenoid is found to be:

$$\frac{1}{\tau_{dep}} = \frac{1}{16\,\tau_x}\left(\frac{\sigma_x^*}{\beta_x^*}\right)^2\left[\frac{\int B_s dz}{<B>R}\right]^2 (1+\nu)^2\,F(\nu,\nu_x) \quad , \tag{39}$$

where the resonance function $F(\nu,\nu_x)$ has been defined by:

$$F(\nu,\nu_x) = \frac{1}{1/N_0^2+t^2} + \frac{1}{1/N_0^2+(2\pi-t)^2} + \frac{1}{1/N_0^2+s^2} + \frac{1}{1/N_0^2+(2\pi-s)^2} \; .$$

$$\left.\begin{aligned} s &= 2\pi(\nu + \nu_x) \\ t &= 2\pi(\nu - \nu_x) \end{aligned}\right\} \text{modulo } 2\pi \quad .$$

Equation (29) exhibits horizontal-betatron-frequency sideband resonances for all the integers but, unless the solenoid is grossly uncompensated or some other pathology arises, depolarization due to a partially compensated solenoid is negligible, except very close to the resonant energies.

As a final example, we use Eqs. (15), (18) and the ensemble-averaging procedure developed above to calculate the depolarization effect of a "low-β" insert.[7] The result is:

$$\frac{1}{\tau_{dep}} = \frac{1}{4\tau_y}\left(\frac{\sigma_y^{*2}}{\beta_y\beta_y^*}\right) \sin^2 \delta/2\,(1+\nu)^2 \quad 1 + \frac{\alpha_y^2}{4} \quad F(\nu, \nu_y) \quad , \tag{40}$$

where β_y, α_y refer to the vertical betatron function and its derivative evaluated at the entrance to the low-β insert. δ is the betatron phase shift of the low-β insert, and the other quantities have been defined previously. Since δ is usually very close to 2π, we again see that the depolarization is negligible except near spin resonances.

It is interesting to note that the three examples of spin depolarization considered above can all be characterized by a "strength" which multiplies a universal function of the storage-ring parameters to give the depolarization rate. The strength associated with the normal vertical betatron motion is approximately:

$$K^2(2\pi\nu_y)^3 \quad .$$

The strength for a nearly compensated solenoid is:

$$\left(\frac{2\pi R}{\beta_x^*}\right) \left[\frac{\int B_s dz}{2 < B > R}\right] \quad ,$$

while the strength of the low-β insert is given by:

$$\left(\frac{2\pi R}{\beta_y}\right) K^{2^-} \sin^2 \delta/2 \left(1 + \frac{\alpha_y^2}{4}\right) \quad .$$

Under most practical conditions, the first expression dominates, so we can conclude that the normal vertical-focusing properties of a storage ring account for most of the beam depolarization and the inclusion of special magnetic elements on a machine, such as low-β insertions or compensated solenoids, will not significantly change the polarization except near spin resonances.

In summary, procedures for calculating the spin motion of particles in storage rings have been developed and applied to certain examples of beam depolarization. Conditions are found where the transverse radiative polarization predicted by Sokolov and Ternov[5] should occur in modern high-energy e^+e^- storage rings.

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