# A RELATIVISTIC QUANTUM DYNAMICAL MODEL BASED 

ON THE CLASSICAL STRING*

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#### Abstract

Motivated by purely geometrical considerations of the string model, we construct a dynamical theory of particles with internal degrees of freedom that is Poincare invariant in four dimensional space-time, with no tachyons and no ghosts. The resulting composite states lie on indefinitely rising trajectories.


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[^0]The advent of the Nambu geometric formulation of the dynamics of the dual resonance model [1] has caused attention in this model to focus on the problem of relativistic invariance. The action on which the theory is based is globally Poincare invariant. Furthermore, the Virasoro ghost climinating operators, [2] which after a long struggle were finally shown to decouple all unphysical states, [3] are seen to be simply the generators of local coordinate transformations, [4] under which the action is also invariant.

Unfortunately, the quantum theory derived canonically from the classical geometric theory suffers from the defects that Lorentz invariance can be realized only in a space of 26 dimensions; and, more seriously, that the spectrum of the (mass) ${ }^{2}$ operator must include a tachyon. [5]

It is reasonable, therefore, to investigate whether a different quantum theory, identical with the string picture at the classical level, can avoid the difficulties of the naively quantized string. The purpose of this letter is to report on one such alternative possibility in the context of a simple model which has several remarkable features. These are that the model has no ghosts; no tachyons; Poincare invariance in four dimensions; fermionic constituent substructure; and indefinitely rising towers of particles.

We motivate our model from the general observation that the string is most easily described [4] in a frame characterized by the gauge conditions [6] $(\mathrm{Y}, \underset{ \pm}{\mu})^{2}=0$. Classically, any such null four-vector can be written in terms of two-component Lorentz spinors as $\mathrm{Y}_{ \pm}^{\mu}=\psi_{ \pm}^{\dagger} \sigma^{\mu} \psi_{ \pm}$, where $\sigma^{\mu}$ are $2 \times 2$ Pauli matrices. The dynamical equation $(\mathrm{Y}, \underset{ \pm}{\mu})=0$ then leads to

$$
\begin{equation*}
\left[\partial_{ \pm}+i \mathrm{gB}_{ \pm}\right] \psi_{\mp}=0 \tag{1}
\end{equation*}
$$

where $B_{ \pm}\left(u_{+}, u_{-}\right)$are arbitrary Hermitian functions. However, the gauge invariance of second kind inherent in the definition of the spinors $\psi_{ \pm}$in terms of $Y,{ }_{ \pm}^{\mu}$ can be preserved in Eq. (1), provided $B_{ \pm}$are chosen to transform as Abelian gauge fields.

These considerations lead us to examine a theory based on the effective Lagrange density

$$
\begin{equation*}
\mathscr{L}=\vec{\psi} \Gamma^{\alpha}\left(\mathrm{i} \partial_{\alpha}-\mathrm{g} \mathrm{~B}_{\alpha}\right) \psi-\frac{1}{4} \mathrm{~F}_{\alpha \beta} \mathrm{F}^{\alpha \beta} \tag{2}
\end{equation*}
$$

where $\mathrm{F}_{\alpha \beta} \equiv \partial_{\beta} \mathrm{B}_{\alpha}-\partial_{\alpha} \mathrm{B}_{\beta}$. Our illustrative model, interesting in its own right, consists of postulating that $\psi(\tau, \theta)$ is a canonical four component fermion field, with no reference to $Y^{\mu}$. The full connection of Eq. (2) with the string model will be dealt with elsewhere. [7]

The structure of the Lagrangian Eq. (2) is identical to that of two dimensional electrodynamics (TDED), which is known to be exactly solvable. [8] Use of four component spinors does not destroy the algebraic properties of the system, so our model is solvable as well. However, the traditional solution, while appropriate for calculation of the Green's functions of the theory, is not convenient for displaying the exact eigenstates and eigenvalues of the Hamiltonian. Consequently, we exhibit an alternate solution. [9]

It is convenient to choose a representation [10] in which $\Gamma^{0}=-\mathrm{i} \gamma^{0} \gamma^{5}$, and $\Gamma^{1}=-\mathrm{i} \gamma^{5}$. Introduce the $\tau=0$ expansion

$$
\begin{align*}
\psi(\theta, 0)= & (2 \pi)^{-1 / 2} \sum_{k>0}\left[e^{i(k-1 / 2) \theta} \quad\left(\begin{array}{l}
b_{1} \\
b_{2} \\
c_{1}+ \\
c_{2^{+}}
\end{array}\right)_{k}\right. \\
& +e^{-i(k-1 / 2) \theta}\left[\begin{array}{c}
c_{1}+ \\
c_{2^{+}} \\
b_{1} \\
b_{2}
\end{array}\right)_{k} \tag{3}
\end{align*}
$$

where $\left\{\mathrm{b}_{\mathrm{i}}(\mathrm{k}), \mathrm{b}_{\mathrm{j}}^{\dagger}\left(\mathrm{k}^{\prime}\right)\right\}=\left\{\mathrm{c}_{\mathrm{i}}(\mathrm{k}), \mathrm{c}_{\mathrm{j}}^{\dagger}\left(\mathrm{k}^{\prime}\right)\right\}=\delta_{\mathrm{ij}} \delta_{\mathrm{kk}}$, ensure canonical anticommutation relations for $\psi$ and $\psi^{\dagger}$. Free vector and axial vector currents are defined in the usual manner, $\mathrm{j}^{\alpha}=\epsilon^{\alpha \beta} \mathrm{a}_{\beta}=: \bar{\psi} \gamma^{\alpha} \psi:$. (Boundary conditions have been chosen for $\psi$ such that the gradient of the charge density vanishes at the edges of the domain. [11])

These definitions of the current may be carried over to the interacting theory as well, provided we work in the axial gauge $\mathrm{B}_{1}=0$. We shall do so, because in this gauge the Maxwell equation $\partial_{\nu} \mathrm{F}^{\mu \nu}=\mathrm{gj}^{\mu}$ can be integrated exactly to obtain [12]

$$
\begin{equation*}
\mathrm{B}_{0}(\theta)=(-\mathrm{g} / 2) \int_{0}^{\pi} \mathrm{d} \theta^{\prime}\left|\theta-\theta^{\prime}\right| \mathrm{j}^{\mathrm{o}}\left(\theta^{\prime}\right) . \tag{4}
\end{equation*}
$$

Now, it follows from Eq. (2) that $\mathrm{B}_{0}$ has no conjugate momentum. Therefore, $\mathrm{B}_{0}$ may be eliminated entirely using Eq. (4). This leads directly to

$$
\begin{equation*}
H=H_{0}-\left(g^{2} / 4\right) \iint d \theta d \theta^{\prime} j^{0}(\theta, 0)\left|\theta-\theta^{\prime}\right| j^{\mathrm{o}}\left(\theta^{\prime}, 0\right), \tag{5}
\end{equation*}
$$

where $H_{0}$ is the massless free two dimensional Dirac Hamiltonian. Our aim is to diagonalize H .

The diagonalization is most elegantly achieved by performing a Bogoliubov transformation on $H$ in momentum space．To this end we introduce the paired fermion operators at $\tau=0$（for $p>0$ ），

$$
\begin{equation*}
\rho(p) \equiv(2 p)^{-1 / 2} \int_{0}^{\pi} d \theta\left[j^{o}(\theta) \cos p \theta-i j^{1}(\theta) \sin p \theta\right] \tag{6}
\end{equation*}
$$

which can be shown to satisfy $\left[\rho(p), \rho^{+}(q)\right]=\delta_{p, q^{*}}$ ．We define also the con－ served charge $Q=\int d \theta j^{\circ}(\theta, t)$ 。 The interaction term in $H$ can be expressed in terms of $Q$ and these＂plasmon＂operators by inverting Eq．（6）to obtain $j^{\circ}(\theta)$ 。

However，before displaying this result，we note an important general fea－ ture of TDED in axial gauge which is true for our model as well．［12，13］It is that careful attention to the Schwinger term in the $j^{\circ}, j^{1}$ commutator leads，by use of the Heisenberg equations of motion，to the Adler anomaly［14］

$$
\begin{equation*}
\partial_{\alpha^{2}}{ }^{\alpha}(\phi, 0)=-\left(\mathrm{g}^{2} / \pi\right) \partial_{\phi} \int \mathrm{d} \theta \mathrm{j}^{\mathrm{o}}(\theta)|\theta-\phi| \tag{7}
\end{equation*}
$$

This equation may be combined with the equation of current conservation to demonstrate that the vector current is a massive free field，with $\mu^{2}=\left(2 g^{2} / \pi\right)$ 。 The most striking consequence of this result is that，for $\mathrm{g} \neq 0$ ，and with our boundary conditions on $\psi(\theta)$ ，the charge $Q$ must vanish．Since $Q \neq 0$ as an operator，it superselects a class of admissible states from the larger fermionic Fock space，

$$
\begin{equation*}
Q \mid \Phi_{\text {phys }}>=0 \tag{8}
\end{equation*}
$$

All physical operators，then，must conserve charge．

Taking account of Eq. (8), the momentum space expansion of the Hamiltonian may be written,

$$
\begin{align*}
\mathrm{H} & =\sum_{\mathrm{n}=1}^{\infty} \sum_{\lambda=1}^{2}\left(\mathrm{n}-\frac{1}{2}\right)\left[\mathrm{b}_{\mathrm{n}}^{\dagger}(\lambda) \mathrm{b}_{\mathrm{n}}(\lambda)+\mathrm{c}_{\mathrm{n}}^{\dagger}(\lambda) \mathrm{c}_{\mathrm{n}}(\lambda)\right] \\
& +\left(\mu^{2} / 4\right) \sum_{\mathrm{p}=1}^{\infty} \mathrm{p}^{-1}\left[\rho^{\dagger} \rho^{\dagger}+\rho \rho+2 \rho^{\dagger} \rho\right]_{\mathrm{p}} . \tag{9}
\end{align*}
$$

It is then a matter of straightforward manipulation to demonstrate that

$$
\begin{align*}
\widetilde{\mathrm{H}} & \equiv \mathrm{e}^{\mathrm{iS}} \mathrm{He}^{-\mathrm{i} S} \\
& =\mathrm{H}_{0}-\mathrm{T}+\epsilon_{0}+\sum_{\mathrm{p}=1}^{\infty} \epsilon(\mathrm{p}) \rho^{\dagger}(\mathrm{p}) \rho(\mathrm{p}) \tag{10}
\end{align*}
$$

where

$$
\begin{aligned}
& \mathrm{S}=(-\mathrm{i} / 2) \sum_{\mathrm{p}=1}^{\infty}\left[\tanh ^{-1}\left(\mu^{2} / \mu^{2}+2 \mathrm{p}^{2}\right)\right]\left[\rho^{\dagger}(\mathrm{p}) \rho^{\dagger}(\mathrm{p})-\rho(\mathrm{p}) \rho(\mathrm{p})\right] \\
& \mathrm{T}=\sum_{\mathrm{p}} \mathrm{p} \rho^{+}(\mathrm{p}) \rho(\mathrm{p}) ; \\
& \epsilon(\mathrm{p})=\sqrt{\mu^{2}+\mathrm{p}^{2}} ; \\
& \epsilon_{0}=\frac{1}{2} \sum_{\mathrm{p}}\left[\epsilon(\mathrm{p})-\mathrm{p}-\left(\mu^{2} / 2 \mathrm{p}\right)\right]
\end{aligned}
$$

In Eq. (10), T has been set together with $\mathrm{H}_{0}$ because $\left[\mathrm{H}_{0}-\mathrm{T}, \rho(\mathrm{p})\right]=0$.
Making use of $\widetilde{H}$, the eigenstates of $H$ may be constructed as follows. We first note that the plasmon operators $\rho$ annihilate neutral filled Fermi sea states of the form ( $\mathrm{j} \neq \mathrm{k}$ )

Thus $\mid F>$ are eigenstates of $\widetilde{H}$ with eigenvalue $\epsilon_{F}=F^{2}+\epsilon_{0}$.

Plasmon excitations may be added onto any sea state to give the general eigenstate of $\widetilde{H}$,

$$
\begin{equation*}
\left|\Phi\left(p_{1}, \ldots p_{m}\right) ; F\right\rangle=\prod_{i=1}^{m} \rho^{\dagger}\left(p_{i}\right)|F\rangle \tag{12}
\end{equation*}
$$

These states have energies $\epsilon=\epsilon{ }_{F}+\sum_{i=1}^{m} \epsilon\left(p_{i}\right)$.

It follows that the states

$$
\begin{equation*}
\left|\Phi\left(p_{1}, \ldots, p_{m}\right) ; F>\equiv e^{-i S}\right| P\left(p_{1}, \ldots, p_{m}\right) ; F> \tag{13}
\end{equation*}
$$

are the eigenstates of H , with energies $\epsilon$. Recall $\epsilon_{0}$ is the ground state energy, and is numerically finite and negative definite. The fact the ground state energy of the interacting system is lower than the energy of the free vacuum state indicates the presence of bound states. Henceforth, we measure energies from the true ground state energy, i.e., drop $\epsilon_{0}$ so $E=\epsilon-\epsilon_{0}$ is the energy.

To exhibit the full four-dimensional Minkowski space content of our theory, it remains to associate the energy eigenstates $|\Phi\rangle$, Eq. (13), with genuine free particle states. We do this in the formalism of null-plane quantum dynamics, [15] labelling a state by its mass, spin, momentum and helicity; and writing ten generators which satisfy the Poincare algebra, and act on the manifold of the states we form.

To construct the Poincare generators, we introduce three canonically conjugate pairs, $\left[\mathrm{X}_{\mathrm{i}}, \mathrm{P}_{\mathrm{j}}\right]=\mathrm{i} \delta_{\mathrm{ij}}$, and $\left[\mathrm{x}^{-}, \mathrm{P}^{+}\right]=-\mathrm{i}$, which commute with all operators $b(n), c(n)$, etc. These are interpreted as center of mass positions and momenta, while the mass, spin, and helicity will depend on the state of internal excitation of the particle. [ 16]

The invariant (mass) ${ }^{2}$ operator $\mathscr{M}^{2}=2 \mathrm{P}^{+} \mathrm{P}^{-}-\mathrm{P}_{\perp}^{2}$, should have an increasing spectrum of positive eigenvalues for composite hadrons, and it is
natural to identify $\mathscr{M}^{2}=\mathrm{f}(\mathrm{H})$. For definiteness, we shall set $\mathscr{A}^{2}=\mathrm{H}$, as in the conventional string model. This, of course, relates the null-plane "energy" operator $\mathrm{P}^{-}$to our dynamical spectrum.

Candidates for little group operators of massive particles are introduced by observing that our model with four component spinors enjoys an $\mathrm{SU}(2)$ symmetry not shared by TDED, which is generated by the canonically obtained operators

$$
\begin{equation*}
\mathrm{J}^{\mathrm{k}}=(1 / 2) \int \mathrm{d} \theta: \psi^{+} \sum^{\mathrm{k}} \psi: \tag{14}
\end{equation*}
$$

where $\sum^{k}=\sigma^{i j}$, (ijk cyclic). These operators commute with $Q$, and are conserved, $[\mathrm{H}, \mathrm{J}]=0$.

It follows from the work of a number of authors [17] that the Poincare algebra may be satisfied with the momenta $\mathrm{P}^{\mu}$ defined above, and the Lorentz generators

$$
\begin{align*}
& \mathrm{K}^{3}=\frac{1}{2}\left\{\mathrm{P}^{+}, \mathrm{x}^{-}\right\} ; \\
& \mathrm{J}^{3}=\left(\mathrm{x}^{1} \mathrm{P}^{2}-\mathrm{x}^{2} \mathrm{P}^{1}\right)+\mathrm{J}^{3} ; \\
& \mathrm{B}^{\perp}=\mathrm{P}^{+} \mathrm{x}^{\perp} ; \\
& \mathrm{S}^{\mathrm{i}}-\frac{1}{2}\left\{\mathrm{x}^{\mathrm{i}}, \mathrm{P}^{-}\right\}+\mathrm{x}^{-} \mathrm{P}^{\mathrm{i}}+\left(2 \mathrm{P}^{+}\right)^{-1} \epsilon^{\mathrm{ij}}\left[J^{3} \mathrm{P}^{\mathrm{j}}+\sqrt{\mathscr{M}^{2}} J^{\mathrm{j}}\right] . \tag{15}
\end{align*}
$$

Note $\sqrt{\mathscr{M}^{2}}$ is well defined, because the spectrum of $H$ is positive definite. (It may be represented in terms of a Gaussian integral.) The correctness of the algebra is easily verified. Unlike the case of the ordinary dual model, there are no ordering problems which necessitate extra spatial dimensions, or inclusion of a tachyon.

We are mindful that our assignments of $\underset{\sim}{J}$ and $\mathscr{M}^{2}$ as spin and mass operators, and of $\mathrm{P}^{\perp}, \mathrm{P}^{+}, \mathrm{x}^{\perp}, \mathrm{x}^{-}$, as center of mass variables, must be tested by
introducing an interaction. (E.g., a magnetic field should split the levels we label as $J^{3}$ degenerage states.) Albeit with this reservation, we label the state vector of a particle in motion as

$$
\begin{equation*}
\left|\mathrm{k}^{+}, \mathrm{k}^{\perp} ; \mathscr{M}^{2}, \mathrm{~J}, \lambda>=\mathrm{e}^{\mathrm{ik} \mu^{\mathrm{x}} \mu}\right| \Phi\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{m}}\right) ; \mathrm{F}> \tag{16}
\end{equation*}
$$

It is clear that $\mathrm{P}^{+}, \mathrm{P}^{\perp}$, and $\mathscr{M}^{2}$ take eigenvalues $\mathrm{k}^{+}, \mathrm{k}^{\perp}$ and E in these states.
Furthermore, $J^{3}$ takes the eigenvalue $\lambda=F$, and the square of the PauliLubanski vector $\mathrm{W}_{\mu}=1 / 2 \epsilon_{\mu \nu \rho \sigma} \mathrm{M}^{\nu \rho_{\mathrm{P}}}{ }^{\sigma}$ takes the value $\mathrm{EF}(\mathrm{F}+1)$ in the state Eq. (16). To demonstrate these properties, note that

$$
\begin{equation*}
[\underset{\sim}{J}, \rho(\mathrm{p})]=0 \tag{17}
\end{equation*}
$$

From this it follows that we may restrict attention to the filled sea factor of the state function, since in this model all plasmons are Lorentz scalar excitations. We pass to the rest frame, in which $\mathrm{k}^{\perp}=0, \mathrm{k}^{+}=\sqrt{\mathrm{E} / 2}$, by finite boost generated by the "non-dynamical" operators [18] $B_{\perp}$ and $K_{3}$. In this frame, $\left(\epsilon_{12}=-\epsilon_{21}=1\right)$

$$
\begin{equation*}
J^{3}\left|F_{i j}\right\rangle=\epsilon_{i j} F\left|F_{i j}\right\rangle \tag{18}
\end{equation*}
$$

The rest of the complete spin $F$ multiplet is obtained from the states $\left|F_{i j}\right\rangle$ through use of the raising and lowering operators $J^{ \pm}=J^{1} \pm i J^{2}$. The proof that $W_{\mu} W^{\mu}$ has the stated eigenvalue is easily completed from these remarks.

It follows from the above observations on the state vectors that our system describes an infinite number of indefinitely rising towers of particles. The filled sea states provide the leading "trajectory", and plasmons displace the trajectory as indicated schematically in Fig. 1.

In conclusion, we have advocated a new way of expressing the classical string variables in terms of fermion fields. This identification makes essential
use of the properties of four dimensional space-time, and is designed to obviate problems with the relativistic invariance of the quantized string. A simple model has been worked out in detail, to illustrate how such a dynamical fermionic theory can be used as a basis for an undiseased free theory of composite hadrons. Our ground state is filled with bare "quarks and antiquarks", which, as eigenstates of $\mathrm{H}_{0}$, are the partons of our model. Excited hadrons correspond to adding dressed pairs to the ground state without altering the charge, and to exciting plasma oscillations. This picture is physically appealling, [19] and we will amplify it to incorporate the full classical string model in a separate, lengthier article. The problems of hadron-hadron and hadron-current interaction must also be faced, and it is hoped our new physical picture will lead to new approaches to these problems.

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FIGURE CAPTION

1. Mass-spin towers in the model with $\mathrm{M}^{2}=\mathrm{II}$ 。 Arrow (a) represents a double excitation of the plasmon with mode number 1 on the $q \bar{q} \bar{q}$ sea, for $\mu^{2} \ll 1$. Arrow (b) represents a single excitation of the second plasmon mode.


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