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APPLICATIONS OF THE INFINITE MOMENTUM METHOD TO QUANTUM ELECTRODYNAMICS AND BOUND STATE PROBLEMS*

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Introduction.

Despite the passage of over a quarter century, the basic rules of calculation in quantum field theory have changed little since the development of the Feynman-Dyson-Schwinger analysis, although there has been extraordinary progress in techniques [1] . The dispersion method [2] represents an important alternative for the calculation of the case of the 3 point vertex graph, but these calculations are often even more arduous than the standard Feynman method and can involve subtle and non-uniform infrared problems. In the case of bound state problems, we must rely on the vigorous Bethe-Salpeter formalism, or alternatively, on the somewhat more tractable quasi-potential method [3] . The latter method, however, has difficulties with non-uniqueness and anomalous analytic properties and it is not clear how to estimate its errors systematically. It should be emphasized that the precision of the experimental measurements of both the positronium and muonium hyperfine splittings are well beyond the present accuracy of our computational methods [4] .

Before the advent of the covariant perturbation theory rules, calculations in quantum electrodynamics relied on straightforward, though extremely arduous, time-ordered perturbation theory. Each Feynman diagram with n vertices requires the computation of $n!$ individually non-covariant TOPTh matrix elements, and the covariant renormalization program was completely obscured. Interest in this method was revived in 1966 when Weinberg [5] (see also ref. [6-8]) realized that by choosing the Lorentz frame of the observer appropriately, only a few of the $n!$ time-ordered contributions survive ; moreover each of the surviving terms has a simple (often covariant) structure paralleling n-body Schrödinger theory. Specifically, in a Lorentz frame chosen such that the total momentum of the initial state $\vec{P} = \sum \vec{p}_i$ is arbitrarily large, then only the relatively few time-ordered graphs in which the momenta $\vec{P}_j = x_j \vec{P} + \vec{k}_\perp$ of all the (one-mass-shell) particles

in every intermediate state have positive components along $\vec{P}$, [i.e. : $0 \leq x_j \leq 1$, $\sum_{int} x_j = 1$] have a surviving contribution in the limit $P \to \infty$. To see this, note that for each intermediate state, the energy denominator of time-ordered perturbation theory becomes

$$\frac{1}{E_{inc}-E_{int}+i\epsilon} = \frac{1}{\sum_{inc}\sqrt{p_i^2+m_i^2} - \sum_{int}\sqrt{p_j^2+m_j^2} + i\epsilon}$$

$$\Rightarrow \frac{1}{P + \sum_{inc}\frac{s_i}{2P} - \sum_{int}\left(|x_i|P + \frac{s_i}{2P}\right) + i\epsilon}$$

$$= \begin{cases} \dfrac{2P}{\sum_{inc} s_i - \sum_{int} s_i + i\epsilon} & \text{for all } x_i > 0 \\[2ex] \dfrac{1}{2\sum_{(x_i<0)} x_i P} & \text{otherwise} \end{cases}$$

where $s_i \equiv \dfrac{k_\perp^{i\,2} + m_i^2}{x_i}$ is the "covariant kinetic energy".

Thus in each case we obtain a non-zero limit only if each intermediate state has all $0 < x_i < 1$. It should be stressed that no invariant quantity is getting large as $P \to \infty$.

The time-ordered perturbation theory which emerges retains the main advantages of the dispersion method since calculations involve physical on-mass-shell intermediate states of fixed particle number, but because of the $P \to \infty$ limit, the complicated square root structure of the phase-space integration is automatically linearized, and the analysis of infrared

divergences is no more difficult than in the corresponding Feynman calculations.

Recently Ralph Roskies, Roberto Suaya and I [9] have found that $TOPTh_{\infty}$ for quantum electrodynamics represents a viable, instructive, and frequently advantageous calculational alternative to the usual Feynman diagram approach. We have shown how to implement the renormalization procedure, simplify spin considerations, and have calculated the electron magnetic moment through fourth order in agreement with the Sommerfield-Peterman [10] results as well as representative contributions to the sixth order moment. Section II reviews the main features of this work.

Over the past few years, it has been found that the infinite momentum method has important advantages for calculations in particle physics, especially in the areas of current algebra [11], parton models [12,4], and eikonal scattering. Because of this similarity to non-relativistic many-body theory one can apply the approximations and concepts such as the impulse and incoherence approximations familiar in atomic and nuclear physics to relativistic field theory and bound state problems. Certain of these applications are reviewed in Section III, especially fixed pole behavior [13], and rearrangement collisions [14]. Conversely, these techniques indicate a new systematic procedure for handling the relativistic and recoil corrections to atomic and nuclear physics problems [15].

An important question is what is the correct formulation of the bound state problem in the infinite momentum frame. In Section IV, we discuss in some detail the connection between the Bethe-Salpeter and infinite momentum frame wave functions [16]. Some comments on the applicability of new approximation procedures to QED within the $TOPTh_{\infty}$ formalism are given in the conclusion.

II - Quantum Electrodynamics in the Infinite Momentum Frame

In this section we will briefly discuss the implementation of the calculational rules for the TOPTh of QED. It is convenient to start with the familiar example of the electron vertex in second order. In a general frame, there are 3! time-ordered contributions which must be summed to yield the Feynman amplitude (see Figure 1). The amplitude for (a) yields by the standard rules for the time-ordered expansion of the S-matrix, (we use the Feynman-Gupta-Bleuler gauge) [17]

$$\mathcal{M}_\mu = -\frac{e^2}{(2\pi)^3}\int \frac{d^3k}{2E_1 2E_2 dE_k} \sum_{s_1,s_2} \frac{\bar{u}(p+q/2)\gamma_\alpha u_{s_1}(p_1)\bar{u}_{s_1}(p_1)\gamma_\mu u_{s_2}(p_2)\bar{u}_{s_2}(p_2)\gamma^\alpha u(p-q/2)}{(E_{p-q/2}-E_k-E_2+i\epsilon)\,(E_{p-q/2}+q_0-E_k-E_1+i\epsilon)}$$

Let us choose the Lorentz frame where

$$p \pm \frac{q}{2} = \left(P + \frac{M^2+\vec{q}_\perp^{\,2}/4}{2P}\,,\ \pm\vec{q}_\perp\,,\ P\right)$$

$$q = (0, \vec{q}_\perp, 0)\ ,\quad q^2 = -\vec{q}_\perp^{\,2} < 0$$

and we parameterize $k = \left(xP + \frac{\vec{k}_\perp^{\,2}}{2xP}\,,\ \vec{k}_\perp\,,\ xP\right)$, $d^3k = d^2k_\perp\, dx\, P$. Here P^2 is assumed to be large compared to M^2, $\vec{q}_\perp^{\,2}$, and $\vec{k}_\perp^{\,2}/x^2$. (Later we must check that we can perform a uniform limit).

Since the photon brings in zero longitudinal momentum in this frame, the only time-ordered diagram with all positive moving particles in each intermediate state is (a), if $0 < x < 1$. We then obtain for $P \to \infty$

$$\mathcal{M}_\mu = -\frac{e^2}{2(2\pi)^3}\int d^2k_\perp \int_0^1 \frac{dx}{x(1-x)^2}\, \frac{\bar{u}(p+q/2)\gamma_\alpha(\not{p}_1+m)\gamma_\mu(\not{p}_2+m)\gamma^\alpha u(p-q/2)}{\left[m^2+\frac{\vec{q}_\perp^{\,2}}{4}-\frac{\vec{k}_\perp^{\,2}}{x}-\frac{(\vec{k}-\vec{q}/2)^2+m^2}{1-x}\right]\left[m^2+\frac{\vec{q}_\perp^{\,2}}{4}-\frac{\vec{k}_\perp^{\,2}}{x}-\frac{(\vec{k}+\vec{q}/2)^2+m^2}{1-x}\right]}$$

where p_1 and p_2 are on the mass shell and are fixed by three-momentum conservation. This result for $\mathcal{M}_\mu$ is in fact the entire covariant result if $\mathcal{M}_\mu = 0$ or 3 . Projection operators for $F_1(q^2)$ and $F_2(q^2)$ can be used to obtain the electron form factors ; the results including $F_2(0) = \alpha/2\pi$ agree with the standard Feynman result [18] . However, as emphasized by Drell, Levy and Yan [6] , for the transverse components of $\mathcal{M}_1$ and $\mathcal{M}_2$, the time-ordered contributions (b) and (c) also contribute ; the spinor algebra produces an extra factor of P^2 which compensates for the single denominator with backward moving particles. In fact this just produces the "seagull" diagrams for the transverse currents characteristic of the $\vec{A}^2$ terms in boson current theories. Fortunately, these are always easy to include by the substitution rule [19]

$$p_1 \longrightarrow \tilde{p}_1 \text{ , with } \tilde{p}_{10} = p_1 + E_{inc} - E_{int} \text{ , } \vec{\tilde{p}} = \vec{p}$$

which formally enforces energy conservation between the external and given intermediate state. This "automatic Z-graph" rule is to be used for each fermion line which extends over a single time-interval. With this substitution rule, only positive-moving graphs such as (a) need to be explicitly considered. This replacement of p_μ by $\tilde{p}$ is very reminiscent of the Feynman approach. One takes the fermion off the mass shell $(\tilde{P}_1^2 \neq m^2)$, but reduces the number of diagrams. However, since not all fermion lines extend over a single time-interval we do not have complete four momentum conservation, and energy conservation is computed between the external and intermediate states , not at each vertex as in the Feynman case. The complete $OFPTH_\infty$ rules are given in the table.

It is straightforward to check that the TOPTh rules and Feynman rules coincide for all Born (tree) calculations. We can also, by using

the summation properties of TOPTh amplitudes, readily demonstrate vertex factorization characteristic of the Feynman theory. For example the sum of all time-orderings for the diagrams of figure (2) yields

$$= \frac{e}{q_F^2 - \lambda^2 + i\epsilon} \; F(q_F^2)$$

where $F(q_F^2)$ is computed from the sum of all time-orderings of the vertex function with a photon of mass $q_F^2 \equiv (p_1 - p_2)^2$. Thus the concept of the off-mass-shell particle plays a natural role in the TOPTh calculations. As a check on our techniques, we can also adopt the following procedure. We start from the Feynman expression for the vertex :

$$\mathcal{M}_\nu = \int d^4k/i \; \frac{\gamma_\mu (\not p_1 + m)\, \gamma_\nu (\not p_2 + m)\, \gamma^\mu}{(p_1^2 - m_1^2 + i\epsilon)(p_2^2 - m_2^2 + i\epsilon)(k^2 - \lambda^2 + i\epsilon)}$$

and parameterize the four momenta as follows $(\vec{q}_\perp^{\,2} = -q^2)$

$$q = (0, \vec{q}_\perp, 0)$$

$$p = \left(P + \frac{m^2 + \frac{1}{4}\vec{q}_\perp^{\,2}}{4P}, \vec{q}, P - \frac{m^2 + \frac{1}{4}\vec{q}_\perp^{\,2}}{4P}\right)$$

$$k = \left(xP + \frac{s_k}{4P}, \vec{k}_\perp, xP - \frac{s_k}{4P}\right)$$

with $s_k = (k^2 + \vec{k}_\perp^{\,2})/x$.

Notice that in this case the mass-shell conditions are satisfied independent of the value of P ; in fact $y = \log 2P/m$ is the rapidity of the incident electron. Of course, in the frame in which $P \to \infty$, the quantity $x = (k_0 - k_3)/2P$ is the fractional longitudinal momentum carried by the photon.

The four degrees of freedom of k^2 are replaced, then, by x, $\vec{k}_\perp$, and s_k ; $d^4k = d^2k\, dx\, ds_k$. Assuming uniform convergence, the integration over s_k may be performed by contour integration [20] . The result is non-zero only for $0 < x < 1$ and agrees with the $TOPTh_\infty$ expression, including the Z-graph substitution rule. Although this one-to-one correspondence does not work for general graphs, one can see here the close relation to the light-cone quantization method in which the variables $(t+z)2P$ and $(t-z)/2P$ conjugate to s_k and x are utilized. In fact the standard rules of calculation are identical in the two theories. For example, the Z-graph contributions of $TOPTh_\infty$ correspond to explicit seagull terms in the interaction using the light-cone quantization method. However, as we shall emphasize below, the direct development of the calculational rules from standard theory with the $P \to \infty$ limit allows one to develop renormalization theory and avoids errors due to non-uniform convergence in the $P \to \infty$ limit. Further, the $P \to \infty$ method allows the use of the Feynman gauge and a straightforward implementation of the renormalization procedure.

Surprisingly, the renormalization procedure is simple and straightforward to apply in $TOPTh_\infty$, and can be applied in parallel to the explicitly covariant Feynman-Dyson approach. Reducible amplitudes with self-energy and vertex insertions are renormalized using subtraction terms corresponding to δ_m , Z_2, and Z_1 counterterms, which can be usually constructed to cancel point-wise the ultraviolet $d^2k_\perp$ phase-space integrations. Since we are subtracting infinite quantities all integrals are assumed to be (covariantly) regulated by using the Pauli-Villars scheme.

As an example, consider the self-energy insertion to Compton scattering in a scalar theory shown in figure 3 . In the usual Feynman approach the renormalized amplitude is constructed by subtracting formally-divergent δ_m and Z_2 subtraction terms in second order. Here we will construct

an integral representation for these constants so that the total integrand of the renormalized amplitude is finite and point-wise convergent. Taking $\vec{p}_a = \vec{P}$, with $P \to \infty$ (only one time-ordering survives) the unrenormalized amplitude is (assuming regularization in the mass λ)

$$\mathcal{M}_u = \frac{g^2}{2(2\pi)^3} \int d^2k_\perp \int_0^1 \frac{dx}{x(1-x)} \left[\frac{1}{s_0-s_1+i\epsilon} \frac{1}{s_0-s_2+i\epsilon} \frac{1}{s_0-s_3+i\epsilon} \right]$$

with $s_0 = (p_1+q_1)^2$, $s_1 = s_3 = m^2$, and

$$s_2 = \frac{\vec{k}_\perp^2 + \lambda^2}{x} + \frac{\vec{k}_\perp^2 + m^2}{1-x} ,$$

The renormalized amplitude is then

$$\mathcal{M}_r = \frac{g^2}{2(2\pi)^3} \int d^2k_\perp \int_0^1 \frac{dx}{x(1-k)}$$

$$\left[\frac{1}{(s_0-s_1)(s_0-s_2)(s_0-s_3)} - \frac{1}{(s_0-s_1)(s_1-s_2)(s_0-s_3)} + \frac{1}{(s_0-s_1)(s_1-s_2)^2} \right]$$

The middle term is seen to be exactly what is required to compute the counterterm if the integration with the denominator $(s_1-s_2)^{-1}$ is performed. Similarly the last term becomes the standard Z_2 counterterm upon integration over $d^2k_\perp$ and dx . Thus the counterterms differ from the unrenormalized term by choice of the denominators ; specifically, the external energy s_{ext} used for the denominator of the subtraction term for a reducible insertion is not the initial energy s_0 but rather the energy s_1 external to that reducible subgraph. We call this the method of "alternate denominators" . The analogue to off-mass-shell behavior in the Feynman approach is precisely this difference between the use of s_0 and s_1 in the energy denominators. Upon combining the terms in $\mathcal{M}_r$, the total

integrand is finite in the ultraviolet $(\vec{k}_\perp^2 \to \infty)$, the single particle pole disappears, so that the location and residue of the pole is still given by the Born term.

For a reducible insertion on a line with momentum $\vec{p}_a$, not necessarily in the z-direction, the use of "scaled" variables $\vec{\ell}_\perp = x\,\vec{p}_a + \vec{k}_\perp$ for the parameterization of the loop integration permits point-wise convergence. In the case of graphs with reducible vertex parts which contribute to more than one surviving time-ordering, some algebraic rearrangement is required to cancel the separate ultraviolet parts of each time-ordered contribution. In the case of spin, the subtraction term numerator of the vertex subgraph must be constructed, as in the Feynman case, to agree with the correct numerator coefficient of γ_μ.

With these rules, the renormalization procedure can be carried out in parallel with the Feynman-Dyson procedure. Because all of the renormalized amplitudes are finite, we always have uniform convergence with respect to the $P \to \infty$ limit once the subtractions are accounted for. In the case of the subtraction terms themselves, there is a subtlety in taking the $P \to \infty$ limit; the calculation of δm_ℓ in second order in QED is an excellent example. There are two contributions (see figure 4) [21]. Because of the singular nature of the integration at $x \sim 0$, the contribution of the integration region where the photon has small fractional longitudinal momentum, and produces an anomalous $\delta(x)$ contribution associated with the Z-graph.

In intermediate stages of the calculation, one has to regulate the individual time-ordered terms to handle up to quadratic ultraviolet $d^2k_\perp$ integrations, although in the final calculation only logarithmic terms appear. The anomalous term also arises in the light-cone quantization method from a "seagull" interaction. In the case of the ds_k integration method discussed before, the anomalous term appears because it is invalid

to perform the dz_k integral first at $x \sim 0$ for part of the δm calculation. Fortunately none of these subtleties are involved in the computation of the physical, renormalized quantities.

It is an interesting question whether the renormalization procedure of quantum electrodynamics could have been discovered in the 1930's had the infinite momentum technique been known. It is, for example, easy to show that by naïve power counting that the degree of divergence in any $TOPTh_\infty$ amplitude only depends on the number of external lines [9] :

$$d_{1/2} = 4 - F_e - B_e$$

Interestingly, the same result is obtained for spin 0 electrodynamics in which δ_m needs quadratic regulation. The result for $d_{1/2}$ is thus an overestimate since, as the Feynman result shows, the portion $F_e/2$ in $d_{1/2}$ cancels when the various time-orderings are combined.

In addition to carrying out the above analysis, Roskies and I have calculated the fourth order anomalous moment and have attempted part of the sixth order calculation. Choosing the frame as above, and employing the automatic Z-graph rule, there are no more than 3 time-ordered graphs for each Feynman graph for the order α^2 corrections, and between 1 and 15 (out of a possible 7!) for the Feynman amplitudes at order α^3. We wrote a simple Fortran program which enumerated all of the contributing $TOPTh_\infty$ amplitudes for each Feynman vertex graph. We then used Hearn's algebraic program REDUCE [22] to

(1) Calculate the numerator Dirac algebra for the projection of the $F_2(q^2)$ invariant amplitude. Due to the Z-graph rule, the form of the trace is independent of the time-ordering ; later specific substitutions for the invariants are required.

(2) Construct all of the four vectors and their parameterization according to each time-ordering.

(3) Construct the s_i-energy denominators and phase-space factors specific to each time-ordered amplitude.

(4) By matching substitutions, perform the denominator expansions to the required order in $\vec{q}_\perp$ and integrate over angles.

The result is an integrand in the $\pi d^2k_i dx$ space, with the trivial $d\phi_i$ integration accounted for. After identification and subtraction of the infrared terms (which occur at $x_i \sim 0$), the results were integrated numerically using the RIWIAD program [23] which employs an adaptive Simpson rule with random sampling. The order α^2 Peterman - Sommerfield [10] results were easily obtained ; the numerical integrations over the $d^2k_i dx_i$ variables appear to be more convergent than the corresponding Feynman results, although we left the integrand in a considerably more complicated form than that which appears in the standard Feynman results. In the case of the sixth order ladder graph, a six-dimensional integrand was integrated in 4 minutes on the SLAC IBM 360-91 to give

$$\frac{g-2}{2} = \frac{\alpha^3}{\pi^3}(1.777 \pm .013)$$

to be compared with the analytic value of Levine and Roskies [1] (see also Levine and Wright [24], Kinoshita and Cvitanović [1])

$$\left(\frac{g-2}{2}\right) = \frac{\alpha^3}{\pi^3}(1.7902778\ldots)$$

The sixth order ladder graph is twice reducible and requires the standard subtraction of vertex subgraphs in second and fourth order. The infrared divergent behavior of the renormalized amplitude is cancelled point-wise in the integrand by employing symmetrization of the integration in the three loop variables. Only one TOPTh_∞ amplitude is required.

We also attempted to calculate a representative irreducible contribution to the sixth order moment, with 8 time-orderings. In this case the algebraic complexity of the integrand makes accurate numerical integration prohibitive ; the results (within large numerical errors) were consistent with the Levine-Wright results for this graph ((12) of ref. [24]).

Our conclusion, then, is that the infinite momentum frame technique is a valid and viable calculational alternative to the usual Feynman parameter method, and can be advantageous in the cases where only a very few time-ordered amplitudes contribute. It is not competitive algebraically when many time-orderings are required. On the other hand, since the techniques are valid in quantum field theoretic perturbation calculations, it opens the way to the application of new approximations in QED. The infinite momentum frame techniques provide a relativistic but Schrödinger-like Fock space (particle number) representation of the electron wave-function and variational, Hartree-Fock, or consistent field approximations may have applicability [25] . One's intuition in dealing with field theory in the infinite momentum frame is further strengthened by the remarkable fact that the (good) components of the current (in the transverse $\vec{q}$ frames) conserve particle number , i.e. have no pair-creation or pair-annihilation matrix elements.

This fact, plus the manifest unitarity, i.e., the intermediate states have a definite number of on-mass-shell particles, makes the infinite momentum frame especially useful for bound state problems. This is discussed in the next section.

III - Bound State Problems in the $P \to \infty$ Frame

Suppose we choose a Lorentz frame such that a bound system has momentum $\vec{P}$ in the direction z. For $P \to \infty$ chosen large enough, we can assume that all of the constituents will be moving in the positive z-direction ; more specifically, we assume the existence of a Fock-space N-particle wave-function amplitude in the infinite momentum frame

$$\lim_{P\to\infty} \psi_P^{(N)}(\vec{p}_i) = \psi^{(N)}(\vec{k}_{i\perp}, x_i) \qquad , i = 1..N$$

where

$$\vec{p}_i = x_i \vec{P} + \vec{k}_{i\perp}$$

$$\sum_{i=1}^{N} \vec{k}_{i\perp} = 0 \qquad , \quad \sum_{i=1}^{N} x_i = 1$$

with $x_i > 0$.

The matrix element of the current is (assuming all scalar particles)

$$\langle p+q | J_\mu(0) | p \rangle = \langle p+q | j_\mu(0) | p \rangle = F(q^2)(2p+q)^\mu$$

where (choosing $\mu = 0$) (see figure 5) .

$$F(q) = \int \frac{d^2k_\perp}{2(2\pi)^3} \int_0^1 \frac{dx}{x(1-x)} \psi^{*(2)}(\vec{k}_\perp + (1-x)\vec{q}_\perp, x)\, \psi^{(2)}(\vec{k}_\perp, x)$$

This is for the special case of the two particle state where the constituent labeled with fractional longitudinal momentum x is charged ; the general formula is given in ref.[6]. Here $\vec{k}_\perp + (1-x)\vec{q}_\perp$ is the component of the vector $\vec{k}_\perp + \vec{q}_\perp$ transverse to the $\overrightarrow{p+q}$ direction of the final state. In the case of the renormalized physical electron in QED, the complete

Fock space representation can be identified to each order in perturbation theory and particle number N from the calculation of the vertex function. Wave-function normalization, i.e. the Ward identity, gives $F(q^2 = 0) = 1$. In the case of bound states (such as atoms, nuclei, or the quark parton model) we can write integral equations or employ the Bethe-Salpeter equation to determine the $\psi^{(n)}$.

Weinberg [5] has proposed an infinite system of integral equations to determine the (n)-particle $P \to \infty$ wave-functions. Truncating at $n = 2$, in $g\phi^3$ theory, (see fig. 6) this yields an eigenvalue equation for the square of the bound state mass m^2:

$$\left[m^2 - \frac{\vec{k}_\perp^2 + m_1^2}{x} - \frac{\vec{k}_\perp^2 + m_2^2}{1-x}\right]\psi_{m^2}(\vec{k}_\perp, x) = \frac{g^2}{2(2\pi)^3}\int d^2\ell_\perp \int_0^1 \frac{dz}{z(1-z)}\, V_W(\vec{k}_\perp, \vec{\ell}_\perp, z, x)\,\psi_{m^2}(\vec{\ell}_\perp, z)$$

where

$$V_W = \frac{\Theta(x-z)}{x-z}\left[m^2 - \frac{\vec{k}_\perp^2 + m_2^2}{1-x} - \frac{\vec{\ell}_\perp^2 + m_1^2}{z} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2 + \lambda^2}{x-z}\right]^{-1}$$

$$+ \frac{\Theta(z-x)}{z-x}\left[m^2 - \frac{\vec{\ell}_\perp^2 + m_2^2}{1-z} - \frac{\vec{k}_\perp^2 + m_1^2}{x} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2 + \lambda^2}{z-x}\right]^{-1}$$

Note that V_W depends on the eigenvalue m^2. This result is of course a drastic approximation since a connection to the three particle state is required in even lowest order perturbation theory. Feldman, Fulton and Townsend [26] have shown that this equation's eigenvalues agree with the Bethe-Salpeter equation in ladder approximation up to terms of order $m\alpha^2$ and $m\alpha^3 \log \alpha^{-1}$, where $\alpha = g^2/4\pi$. Note, however, that terms of order $\alpha^3 \log \alpha^{-1}$ are anomalous ; they do not appear in the full Bethe-Salpeter equation (with the crossed-graph kernels) which has an eigenvalue expansion proceeding as α^2, α^4 [27].

More recently Namyslowski [28] and also Atanasov [29] have used

the light-cone or infinite momentum variables and the simple two-body propagator of the Weinberg equation to construct two-body quasi-potential equations for the bound state. In fact, by replacing the x variable by a third component momentum variable k_3 in order to recover explicit rotational invariance in the cm system, their results become essentially equivalent to the quasi-potential equations of Todorov [3],[30].

The Todorov and the related Grotch-Yennie [3] equations may in fact provide the best attack on the two-body problem. However, it could be more desirable to have a systematic covariant procedure and, in principle, exact approach to the bound state problem.

In a new approach that I have been working on recently, one starts with the exact Bethe-Salpeter equation [31] and systematically deduces an infinite momentum frame two particle equation. The Bethe-Salpeter equation for two spin $\frac{1}{2}$ particles is

$$(\not{k}-m_1)(\not{p}-\not{k}-m_2)\Psi_p(\ ) = i\int\frac{d^4\ell}{(2\pi)^4}K(p,\ell,k)\Psi_p(\ell)\equiv\Gamma(k)$$

where k is the sum of all two-particle irreducible kernels. The wave-function Ψ_p is the amplitude for the coupling of the bound state to two off-shell particles. Let us use the general Lorentz frame parameterization

$$p = (P + \frac{m^2}{4P}, \vec{0}_\perp, P - \frac{m^2}{4P})$$

$$k = (xP + \frac{s_k}{4P}, \vec{k}_\perp, xP - \frac{s_k}{4P})$$

$$\ell = (zP + \frac{s_\ell}{4P}, \vec{\ell}_\perp, zP - \frac{s_\ell}{4P})$$

where again P is arbitrary ($P = M/2$ is the cm system) and $s_k = \frac{k^2+\vec{k}_\perp^2}{x}$, $s_\ell = \frac{\ell^2+\vec{\ell}_\perp^2}{z}$.

Let us write

$$K = \bar{K}(\vec{\ell}_\perp, z, \vec{k}_\perp, x) + \delta K$$

so that $\bar{K}$ is independent of s_k and s_ℓ. We can treat δk later by perturbation theory.

There are many ways of choosing the approximate kernel $\bar{K}$. For example we can ignore the dependence of the kernel on the off-mass-shell mass of particle one and take

$$s_k \to (m_1^2 + \vec{k}_\perp^2)/x \quad , \quad s_\ell \to (m_1^2 + \vec{\ell}_\perp^2)/z$$

This is clearly an excellent approximation for $m_2^2 >> m_1^2$, or weakly bound atomic systems. In any case, using $\bar{K}$, $\Gamma \to \bar{\Gamma}(\vec{k}_\perp, x)$ we can define a new wave-function [32]

$$\Psi_{m^2}(\vec{k}_\perp, x) \equiv i \int_{-\infty}^{\infty} \frac{ds_k}{2\pi} \Psi_p(k)\, x(1-x)$$

$$= i \int_{-\infty}^{\infty} \frac{ds_k}{2\pi} \frac{(\not p - \not k + m_2)(\not k + m_1)\bar{\Gamma}(\vec{k}_\perp, x)\; x(1-x)}{[(p-k)^2 - m_2^2 + i\epsilon][k^2 - m_1^2 + i\epsilon]}$$

$(0 < x < 1)$

$$= \left. \frac{(\not p - \not k + m_2)(\not k + m_1)\bar{\Gamma}(\vec{k}_\perp, x)}{m^2 - \frac{\vec{k}_\perp^2 + m_1^2}{x} - \frac{\vec{k}_\perp^2 + m_2^2}{1-x} + i\epsilon} \right|_{s_k = \frac{m^2 + \vec{k}_\perp^2}{x}}$$

We thus have the simple 3-dimensional equation

$$\left[m^2 - \frac{\vec{k}_\perp^2 + m_1^2}{x} - \frac{\vec{k}_\perp^2 + m_2^2}{1-x}\right] \Psi_{m^2}(\vec{k}_\perp, x) = \mathcal{P} \int \frac{d^2\ell_\perp}{(2\pi)^3} \int_0^1 \frac{dz\, \bar{K}(\vec{k}_\perp, x, \vec{\ell}_\perp, z)}{z(1-z)} \Psi_{m^2}(\vec{\ell}_\perp, z)$$

with

$$\mathcal{P} \equiv (\not p - \not k + m_2)(\not k + m_1)\Big|_{k^2 = m_1^2}$$

We can also write this in the Dirac form

$$(\not{p}-\not{k}-m_2)\,\Psi^{(2)}_{m^2}(\vec{k}_\perp,x) = (\not{k}+m_1)\int \frac{d^2\ell_\perp}{(2\pi)^3}\int_0^1 \frac{dz}{z}\,\overline{K}\,\Psi^{(2)}_{m^2}(\vec{\ell}_\perp,z)$$

with $k^2 = m_1^2$, $p^2 = m^2$ and $\Psi^{(2)} = \Psi_{m^2}(\vec{k}_\perp,x)/(1-x)$.

In either case one has a simple eigenvalue equation for m^2 . Since the potential does not depend on m^2 , perturbation theory is Schrödinger-like, and unlike the Bethe-Salpeter equation, the wave-function normalization condition is trivial. The infinite momentum frame form factor expression given above is correct for the wave-function $\Psi_{m^2}(\vec{k}_\perp,x)$. The Dirac form of the equation shows that we have a covariant equation for particle (2) in the field of the one-shell particle (1) where all of the relativistic kinematics of particle (1) are retained. The equations can be rewritten in terms of standard cm 3-dimensional coordinates by utilizing the <u>definition</u>

$$k_z = xP - \frac{m_1^2+\vec{k}_\perp^{\,2}}{4xP} \qquad , \text{ with } P = m/2 .$$

The Dirac-Coulomb equation is recovered for $m_1 \to \infty$.

Using various approximation for $\overline{K}$ we can also obtain quasi-potential equations for Ψ_{m^2} . However, since we start from the exact Bethe-Salpeter formalism we have a systematic procedure which can be carried out to any order in the interaction.

We plan to investigate the usefulness of this approach to positronium in the near future. It is also clear that the n-particle generalization of this equation will be useful for general atomic physics problems where relativistic and recoil corrections are important.

Among the examples of such applications are

(1) The high energy limit ($\gamma >> B,E$) of forward Compton

scattering on the bound electrons of an atom One finds that the spin-averaged amplitude is asymptotically constant and real [33]:

$$f(\nu) \Rightarrow -Z\,\frac{e^2}{M_T}\int_0^1 \frac{f_e(x)}{x}\,dx \equiv -\frac{Ze^2}{m_{eff}}$$

where $f_e(x)$ is the normalized probability of finding the electron with fractional longitudinal momentum ; it can be computed from a wave-function integration, or in fact be directly measured in deep inelastic electron-atom scattering.

Note that $x\,M_T$ plays the role of the effective electron mass ; m_{eff} contains corrections from atomic binding and finite nuclear mass corrections. The result is derived simply from the Z-graph contributions to the Compton amplitude.

(2) Rearrangement collisions in atomic, nuclear or quark model bound state scattering theory. The rearrangement amplitude for the elastic collision of two-particle bound state systems (e.g. : electron exchange in H - H atom collisions) is simply (see figure 6) [34]

$$\mathcal{M}(t,u) = \int \frac{d^2k_\perp}{2(2\pi)^3}\int_0^1 \frac{dx}{x^2(1-x)^2}\,\Delta$$
$$\Psi_B(\vec{k}_\perp)\Psi_D(\vec{k}_\perp + (1-x)\vec{q}_\perp)\,\Psi_A(\vec{k}_\perp + (1-x)\vec{q} - x\vec{r}_\perp)\,\Psi(\vec{k}_\perp - x\vec{r}_\perp)$$

where $\Delta = \sum_{inc} s_i - \sum_{int} s_j$

and $\vec{q}_\perp^{\,2} = -t = 2p_{cm}^2(1-\cos\theta_{cm})$

$\vec{r}_\perp^{\,2} = -u = 2p_{cm}^2(1+\cos\theta_{cm})$

$\vec{q}_\perp\cdot\vec{r}_\perp = 0$ (assuming $m_A^2 + m_B^2 = m_C^2 + m_D^2$).

This result ignores the Coulomb interactions between the electrons and between the atoms , but includes the binding forces correctly (including all recoil and relativistic terms).

Spin corrections are discussed in ref. [14]. This result assumes the wave-functions obey approximate equations $\Psi_{m^2}(\vec{k}_\perp, x)$ with $\bar{\Gamma} \neq \Gamma(s_\ell)$. It is then obtained immediately from the Feynman loop integration using the frame $p_z = P - \frac{m^2}{4P}$ and explicit integration over s_ℓ .[35]

In conclusion, we have seen that the $P \to \infty$ method is a valid and useful calculational alternative to standard perturbation theory methods. The most exciting future applications may be in bound state problems in quantum electrodynamics. The single-time equation derived above for $\Psi_{m^2}(\vec{k}_\perp, x)$ provides a rigorous, systematic basis for calculations of the positronium and muonium spectra. However, because the infinite momentum equations so closely resemble non-relativistic many-body theory, there is also the exciting potential of applying new types of perturbation methods, including variational and self-consistent approximation to problems in relativistic field theory.

Acknowlegements

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We assume at this point that contributions from the numerator dependence in s_k can be ignored. A more complete discussion of these equations will be given elsewhere.

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Figure and Table Captions

1. Illustration of Time-Ordered Perturbation Theory (TOPTh) graphs for the electron vertex function. The covariant Feynman amplitude requires in general the sum of 3! time-ordered contributions. In the $P \to \infty$ Lorentz Frame, only (a) needs to be explicitly computed.
2. Factorization of the Vertex Function. Upon summation over all time-orders, the Feynman vertex function $F(q^2)$ factorizes, with $q = p_1 - p_2$.
3. Unrenormalized Compton Amplitude with a self-energy insertion.
4. Time-ordered contributions to the electron-self-energy. The contributions of the vacuum disconnected graph (c) with $p' = p$ replaces that of (b) if the Pauli principle is taken into account.
5. Time-ordered Perturbation Theory contribution to the Bound-State Form factor. In the $P \to \infty$ Frame, with $q_z = 0$, the other graphs, involving pair creation and annhilation do not contribute to the j_0 and j_z matrix elements.
6. The Weinberg equation truncated to the two particle state.
7. Rearrangement collision of two bound states. In the case of H atom-H-atom collisions, the electrons are interchanged.

Table Caption

Rules for calculation using time-ordered perturbation theory in the infinite momentum frame (TOPTh_∞). See ref.[9] for derivations. The atomic Z-graph rule indicated in step 4 states that energy conservation between the initial and intermediate state rather than the mass-shell condition is to be used for the spin projection sum if the fermion spans one and only one time interval.

- TABLE I -

Calculation Rules for Q.E.D. in TOPTh$_\infty$.

$$S = 1-(2\pi)^4 i\,\delta^{(4)}(p_{final} - p_{init}) \quad \pi(2\pi)^{-3/2}(2E_i)^{-1/2}$$

To compute $\mathcal{M}$, the invariant amplitude

1. For each Feynman graph of order n , write all time-ordered forward moving graphs
2. Assign $\vec{k}_\perp$,x for each internal line
3. At each vertex but the last, assign a factor $(e^2 = 4\pi\alpha)$

$$(2\pi)^3\delta(\Sigma x_i)\,\delta^2(\Sigma \vec{k}_{i\perp}) \begin{cases} e\,\bar{u}\,\not{\varepsilon}\,u \\ e\,\bar{u}\,\not{\varepsilon}\,v \\ -e\,\bar{v}\,\not{\varepsilon}\,v \\ -e\,\bar{v}\,\not{\varepsilon}\,u \end{cases}$$

with just the spinor factor at the last vertex.

4. Sum over polarizations

$$\sum_{pol} \varepsilon_\mu \varepsilon_\nu = -g_{\mu\nu} \qquad \sum_{spin}\begin{Bmatrix} u\,\bar{u} \\ v\,\bar{v} \end{Bmatrix} = (\pm\not{p}+m) \quad \text{or} \quad (\pm\tilde{\not{p}}+m)$$

for fermions spanning one time interval

5. For each internal state assign the factor

$$\frac{2}{\sum_{inc} s_i - \sum_{int} s_i + i\epsilon}$$

6. Integrate $\int d^2k_i\, dx_i\, \theta(x_i)\, (2\pi)^{-3}\, (2x_i)^{-1}$ for each internal line.

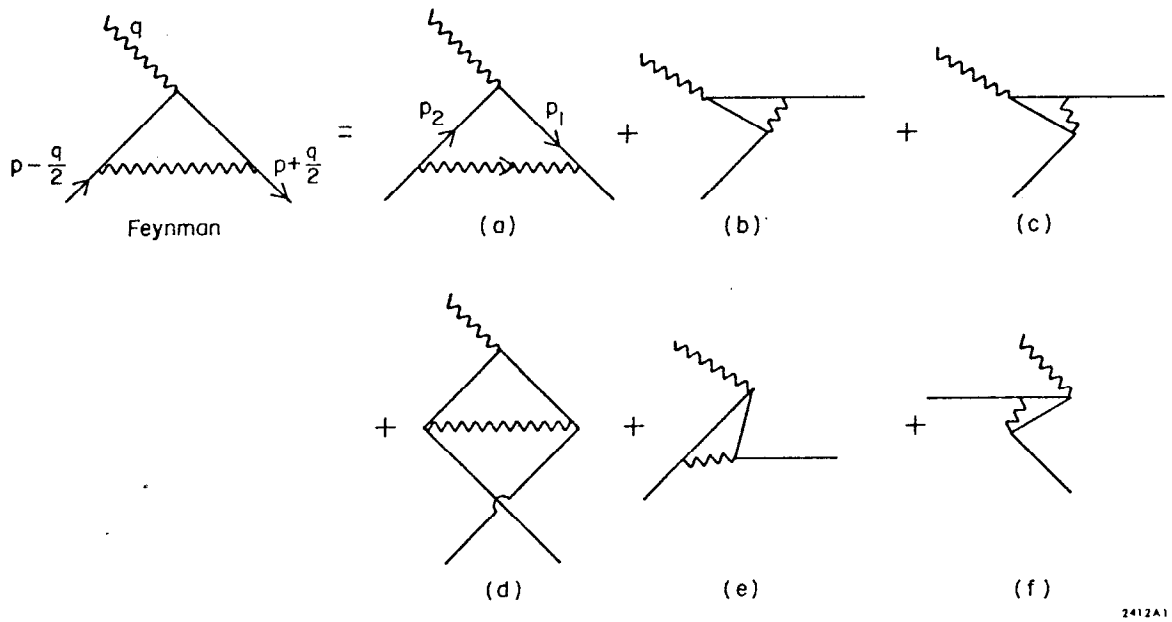


Fig. 1

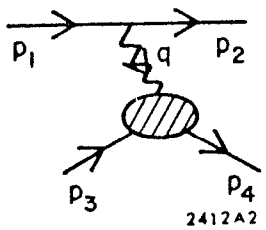


Fig. 2

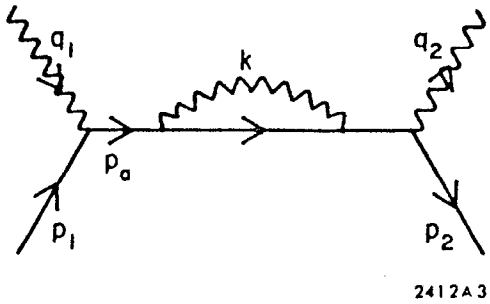


Fig. 3

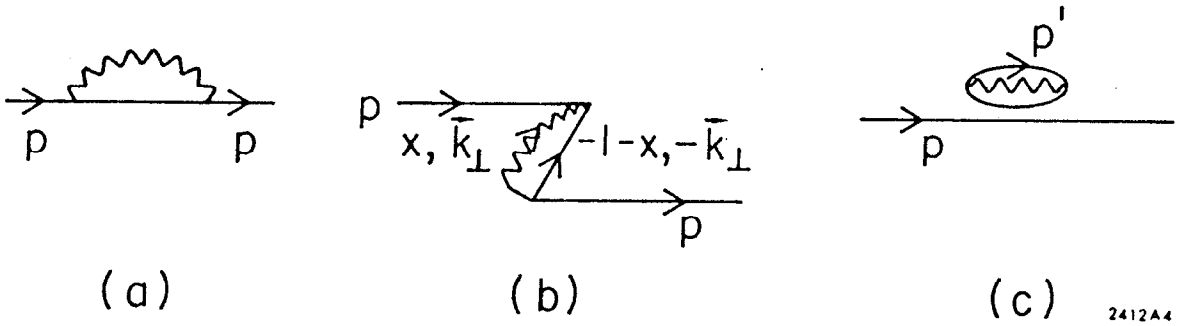


Fig. 4

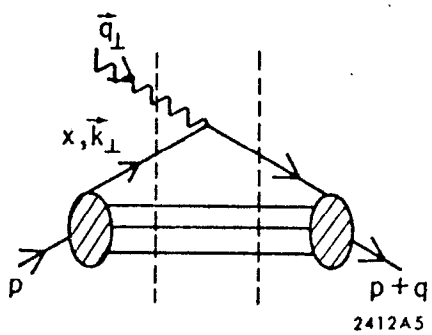


Fig. 5

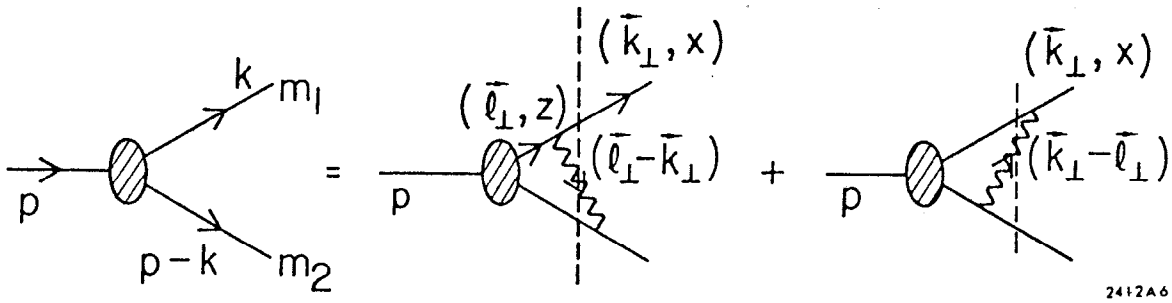
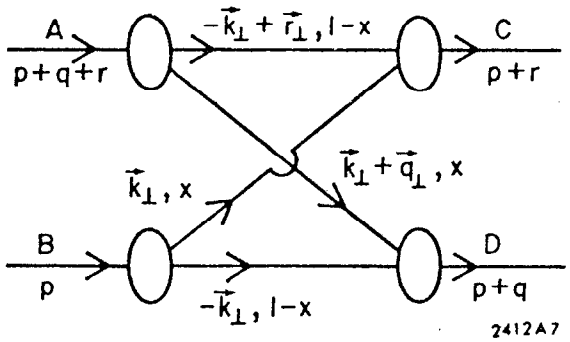

p
k
m1
p−k
m2
=
p
(ℓ⊥, z)
(k⊥, x)
(ℓ⊥−k⊥)
+
p
(k⊥, x)
(k⊥−ℓ⊥)
2412A6

Fig. 6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p+q+r
−k⊥ + r⊥, 1−x
C
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k⊥, x
k⊥ + q⊥, x
B
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−k⊥, 1−x
D
p+q
2412A7

Fig. 7