# AMPLITUUE ANALYSES IN THREE BODY FINAL STATES ${ }^{1}$ 

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## I. INTRODUCTION

In these lectures I am going to attempt to review the status and methods of analysis of three body final states. In the last few years there has been a great deal of activity in the phenomenological analysis of these systems motivated by a variety of reasons. The three body states considered have been obtained in both formation and production experiments as indicated in Fig. 1 and the main emphasis has been in the "spectroscopic" aspects of the results.

In the formation experiments work has centered primarily around the reactions (Herndon et al., 1972; Mast et al. , 1972)

$$
\begin{gather*}
\pi \mathrm{N} \rightarrow \pi \pi \mathrm{~N}  \tag{1}\\
\mathrm{k}^{-} \mathrm{p} \rightarrow \Lambda \pi \pi \tag{2}
\end{gather*}
$$

Such reactions constitute a large part of the inelastic cross sections derived from the incident two body systems. Indeed for $\pi \mathrm{N}$ collisions with $\mathrm{E}_{\mathrm{c} . \mathrm{m} .}<2000 \mathrm{MeV}$, the $\pi \pi \mathrm{N}$ channel accounts almost completely for the inelastic cross section which itself is approximately $50 \%$ of the total. Thus if we are ever to understand the $\pi N$ interaction it will be essential to have a description of these three body states. Furthermore, this low energy region is dominated by overlapping resonances ("the resonance region") many of which are highly inelastic. In our present attempts to understand the systematics of these resonances, the existence and branching fractions of each state are required. Of particular interest are the decays into $\pi \Delta, N \rho, \pi Y_{1}^{*}(1385)$, etc., which can be related to $\pi N$,
$\overline{\mathrm{K}} N, \Lambda \pi$, etc., in any symmetry scheme higher than SU 3 . The fact that the parent resonances are overlapping necessitates partial wave analyses which would be comparatively easy were the $\Delta, \rho$, ctc., stable particles (although of course $0^{-}+1 / 2^{+} \rightarrow 0^{-}+3 / 2^{+}$is necessarily more complicated than $0^{-}+1 / 2^{+} \rightarrow 0^{-}+1 / 2^{+}$). Unfortunately they are not and their widths are sufficiently large that for much of the Dalitz plot their amplitudes (Breit-Wigner) are large, as is clear from Fig. 2. In these circumstances it is difficult to select the data to produce a pure sample of, e.g., $\pi N \rightarrow \pi \Delta$ and any detailed analysis of such selected data will be correspondingly suspect. Thus one must resort to other techniques which can take into account these overlapping resonance bands and perhaps even exploit their presence. Such methods and one in particular, the isobar model, together with their results will be the subject of these lectures. At this time the most important results have emerged from the analysis of reaction (1) but one should expect in the future valuable results from $\overline{\mathrm{K}} N$ induced reactions, the $\mathrm{Y}^{*}$ situation being at least as complicated as the N*.

The activity in production reactions is essential for a variety of reasons (Ascoli et al., 1971). For meson systems parity conservation forbids the decay into two pseudoscalar mesons of any resonance belonging to the unnatural spin parity series $0^{-}, 1^{+}, 2^{-}$. The first available state is then usually the three pseudoscalar meson system and thus we expect to observe such resonances there. The requirement of G-parity conservation means that for $G=-1, S=0$ meson resonances the decay into
two pions is forbidden. We then once again expect to see these states first in three pion systems, e.g., $\mathrm{A}_{2} \rightarrow 3 \pi$. The whole discussion of the existence of many of these resonances is further complicated by dynamical processes which may occur. The diffractive excitation of a $\pi$ or K (see Fig. 3) will automatically lead to a system of particles with unnatural spin parity if the following rule is obeyed

$$
\begin{equation*}
\Delta P=(-1)^{\Delta J} \tag{3}
\end{equation*}
$$

where $\Delta \mathrm{P}$ is the change in parity and $\Delta \mathrm{J}$ the change in angular momentum. Hence we have long had the debate as to whether the low mass enhancement in the three pion system is due to a resonance or a dynamical effect (Deck mechanism). The resolution of this problem is vitally important to meson spectroscopy and awaits study of these three particle states in reactions in which diffraction dissociation is not possible, e.g., the charge exchange or hypercharge exchange reactions of Fig. 4

$$
\begin{align*}
& \mathrm{K}^{-} \mathrm{n} \rightarrow \pi^{-} \pi^{+} \pi \Lambda  \tag{4}\\
& \pi^{+} \mathrm{n} \rightarrow \pi^{+} \pi^{-} \pi^{o} \mathrm{p} \tag{5}
\end{align*}
$$

Of course we would also like a clear understanding of the process of diffraction excitation and as already noted this leads naturally to the consideration of three particle final states (these are in general the easiest to observe experimentally). Thus we will continue to study the reactions of Fig. 3, but we will in addition direct our attention to the diffractive excitation of the nucleon as indicated in Fig. 5. Indeed the latter is possibly the best system in which to study this process as one
already has detailed information on the baryon resonances spectrum. In general the analysis of these states in production reactions will always give information on the production dynamics.

In the following sections, I will discuss the various techniques of analysis (Section II) leading to a very detailed description of the isobar model (Section III), in order to allow people both to use the method themselves but also to be familiar with the approximations, definitions, etc., involved in any of the quoted results. Sections III. A and III. B will be concerned with the calculation of matrix elements while Sections III.C and III. D will contain the applications to formation and production reactions respectively. In Sections IV and V, I will describe and try to evaluate the results which have so far been obtained and finally in Section VI I will attempt to summarize the situation and indicate in which directions we will proceed.
II. THE METHODS OF ANALYSING THREE BODY FINAL STATES

In the introduction I have already hinted at some of the methods one might apply and I would now like to discuss them in more detail. I will first deal with the approach in which one attempts to reduce the problem to one of the stable two body scattering by suitable cuts on the data. This will be followed by a discussion of the general methods of analysing three body final states. .However the ambiguities, the variety of experimental data required, and the difficulties of interpretation render these approaches of little value at present making it necessary to use somewhat more model dependent methods. The most successful of these models is introduced and dealt with in detail in Section III.

## A. SELECTION OF TWO BODY REACTIONS

The oldest approach to analysing three body final states is to attempt to isolate specific two body reactions, e.g.,

$$
\begin{align*}
& \pi^{-} p \rightarrow \pi^{+} \Delta^{-}  \tag{6}\\
& \pi^{+} p \rightarrow \pi^{o} \Delta^{++} \tag{7}
\end{align*}
$$

or

$$
\begin{equation*}
\mathrm{Q} \rightarrow \pi \mathrm{~K}^{*} \tag{8}
\end{equation*}
$$

This is done by selecting events in which the invariant mass of a pair of particles lies in the required resonance band. As I have already indicated this can be a dangerous business. In general the resonances are wide and this means that at almost any point in the Dalitz plot their amplitudes are still quite large. Thus even though we believe we are selecting a clean
sample of reaction (6) from the final state $\pi^{+} \pi^{-} n$ the presence of the reaction (see Fig. 2)

$$
\begin{equation*}
\pi^{-} \mathrm{p} \rightarrow \pi^{-} \Delta^{+} \tag{9}
\end{equation*}
$$

can produce appreciable interference effects within the $\Delta^{-}$band. Since the helicity angle in the decay of the $\Delta^{-}$is linearly related to the position along the $\Delta^{-}$band, the decay distributions (and hence density matrix elements) are particularly susceptible to these interference effects. Of course the higher the centre of mass energy the smaller the proportion of resonance overlap and thus this technique becomes more reliable at higher energies.

The reason for attempting this isolation is that the calculations are comparatively simple (even if somewhat tedious). The complete expressions for the production differential cross section and decay density matrix in terms of partial wave amplitudes in the reaction

$$
\begin{equation*}
0^{-}+1 / 2^{+} \rightarrow 0^{-}+3 / 2^{+} \tag{10}
\end{equation*}
$$

have been published (Brody and Kernan, 1969) and this has been the reaction most exhaustively studied in this manner. The limited data has necessitated energy dependent analyses, i.e., the partial wave amplitudes are parametrized in terms of Breit-Wigner resonances and energy dependent backgrounds and data at a variety of energies fitted simultaneously. This ensures continuity and hopefully reduces ambiguities which might exist by specifying the partial wave energy dependence.

The results from analyses of this type are probably satisfactory for the large dominant waves whereas the smaller partial waves are
poorly determined. Such analyses do however provide a useful guide to and check on the results of more sophisticated methods.

The final objection to this type of analysis is that one does not obtain the relative phases of partial wave amplitudes for different channels, e.g.,

$$
\begin{align*}
\pi N & \rightarrow \pi \Delta \\
& \rightarrow N \rho \tag{11}
\end{align*}
$$

because the regions of the Dalitz plot which would allow this are explicitly removed from the analysis. As we shall see such quantities are of interest in higher symmetry schemes.
B. MODEL INDEPENDENT ANALYSES OF THE THREE BODY

FINAL STATE
In general five variables are required to describe a three body final state. These variables are usually three Euler angles specifying the orientation of the three particle state with respect to some co-ordinate system together with two Dalitz plot variables $\left(\omega_{1}^{2}\right.$ and $\left.\omega_{2}^{2}\right)$ giving effectively the energies of the three particles in their c.m. system. In the case of formation reactions using unpolarized targets one of these angles does not appear as it corresponds to arbitrary rotations about the beam direction. In order to use the data to maximum advantage it would be best to exploit all the correlations that exist. However the correlation between Dalitz plot populations and the angles requires specific dynamical assumptions, e.g., a resonance is produced in a particular angular momentum state.

In model independent analyses one selects regions of the Dalitz plot and attempts to find the $J^{P}$ states associated with that area. The formalism for these types of analysis has been discussed exhaustively both for formation (Cashmore and Hey, 1972) and production reactions (Berman and Jacob, 1965). In both cases the angular orientation of the three body place is considered, the form of the distributions and the correlations present indicating the partial wave structure.

I will demonstrate some of the properties of this type of analysis by using the reaction

$$
\begin{equation*}
\pi \mathrm{N} \rightarrow \pi \pi \mathrm{~N} \tag{12}
\end{equation*}
$$

as an example. In this case the differential cross section from an unpolarized target is written as (Cashmore and Hey, 1972)

$$
\begin{equation*}
\frac{d^{4} \sigma}{d \omega_{1}^{2} d \omega_{2}^{2} d \cos \theta d \phi}=\sum_{L, M} W_{L}^{M}\left(\omega_{1}^{2}, \omega_{2}^{2}\right) Y_{L}^{M}(\theta, \phi) \ldots \tag{13}
\end{equation*}
$$

where an integration has been performed over the angle about the incident beam direction, $\theta$ and $\phi$ the polar angles of the incident pion within a co-ordinate system defined by the final state particles. The object of any analysis of this type is then to determine the $W_{L}^{M}$ as functions of Dalitz plot position. The $W_{L}^{M}$ are then analogous to the expansion coefficients $A$ and $B$ of two body scattering differential cross sections and polarization distributions. These $W_{L}^{M}$ possess many similar properties, e.g., (a) interference of waves with opposite (same) parity lead to terms with odd (even) values of $L$. (b) If $J_{\text {max }}$ is the maximum angular momentum contributing to the reaction, and only one parity is present
corresponding to this value, then $L_{\max }=2 \mathrm{~J}_{\max }-1$ (for J half integral). However if waves of opposite parity are present with this value of $J_{\max }$ then $L_{\max }=2 J_{\max }$. (c) The $W_{L}^{M}$ are bilinear products of the partial wave amplitudes $\mathrm{B}_{\mathrm{J} \Lambda}^{\mu \tau}\left(\omega_{1}^{2}, \omega_{2}^{2}\right)$ (Cashmore and Hey, 1972)

$$
\begin{equation*}
\mathrm{W}_{\mathrm{L}}^{\mathrm{M}} \sim \sum_{\tau} \sum_{J J^{\prime}} \mathrm{F}\left(J, \Lambda, J^{\prime}, \Lambda^{\prime}, \mathrm{L}\right) \mathrm{B}_{\mathrm{J} \Lambda}^{\mu \Lambda^{\prime}} \mathrm{B}_{J^{\prime} \Lambda^{\prime}}^{\mu \tau} \delta_{\mathrm{M} \Lambda-\Lambda^{\prime}} \tag{14}
\end{equation*}
$$

A complete parity ambiguity exists unless the final state nucleon polarization is observed (Cashmore and Hey, 1972). This is equivalent to the Minani ambiguity of two body scattering.

In Figs. 6a,b as an example, we can see the $W_{L}^{M}$ averaged over the Dalitz plot for the final state $\pi^{+} \pi^{-} n$. The absence of moments with $\mathrm{L}+\mathrm{M}$ odd is required by parity conservation, while the large moments with L odd indicates the presence of waves of opposite parity. At a given energy the maximum $L$ present does give an indication of the largest angular moment wave present. However one should not put too great a reliance on this since in the region of 1700 MeV , the $\mathrm{L}=4$ and 5 moments are small where we know the F15 and D15 resonances are important.

From the measured values of the $\mathrm{W}_{\mathrm{L}}^{\mathrm{M}}\left(\omega_{1}^{2}, \omega_{2}^{2}\right)$ we would like to extract the partial wave amplitudes for this Dalitz plot position. It would then be possible to study the variation of these amplitudes with c.m. energy just as one does in conventional elastic scattering. Unfortunately there are many factors which reduce the value of this method.
(a) It is essential to make polarization measurements (preferably of the final baryon) to give enough $W_{L}^{M}$ to allow the extraction of partial wave amplitudes.
(b) The partial wave amplitudes are functions of the Dalitz plot position, and ideally we would like to know them everywhere. The determination of the moments $W_{L}^{M}$ requires $\sim 1000-2000$ events and thus we can only hope to obtain these $W_{L}^{M}$ integrated over regions of the Dalitz plot. At this time the lack of experimental data does not allow the use of even a coarse grid (characteristically there are $\sim 1000-10,000$ events in a given channel at a given energy).
(c) The overall phase of the amplitudes is undetermined from one Dalitz plot position to the next.
(d) Optimistically one would hope to see variations of the partial wave amplitudes as a function of the Dalitz plot variables indicating the association of a $J^{P}$ state with a particular decay decimal. However to extract couplings to these decay channels still requires a detailed model which predicts the variation of these amplitudes with Dalitz plot position.

However one must point out that the results of such an analysis are model independent and do represent a permanent record of the correlations which exist in the data. Furthermore these methods are useful when considering model dependent analyses in that they can provide an indication of waves which should be present (e.g., of opposite parity) and do sometimes limit the maximum angular momentum it is necessary to consider (although this can be misleading).

In the case of production reactions one usually considers the angular distribution in $\alpha, \beta$ and $\gamma$, the three angles of the problem (Berman and Jacob, 1965), for a given region of the Dalitz plot

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} \alpha \mathrm{~d} \cos \beta \mathrm{~d} \gamma} \sim \sum \mathrm{~W}_{\mathrm{JM} \Lambda}\left(\omega_{1}^{2}, \omega_{2}^{2}\right) \mathscr{D}_{\mathrm{M} \Lambda}^{\mathrm{J}}(\alpha, \beta, \gamma) \ldots \tag{1.5}
\end{equation*}
$$

and the $W_{J M \Lambda}$ are again products of the partial wave amplitudes. These results are published (Berman and Jacob, 1965) and we will see many of the general properties later in our discussion of model dependent analyses.

If the object of our analysis is finally to measure the couplings to decay channels then point (d) above suggests that it would be more sensible to make a specific model which will do this from the very beginning. This will immediately specify the correlations between the angular variables and the Dalitz plot position which we can exploit in our analysis. Of course the drawback is that it does become a model dependent analysis. However the situation is somewhat analogous to the state of $\pi \mathrm{N}$ partial wave analyses of 10 years ago. Then one performed energy dependent partial wave analyses on small quantities of data. In this case we are performing mass dependent (in the sense of Dalitz plot variables) partial wave analyses on similarly small quantities of data.

In the following section I will introduce such a model dependent analysis and subsequently discuss it at length.
C. MODEL DEPENDENT ANALYSIS - THE ISOBAR MODEL

If we study Fig. 2 or Fig. 7 we immediately notice the existence of strong resonance bands, the $\Delta$ and the $\rho$. Since we are not in a position
fruitfully to follow a model independent discussion, this suggests a course in which we insert as an integral part of our analysis the presence of these resonances. The method then consists of writing the transition amplitude for reaching a given final state as a coherent sum of two body processes as indicated in Fig. 8. The transition matrix is then written as (Deler and Valladas, 1966; Cashmore et al., 1972)
$\mathrm{T}\left(\mathrm{W}, \omega_{1}, \omega_{2}, \alpha, \beta, \gamma\right)=$

$$
\begin{equation*}
=\sum_{\substack{\mathrm{JLSI} \\ \ell}} A^{\mathrm{IJLS} \ell}(W) C^{\mathrm{I}} X^{\mathrm{JLS} \ell}\left(W, \omega_{1}, \omega_{2}, \alpha, \beta, \gamma\right) \mathrm{B}^{\mathrm{L}}\left(\omega_{1}, \omega_{2}\right) \ldots \tag{16}
\end{equation*}
$$

where $\omega_{1}, \omega_{2}, \alpha, \beta, \gamma$ are the kinematical variables required to specify the reaction, $\mathrm{C}^{\mathrm{I}}$ the product of all isospin Clebsch-Gordan coefficients, $\mathrm{X}^{\text {JLSL }}$ contains all factors related to angular momentum decompositions, $B^{L}\left(\omega_{1}, \omega_{2}\right)$ is the final state enhancement factor, e.g., a Breit-Wigner, where $\ell$ is the orbital angular momentum in the decay of the isobar. The partial wave amplitudes $A^{I J L S \ell}(W)$ are then obtained in fits to the data. One considers many intermediate states in analysing the different reactions:

$$
\begin{aligned}
\pi \mathrm{N} & \rightarrow \pi \pi \mathrm{~N} \\
\mathrm{~A} & \rightarrow 3 \pi \\
\mathrm{Q}, \mathrm{~L} & \rightarrow \mathrm{~K} \pi \pi
\end{aligned} \quad \mathrm{~A} \rightarrow \pi \Delta, \mathrm{~N} \rho, \mathrm{~N} \epsilon \mathrm{~K}, \pi \rho, \pi \mathrm{f},
$$

and in order to unravel these, one would like to use all of the correlations present in the data.

However it is possible to obtain more limited results by analysis of the Dalitz plot distributions done. In this case there are no longer terms
corresponding to the interference of waves of different $J$ or different parity $P$ and moreover the different magnetic substates ( $J_{z}$ substates) in which a resonance may be formed all lead to the same Dalitz plot distribution. Thus it is now only possible to determine the contribution of each J, P state. Furthermore, ignoring the $\alpha, \beta$ and $\gamma$ dependences of the cross section will lead to a less reliable estimate of these partial wave amplitudes.

The following section is devoted to a detailed discussion of the calculations of these cross section formulae and their properties while the results are reserved for Sections IV and V.

## III. THE CALCULATION OF TRANSITION AMPLITUDES TO THREE

## PARTICLE STATES

In this section I will develope the formulae for the transition to a three particle state in formation reactions, e.g.,

$$
\begin{equation*}
\pi N \rightarrow \pi \pi N \tag{17}
\end{equation*}
$$

together with the similar results for the decay to three bodies of states obtained in production, e.g.,

$$
\begin{array}{r}
\pi \mathrm{N} \rightarrow \mathrm{~A}_{2}+\mathrm{N}  \tag{18}\\
L \pi \pi \pi
\end{array}
$$

In Section III. A I will deal with that part of the calculation which is common to both results - the decay of a spin parity state $\mid J^{P} M>$ which is defined in a co-ordinate system related to the final three particles. Section III. B deals with the presence of two identical particles, e.g., two $\pi^{+}{ }^{1}$ s and the isospin decompositions which occur. In Section III. C I specifically deal with formation reactions while Section III. D contains a similar discussion of production reactions. Finally in Section III. E I remark on the methods of applying this formalism in the analysis of experimental data.

Before I begin I would like to summarize the notation and symbols I use in order to prevent their introduction in a random manner. It is not possible to do this completely but I hope the confusion will be reduced. I have also committed to the appendix the definition of states, angular momentum projections, normalizations, phase spaces, properties of $\mathscr{D}$
functions and any detail manipulations in an attempt to keep the text as clear as possible.

## Notation

Final three particle system. Let $j, k$ and 1 represent the final three particles. Let the diparticle be composed of particles $k$ and 1 . All quantities pertaining to the diparticle are indexed by the subscript $j$. All quantities are defined in the three particle c.m. system $S$.
(a) Total three particle c.m. energy, total angular momentum and

$$
\text { parity }-\mathrm{W}, \mathrm{~J}, \mathrm{P}
$$

(b) Z-component of $J$ in system $S-M$
(c) c.m. system four momenta $-Q_{j}, Q_{k}, Q_{1}$
(d) Particle spins $-\sigma_{j}, \sigma_{k}, \sigma_{1}$
(e) c.m. system helicities $-\mu_{\mathrm{j}}, \mu_{\mathrm{k}}, \mu_{1}$
(f) Intrinsic parities $-\eta_{j}, \eta_{k}, \eta_{1}$
(g) Mass of diparticle $-\omega_{j}$
(h) Spin of c.m.s. helicity of the diparticle $-j_{j}, \lambda_{j}$
(i) Outgoing orbital angular momentum and total spin $-L_{j}, S_{j}$

In the diparticle rest frame we have the quantities
(j) Four momenta of decay particles $-q_{k}, q_{l}$
(k) Helicities of decay particles $-\nu_{k}, \nu_{1}$
(l) Orbital angular momentum and tolal spin of the decay particles $-1_{j}, s_{j}$

We then use LS coupling to give

$$
\left.\begin{array}{l}
s_{j}=\sigma_{k}+\sigma_{1} \\
j_{j}=l_{j}+s_{j} \\
S_{j}=\sigma_{j}+j_{j}  \tag{19}\\
J=L_{j}+S_{j}
\end{array}\right\}
$$

We assume that $L_{j}$ and $l_{j}$ are chosen to conserve parity

$$
\begin{equation*}
P=\eta_{\mathrm{j}} \eta_{\mathrm{k}} \eta_{\mathrm{l}}^{(-1)^{\mathrm{L}_{\mathrm{j}}+\mathrm{l}_{\mathrm{j}}}} \tag{20}
\end{equation*}
$$

These definitions are summarized in Fig. 9.
Incident two particle state in formation reactions
Let a and b represent the incident particles. In this case the $\mathrm{c} . \mathrm{m}$. system for the two particle state is the same as that for the three particle state.
(a) c.m.s. four momenta $-p_{a}, p_{b}$
(b) Particle spins $-\sigma_{a}, \sigma_{b}$
(c) c.m.s. helicities $-\mu_{\mathrm{a}}, \mu_{\mathrm{b}}$
(d) Intrinsic particles $-\eta_{\mathrm{a}}, \eta_{\mathrm{b}}$
(e) Incident orbital angular momentum and total spin $-\mathrm{L}, \mathrm{S}$

Then

$$
\left.\begin{array}{l}
\mathrm{S}=\sigma_{\mathrm{a}}+\sigma_{\mathrm{b}}  \tag{21}\\
J=\mathrm{L}+\mathrm{S}
\end{array}\right\}
$$

and we assume parity is conserved so that

$$
\begin{equation*}
\mathrm{P}=\eta_{\mathrm{a}} \eta_{\mathrm{b}}(-1)^{\mathrm{L}} \tag{22}
\end{equation*}
$$

These are summarized in Fig. 10.
Production reactions
In this case we have three other particles to describe besides $\mathrm{j}, \mathrm{k}$, and 1 of the three particle system we are considering. In this case the $\mathrm{j}, \mathrm{k}, \mathrm{l}$ c.m. system is not the same as the overall (ab) c.m. system. In general all quantities pertaining to $a, b$ and $c$ will be measured in the overall c.m. system.
(a) c.m.s. four momenta $-p_{a}, p_{b}, p_{c}$
(b) Particle spins $-\sigma_{a}, \sigma_{b}, \sigma_{c}$
(c) Helicities $-\mu_{a}, \mu_{b}, \mu_{c}$
(d) Intrinsic parities $-\eta_{\mathrm{a}}, \eta_{\mathrm{b}}, \eta_{\mathrm{c}}$

These are summarized in Fig. 11.
For simplification in many of the following formulae $n$ will be used to represent a set of quantities

Decay of three particle system: $n \equiv\left\{j, J, P, M ; L_{j}, S_{j} ; j_{j}, 1, S_{j}\right\}$
Formation reactions: $\quad n \equiv\left\{j, J, P, M ; L, S ; L_{j}, S_{j} ; j_{j}, l_{j}, s_{j}\right\}$
Production reactions:
$n \equiv\left\{j, J, P, M ; L_{j}, S_{j} ; j_{j}, l_{j}, s_{j}\right\}$
In some cases it may be necessary to display one particular quantity of this set, in which circumstances I will continue to use a to represent the remaining quantities.
A. THE DECAY OF AN INTERMEDIATE STATE INTO THREE FINAL

## PARTICLES

We first define a co-ordinate system $S$ with respect to $j, k$ and 1 as shown in Fig. 12. We now wish to consider the decay of a state |nJM> defined in this system to a state whish is the product of three usual helicity states

$$
\begin{equation*}
\left.\left|Q_{j} \mu_{j} Q_{k} \mu_{k} Q_{l} \mu_{1}\right\rangle=\left|Q_{j} \mu_{j}\right\rangle\left|Q_{k} \mu_{k}>\right| Q_{1} \mu_{l}\right\rangle \tag{23}
\end{equation*}
$$

i.e., we wish to calculate the transition

$$
\begin{equation*}
\mathrm{f}_{\mu}^{\mathrm{nJM}}=\left\langle\overrightarrow{\mathrm{Q}}_{\mathrm{j}} \mu_{\mathrm{j}} \overrightarrow{\mathrm{Q}}_{\mathrm{k}} \mu_{\mathrm{k}} \overrightarrow{\mathrm{Q}}_{1} \mu_{\mathrm{l}}\right| \mathrm{T}|\mathrm{nJM}\rangle \tag{24}
\end{equation*}
$$

where we envisage the reaction as first proceeding through a quasi-two body state followed by the decay of one of these particles

$$
\begin{align*}
\mathrm{f}_{\mu}^{\mathrm{nJM}}=\sum_{\mu_{\mathrm{m}}} \int \frac{\mathrm{~d}^{3} \mathrm{Q}_{\mathrm{m}}}{2 \mathrm{E}_{\mathrm{m}}} \frac{\mathrm{~d}^{3} \mathrm{Q}_{\mathrm{n}}}{2 \mathrm{E}_{\mathrm{n}}} & <\mathrm{Q}_{\mathrm{j}} \mu_{\mathrm{j}} \mathrm{Q}_{\mathrm{k}} \mu_{\mathrm{k}} \mathrm{Q}_{1} \mu_{1}\left|\mathrm{~T}_{2}\right| \mathrm{Q}_{\mathrm{m}} \mu_{\mathrm{m}} Q_{\mathrm{n}} \mu_{\mathrm{n}}> \\
& <\mathrm{Q}_{\mathrm{m}} \mu_{\mathrm{m}} \mathrm{Q}_{\mathrm{n}} \mu_{\mathrm{n}}\left|\mathrm{~T}_{1}\right| \mathrm{nJM}>
\end{align*}
$$

The isobar model then consists of writing

$$
\begin{align*}
\left\langle Q_{j} \mu_{j} Q_{k} \mu_{k} Q_{1} \mu_{1}\right| T_{2}\left|Q_{m} \mu_{m} Q_{n} \mu_{n}\right\rangle= & \delta^{3}\left(Q_{j}-Q_{m}\right) \delta_{\mu_{j} \mu_{m}} \delta^{3}\left(Q_{n}-Q_{1 k}-Q_{1}\right) \\
& \left.<Q_{k} \mu_{k} Q_{1} \mu_{1}\left|T_{2}\right| Q_{n} \mu_{n}\right\rangle \tag{26}
\end{align*}
$$

so that

$$
\begin{equation*}
\mathrm{f}_{\mu}^{\mathrm{nJM}}=\sum_{\lambda_{j}}<\mathrm{Q}_{\mathrm{k}} \mu_{\mathrm{k}} \mathrm{Q}_{1} \mu_{1}\left|\mathrm{~T}_{2}\right|-Q_{j} \lambda_{j}><Q_{j} \mu_{j}-Q_{j} \lambda_{j}\left|T_{1}\right| J M n> \tag{27}
\end{equation*}
$$

This is summarized in Fig. 13.

## 1. Primary Decay $\mathrm{T}_{1-}$

We write this using the decomposition into angular momentum states (see appendix)

$$
\begin{equation*}
\left.\left\langle Q_{j} \mu_{j}-Q_{j} \lambda_{j}\right| T_{1}\left|n J M>=\sqrt{\frac{2 J+1}{4 \pi}}\left(\frac{4 W}{Q_{j}}\right)^{1 / 2} \mathscr{D}_{M, \mu_{j}-\lambda_{j}}^{J_{j}^{*}}\left(\Omega_{j}\right)<Q_{j} J M \mu_{j}\right| T_{1} \right\rvert\, n J M> \tag{28}
\end{equation*}
$$

where the arguments of the $\mathscr{D}$ function are angles $\Omega_{j}=\left(\Phi_{j}, \Theta_{j},-\Phi_{j}\right)$, the production angles of particle $j$ in frame $S$. We can then perform a further partial wave decomposition converting from helicity states to LS states (see appendix)

$$
\begin{gather*}
=\left(\frac{4 \pi}{Q_{j}}\right)^{1 / 2}\left(\frac{2 L_{j}+1}{4 \pi}\right)^{1 / 2} C\left(\sigma_{j}, j_{j}, S_{j} \mid \mu_{j}-\lambda_{j}\right) C\left(L_{j}, S_{j}, J \mid 0, \mu_{j}-\lambda_{j}\right) \\
\mathscr{D}_{M, \mu_{j}-\lambda_{j}}^{J^{*}}(j)<Q_{j} J M L_{j} S_{j}\left|T_{1}\right| n J M> \tag{29}
\end{gather*}
$$

The last partial wave amplitude is independent of $M$, due to rotational invariance and we will write it as

$$
\begin{equation*}
\left\langle Q_{j} J M L_{j} S_{j}\right| T_{1}|n J M\rangle=T_{1 L_{j}} S_{j}\left(W, \omega_{j}\right) \tag{30}
\end{equation*}
$$

## 2. Decay of Isobar T ${ }_{2}$

We wish to evaluate the decay of the diparticle and this is most easily done in the isobar rest frame

$$
\begin{equation*}
\left\langle Q_{k} \mu_{k} Q_{1} \mu_{1}\right| T_{2}\left|-Q_{j} \lambda_{j}\right\rangle=\sum_{\nu_{k} \nu_{1}} \mathscr{D}_{\nu_{k} \mu_{k}}^{v_{k}^{*}}\left(\theta_{j}^{k} \hat{n}_{k}\right) \mathscr{D}_{\nu_{1} \mu_{1}}^{\sigma_{1}^{*}}\left(\theta_{j}^{1} \hat{n}_{1}\right)\left\langle q_{k} \nu_{k} q_{1} \nu_{1}\right| T_{2}\left|j_{j}-\lambda_{j}\right\rangle \tag{31}
\end{equation*}
$$

The $\mathscr{D}$ functions correspond to the rotations introduced by Lorentz transformations not along the directions of k and 1 (see appendix). The isobar when at rest is in a magnetic substate $\left|j_{j}-\lambda_{j}\right\rangle$ with respect to the direction $Q_{j}$ since it was initially the $\chi$ state in the two particle helicity state (see appendix). We finally require the matrix element to a product of three helicity states as defined in Eq. (23). Thus when we perform the angular momentum decomposition of the helicity matrix element in Eq. (30) we must convert the second particle 1 from the conventional $\chi$ state to an $\psi$ state (see appendix). Inserting the angular momentum decomposition gives

$$
\begin{align*}
& \left\langle q_{k} \nu_{k} q_{1} \nu_{1}\right| T_{2}\left|j_{j}-\lambda_{j}\right\rangle=\left(\frac{4 \omega_{j}}{q_{k}}\right)^{1 / 2}\left(\frac{2 L_{j}+1}{4}\right)^{1 / 2} C\left(\sigma_{k}, \sigma_{1}, s_{j} \mid \nu_{k},-\nu_{1}\right) \\
& C\left(l_{j} s_{j} j_{j} \mid 0, \nu_{k}-\nu_{1}\right) \mathscr{D}_{-\lambda_{j}}^{j_{j}^{*}}, \nu_{k}-\nu_{1}\left(\Omega_{k}^{d}\right)(-1)^{\sigma_{1}-\nu_{1}}\left\langle q_{k} j_{j}-\lambda_{j} l_{j} s_{j}\right| T_{2}\left|j_{j}-\lambda_{j}\right\rangle \tag{32}
\end{align*}
$$

where the arguments of the $\mathscr{D}$ function are $\Omega_{\mathrm{k}}^{\mathrm{d}}=\left(+\phi_{\mathrm{j}}, \theta_{\mathrm{j}}-\phi_{\mathrm{j}}\right)$, the decay angles of isobar j in its rest frame defined using particle k. Due to rotational invariance we can then write

$$
\begin{equation*}
\left\langle q_{k} j_{j}-\lambda_{j} l_{j} s_{j}\right| T_{2}\left|j_{j}-\lambda_{j}\right\rangle=\mathbb{D}_{l_{j} s_{j}}^{j_{j}}\left(\omega_{j}\right) \tag{33}
\end{equation*}
$$

Thus the final amplitude is then written as

$$
\begin{equation*}
\mathrm{f}_{\mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}}^{\mathrm{nJM}}=\mathrm{G}_{\mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}}^{\mathrm{nJM}} \mathrm{~T}_{1 \mathrm{~L}_{\mathrm{j}} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{nJ}}\left(\mathrm{~W}, \omega_{\mathrm{j}}\right)_{\mathrm{D}_{\mathrm{j} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{j}}\left(\omega_{\mathrm{j}}\right)} \tag{34}
\end{equation*}
$$

where

$$
\begin{aligned}
& G_{\mu_{j} \mu_{k} \mu_{1}}^{n J M}=\frac{1}{4 \pi}\left[\frac{4 W}{Q_{j}} \frac{4 \omega_{j}}{q_{k}}\right]^{1 / 2}\left[\sum_{\lambda_{j}} C\left(\sigma_{j} j_{j} S_{j} \mid \mu_{j}-\lambda_{j}\right) C\left(L_{j} S_{j} J \mid 0 \mu_{j}-\lambda_{j}\right)\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\left.\mathscr{D}_{-\lambda_{\mathrm{j}}, \nu_{\mathrm{k}}-\nu_{1}}^{\mathrm{j}_{\mathrm{j}}^{*}}\left(\phi_{\mathrm{j}}, \theta_{\mathrm{j}},-\phi_{\mathrm{j}}\right) \mathscr{D}_{\nu_{\mathrm{k}} \mu_{\mathrm{k}}}^{\sigma_{\mathrm{k}}^{*}}\left(\theta_{\mathrm{j}}^{\mathrm{k}}\right) \mathscr{D}_{\nu_{1} \mu_{1}}^{\sigma_{1}^{*}}\left(\theta_{\mathrm{j}}^{1}\right)(-1){ }^{\sigma}-\nu_{1}\right\}\right] \tag{35}
\end{align*}
$$

Note we do not include sums over $L_{j}, S_{j}, l_{j}, s_{j}$ as these are specified in $n$, i.e., we are considering the transition by a very specific intermediate state.

The angles of the $\mathscr{D}$ functions are summarized in Figs, 12 and 14.
3. The Reduced Matrix Elements
(a) Initial Decay Matrix Element: $T_{1}^{n}\left(W, \omega_{j}\right)$. It is usual to extract from this the barrier penetration factors

$$
\frac{Q_{j}^{L_{j}+1 / 2}}{\sqrt{4 W}}
$$

which have the correct threshold dependence.
The charge dependence may also be removed by including the isospin Clebsch-Gordan coefficients. Thus

$$
\begin{equation*}
T_{1}^{n}\left(W, \omega_{j}\right)=\frac{Q_{j}^{L_{j}^{+1 / 2}}}{\sqrt{4 W}} C\left(I^{D}, I^{j}, I \mid I_{z}^{D_{z}^{j}}\right) \tau_{1}^{n}\left(W, \omega_{j}\right) \tag{36}
\end{equation*}
$$

$I^{j}, I_{z}^{j}$ isospin and $z$ component of isospin for $j$
$\mathrm{I}^{\mathrm{D}}, \mathrm{I}_{\mathrm{z}}^{\mathrm{D}}$ isospin and z component of isospin for the isobar $I, I_{z}=I_{z}^{j}+I_{z}^{D} \quad$ total isospin and $z$-component of isospin .

It is often common to introduce further $Q_{j}$ dependence through the Blatt and Weiskoff barriers $B_{L_{j}}\left(Q_{j}\right)$. These have the desirable property of damping the $Q_{j}{ }_{j}$ factors at large $Q_{j}$. Other parametrizations may be used which possess this threshold behaviour, but in all cases we expect the $\tau_{1}^{n}\left(W, \omega_{j}\right)$ to be essentially independent of $\omega_{j}$. All analyses assume this to be the case.
(b) Decay Matrix Element: $\mathbb{0}_{1_{j} \mathbf{s}_{j}}^{\mathbf{j}_{j}}\left(\omega_{j}\right)$. We can again remove the charge dependence from the decay term

$$
\begin{equation*}
\mathbb{D}_{l_{j} S_{j}}^{j_{j}}\left(\omega_{j}\right)=C\left(I^{k}, I^{l}, I^{D} I_{z}^{k} I_{z}^{l}\right) D_{l_{j} S_{j}}^{j_{j}}\left(\omega_{j}\right) \tag{37}
\end{equation*}
$$

where $I^{k}, I_{z}^{k}$ isospin and $z$-component of isospin for $k$ $I^{1}, I_{Z}^{l}$ isospin and $z$-component of isospin for 1 .

To evaluate $D_{1}^{j_{j}}{ }_{j}\left(\omega_{j}\right)$ one uses either the Watson final state interaction theorem or a modified Breit-Wigner.

Watson:

$$
\begin{align*}
& D_{l_{j} S_{j}}^{j_{j}}\left(\omega_{j}\right)=\frac{e^{i \delta} \sin \delta}{{l_{j}+1}_{q_{k}}^{\frac{q_{k}}{4 \omega_{j}}}}  \tag{38}\\
& \delta \text { - elastic scattering phase shift at the energy } \omega_{j} \\
& \sqrt{\frac{q_{k}}{4 \omega_{j}}}-\text { is a factor added to ensure correct threshold } \\
& \text { behaviour. }
\end{align*}
$$

## Relativistic Breit-Wigner:

$$
\begin{align*}
& D_{i_{j} \mathbf{s}_{j}}^{\mathbf{j}_{j}}\left(\omega_{\mathfrak{j}}\right)=\frac{1}{\sqrt{\pi}} \frac{\sqrt{\omega_{0} \Gamma_{j}\left(\omega_{\mathrm{j}}\right)}}{\left(\omega_{0}^{2}-\omega_{\mathrm{j}}^{2}\right)-\mathrm{i} \omega_{0} \Gamma_{\mathrm{j}}\left(\omega_{\mathrm{j}}\right)}  \tag{39}\\
& \Gamma_{\mathrm{j}}\left(\omega_{\mathrm{j}}\right)=\Gamma_{\mathrm{j}}\left(\omega_{0}\right)\left[\frac{q_{k}\left(\omega_{j}\right.}{q_{k}\left(\omega_{0}\right)}\right]^{21_{j}} \frac{q_{\mathrm{k}}\left(\omega_{\mathrm{j}}\right)}{q_{\mathrm{k}}\left(\omega_{0}\right)} \frac{\omega_{0}}{\omega_{\mathrm{j}}} \tag{40}
\end{align*}
$$

and $\omega_{0}$ is the resonance energy.
Non-relativistic Breit-Wigner:

$$
\begin{equation*}
\mathrm{D}_{\mathrm{l}_{\mathrm{j}} \mathrm{~s}_{\mathrm{j}}}^{\mathrm{j}_{\mathrm{j}}}\left(\omega_{j}\right)=\frac{1}{\sqrt{2 \pi \omega_{0}}} \frac{\sqrt{\Gamma_{j}\left(\omega_{\mathrm{j}}\right) / 2}}{\left(\omega_{0}-\omega_{\mathrm{j}}\right)-\mathrm{i} \Gamma_{j}\left(\omega_{j}\right) / 2} \tag{41}
\end{equation*}
$$

and $\Gamma_{j}\left(\omega_{j}\right)$ is defined as above.
Both of these forms are defined so that in the limit of zero width

$$
\begin{equation*}
\operatorname{Lim}_{\Gamma_{j} \rightarrow 0} \int\left|D_{l_{j} s_{j}}^{\mathrm{j}_{\mathrm{j}}}\left(\omega_{\mathrm{j}}\right)\right|^{2} \mathrm{~d} \omega_{\mathrm{j}}^{2}=\delta\left(\omega_{0}^{2}-\omega_{\mathrm{j}}^{2}\right) \tag{42}
\end{equation*}
$$

In general the dependences of $T_{1}$ and $D$ on the subenergy $\omega_{j}$ constitute the major assumptions and approximations of the isobar model.

## 4. Other Isobars

In the cases in which we are interested, each two body subsystem may contain a number of isobars, e.g., in $\pi_{1}^{+} \pi_{2}^{+} \pi^{-}$we can have $\rho(I=1, J=1)$ or $\epsilon(\mathrm{I}=0, \mathrm{~J}=0)$ states in either $\left(\pi_{1}^{+} \pi^{-}\right)$or $\left(\pi_{2}^{+} \pi^{-}\right)$. The total amplitude for the decay of the state $J M$ is then given by

$$
\mathrm{f}_{\mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{\mathrm{l}}}^{\mathrm{JM}}=\sum_{\mathrm{n}}^{\mathrm{j}} \mathrm{G}_{\mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}}^{\mathrm{JMn}}(\mathrm{j}) \mathrm{T}_{1 L_{\mathrm{j}} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{Jn}}\left(W, \omega_{\mathrm{j}}\right) \mathrm{D}_{\mathrm{l}_{\mathrm{j}} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{j}_{\mathrm{j}}}\left(\omega_{\mathrm{j}}\right)
$$

i.e., we add all of the amplitudes together coherently. This does result in some slight double counting but this has been shown to be small and in practice is far less significant than the assumptions within the decay matrix elements.

In making this addition if identical particles are present care has to be made in making this addition in order to observe the correct symmetry. This is discussed in the next section.
B. THE PRESENCE OF TWO OR MORE ISOBARS, ISOSPIN

DECOMPOSITION AND BOSE SYMMETRY
We are mainly interested in situations with two or more pions present in the final state and hence we require the final amplitude to be Bose-symmetric. Furthermore we will concentrate on the reactions

$$
\begin{align*}
\pi \mathrm{N} & \rightarrow \pi \pi \mathrm{~N}  \tag{43}\\
\mathrm{~A} & \rightarrow \pi \pi \pi \tag{44}
\end{align*}
$$

and use these to demonstrate a method of calculation which always ensures the correct result. In the following discussion, I explicitly demonstrate the Clebsch-Gordan coefficients of Eqs. (36) and (37) and derive the necessary factors of $\sqrt{2}$, etc., which must be inserted when we have identical particles present.

The isospin decomposition also allows the determination of the same partial wave amplitudes in different charge states where the interference phenomena are different. In practice this is a valuable point.

1. $\pi \mathrm{N} \rightarrow \pi \pi \mathrm{N}$

Consider

$$
\begin{align*}
\pi^{+} \mathrm{p} & \rightarrow \pi^{+} \pi^{o} \mathrm{p}  \tag{45}\\
& \rightarrow \pi^{+} \pi^{+} \mathrm{n} \tag{46}
\end{align*}
$$

and decays through intermediate $\Delta \pi$ states.

$$
\begin{equation*}
\left|\frac{3}{2} \frac{3}{2}\right\rangle \rightarrow \frac{1}{\sqrt{2}}\left\{\mathrm{~A}_{1}\left|\Delta_{1} \pi_{1}\right\rangle+\mathrm{A}_{2}\left|\Delta_{2} \pi_{2}\right\rangle\right\} \tag{47}
\end{equation*}
$$

Here I use the pion not resonating to label the $\Delta$ state and the decay amplitude. The expression is clearly symmetric in $\pi_{1}$ and $\pi_{2}$. A $A_{1}$ and $A_{2}$ will contain the kinematical information associated with the decay. Now

$$
\begin{equation*}
\left|\frac{3}{2} \frac{3}{2}\right\rangle=\sqrt{\frac{3}{5}} \Delta^{++} \pi^{o}-\sqrt{\frac{2}{5}} \Delta^{+} \pi^{+} \tag{48}
\end{equation*}
$$

We next consider the decay of the $\Delta$ and introduce a transition matrix element D which contains all the kinematics.

$$
\left.\begin{array}{l}
\Delta^{++} \rightarrow \pi^{+} \mathrm{p}  \tag{49}\\
\Delta^{+} \rightarrow \sqrt{\frac{1}{3}} \pi^{+} \mathrm{n}+\sqrt{\frac{2}{3}} \pi^{o} \mathrm{p}
\end{array}\right\}
$$

and if we write $G_{i}=A_{i} D_{i}$ we have

$$
\begin{align*}
\left\lvert\, \frac{3}{2} \frac{3}{2}>\rightarrow\right. & \frac{1}{\sqrt{2}}\left[\mathrm{~A}_{1}\left\{\sqrt{\frac{3}{5}} \Delta_{1}^{++} \pi_{1}^{o}-\sqrt{\frac{2}{5}} \Delta_{1}^{+} \pi_{1}^{+}\right\}+\mathrm{A}_{2}\left\{\sqrt{\frac{3}{5}} \Delta_{2}^{++} \pi_{2}^{o}-\sqrt{\frac{2}{5}} \Delta_{2}^{+} \pi_{2}^{+}\right\}\right] \\
= & \frac{1}{\sqrt{2}}\left[\mathrm{~A}_{1} \mathrm{D}_{1}\left\{\sqrt{\frac{3}{5}} \pi_{2}^{+} \mathrm{p} \pi_{1}^{o}-\sqrt{\frac{2}{5}}\left(\sqrt{\frac{1}{3}} \pi_{2}^{+} \mathrm{n} \pi_{1}^{+}+\sqrt{\frac{2}{3}} \pi_{2}^{o} \mathrm{p} \pi_{1}^{+}\right)\right\}\right. \\
& \left.+\mathrm{A}_{2} \mathrm{D}_{2}\left\{\sqrt{\frac{3}{5}} \pi_{1}^{+} \mathrm{p} \pi_{2}^{o}-\sqrt{\frac{2}{3}}\left(\sqrt{\frac{1}{3}} \pi_{1}^{+} \mathrm{n} \pi_{2}^{+}+\sqrt{\frac{2}{3}} \pi_{1}^{o} \mathrm{p} \pi_{2}^{+}\right)\right\}\right] \\
= & \frac{1}{\sqrt{30}}\left\{\left[3 \mathrm{G}_{1}-2 \mathrm{G}_{2}\right] \pi_{1}^{o} \pi_{2}^{+} \mathrm{p}+\left[3 \mathrm{G}_{2}-2 \mathrm{G}_{1}\right] \pi_{1}^{+} \pi_{2}^{o} \mathrm{p}-\sqrt{2}\left[\mathrm{G}_{1}+\mathrm{G}_{2}\right] \pi_{1}^{+} \pi_{2}^{+} \mathrm{n}\right\} \tag{50}
\end{align*}
$$

Thus the amplitude for obtaining a $\pi^{\circ} \pi^{+} p$ final state is then

$$
\begin{equation*}
\frac{\sqrt{2}}{\sqrt{30}}\left[3 G_{1}-2 G_{2}\right] \tag{51}
\end{equation*}
$$

Since there are two distinguishable ways of obtaining this and these must be added incoherently ( $1 \equiv \pi^{\mathrm{o}}, 2 \equiv \pi^{+}$). The amplitude for obtaining $\pi^{+} \pi^{+} \mathrm{n}$ is

$$
\begin{equation*}
-\frac{\sqrt{2}}{\sqrt{30}}\left[G_{1}+G_{2}\right] \tag{52}
\end{equation*}
$$

We now have the correct factors of $\sqrt{2}$ which account for the presence of the two identical $\pi^{+}$'s in the $\pi^{+} \pi^{+} n$ final state.

Similar calculations can be performed for all of the $\pi \pi \mathrm{N}$ states obtained from $\pi^{ \pm} \mathrm{p}$ incident particles and these results are contained in Table I. This table includes the isospin decomposition of the incident state.

$$
\begin{equation*}
\left|\pi^{+} p\right\rangle=\left|\frac{3}{2} \frac{3}{2}\right\rangle \tag{53a}
\end{equation*}
$$

$$
\begin{equation*}
\left|\pi^{-} p\right\rangle=\sqrt{\frac{1}{3}}\left|\frac{3}{2}-\frac{1}{2}\right\rangle-\sqrt{\frac{2}{3}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle \tag{53b}
\end{equation*}
$$

## 2. $I=0, I=1$ Meson Decay into $3 \pi^{\prime} \mathrm{S}$

The other three particle final state so far analysed by the methods I describe is the $3 \pi$ final state. Consider an $A_{1}$ or $A_{2}(I=1)$ state decaying via the intermediate states $\pi \rho$ and $\pi \epsilon$. Then we can write as above

$$
\begin{equation*}
|\mathrm{A}\rangle \rightarrow \frac{1}{\sqrt{3}} \mathrm{~A}_{\rho}\left[\rho_{1} \pi_{1}+\rho_{2} \pi_{2}+\rho_{3} \pi_{3}\right]+\frac{1}{\sqrt{3}} \mathrm{~A}_{\epsilon}\left[\epsilon_{1} \pi_{1}+\epsilon_{2} \pi_{2}+\epsilon_{3} \pi_{3}\right] \tag{54}
\end{equation*}
$$

Then for

$$
\begin{align*}
& |1|\rangle=\sqrt{\frac{1}{2}} \rho^{+} \pi^{0}-\sqrt{\frac{1}{2}} \rho^{o} \pi^{+}  \tag{55}\\
& |11\rangle=\epsilon^{0} \pi^{+}
\end{align*}
$$

and subsequent $\rho, \epsilon$ decay we obtain

$$
\begin{align*}
& \left\lvert\, \mathrm{A}>\rightarrow \frac{1}{\sqrt{3}} \mathrm{~A}_{\rho}\left\{\sqrt{\frac{1}{2}} \pi_{1}^{o}\left[\sqrt{\frac{1}{2}} \pi_{2}^{+} \pi_{3}^{o}-\sqrt{\frac{1}{2}} \pi_{2}^{o} \pi_{3}^{+}\right] \mathrm{B}_{\rho}(2,3)\right.\right. \\
& \\
& \quad-\sqrt{\frac{1}{2}} \pi_{1}^{+}\left[\sqrt{\frac{1}{2}} \pi_{2}^{+} \pi_{3}^{-}-\sqrt{\frac{1}{2}} \pi_{2}^{-} \pi_{3}^{+}\right] \mathrm{B}_{\rho}(2,3) \\
& \\
& \left.+ \text { similar terms for } \pi_{2}, \pi_{3}\right\}  \tag{56}\\
& +\sqrt{\frac{1}{3}} \mathrm{~A}_{\epsilon}\left\{\frac{1}{\sqrt{3}} \pi_{1}^{+}\left[\pi_{2}^{+} \pi_{3}^{-}-\pi_{2}^{o} \pi_{3}^{o}+\pi_{2}^{-} \pi_{3}^{+}\right] \mathrm{B}_{\epsilon}(2,3)\right. \\
& \\
& \left.\quad+\text { similar terms for } \pi_{2}, \pi_{3}\right\}
\end{align*}
$$

Gathering terms together we eventually have

$$
\begin{align*}
\frac{1}{6} & {\left[\pi_{1}^{+} \pi_{2}^{+} \pi_{3}^{-}\left\{\sqrt{3} \mathrm{~A}_{\rho}\left(-\mathrm{B}_{\rho}(2,3)+\mathrm{B}_{\rho}(3,1)\right)+2 \mathrm{~A}_{\epsilon}\left(\mathrm{B}_{\epsilon}(2,3)+\mathrm{B}_{\epsilon}(3,1)\right)\right\}\right.} \\
& +\pi_{1}^{-} \pi_{2}^{+} \pi_{3}^{+}\left\{\sqrt{3} \mathrm{~A}_{\rho}\left(-\mathrm{B}_{\rho}(3,1)+\mathrm{B}_{\rho}(1,2)\right)+2 \mathrm{~A}_{\epsilon}\left(\mathrm{B}_{\epsilon}(3,1)+\mathrm{B}_{\epsilon}(1,2)\right)\right\} \\
& +\pi_{1}^{+} \pi_{2}^{-} \pi_{3}^{+}\left\{\sqrt{3 \mathrm{~A}_{\rho}}\left(-\mathrm{B}_{\rho}(1,2)+\mathrm{B}_{\rho}(2,3)\right)+2 \mathrm{~A}_{\epsilon}\left(\mathrm{B}_{\epsilon}(1,2)+\mathrm{B}_{\epsilon}(2,3)\right)\right\} \\
& -\pi_{1}^{+} \pi_{2}^{o} \pi_{3}^{\mathrm{o}}\left\{\sqrt{3} \mathrm{~A}_{\rho}\left(-\mathrm{B}_{\rho}(1,2)+\mathrm{B}_{\rho}(3,1)\right)+2 \mathrm{~A}_{\epsilon} \mathrm{B}_{\epsilon}(2,3)\right\} \\
& -\pi_{1}^{\mathrm{o} \pi_{2}^{+} \pi_{3}^{\mathrm{o}}\left\{\sqrt{3} \mathrm{~A}_{\rho}\left(-\mathrm{B}_{\rho}(2,3)+\mathrm{B}_{\rho}(1,2)\right)+2 \mathrm{~A}_{\epsilon} \mathrm{B}_{\epsilon}(3,1)\right\}} \\
& -\pi_{1}^{o} \pi_{2}^{\left.\mathrm{o} \pi_{3}^{+}\left\{\sqrt{3} \mathrm{~A}_{\rho}\left(-\mathrm{B}_{\rho}(3,1)+\mathrm{B}_{\rho}(2,3)\right)+2 \mathrm{~A}_{\epsilon} \mathrm{B}_{\epsilon}(1,2)\right\}\right]} \tag{57}
\end{align*}
$$

where the arguments of $B$ represent the pions of the intermediate isobar, the first labelling the decay. This expression is symmetrical in any pair of pion labels because

$$
\left.\begin{array}{c}
\mathrm{B}_{\rho}(1,2)=-\mathrm{B}_{\rho}(2,1)  \tag{58}\\
\mathrm{B}_{\epsilon}(1,2)=\mathrm{B}_{\epsilon}(2,1)
\end{array}\right\}
$$

We see there are three ways of obtaining $\pi^{+} \pi^{+} \pi^{-}$and thus the amplitude for obtaining this final state is just $\sqrt{3}$ times an individual expression. Similar calculations can be performed for the other charge states and also the $I=0$ state. These results are summarized in Table II.

## 3. Bose Symmetry

If one now follows the same prescription for calculating the angular momentum factors, etc. in the transition matrix element (in the same overall co-ordinate system) then Bose symmetry will be automatically satisfied, i.e., we use the formalism described in Section III. A.
4. Summary

In Tables I, II, and III which summarize the results we associate the product of the Clebsch-Gordan coefficients with the pair of particles resonating in the intermediate state. This is slightly different from the text where we used the particle not resonating as the label. However the charge is obvious. Furthermore the subscripts 1, 2, etc., label specific charge states.
C. FORMATION REACTIONS

In this section we will develope the formalism to deal with reactions

$$
\begin{equation*}
a+b \rightarrow j+k+1 \tag{59}
\end{equation*}
$$

We begin with a calculation of the transition amplitudes, then derive formulae for all the observable quantities in reactions of type (17) and discuss the unpolarized cross section in some detail. Finally we review the quantities of physical interest we wish to obtain using such an analysis. 1. Transition Amplitude

Suppose the initial beam has polar angles $\Theta, \Phi$ in co-ordinate system S. We then consider the transition matrix element and make an LS partial wave decomposition

$$
\begin{align*}
& s^{\text {nJM }\left|T_{p}\right| \vec{p}_{a} \mu_{a} \vec{p}_{b} \mu_{b}>} \\
& =\left(\frac{2 J+1}{4 \pi}\right)^{1 / 2}\left(\frac{4 \mathrm{~W}}{\mathrm{p}}\right)^{1 / 2} \mathscr{D}_{\mathrm{M} \mu_{\mathrm{a}}-\mu_{\mathrm{b}}}^{\mathrm{J}}(\Phi, \oplus,-\Phi) \mathrm{S}^{\left\langle\mathrm{nJM} / \mathrm{T}_{\mathrm{p}} \mid \mathrm{pJM} \mu_{\mathrm{a}} \mu_{\mathrm{b}}\right\rangle} \\
& =\left(\frac{4 \mathrm{~W}}{\mathrm{p}}\right)^{1 / 2}\left(\frac{2 \mathrm{~L}+1}{4 \pi}\right)^{1 / 2} \mathrm{C}\left(v_{\mathrm{a}}, v_{\mathrm{b}}, \mathrm{~S} \mid \mu_{\mathrm{a}},-\mu_{\mathrm{b}}\right) \mathrm{C}\left(\mathrm{LSJ} \mid 0 \mu_{\mathrm{a}}-\mu_{\mathrm{b}}\right) \\
& \mathscr{D}_{\mathrm{M}, \mu_{\mathrm{a}}-\mu_{\mathrm{b}}^{\mathrm{J}}}^{(\Phi, \Theta,-\Phi)} \mathrm{S}^{\langle\mathrm{nJM}| \mathrm{T}_{\mathrm{p}}|\mathrm{pJMLS}\rangle} \tag{60}
\end{align*}
$$

where rotational invariance implies that we can write

$$
\begin{equation*}
<\mathrm{nJM}\left|\mathrm{~T}_{\mathrm{p}}\right| \mathrm{pJMLS}>=\mathrm{T}_{\mathrm{p}}^{J L S}(\mathrm{~W}) \tag{61}
\end{equation*}
$$

It is often convenient to again remove the charge dependence from this and also display explicitly some of its kinematical properties

$$
\begin{equation*}
\mathrm{T}_{\mathrm{p}}^{\mathrm{JLS}}(\mathrm{~W})=\frac{\mathrm{p}^{\mathrm{L}+1 / 2}}{\sqrt{4 \mathrm{~W}}} \mathrm{C}\left(\mathrm{I}^{\mathrm{a}} \mathrm{I}^{\mathrm{b}} \mathrm{I} \mid \mathrm{I}_{\mathrm{z}}^{\mathrm{a}} \mathrm{I}_{\mathrm{z}}^{\mathrm{b}}\right) \tau_{\mathrm{p}}^{\mathrm{JLS}}(\mathrm{~W}) \tag{62}
\end{equation*}
$$

These isospin Clebsch-Gordan coefficients have been explicitly included in Table I. Also note that Eq. (60) does not contain a sum over L, S as n implies a specific choice of these.

The final amplitude for the process (59) is then given by a coherent sum over all intermediate states

$$
\begin{align*}
\mathrm{A}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}}= & \sum_{\mathrm{n}} \mathrm{~A}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{\mathrm{l}}} \\
= & \sum_{\mathrm{n}, J, \mathrm{M}}\left(\frac{4 \mathrm{~W}}{\mathrm{p}}\right)^{1 / 2}\left(\frac{2 \mathrm{~L}+1}{4 \pi}\right)^{1 / 2} \mathrm{C}\left(\sigma_{\mathrm{a}} \sigma_{\mathrm{b}} \mathrm{~S} \mid \mu_{\mathrm{a}} \mu_{\mathrm{b}}\right) \mathrm{C}\left(\mathrm{LSJ} \mid 0 \mu_{\mathrm{a}}-\mu_{\mathrm{b}}\right) \\
& \mathrm{G}_{\mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}} \mathscr{D}_{\mathrm{M}, \mu_{\mathrm{a}}-\mu_{\mathrm{b}}}^{J}(\Phi, \oplus,-\Phi) \mathrm{T}_{\mathrm{p}}^{\mathrm{JLS}}(\mathrm{~W}) \mathrm{T}_{1 L_{\mathrm{j}} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{nJ}}\left(\mathrm{~W}, \omega_{\mathrm{j}}\right) \mathbb{D}_{l_{j} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{j}_{\mathrm{j}}}\left(\omega_{\mathrm{j}}\right) \tag{63}
\end{align*}
$$

where $\mathrm{G}_{\mu_{j} \mu_{k} \mu_{\mathrm{l}}}^{\mathrm{nJM}}$ is given by (35). It is then convenient to define

$$
\begin{equation*}
T^{n}\left(W, \omega_{j}\right)=T_{p}^{J L S}(W) T_{1 l_{j} S_{j}}^{n J}\left(W, \omega_{j}\right) \tag{64}
\end{equation*}
$$

It is important to note that $\sum_{\mathrm{M}}$ may be removed since

$$
\begin{gather*}
\sum_{M} \mathscr{D}_{M, \mu_{j}-\lambda_{j}}^{J *}\left(\Phi_{j}, \Theta_{j},-\Phi_{j}\right) \mathscr{D}_{\mathrm{M}, \mu_{a}-\mu_{b}}^{\mathrm{J}}(\Phi, \Theta,-\Phi)=\sum_{M} \mathscr{D}_{\left.\mu_{j}-\lambda_{j}, M^{\left(+\Phi_{j},-\Theta_{j}\right.}{ }_{j}^{J} \Phi_{j}\right)} \\
\mathscr{D}_{\mathrm{M} \mu_{\mathrm{a}}-\mu_{b}}^{\mathrm{J}}(\Phi, \Theta,-\Phi)=\mathscr{D}_{\mu_{j}-\lambda_{j}, \mu_{a}-\mu_{b}}^{J}(\alpha, \beta, \gamma) \tag{65}
\end{gather*}
$$

where

$$
\begin{equation*}
R(\alpha, \beta, \gamma)=R\left(\Phi_{j},-\Theta{ }_{j},-\Phi_{j}\right) R(\Phi, \Theta,-\Phi) \tag{66}
\end{equation*}
$$

2. Transition Amplitude when using a Polarized Target or Beam

In this case we have an initial co-ordinate system defined when there is transverse polarization. If the polar angles of the beam are again $\Theta, \Phi$ in system $S$, then the initial co-ordinate system will be related by an extra rotation $\alpha$ about the $0 Z$ axis of $S^{\prime}$ ( $S^{\prime}$ is the co-ordinate system obtained by rotation of S through the Euler angles $\Phi, \Theta,-\Phi$ ), i.e., we have
$\left.\begin{array}{lll}0 X Y Z & \rightarrow 0 X^{\prime} Y^{\prime} Z^{\prime} & \text { Euler angles } \Phi, \Theta,-\Phi \\ 0 X^{\prime} Y^{\prime} Z^{\prime} & \rightarrow 0 \mathrm{xyz} & \text { Euler angles } \alpha, 0,0 \\ 0 \mathrm{XYZ} \rightarrow 0 \mathrm{Xy} & \text { Euler angles } \Phi, \Theta, \alpha-\Phi\end{array}\right\}$
$0 x y z$ is the frame in which the target polarization is defined. Then

$$
\begin{equation*}
A_{\mu}^{n^{\prime}}=A_{\mu}^{n} e^{-i\left(\mu_{a}-\mu_{b}\right) \alpha} \tag{68}
\end{equation*}
$$

3. Transition Amplitude when a Final State Particle Decays

If one of the final particles is a baryon which can undergo weak decay, e.g., $\Lambda \rightarrow p \pi^{-}$then the decay angular distribution will give information on the parent baryon polarization. This decay amplitude can then be directly introduced into the transition amplitude.

Suppose particle $j$ of (59) undergoes weak decay. We express the decay matrix in terms of canonical spin states with respect to the helicity frame axes of particle j (see Fig. 15), i.e., the matrix element for

$$
\begin{equation*}
j \rightarrow 1+2 \tag{69}
\end{equation*}
$$

is written as

$$
\begin{align*}
\mathrm{B}_{\mathrm{m}_{1} \mathrm{~m}_{2} \mu_{j}}^{\mathrm{D}} & =\left\langle\sigma_{1} \mathrm{~m}_{1} \sigma_{2} \mathrm{~m}_{2}\right| \mathrm{T}_{\mathrm{D}}\left|\sigma_{j} \mu_{\mathrm{j}}\right\rangle \\
& =\sum_{\mathrm{L}_{d} S_{d}} B^{L_{d} S_{d}} \mathrm{C}\left(\sigma_{1} \sigma_{2} S_{d} \mid m_{1} m_{2}\right) \mathrm{C}\left(\mathrm{~L}_{d} S_{d} \sigma_{j} \mid \mu_{j}-m m\right) Y_{L_{d}}^{\mu_{j}-\mathrm{m}_{d}}\left(\theta_{d}, \phi_{d}\right) \tag{70}
\end{align*}
$$

where

$$
\begin{equation*}
B^{\mathrm{L}_{\mathrm{d}} \mathrm{~S}_{\mathrm{d}}}=\left\langle\sigma_{1} \sigma_{2} \mathrm{~L}_{\mathrm{d}} \mathrm{~S}_{\mathrm{d}}\right| \mathrm{T}_{\mathrm{D}}\left|\sigma_{\mathrm{j}}\right\rangle \tag{71}
\end{equation*}
$$

Thus the final transition amplitude becomes

$$
\begin{equation*}
\mathrm{A}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mathrm{~m}_{1} \mathrm{~m}_{2} \mu_{\mathrm{k}} \mu_{1}}^{\mathrm{D}}=\sum_{\mu_{\mathrm{j}}} \mathrm{~B}_{\mathrm{m}_{1} \mathrm{~m}_{2} \mu_{\mathrm{j}}}^{\mathrm{D}} \mathrm{~A}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{j}} \mu_{\mathrm{k}} \mu_{1}} \tag{72}
\end{equation*}
$$

In the case of $\Lambda$ decay obtained for instance in the reaction

$$
\begin{equation*}
\mathrm{K}^{-} \mathrm{p} \rightarrow \Lambda \pi^{+} \pi^{-} \tag{73}
\end{equation*}
$$

many simplifications occur; $\sigma_{2}=m_{2}=0$ and

$$
\begin{equation*}
B_{m_{1} m_{2} \mu_{j}}^{D}=\sum_{L_{d}} B^{L_{d}} C\left(l_{d} \sigma_{1} \sigma_{j} \mid \mu_{j}-m_{1} m_{1}\right) Y_{L_{d}}^{\mu_{j}-m_{1}}{ }_{\left(\theta_{d}, \phi_{d}\right)} \tag{74}
\end{equation*}
$$

Further if we perform scattering from polarized targets the total transition amplitude will be (using (68))

$$
\begin{equation*}
A_{\mu_{a}}^{\mathrm{D}} \mu_{\mathrm{b}} \mathrm{~m}_{1} \mathrm{~m}_{2} \mu_{k} \mu_{1}=A_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mathrm{~m}_{1} \mathrm{~m}_{2} \mu_{k} \mu_{1}}^{\mathrm{D}} \mathrm{e}^{-\mathrm{i}\left(\mu_{\mathrm{a}}-\mu_{\mathrm{b}}\right) \alpha} \tag{75}
\end{equation*}
$$

## 4. Experimental Observables

In order to simplify this discussion, I will specialize to

$$
\begin{equation*}
\mathrm{M}_{1} \mathrm{~B}_{1} \rightarrow \mathrm{~B}_{2} \mathrm{M}_{2} \mathrm{M}_{3} \tag{76}
\end{equation*}
$$

where

$$
B_{1} B_{2}=\frac{1}{2}^{+}, \quad M_{1}, M_{2}, M_{3}=0^{-}
$$

e.g., reactions

$$
\left.\begin{array}{l}
\pi \mathrm{N} \rightarrow \mathrm{~N} \pi \pi  \tag{77}\\
\mathrm{KN} \rightarrow \Lambda \pi \pi
\end{array}\right\}
$$

We assume the initial polarization is specified in $0 x y z$ and final baryon polarization in the helicity frame (see Fig. 15). The transition amplitudes are then given by Ea. (68).
(a) Unpolarized Cross Sections. The initial density matrix is

$$
\begin{equation*}
\rho^{i}=\frac{1}{2} \tag{78}
\end{equation*}
$$

and the differential cross section is given by

$$
\begin{equation*}
I_{0}=\operatorname{Tr}\left[A^{\prime} \rho^{\mathrm{i}} \mathrm{~A}^{\prime} \dagger\right]=\frac{1}{2} \sum_{\mu}\left|\mathrm{A}_{\mu}^{\prime}\right|^{2}=\frac{1}{2} \sum_{\mu}\left|\mathrm{A}_{\mu}\right|^{2} \tag{79}
\end{equation*}
$$

(b) Asymmetry from a Polarized Target. In this situation the initial density matrix is given by

$$
\begin{equation*}
\rho^{\mathrm{i}}=\frac{1}{2}\left[1+\mathrm{p}_{\mathrm{b}} \cdot \sigma_{-\mathrm{b}}\right] \tag{80}
\end{equation*}
$$

and the differential cross section is

$$
\begin{equation*}
I_{p}=\operatorname{Tr}\left[A^{\prime} \rho^{i} A^{\prime}{ }^{\dagger}\right]=I_{0}\left[1+\vec{p}_{b} \cdot \vec{a}\right] \tag{81}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{0} \vec{a}=\frac{1}{2} \operatorname{Tr}\left[A^{\prime} \sigma_{b} A^{\prime} \dagger\right] \tag{82}
\end{equation*}
$$

(c) Final Baryon Polarization in Scattering from an Unpolarized Target. In this situation the final baryon density matrix is given by

$$
\begin{equation*}
\mathrm{I}_{0} \rho^{\mathrm{f}}=\frac{1}{2} \mathrm{~A}^{\prime} \mathrm{A}^{\mathrm{t}} \tag{83}
\end{equation*}
$$

Then the polarization, $P_{L}$, of $B_{2}$ in a direction $L$ is given by

$$
\begin{equation*}
\mathrm{I}_{0} \mathrm{P}_{\mathrm{L}}=\frac{1}{2} \operatorname{Tr}\left[\mathrm{~A}^{\prime} \mathrm{A}^{\prime} \dagger_{\sigma_{L}}\right] \tag{84}
\end{equation*}
$$

(d) Depolarization Tensor, i.e., Final Polarization from a Polarized Target. If $P_{L j}$ is the polarization of particle $j$ in direction $L$ then

$$
\begin{equation*}
P_{L_{j}}=\operatorname{Tr}\left[\rho^{f} \sigma_{L_{j}}\right] \tag{85}
\end{equation*}
$$

and we obtain

$$
\begin{align*}
I_{p} P_{L j} & =\operatorname{Tr}\left[A^{\prime} \rho^{i} A^{\prime}{ }^{\dagger} \sigma_{L j}\right] \\
& =I_{0}\left[P_{L j}+\sum_{i} P_{i b} D_{i b, L j}\right] \tag{86}
\end{align*}
$$

where

$$
\begin{equation*}
\mathrm{I}_{0} \mathrm{D}_{\mathrm{ib}, \mathrm{Lj}}=\frac{1}{2} \operatorname{Tr}\left[\mathrm{~A}^{\prime} \sigma_{i b} \mathrm{~A}^{\dagger^{\dagger}} \sigma_{\mathrm{Lj}}\right] \tag{87}
\end{equation*}
$$

These results are summarized in Table IV.
We now turn to a detailed discussion of the scattering from an unpolarized incident state.
5. The Differential Cross Section from an Unpolarized Incident State

In the appendix I give the phase space which corresponds to our normalization of states and this leads to the following differential cross section for reaction (59)

$$
\begin{equation*}
\mathrm{d} \sigma=\frac{\pi^{2}}{\mathrm{Wp}} \frac{1}{\left(2 \sigma_{\mathrm{a}}+1\right)\left(2 \sigma_{\mathrm{b}}+1\right)} \sum_{\mu}\left|\mathrm{A}_{\mu}\right|^{2} \mathrm{~d} \rho \tag{88}
\end{equation*}
$$

where $A_{\mu}$ is given by Eq: (63).
(a) Conservation of Parity. The variety of terms in this sum can be reduced by understanding the effects of parity. This is equivalent to setting $\mu \rightarrow-\mu$ in (63). We write

$$
\begin{equation*}
A_{\mu_{a} \mu_{b} \mu_{j} \mu_{k} \mu_{l}}=g_{\mu}^{\mathrm{n}} \mathrm{~T}^{\mathrm{n}} \mathrm{D}_{\mathrm{l}_{\mathrm{j}} \mathrm{~s}_{\mathrm{j}}}^{\mathrm{j}_{\mathbf{j}}} \tag{89}
\end{equation*}
$$

and it can be shown (Cashmore et al., 1972) that

$$
\begin{equation*}
\mathrm{g}_{-\mu}^{\mathrm{n}}=\eta_{\mathrm{a}} \eta_{\mathrm{b}} \eta_{\mathrm{j}} \eta_{\mathrm{k}} \eta_{\mathrm{l}}(-\mathrm{I})^{\sigma_{\mathrm{a}}+\mu_{\mathrm{a}}+\sigma_{\mathrm{b}}-\mu_{\mathrm{b}}}{ }_{(-1)^{\sigma_{\mathrm{j}}+\mu_{\mathrm{j}}+\sigma_{\mathrm{k}}+\mu_{\mathrm{k}}+\sigma_{\mathrm{l}}+\mu_{1}}\left(\mathrm{~g}_{\mu}^{\mathrm{n}}\right)^{*}} \tag{90}
\end{equation*}
$$

For any specific problem this reduces the number of independent $\mathrm{g}_{\mu}^{\mathrm{n}}$. In the case of reaction (76) we have

$$
\begin{equation*}
\mathrm{g}_{-\mu}^{\mathrm{n}}=(-1)^{\mu_{\mathrm{f}}-\mu_{\mathrm{i}}}\left(\mathrm{~g}_{\mu}^{\mathrm{n}}\right)^{*} \tag{91}
\end{equation*}
$$

where $\mu_{\mathrm{i}}$ and $\mu_{\mathrm{f}}$ are the helicities of the incident and final baryons. Since $\mathrm{T}^{\mathrm{n}}$ is independent of $\mu$ we finally obtain in this case

$$
\begin{equation*}
A_{-\mu}^{\mathrm{n}}=(-1)^{\mu_{\mathrm{f}}-\mu_{1}} \mathrm{~g}_{\mu}^{\mathrm{n} *} \mathrm{~T}_{\mathrm{n}} \mathrm{D}_{\mathrm{l}_{\mathrm{j}} \mathrm{~s}_{\mathrm{j}}}^{\mathrm{j}_{\mathrm{j}}} \tag{92}
\end{equation*}
$$

(b) The Cross Section due to Isobars in the (kl) system and the Relation to Two Body Scattering. We first calculate the cross section due to isobars
in the ( kl ) system and then relate this to the equivalent expressions for stable two body final states through the zero width approximation for these isobars.

Using the fact that we can integrate over $\alpha$ (the angle of rotation about the incident beam) we can write

$$
\begin{equation*}
\mathrm{d}_{\rho}=\frac{\pi \mathrm{q}_{\mathrm{k}} \mathrm{Q}_{\mathrm{j}}}{8 \mathrm{~W} \omega_{\mathrm{j}}} \mathrm{~d} \omega_{\mathrm{j}}^{2} \mathrm{~d} \cos \theta_{\mathrm{j}} \mathrm{~d} \cos \Theta \mathrm{~d} \Phi \tag{93}
\end{equation*}
$$

The total cross section is then
$\sigma=\int \underset{W p}{2} \sum_{\mu} \sum_{n m} g_{\mu} \mathrm{n}_{\mu} \mathrm{g}_{\mu}^{*} \mathrm{X}^{\mathrm{n}}\left(\mathrm{W}, \omega_{\mathrm{j}}\right) \mathrm{X}^{\mathrm{n}^{*}}\left(\mathrm{~W}, \omega_{\mathrm{j}}\right) \frac{\pi \mathrm{q}_{\mathrm{k}} \mathrm{Q}_{\mathrm{j}}}{8 \mathrm{~W} \omega_{\mathrm{j}}} \mathrm{d} \omega_{\mathrm{j}}^{2} d \cos \theta_{\mathrm{j}} \mathrm{d} \cos \Theta \mathrm{d} \Phi$
where

$$
\sum_{\mu}=\frac{1}{\left(2 \sigma_{\mathrm{a}}+1\right)\left(2 \sigma_{\mathrm{b}}+1\right)} \sum_{\mu}
$$

and

$$
x^{n}\left(W, \omega_{j}\right)=T^{n}\left(W, \omega_{j}\right){ }_{1_{j} s_{j}}^{j_{j}}\left(\omega_{j}\right)
$$

The above expression can then be reduced to give (Cashmore et al., 1972)

$$
\begin{equation*}
\sigma=\frac{\pi}{\mathrm{p}^{2}} \sum_{\mathrm{n}} \frac{(2 \mathrm{~J}+1)}{\left(2 \sigma_{\mathrm{a}}+1\right)\left(2 \sigma_{\mathrm{b}}+1\right)} \int\left|\mathrm{X}^{\mathrm{n}}\left(\mathrm{~W}, \omega_{\mathrm{j}}\right)\right|^{2} \mathrm{~d} \omega_{\mathrm{j}}^{2} \tag{95}
\end{equation*}
$$

where we see that isobars of different quantum numbers in the (kl) system do not interfere.

If we now use Breit-Wigner form for $D_{l_{j} S_{j}}^{j_{j}}\left(\omega_{j}\right)$ and take the zero width limit the cross section becomes

$$
\begin{equation*}
\sigma=\frac{\pi}{p^{2}} \sum_{\mathrm{n}} \frac{2 \mathrm{~J}+1}{\left(2 \sigma_{a}+1\right)\left(2 \sigma_{\mathrm{b}}+1\right)}\left|\mathrm{T}_{\mathrm{LS}_{j} \mathrm{j}_{\mathrm{j}}}\left(\mathrm{~W}, \omega_{0}\right)\right|^{2} \tag{96}
\end{equation*}
$$

which is the usual form in two body scattering.
Finally we note that our forms of the kinematical factors (see Eqs. (72) and (62)) mean that the total cross section has the correct threshold dependence, i.e.,

$$
\begin{equation*}
\sigma \propto \frac{1}{p^{2}} p^{2 L+1} Q_{j}^{2 L j+1} \tag{97}
\end{equation*}
$$

6. The Quantities of Physical Interest

The object of analysing reaction (59) is
(i) to measure the partial wave amplitudes $\tau^{\mathrm{n}}$ where

$$
\begin{align*}
T^{n} & =T_{p}^{J L S}(W) T_{1 L_{j} S_{j}}^{n J}\left(W, \omega_{j}\right) \\
& \sim \frac{p^{L+1 / 2}}{\sqrt{4 W}} \frac{Q_{j}^{L}+1 / 2}{\sqrt{4 W}} \tau_{p}^{J L S}(W) \tau_{1 L_{j} S_{j}}^{n J}(W) \tag{98}
\end{align*}
$$

and we define

$$
\begin{equation*}
\tau^{\mathrm{n}}=\tau_{\mathrm{p}}^{\mathrm{JLS}}(\mathrm{~W}) \tau_{1 \mathrm{~L}_{\mathrm{j}} \mathrm{~S}_{\mathrm{j}}}^{\mathrm{nJ}}(\mathrm{~W}) \tag{99}
\end{equation*}
$$

The variation of this quantity will then indicate the presence of resonances (together with their decay channels) and its values can be compared with theoretical predictions.
(ii) to measure $D_{1}{ }_{j}^{j}\left(\omega_{j}\right)$ in some specific cases. This has been used to try to determine the $\mathrm{I}=0(\pi \pi)$ phase shifts at low energies, a difficult thing to do by other techniques. Unfortunately the results are very dependent on the form of the model and there are some problems of interpretation making this a much less rewarding application.

In the rest of these lectures I will concentrate on (i) as these are the most reliable results of this type of analysis and are providing the most excitement in this area of physics.

I want finally in this section to list the isobar states that we have considered in our analysis of

$$
\begin{equation*}
\pi \mathrm{N} \rightarrow \pi \pi \mathrm{~N} \tag{100}
\end{equation*}
$$

for $E_{c . m}<2000 \mathrm{MeV}$ together with their partial waves. We have so far only considered the prominent $\Delta, \rho$ and $\epsilon(\mathrm{I}=0 \pi \pi)$ final state interactions and their partial waves are listed in Table $V$, where the notation is $L_{\text {in }}, L_{\text {out }}$, $2 \mathrm{I}, 2 \mathrm{~J}$. One can easily see how rapidly the number of partial waves grows. Furthermore certain aspects of the data already indicate that an improvement will be obtained by inclusion of $\mathrm{P}_{11}, \mathrm{D}_{13}$ and $\mathrm{F}_{15}$ isobars particularly in the $\pi^{+} \pi^{+} \mathrm{n}$ final state.
D. PRODUCTION REACTIONS

In these types of reactions we are concerned with three particle subsystems in final states containing four or more particles, i.e., reactions

$$
\begin{align*}
& \mathrm{a}+\mathrm{b} \rightarrow \mathrm{c}+\mathrm{X} \\
& \bigsqcup^{\mathrm{X}} \mathrm{j}+\mathrm{k}+1 \tag{101}
\end{align*}
$$

of which there are many examples at present being studied, e.g.,

$$
\begin{align*}
\pi+\mathrm{p} & \rightarrow \mathrm{p}+\left(\mathrm{A}_{1}, \mathrm{~A}_{2}, \mathrm{~A}_{3}\right)  \tag{102}\\
\mathrm{Kp} & \rightarrow \mathrm{p}+(\mathrm{Q}, \mathrm{~L})  \tag{103}\\
\pi, \mathrm{K}, \mathrm{p}+\mathrm{p} & \rightarrow \pi, \mathrm{~K}, \mathrm{p}+\mathrm{N}^{*} \tag{104}
\end{align*}
$$

These are sketched in Fig. 16.
The variables we require to define the particles $a, b$ and $c$ have already been given at the beginning of Section III. We do however have to introduce a new co-ordinate system $\mathrm{S}^{\prime}$ in which to describe the production properties of the state $X . S^{\prime}$ will be related to the other particles $\mathrm{a}, \mathrm{b}$ and c and will reflect our prejudices about the type of production process occurring, e.g.,
(i) $S^{\prime}$ will be the Gottfried-Jackson system if we are interested in one particle exchange (see Fig. 17). 0Z' is defined as $\vec{p} \pi$ and $0 Y^{t}$ by $\vec{p}_{\text {in }} \wedge \vec{p}_{\text {out }}$ in the jkl rest system.
(ii) $S^{\prime}$ will be the helicity frame if we are concerned with S-channel helicity conservation or absorption model predictions. In this case 0Z' is defined as $-\overrightarrow{\mathrm{p}}_{\text {out }}$ in the jkl system. $0 \mathrm{Y}^{\prime}$ has the previous definition. 1. Amplitude for Production of $X$ and its Subsecuent Decay

We assume that X is produced in an angular momentum substate |nJM> with respect to $S^{\prime}$. Let the amplitude for the production of this state together with a final helicity $\mu_{c}$ of particle c from particles a and b be

$$
\begin{equation*}
\mathrm{R}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}}^{\mathrm{J} \mu_{\mathrm{c}}^{\mathrm{p}}}(\mathrm{E}, \mathrm{~W}, \mathrm{t})} \tag{105}
\end{equation*}
$$

where W is the mass of the three particle system, t is the four-momentum transfer to the three particle system and E is the total c.m. energy (i.e., in the $a b$ system). We also include $n$ in the state definition at this point to account for the fact that different channels of the same JPM may be produced with different helicity changes at the other vertex. In general, I will not include the total c.m. energy $E$ in the arguments of $R$ but it should always be understood to be there.

In order to describe the subsequent decay of $X$ we require the following transition matrix element

$$
\begin{align*}
A_{\mu_{j} \mu_{k} \mu_{I}}^{\mathrm{nJPm}} & =S^{<n Q_{j} \mu_{j} Q_{k} \mu_{k} Q_{1} \mu_{1}|T| J P M n>} S^{\prime} \\
& \left.=\sum_{m} S^{<Q_{j} \mu_{j} Q_{k} \mu_{k} Q_{1} \mu_{1}|T| J P m n>} S \quad S^{\langle J P m n| J P M n}\right\rangle_{S^{\prime}}  \tag{106}\\
& =\sum_{m} S^{\left\langle Q_{j} \mu_{j} Q_{l} \mu_{k} Q_{1} \mu_{1}\right| T \mid J P m n>S} \mathscr{D}_{m M}^{J}(\alpha, \beta, \gamma) \tag{107}
\end{align*}
$$

where $\alpha, \beta, \gamma$ are the Euler angles defining the transformation from $S$ to $S^{\prime}$. We have already calculated the first factor in this sum - it is just the $f_{\mu_{j} \mu_{k} \mu_{l}}^{\text {nJM }}$ of Section III.A. Thus we can write

$$
\begin{align*}
A_{\mu_{j} \mu_{k} \mu_{l}}^{\mathrm{nJPM}} & =\sum_{\mathrm{m}} G_{\mu_{j} \mu_{k} \mu_{l}}^{\mathrm{nJm}} \mathrm{~T}_{1 L_{j} S_{j}}^{\mathrm{nJ}}\left(W, \omega_{j}\right) D_{l_{j} S_{j}}^{\mathrm{j}_{j}}\left(\omega_{j}\right) \mathscr{D}_{\mathrm{mM}}^{J}(\alpha, \beta, \gamma)  \tag{108}\\
& =F_{\mu_{j} \mu_{k} \mu_{l}}^{\mathrm{JJM}} \mathrm{~T}_{1 L_{j} S_{j}}^{\mathrm{nJ}}\left(W, \omega_{j}\right) D_{l_{j} S_{j}}^{\mathrm{j}_{j}}\left(\omega_{j}\right) \tag{109}
\end{align*}
$$

Note that I have just written the amplitude for reaching the final state via one intermediate isobar state. The total amplitude is again obtained by coherently adding the terms due to other isobars ensuring that the correct
symmetry properties are obtained. The forms of $T_{1 L_{j} S_{j}}^{J n}$ and $D_{l_{j} \mathbf{S}_{j}}^{j_{j}}\left(\omega_{j}\right)$ have been discussed in Section III. A.

## 2. Properties of Production and Decay Amplitudes under Parity

(a) Production Amplitude: $\mathrm{R}_{\mu_{\mathrm{a}} \mu_{b} \mu_{\mathrm{c}}}^{\mathrm{nJM}}$. Consider a co-ordinate system $S^{\prime}$ in which the Z axis is defined by a polar vector and the Y axis by an axial vector, e.g., the Gottfried-Jackson system as defined in Fig. 17. Then

$$
\begin{aligned}
& =\left\langle\phi_{J M^{*}}^{\psi_{\mathrm{q}_{c}} \mu_{\mathrm{c}}}\right| \mathrm{P}^{-1} \mathrm{PTP}^{-1} \mathrm{P} \mid \psi_{\vec{q}_{a}} \mu_{\mathrm{a}} \psi_{\overrightarrow{\mathrm{q}}_{\mathrm{b}}} \mu_{\mathrm{b}}>
\end{aligned}
$$

where $\mathrm{PT} \mathrm{P}^{-1}=\mathrm{T}$ and $\mathrm{P} \phi_{\mathrm{JM}}=\eta_{\mathrm{J}} \phi_{\mathrm{JM}}$, i.e., $\eta_{\mathrm{J}}$ is the parity of state X . If we now use an operator 0 which produces a rotation $\pi$ about the Y axis we have

$$
\begin{aligned}
& 0 \psi_{-\overrightarrow{\mathrm{q}}-\mu}=\psi_{\overrightarrow{\mathrm{q}}-\mu} \\
& 0 \psi_{\mathrm{JM}}={ }_{(-1)^{\mathrm{J}-\mathrm{M}_{\phi_{\mathrm{J}-\mathrm{M}}}}} .
\end{aligned}
$$

N.B. this assumes the Z-axis is in the production plane.

Thus eventually we have
(b) Decay Amplitude: $\mathrm{A}_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{nJM}}$. In this case the properties under parity are most easily seen by setting $\mu \rightarrow-\mu$ and following a similar calculation to that for the formation reaction (Cashmore et al., 1972). The result we obtain is that

$$
\begin{equation*}
A_{-\mu_{j}-\mu_{k}-\mu_{1}}^{\mathrm{nJ}-M}=(-1)^{J-M}(-1)^{l_{j}+L_{j}}{ }_{(-1)}^{\sigma_{j}+\mu_{j}+\sigma_{k}+\mu_{k}+\sigma_{1}+\mu_{1}}\left[F_{\mu_{j} \mu_{k} \mu_{1}}^{n J M}\right]^{*} \mathrm{~T}_{1 L_{j} S_{j}}^{J n}\left(X, \omega_{j}\right) D_{1_{j} S_{j}}^{\mathrm{j}_{j}}\left(\omega_{j}\right) \tag{111}
\end{equation*}
$$

3. The Cross Section and its Properties for the Process $a+b \rightarrow c+j+k+1$

The differential cross section for the process is then given as

$$
\begin{equation*}
\frac{d^{5} \sigma\left(W, t, \mu_{a} \mu_{b} \mu_{c} \mu_{j} \mu_{k} \mu_{1}\right)}{d \alpha d \cos \beta d \gamma d \omega_{1}^{2} d \omega_{2}^{2}} \propto\left|\sum_{n J M} R_{\mu_{a} \mu_{b} \mu_{c}}^{n J M}(W, t) A_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{nJM}}\left(\mathrm{~W}, \alpha, \beta, \gamma, \omega_{1}^{2}, \omega_{2}^{2}\right)\right|^{2} \tag{112}
\end{equation*}
$$

i.e., the production amplitude $R$ is a function of $W$ (the mass of the intermediate system) and t , while A is a well defined function of $\mathrm{W}, \alpha, \beta, \gamma, \omega_{1}^{2}$ and $\omega_{2}^{2}$, the parameters necessary to describe the final three particle state.

Note here we have five variables describing the decay whereas in the formation reaction (for unpolarized incident particles) we only require four. This is due to the fact that $a, b$ and $c$ define very specifically the initial co-ordinate frame (just as a polarized target would).

Thus if we consider the situation in which we do not observe the polarizations of the final particles in seattering from an unpolarized target we have

$$
\begin{align*}
& \mathrm{d} \sigma \propto \sum_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{c}}}\left|\sum_{\mathrm{nJM}} \mathrm{R}_{\mu_{\mathrm{a}} \mu_{b} \mu_{c}}^{\mathrm{nJM}}(\mathrm{~W}, \mathrm{t}) A_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{nJM}}\left(\mathrm{~W}, \alpha, \beta, \gamma, \omega_{1}^{2}, \omega_{2}^{2}\right)\right|^{2}  \tag{113}\\
& \mu_{j} \mu_{k} \mu_{1} \\
& \propto \sum_{\substack{\mu_{a} \mu_{b} \mu_{c} \\
\mu_{j} \mu_{k} \mu_{1}}} \sum_{\substack{J M n \\
J^{\prime} M^{\prime} n^{\prime}}} R_{\mu_{a} \mu_{b} \mu_{c}}^{n J M} R_{\mu_{a} \mu_{b} \mu_{c}}^{n^{\prime} J^{\prime} M^{+*}} A_{\mu_{j} \mu_{k} \mu_{1}}^{n J M} A_{\mu_{j} \mu_{k} \mu_{1}}^{n^{\prime} J^{\prime} M^{*}}  \tag{114}\\
& \propto \sum_{\mu_{j} \mu_{k^{\prime}} \mu_{1}} \sum_{\substack{n^{\prime} J M \\
n^{\prime} \mathrm{J}^{\prime}}} \rho_{\mathrm{MM}} \mathrm{nn}^{\prime} A_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{nJM}} A_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{n}^{\prime} J^{\prime} \mathrm{M}^{\prime *}} \tag{115}
\end{align*}
$$

where we have defined an unnormalized density matrix

$$
\begin{equation*}
\rho_{M_{M}^{\prime}}^{\mathrm{nn}^{\prime}}=\sum_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{c}}} \mathrm{R}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}} \mathrm{R}_{\mu_{\mathrm{a}} \mu_{\mathrm{b}} \mu_{\mathrm{c}}^{\mathrm{n}^{\prime} \mathrm{J}^{\prime} \mathrm{M}^{\prime} *}} \tag{116}
\end{equation*}
$$

This density matrix possesses
(i) the usual property of hermiticity

$$
\begin{equation*}
\rho_{\mathrm{MM}^{\prime}}^{\mathrm{nn}^{\prime}}=\frac{\mathrm{n}^{\prime} \mathrm{n}^{\prime} \mathrm{M}^{\prime} \mathrm{M}}{} \tag{117}
\end{equation*}
$$

(ii) the property from conservation of parity of

$$
\begin{equation*}
\rho_{\mathrm{Mn}^{\prime}}^{\mathrm{nn}^{\prime}}=\eta_{\mathrm{n}^{\prime}} \eta_{\mathrm{n}^{\prime}}(-1)^{\mathrm{J}+\mathrm{J}^{\prime}-\mathrm{M}-\mathrm{M}^{\prime}} \rho_{-\mathrm{M}-\mathrm{M}^{\prime}}^{\mathrm{nn}^{\prime}} \tag{118}
\end{equation*}
$$

If $\mathrm{J}=\mathrm{J}$ ' and $\mathrm{n} \equiv \mathrm{n}^{\prime}$

$$
\begin{equation*}
\rho_{\mathrm{MM}}^{\mathrm{nn}}=(-1)^{\mathrm{M}-\mathrm{M}^{\prime}} \rho_{-\mathrm{M}-\mathrm{M}^{\prime}}^{\mathrm{nn}} \tag{119}
\end{equation*}
$$

## 4. Structure of Cross Section

We can now demonstrate some aspects of the structure of these cross sections which will lead us to well known results.

We first display explicitly the $\alpha, \beta, \gamma$ dependence of the distributions

$$
\mathrm{d}^{5} \sigma \propto \sum_{\mathrm{Mn}} \rho_{\mathrm{MM}^{\prime}}^{\mathrm{nn}^{\prime}}(\sum_{\mathrm{m}, \mathrm{~m}^{\prime}} \underbrace{\mathscr{D}_{\mathrm{mM}}^{J}(\alpha, \beta, \gamma) \mathscr{D}_{\mathrm{m}^{\prime} \mathrm{M}^{\prime}}^{J^{\prime *}}(\alpha, \beta, \gamma), \ldots}_{\begin{array}{c}
\text { The only terms containing }  \tag{120}\\
\alpha, \beta, \gamma, \mathrm{M} \text { and } \mathrm{M}^{\prime}
\end{array}})
$$

If we first integrate over $\alpha$ we obtain the result that $m=m^{\prime}$, i.e., the dependence of the differential cross section on $\beta$ and $\gamma$ is given by

$$
\begin{align*}
\mathrm{d}^{4} \sigma & \propto \sum_{\substack{\mathrm{nn}^{\prime} \\
\mathrm{MM}}} \rho_{\mathrm{MM}} \mathrm{nn}^{\prime},\left\{\sum_{\mathrm{m}} \mathscr{D}_{\mathrm{mM}} \mathrm{~J}^{(0, \beta, \gamma)} \mathscr{D}_{\mathrm{mM}^{\prime}}^{\mathrm{J}^{\prime}}(0, \beta, \gamma)\right\} \\
& \propto \sum_{\mathrm{Mn}^{\prime}} \rho_{\mathrm{MM}^{\prime}} \rho_{\mathrm{Mn}^{\prime}}\left\{\sum_{\mathrm{m}} \mathscr{D}_{\mathrm{mM}^{\prime}}^{\mathrm{J}}(0, \beta, \gamma) \mathscr{D}_{-\mathrm{m}^{\prime}-\mathrm{M}^{\prime}}^{\mathrm{J}^{\prime}}(0, \beta, \gamma)(-1)^{\mathrm{m}-\mathrm{M}^{\prime}}\right\} \tag{12I}
\end{align*}
$$

We can now use the result that

$$
\begin{align*}
\mathscr{D}_{\mathrm{mM}}^{\mathrm{J}}(0, \beta, \gamma) \mathscr{D}_{-\mathrm{m}-\mathrm{M}^{\prime}}^{\mathrm{J}^{\prime}}(0, \beta, \gamma)= & \sum_{\mathrm{L}} \mathrm{C}\left(J, J^{\prime} \mathrm{L} \mid \mathrm{m},-\mathrm{m}\right) \mathrm{C}\left(\mathrm{~J} J^{\prime} \mathrm{L} \mid \mathrm{M},-\mathrm{M}^{\prime}\right) \\
& \mathscr{D}_{0, \mathrm{M}-\mathrm{M}^{\prime}}^{\mathrm{L}}(0, \beta, \gamma) \\
= & \sum_{\mathrm{L}} \mathrm{C}\left(\mathrm{~J}, J^{\prime} \mathrm{L} \mid \mathrm{m},-\mathrm{m}\right) \mathrm{C}\left(\mathrm{~J}^{\prime} \mathrm{L} \mid \mathrm{M},-\mathrm{M}^{\prime}\right)\left(\frac{4 \pi}{2 \mathrm{~L}+1}\right)^{1 / 2} \mathrm{Y}_{\mathrm{L}}^{\mathrm{M}^{\prime}-\mathrm{M}_{(\beta, \gamma)}} \tag{122}
\end{align*}
$$

to display the $(\beta, \gamma)$ dependence of the cross section (121), i. e., the presence of spherical harmonics $Y_{L}^{m}$ with $m \neq 0$ immediately implies the
presence of off diagonal terms in the density matrix and furthermore the maximum value of $L$ can be a guide to the maximum $J$ present ( $L_{\leq} \leq{ }^{2} \max$ ).

If we now integrate over $\alpha, \beta$ and $\gamma$ (i.e., we just look at the Dalitz plot population) we can use the orthogonality properties of the $\mathscr{D}$ function to write

$$
\begin{equation*}
\frac{d^{2} \sigma}{d \omega_{1}^{2} d \omega_{2}^{2}} \propto \sum_{\substack{n, n^{\prime} \\ J=J^{\prime} \\ M=M^{\prime}}} \rho_{\mathrm{MM}}^{\mathrm{nn}}\left(\sum_{\mathrm{m}} \ldots\right. \tag{123}
\end{equation*}
$$

Thus the Dalitz plot population depends only on $\rho_{\mathrm{MM}}^{\mathrm{JJ}}$ and does not contain any terms corresponding to interferences between states which differ in $J$ or M. Moreover since M does not appear within the parentheses we find that the Dalitz plot population is independent of $M$, the magnetic substate in which the state is initially produced. This result is expected since by integrating over $\alpha, \beta$, and $\gamma$ we have effectively destroyed any knowledge of the initial orientation.

The other result we would like to demonstrate is that waves of different parity do not interfere in the Dalitz plot distribution. To do this consider the interference of an isobar of type $j$ with one of type $k$. We already know that we only have to consider states of the same $J$ and $M$,
and thus the general term is

$$
\begin{align*}
& \frac{\mathrm{d}^{2} \sigma}{\mathrm{~d} \omega_{1}^{2} \mathrm{~d} \omega_{2}^{2}} \propto \int \sum_{\mu_{\mathrm{j}} \mu_{k} \mu_{1}}\left[\sum_{M} \rho_{M M}^{\mathrm{nn}^{\prime}} A_{\mu_{j} \mu_{k} \mu_{1}}^{\mathrm{nJM}}(\mathrm{j}) A_{\mu_{\mathrm{j}} \mu_{k} \mu_{1}}^{\left.\mathrm{n}^{\prime} \mathrm{J}\right)}\right] \mathrm{d} \alpha \mathrm{~d} \cos \beta \mathrm{~d} \gamma \\
& \propto \int \sum_{\mu_{j} \mu_{k} \mu_{1}} \sum_{\mathrm{M} \geq 0}\left(1-\frac{\delta_{M, 0}}{2}\right)\left[\rho_{M M}^{n n^{\prime}} A_{\vec{\mu}}^{\mathrm{nJM}}(\mathrm{j}) \mathrm{A}_{\vec{\mu}}^{\mathrm{n}^{\prime} J M^{*}}(\mathrm{k})\right. \\
& \left.+\rho_{-M-M}^{n n^{\prime}} A_{\vec{\mu}}^{\mathrm{nJ}-M^{2}} A_{\vec{\mu}}^{\mathrm{n}^{\prime} J-M^{*}}\right] \mathrm{d} \alpha \operatorname{d} \cos \beta d \gamma \\
& \propto \int \sum_{\mu_{j} \mu_{k} \mu_{1}} \sum_{M \geq 0}\left(1-\frac{\delta_{M, 0}}{2}\right) \rho_{M M}^{n n^{\prime}}\left[A_{\vec{\mu}}^{n J M}(j) A_{\vec{\mu}}^{n^{\prime} J M^{*}}(k)\right. \\
& +\eta \eta^{\prime} A_{\vec{\mu}}^{\left.\left.\mathrm{nJ}-\mathrm{M}_{(\mathrm{j})} \mathrm{A}_{\vec{\mu}}^{\mathrm{n}^{\prime} J-\mathrm{M}^{*}}(\mathrm{k})\right] \mathrm{d} \alpha \mathrm{~d} \cos \beta \mathrm{~d} \gamma,{ }^{*}\right)} \tag{124}
\end{align*}
$$

But we know that
i.e., independent of M

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \sigma}{\mathrm{~d} \omega_{1}^{2} \mathrm{~d} \omega_{2}^{2}} \propto \rho_{\mathrm{MM}}^{\mathrm{nn}}\left[1-\frac{\delta_{\mathrm{M}, 0}}{2}\right] \Delta^{\mathrm{Jnn}}(\mathrm{j}, \mathrm{k})\left[1+\eta \eta^{\prime}\right] \tag{126}
\end{equation*}
$$

Thus if $\eta=-\eta^{\prime}$, i.e., waves of opposite parity

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \sigma}{\mathrm{~d} \omega_{1}^{2} \mathrm{~d} \omega_{2}^{2}}=0 \tag{127}
\end{equation*}
$$

Thus we have the well known results that
(i) states of different $J$ do not interfere in the Dalitz plot
(ii) states of different parity do not interfere in the Dalitz plot
(iii) the Dalitz plot density is independent of $M$, the magnetic substate in which a state is produced.
5. Application of Analysis to (K) $\pi+N \rightarrow(\mathrm{~K}) \pi+\pi+\pi+\mathrm{N}$

Here we want specifically to consider the ease one of the incident particles has spin $0^{-}$and is transformed into a three body state, i.e., production of the $A_{1}, A_{2}$, etc. We can write the cross section

$$
\begin{equation*}
\mathrm{d} \sigma \propto \sum_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}\left|\sum_{\mathrm{nJM}} \mathrm{R}_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{nJM}}(\mathrm{~W}, \mathrm{t}) \mathrm{A}^{\mathrm{nJM}}\left(\mathrm{~W}, \alpha, \beta, \gamma, \omega_{1}^{2}, \omega_{2}^{2}\right)\right|^{2} \tag{128}
\end{equation*}
$$

where $\mu_{i}$ and $\mu_{f}$ are the initial and final proton helicities and $\mu_{j}=\mu_{k}=\mu_{l}=0$.
As before we can define a density matrix

$$
\begin{equation*}
\cdot \rho_{M_{M}}^{n^{\prime}}=\sum_{\mu_{i} \mu_{\mathrm{f}}} R_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{nJM}} R_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{n}^{\mathrm{j}} \mathrm{~J}^{\prime} \mathrm{M}^{\prime *}} \tag{129}
\end{equation*}
$$

Clearly the object of any analysis is to determine the elements of the density matrix

$$
\begin{equation*}
\mathrm{d} \sigma \propto \sum_{\substack{\mathrm{nJM} \\ \mathrm{n}^{\prime} J^{\prime} M^{\prime}}} \rho_{M M^{\prime}}^{\mathrm{nn}} \mathrm{~A}^{\mathrm{nJM}} A^{\mathrm{n}^{\prime} J^{\prime} \mathrm{M}^{\prime *} *} \tag{130}
\end{equation*}
$$

This equation will always hold even if we bin the data in $W$ and $t$, i.e., we integrate over certain regions

$$
\begin{equation*}
\rho_{M_{M}^{\prime}}^{\mathrm{nn}^{\prime}}(\mathrm{W}, \mathfrak{t})=\sum_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}} \int_{\mathrm{t}_{1}}^{\mathrm{t}^{2}} R_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{nJM}}(W, t) \mathrm{R}_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{n}^{\prime} \mathrm{M}^{\prime} *}(W, \mathrm{t}) \mathrm{dt} \tag{131}
\end{equation*}
$$

In Table VI I have listed all the waves one might use in studying $\mathrm{A}_{1}, \mathrm{~A}_{2}$ decays considering only well known strong final state interactions

$$
\begin{align*}
\mathrm{A}_{1}, \mathrm{~A}_{2} & \rightarrow \pi \epsilon  \tag{132}\\
& \rightarrow \pi \rho
\end{align*}
$$

For the strange particles we have

$$
\begin{align*}
Q, \mathrm{~K}^{*}(1400) & \rightarrow \mathrm{K} \epsilon, \quad \pi(\mathrm{~K} \pi) \text { S-wave }  \tag{133}\\
& \rightarrow \mathrm{K}_{\rho}, \quad \pi \mathrm{K}^{*}
\end{align*}
$$

and there will be twice the number of amplitudes to consider. If one wishes to consider higher mass states, the $A_{3}$ and $L$, then we have to include amplitudes corresponding to decay into pseudoscalar and tensor mesons. Even if we restrict ourselves to $J \leq 2$ we realise that we have an enormous density matrix to determine, e.g., a $29 \times 29$ density matrix. This will contain 841 complex clements which parity and hermiticity reduce to approximately 841 real numbers. These must also satisfy certain constraints, e.g.,

$$
\left|\rho_{\mathrm{ij}}\right|^{2} \leq \rho_{\mathrm{ii}} \rho_{\mathrm{ij}}
$$

The imposition of these constraints is not entirely trivial and the task is one of certain difficulty! Instead one makes simplifying
assumptions. These fall into two classes
(i) Resonance approximation - here one writes

$$
\begin{equation*}
\rho_{\mathrm{MM}}^{\mathrm{nn}}{ }^{\prime}=\rho_{\mathrm{MM}} \mathrm{JJ}^{\mathrm{JJ}}, \mathrm{C}^{\mathrm{n}} \mathrm{C}^{\mathrm{n}^{\prime *}} \tag{134}
\end{equation*}
$$

e.g.,

$$
\left.\begin{array}{rl}
1^{+} & \rightarrow \pi \rho \\
& \rightarrow \pi \epsilon
\end{array}\right\} \begin{aligned}
& \text { have the same density matrix just different } \\
& \text { couplings to the final states } .
\end{aligned}
$$

Furthermore these couplings are assumed to be independent of $M$, i.e., $1^{+} \mathrm{M}=1,1^{+} \mathrm{M}=0$ decays have the same C . This immediately reduces the number of parameters to $(17 \times 17)+9 \times 2 \sim 300$. This is the approach followed by the Illinois group.
(ii) One parametrizes the density matrix in terms of the production amplitudes

$$
\begin{equation*}
\rho_{\mathrm{MM}^{\prime}}^{\mathrm{nn}^{\prime}}=\mathrm{R}^{\mathrm{nJM}} R^{\mathrm{n}^{\prime} J^{\prime} \mathrm{M}^{\prime} *} \tag{135}
\end{equation*}
$$

and then attempts to determine the $\mathrm{R}^{\mathrm{nTM}}$. This has the advantage that we now only have 29 complex numbers to determine (58, if we allow both spin flip terms and spin-nonflip terms at the proton vertex) and the relations amongst density matrix elements are automatically ensured. However there is a disadvantage in that we have to assume that $\mathrm{R}_{2}^{n J M}$ are constant in any bin we consider or at least have the same $t$ dependence. In reality

$$
\begin{equation*}
\rho_{M_{M}}^{n^{\prime}}(\overline{\mathrm{W}}, \overline{\mathrm{t}})=\iint R^{\mathrm{nJM}} R^{n^{\prime} J^{\prime} M^{\prime} *} d W d t \neq R^{\mathrm{nJM}}(\overline{\mathrm{~W}}, \overline{\mathrm{t}}) R^{\mathrm{n}^{\prime} J^{\prime} M^{\prime}}(\overline{\mathrm{W}}, \overline{\mathrm{t}}) \Delta W \Delta t \tag{136}
\end{equation*}
$$

Even after these simplifications the number of parameters remains large and in order to make the problem tractable it is then essential to choose a subset of the partial waves. This is done in all analyses so far.

In using the last approach it is essential to realise that we really only determine the density matrix, the production amplitudes although having a physical foundation at this point only serve as a parametrization. Of course with more polarization experiments one will eventually be in a position to measure these.

## 6. Results of These Analyses

I want finally to list the results that one usually sees quoted from these analyses.
(i) The cross section in a given $J^{P}$ state is given by

$$
\begin{align*}
\sigma^{J^{P}} & =\int \sum_{M M M^{\prime}} \rho_{n^{\prime}}^{n n^{\prime}} A^{n J M} A^{n^{\prime} J M^{\prime *}} d \alpha d \cos \beta d \gamma d \omega_{1}^{2} d \omega_{2}^{2} \\
& =\int \sum_{M} \rho_{n, n^{\prime}}^{n n^{\prime}} \rho_{M M}^{n J M_{1}} A^{n^{\prime} J M^{*}} d \alpha d \cos \beta d \gamma d \omega_{1}^{2} d \omega_{2}^{2} \tag{137}
\end{align*}
$$

Note that we retain $\mathrm{n}, \mathrm{n}^{\prime}$ even though $\mathrm{J}=\mathrm{J}^{\prime}, \mathrm{P}=\mathrm{P}^{\prime}, \mathrm{M}=\mathrm{M}^{\prime}$ since these still contain the different possible decay channels, e.g.,

$$
\begin{aligned}
1^{+} & \rightarrow \pi \rho-s \text {-wave } \\
& \rightarrow \pi \epsilon-\text { p-wave }
\end{aligned}
$$

(ii) The cross section for a given $J^{P}$ and a given decay channel

$$
\begin{equation*}
\sigma^{\mathrm{nJ}}=\int \sum_{\mathrm{M}} \rho_{\mathrm{MM}}^{\mathrm{nn}}\left|\mathrm{~A}^{\mathrm{nJM}}\right|^{2} \mathrm{~d} \alpha \mathrm{~d} \cos \beta \mathrm{~d} \gamma \tag{138}
\end{equation*}
$$

Note $\sigma^{\mathrm{J}^{\mathrm{P}}} \neq \sum_{\mathrm{n}} \sigma^{n J^{\mathrm{P}}}$ because of the presence of interference terms between different $\mathrm{n}, \mathrm{e} . \mathrm{g} .$,

$$
\left.\begin{array}{rl}
1^{+} & \rightarrow \pi \rho-\text { s-wave } \\
\rightarrow \pi \epsilon-\text { d-wave }
\end{array}\right\}
$$

(iii) Normalized density matrix

$$
\begin{equation*}
\widetilde{\rho}_{\mathrm{MM}} \mathrm{nn}^{\prime}{ }^{\prime}=\frac{\rho_{\mathrm{MM}} \mathrm{nn}^{\prime}}{\sum_{\mathrm{n}, \mathrm{M}} \rho_{\mathrm{MM}}^{\mathrm{nn}}} \tag{139}
\end{equation*}
$$

Of course within any $J^{P}$ state $\sum_{M}{ }_{\rho_{\mathrm{J}}{ }^{\mathrm{P}}{ }_{\mathrm{M}}{ }^{\mathrm{P}}} \neq 1$ but these can be suitably renormalized.
(iv) The complex coupling constants $\mathrm{C}^{\mathrm{n}}$. As we will see the phases of these are important and from the $\mathrm{C}^{\mathrm{n}}$ can be deduced the branching functions for resonances.
7. Application to $(\mathrm{K}, \mathrm{p}) \pi+\mathrm{N} \rightarrow \mathrm{K}(\pi)+\pi \pi \mathrm{N}$

The next important application of this type of analysis will be to those types of reactions drawn schematically in Fig. 16b. I just want to remark at this point that for $K, \pi$ induced reactions the problem is somewhat simpler. Since these particles are spinless we do not have to consider spin flip terms at this vertex. We do of course have to consider the transition from protons of different helicity to the various $n, J^{P}, M$ states. However as we have seen these amplitudes are simply related by parity

$$
\begin{equation*}
\mathrm{f}_{-}^{\mathrm{nJM}}=\mathrm{f}_{+}^{\mathrm{nJ}-\mathrm{M}} \eta(-1)(-1)^{\mathrm{J}-\mathrm{M}} \tag{140}
\end{equation*}
$$

where,-+ are the initial helicities of the proton. For pp interactions this simplification clearly does not occur .

## 8. Application to Polarized Target Experiments

In this case it is most convenient to consider the helicity system. If the target proton is transversely polarized then this will define an initial co-ordinate system $S^{\prime \prime}$. The co-ordinate system $S^{\prime}$ will then be related to $S^{\prime \prime}$ by Euler angles $0,-\theta_{\mathrm{p}},-\phi_{\mathrm{p}}$ where $\theta_{\mathrm{p}}$ and $\phi_{\mathrm{p}}$ are production angles of the final multiparticle system with respect to $S^{\prime \prime}$. This is illustrated in Fig. 18. Note $\theta_{p}$ is defined for a given $W$ and $t$. We then have amplitudes

$$
\begin{equation*}
R_{\mu_{i} \mu_{f}}^{\mathrm{nJM}}\left(\mathrm{~W}, \mathrm{t}, \phi_{\mathrm{p}}\right)=\mathrm{R}_{\mu_{\mathrm{i}} \mu_{\mathrm{f}}}^{\mathrm{nJM}}(\mathrm{~W}, \mathrm{t}) \mathrm{e}^{-\mathrm{i} \mu_{\mathrm{i}} \phi_{\mathrm{p}}} \tag{141}
\end{equation*}
$$

where $\mu_{\mathrm{i}}$ is the initial proton helicity.
We can then repeat the calculations of Section III. C in order to obtain expressions for the asymmetries from polarized targets, the polarization of the final recoil nucleon and the depolarization tensor of the nucleon. We will again obtain Table IV with the charge $\alpha \rightarrow-\phi_{p}$. It is also important to remember that the order of the proton helicities is reversed in the table.
9. Structure of the Production Amplitude $R^{n J M} \mu_{b} \mu_{c}(W, t)$

It is often convenient to extract the minimum angular momentum structure which must exist in this amplitude. Again the calculation is
most easily performed in the helicity system in which we have

$$
\begin{equation*}
R_{\mu_{b} \mu_{c}}^{\mathrm{nJM}}(W, t)={ }_{S^{\prime}}^{<n J M} q_{c} \mu_{c}|T| q_{a} \mu_{a} q_{b} \mu_{b}>S^{\prime \prime} \tag{142}
\end{equation*}
$$

where $S^{\prime \prime}$ is a co-ordinate system defined with respect to the incident particles. If the production angle is $\theta_{\mathrm{p}}$ and the initial state unpolarized we have

$$
\begin{align*}
\mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}}(\mathrm{~W}, \mathrm{t}) & \sim \sum_{\mathrm{r}} \mathscr{D}_{\mathrm{M}-\mu_{\mathrm{c}}, \mu_{\mathrm{a}}-\mu_{\mathrm{b}}}^{\mathrm{r}}\left(0,-\theta_{\mathrm{p}}, 0\right) \mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJMr}} \\
& \sim \sum_{\mathrm{r}} \mathrm{~d}_{\mathrm{M}-\mu_{\mathrm{c}},-\mu_{\mathrm{b}}}^{\mathrm{r}}\left(0,-\theta_{\mathrm{p}}, 0\right) \mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJMr}} \tag{143}
\end{align*}
$$

where $r$ is the total angular momentum. For small $\theta_{p}$ we can write

$$
\begin{gather*}
d_{\mathrm{M}-\mu_{c^{\prime}}-\mu_{\mathrm{b}}}^{\mathrm{r}}\left(0,-\theta_{\mathrm{p}}, 0\right) \sim \cos \left(\frac{\theta_{\mathrm{p}}}{2}\right)^{\left|\mathrm{M}-\mu_{c}-\mu_{\mathrm{b}}\right|} \sin \left(\frac{\theta_{\mathrm{p}}}{2}\right)^{\left|\mathrm{M}-\mu_{\mathrm{c}}+\mu_{\mathrm{b}}\right|}  \tag{144}\\
\mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}}(\mathrm{~W}, \mathrm{t}) \sim\left[\cos \frac{\theta_{\mathrm{p}}}{2}\right]^{\left|\mathrm{M}-\mu_{\mathrm{c}}-\mu_{\mathrm{b}}\right|}\left[\sin \frac{\theta_{\mathrm{p}}}{2}\right]^{\mid \mathrm{M}-\mu_{c^{+}+\mu_{b} \mid}} R_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{J M n}(\mathrm{~W}, \mathrm{t}) \tag{115}
\end{gather*}
$$

If $\mathrm{M}-\mu_{\mathrm{c}}=-\mu_{\mathrm{b}}$ (no helicity flip)

$$
\begin{equation*}
\mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}} \sim\left(\cos \frac{\theta_{\mathrm{p}}}{2}\right)^{12 \mu_{\mathrm{b}} \mathrm{l}} \mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}} \tag{146}
\end{equation*}
$$

If $\mathrm{M}-\mu_{\mathrm{c}}=-\mu_{\mathrm{b}}+1$

$$
\begin{equation*}
\mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}}^{\mathrm{nJM}} \sim\left(\cos \frac{\theta_{\mathrm{p}}}{2}\right)^{\mid-2 \mu_{\mathrm{b}}^{+1 \mid}} \sin \frac{\theta_{\mathrm{p}}}{2} \mathrm{R}_{\mu_{\mathrm{b}} \mu_{\mathrm{c}}^{\mathrm{nJM}}} \tag{147}
\end{equation*}
$$

Now since

$$
\begin{aligned}
t & =\left(E_{c}-E_{b}\right)^{2}-\left(\vec{p}_{c}-\vec{p}_{b}\right)^{2} \\
& =m_{c}^{2}+m_{b}^{2}-2 E_{b} E_{c}+2 p_{b} p_{c} \cos \theta_{p}
\end{aligned}
$$

and

$$
\begin{align*}
t_{\min } & =m_{c}^{2}+m_{b}^{2}-2 E_{b} E_{c}+2 p_{b} p_{c} \\
t-t_{\min } & =2 p_{b} p_{c}\left(\cos \theta_{p}-1\right) \\
& =2 p_{b} p_{c} 2 \sin ^{2} \theta_{p} / 2 \\
\sin \theta_{p} / 2 & \sim\left(t-t_{\min }\right)^{1 / 2} \tag{148}
\end{align*}
$$

We have the well known result that amplitudes with a net helicity flip of $\lambda$ behave as $\sim\left(t-t_{\text {min }}\right)^{\lambda / 2}$. This minimum $t$ dependence can then be displayed explicitly and we then have to determine the $R_{\mu_{b}}^{n J M}$. If such a technique is employed then the approximations used in the parametrization of the density matrix (see Eq. (136)) can be improved.

Of course this will not necessarily be the total $t$-dependence of the amplitude but it is the minimum structure required by angular momentum considerations.

## E. FITTING TIE EXPERIMENTAL DATA

In fitting the experimental data, either from formation or production reactions, it is quite common to integrate over some of the variables, i.e., one fits projections of the data such as the Dalitz plot population. However the best results will be obtained by making maximum use of the correlations that exist in the 4 (formation) or 5 (production) dimension
variable spaces. This is most easily done by the maximum likelihood technique. One constructs a likelihood function

$$
\begin{equation*}
\mathscr{L}=\prod_{\mathrm{i}=1}^{\mathrm{N}} \sigma\left(\vec{\tau}, \overrightarrow{\mathrm{x}}_{\mathrm{i}}\right) \tag{149}
\end{equation*}
$$

where $\vec{x}_{i}$ is the vector corresponding to the ith event in the multidimensional space, $\vec{\tau}$ is the vector of partial wave amplitudes and $\sigma\left(\vec{\tau}, \overrightarrow{\mathrm{x}}_{\mathrm{i}}\right)$ is the probability (cross section) for such an event as calculated in Sections III. A, III. C, and III.D. Then $\log \mathscr{R}$ is maximized by varying the parameters $\vec{\tau}$.

Finally a check is usually made that these set of amplitudes $\vec{\tau}$ correspond to a good fit to the data by calculating the $\chi^{2}$ for fits to the projections.

The results I will quote in Sections IV and V are obtained using this technique.

## IV. THE RESULTS OBTAINED FROM FORMATION EXPERIMENT

In this section I will attempt to summarize the situation that exists in the analysis of formation experiments. The results at this time bear mainly on the classification of resonance states and on understanding of their various decays within these classification schemes. The major results have come from analyses (Herndon et al., 1972) of

$$
\begin{equation*}
\pi \mathrm{N} \rightarrow \pi \pi \mathrm{~N} \tag{150}
\end{equation*}
$$

but significant measurements have been made in the study of (Mast et al., 1972)

$$
\begin{equation*}
\mathrm{K}^{-} \mathrm{p} \rightarrow \Lambda \pi \pi \tag{151}
\end{equation*}
$$

and (Bland et al., 1969)

$$
\begin{equation*}
\mathrm{K}^{+} \mathrm{p} \rightarrow \mathrm{~K} \pi \mathrm{~N} \tag{152}
\end{equation*}
$$

Before beginning detail discussion of each set of results I would like briefly to review the motivations, some of which I have already noted in Section I, for pursuing such a study.
(a) Identification of Resonant States and Classification in Supermultiplets In the past few years the existence of many resonance states has been claimed from the partial wave analysis of two body reactions

$$
\begin{align*}
\pi^{+} \mathrm{p} & \rightarrow \pi^{+} \mathrm{p} \\
\pi^{-} \mathrm{p} & \rightarrow \pi^{-} \mathrm{p}, \pi^{o} \mathrm{n} \tag{153}
\end{align*} \quad \mathrm{I}=1 / 2 \text { and } \mathrm{I}=3 / 2 \mathrm{~N}^{*}
$$

Many of these states are very inelastic. Furthermore the identification of low spin resonances tends to be difficult in the presence of high spin states. For example in the resonance region of $\pi \mathrm{N}$ scattering at $\mathrm{E}_{\text {c. } \mathrm{m} .} \sim 1600-1700 \mathrm{MeV}$ there are two spin $5 / 2$ resonances, the F 15 and D15. However only after extended analysis were the lower spin states S31, S11, D33 finally identified. Both of these comments imply the need to study inelastic reactions. The first because this is where the resonances may be prolifically found and the second because momenta are on average lower and thus higher angular momentum states are expected to be suppressed ( $\mathrm{L} \sim \mathrm{qR}$, with $\mathrm{R} \sim 1$ fermi, is smaller). Indeed the $\mathrm{Y}^{*}$ situation would be in very poor shape were it not for the analyses of the inelastic two body final states $\Lambda \pi, \Sigma \pi$. Since the cross sections for $\pi \pi N$ and $\Lambda \pi \pi, \Sigma \Sigma \pi, \overline{\mathrm{K}} \pi N$ are large one hopes for further improvements from study of these states.

With the identification of many of these states classification within supermultiplets becomes possible. We are all familiar with the $\operatorname{SU}(3)$ classification of states into singlets, octets and decuplets. Of course there are many other irreducible representations of $\operatorname{SU}(3)$ but these are the ones predicted by simple quark models (qqq for baryons and $q \bar{q}$ for mesons). For $\mathrm{N}^{*}$ state with a given $\mathrm{J}^{\mathrm{P}}$ all the other members of the octet must exist, just as for a $\Delta$ state all the other docuplet members must be present. At this point $\operatorname{SU}(3)$ classification and the quark model works surprisingly well - many octets $\left(\frac{1}{2}^{-}, \frac{3}{2}^{-}, \frac{1}{2}^{-}, \frac{5}{2}^{-}, \frac{5}{2}^{+}\right)$
decuplets $\left(\frac{3}{2}^{+}, \frac{7}{2}^{+}\right)$are almost complete. The analysis of reactions

$$
\begin{equation*}
\mathrm{KN} \rightarrow \mathrm{KN}, \mathrm{~K} \pi \mathrm{~N} \tag{155}
\end{equation*}
$$

if they demonstrated the existence of $Z^{* r_{s}}$ ( $S=+1$ baryon resonances) could be fatal to simple quarls models (requiring qqqqq ) and allow myriad of other resonant states. However it is possible that such states might be dynamical in origin thus allowing the quark model a reprieve.

Another highly successful symmetry scheme is that of $\operatorname{SU}(6) \times 0(3)$ ( $\operatorname{SU}(3) \times \operatorname{SU}(2) \times 0(3)$ of the quark states) (Dalitz, 1969; Greenberg, 1969). This attempts to group together known states of different $J^{P}$ into yet larger multiplets. Of courst fewer of these supermultiplets are identified as they contain more resonarit states. Again the most popular model for baryons is the $L$ excitation $s y$ mmetric quark model of which the $\left[56, \mathrm{~L}=0^{+}\right]$, $\left[70, \mathrm{~L}=1^{-}\right],\left[56, \mathrm{~L}=2^{+}\right]$multir, lets appear on a sure footing.

However many states still need to be identified and the analysis of the inelastic reactions should provide the information with which to confirm or confound these schemes.
(b) Couplings to Decay Chari:t:1s

The success of the clas:inication schemes suggests that we might attempt to apply the same sy:mmetry groups to the coupling constants. The application of $\operatorname{SU}(3)$ to the derays

$$
\begin{aligned}
\mathrm{N}^{*}, \Delta & \rightarrow \mathrm{~N} \pi \\
Y_{0}^{*}, \mathrm{Y}_{1}^{*} & \rightarrow \Lambda \pi, \Sigma \pi, \overline{\mathrm{~K}} \mathrm{~N} \\
\Xi^{*} & \rightarrow \Xi \pi
\end{aligned}
$$

has already been discussed exhaustively by many authors (Levi Setti, 1969; Plane et al., 1970). The undoubted success is summarized in Fig. 19 where the agreement is perfect (the one discrepancy has later been resolved (Plane et al., 1970)).

With the apparent success of $\operatorname{SU}(6)$ classification schemes one would like to apply this symmetry group to the decays as well (Faiman, 1971; Faiman and Plane, 1972). This means that we can now relate

$$
\mathrm{N}^{*} \rightarrow \mathrm{~N} \pi \quad \text { and } \quad \mathrm{N}^{*} \rightarrow \Delta \pi
$$

(since the $N$ and $\Delta$ belong to the same $\operatorname{SU}(6)$ multiplet) as well as the decays of different $J^{P}$ states. Furthermore since the $\rho$ and $\pi$ reside within the same 35 of $\mathrm{SU}(6)$ we have relations for $N^{*} \rightarrow N_{\rho}$ decays. It is immediately apparent that one needs to analyse the $\pi \pi N$ final states and with successful analysis of $\Lambda \pi \pi, \Sigma \pi \pi, \overline{\mathrm{K}} \pi \mathrm{N}$ one could again make the SU(3) tests for

$$
\begin{aligned}
\mathrm{N}^{*} & \rightarrow \pi \Delta \\
\mathrm{Y}_{0}^{*}, \mathrm{Y}_{1}^{*} & \rightarrow \pi \mathrm{Y}_{1}^{*}(1385) \quad, \text { etc. }
\end{aligned}
$$

The measurement of these decay couplings is clearly the next step in the developement of our understanding of the resonance region.
(c) Multichannel Analyses

If one can obtain a description of the inelastic states (i.e., the three body states for $\pi \mathrm{N}$ ) for $\mathrm{E}<2000 \mathrm{MeV}$ we will essentially have all the information possibie on the $\pi N$ interaction. In any analysis we will be able to exploit the constraints of unitarity to the maximum effect. This should allow us to give not only resonance parameters but the complete analytic
structure of the T-matrix. This will then be a sensitive test of any dynamical theory of resonances just as the coupling constants described in (b) are a fundamental test of the symmetry schemes.
(d) Dynamical Results

With the advent of inelastic partial wave amplitudes it will be possible to construct finite energy sum rules relating the low and high energy behaviour of the scattering amplitudes, as was done in $\pi \mathrm{N}$ scattering. This should allow a new study of duality in a different set of reactions.

Clearly I have not been exhaustive in listing all the motivations but these few emphasize the great importance of this type of analysis.
A. $\pi \mathrm{N} \rightarrow \pi \pi \mathrm{N}$

This reaction has been extensively studied by the SLAC/LBL collaboration (Herndon et al., 1972; Cashmore, 1973) and many of its results are confirmed by or consistent with other analyses (Bowler and Cashmore, 1970; Chinowsky et al., 1970; DeBeer et al., 1969a; 1969b; Brody et al., 1971; Mehtani et al., 1972). I will concentrate on the results of this analysis for a number of reasons.
(i) It spans the c. m . energy range $1300<\mathrm{E}<2000$ except for a 100 MeV gap $1540<\mathrm{E}<1650$ where the data is not yet available for analysis.
(ii) It utilizes the data in the most efficient manner making simultaneous maximum likelihood fits to the three major channels at each energy

$$
\begin{equation*}
\pi^{-} p \rightarrow \pi^{+} \pi^{-} n \tag{156}
\end{equation*}
$$

$$
\begin{align*}
& \pi^{-} \mathrm{p} \rightarrow \pi^{-} \pi^{\circ} \mathrm{p}  \tag{157}\\
& \pi^{+} \mathrm{p} \rightarrow \pi^{+} \pi^{0} \mathrm{p} \tag{158}
\end{align*}
$$

(iii) Excellent agreement with the inelastic cross sections predicted by elastic phase shift analyses (EPSA).
(iv) Continuous solutions have been identified throughout the energy range from the independent solutions at each energy. This allows the presentation of reliable Àrgand diagrams for the first time.
(v) The analysis is of the type described in Section II. C the intermediate states considered being

$$
\begin{aligned}
\pi N & \rightarrow \pi \Delta \\
& \\
& \text { S(the total } N_{\rho} \text { intrinsic spin) }=1 / 2 \text { or } 3 / 2 \\
& \text { denoted as } \rho_{1} \text { or } \rho_{3}
\end{aligned}
$$

$\rightarrow \mathrm{N} \epsilon$
The results are presented as partial wave amplitudes in an LS representation of the $T$ matrix. The energies and the data considered are listed in Table $\cdot$ VII where its lack of data from 1540 to 1650 is particularly conspicuous.
(vi) The fits to the experimental data are of good equality although they deteriorate at the higher energies. As can be seen in Fig. 20, a 4-D representation of the fit to $\pi^{+} \pi^{-} \mathrm{n}$ at 1690 MeV , the enormous variations of structure are extremely well reproduced. Furthermore the partial wave amplitudes make excellent predictions of the $\pi^{\circ} \pi^{\circ} \mathrm{n}$ cross section as demonstrated in Fig. 21.

## 1. Description of Partial Wave Amplitudes

Before showing the Argand diagrams of the T matrix elements leading to inelastic channels I have to emphasize one point. In analysing inelastic reactions at a given energy $E$ the relative phases of the individual partial waves are only determined and not the absolute phase (a distinct difference from elastic phase shift analyses - there one essentially has the unscattered wave to define the absolute phase). Thus in order to present Argand diagrams it is necessary to specify the absolute phase at each energy (i.e., one free parameter for all waves, since their relative position is fixed). This has been done by using K-matrix fits to the P11, D15 and F15, F35 waves in the low, middle and high energy regions of the data. Now unfortunately the absence of data for $1540<\mathrm{E}<1650$ makes it difficult to be certain of the correct continuity. Indeed the presence of two marginally different high encrgy solutions indicates that it would be possible to obtain radically different continuity through this region. I stress this point as it is of crucial significance in the comparison with theory at the present time. The Argand diagrams I will show correspond to one solution (A). Solution B differs qualitatively in P11 waves at the higher energies but at lower energies (<1540) it is possible that the partial wave amplitudes may be rotated by $180^{\circ}$. I will not discuss solution B as the work on it is far from complete, except to point out the changes of interpretation it would produce. Of course this would be unnecessary had the data in the 'gap' been analysed!
(a) I=1/2 States. In Fig. 22 are the Argand diagrams of all the $I=1 / 2$ waves determined in the analysis. The figures contain a great deal of information and I will only point out the most important and striking features.
(i) The considerable motion in the Argand diagrams is not surprising, essentially all the structure is associated with the existence of known resonance states.
(ii) The following resonant states are clearly observed (the notation is $L, L^{\prime}, 2 \mathrm{I}, 2 \mathrm{~J}$, where L is the incoming orbital angular momentum, $\mathrm{L}^{\prime}$ the final state orbital angular momentum, I the isotopic spin, and $J$ the total angular momentum)
$S 11(1700) \quad$ in $N_{\rho_{1}}(S S 11)$ and $N \epsilon(S P 11)$
$\left.\begin{array}{l}\text { P11(1470) } \\ \operatorname{P11(\sim 1780)}\end{array}\right\}$ in $\pi \Delta\left(\mathrm{DS13}\right.$ and DD13) and $\mathrm{N}_{3}(\mathrm{DS} 13) \quad$ (see Fig. 23)
D13(1520) in $\pi \Delta\left(\mathrm{DS13}\right.$ and $\mathrm{DD13)}$ and $\mathrm{N}_{3}$ (DS13) (see Fig. 24)
D 15 (1680) $\quad$ in $\pi \Delta(\mathrm{DD} 15)$
F15(1680) in $\pi \Delta($ FP15 $), N \rho_{3}$ (FP15), $N \in$ (FD15) (see Fig. 25)
$\operatorname{P13(\sim 1860)} \quad$ in $\mathrm{N}_{1}(\mathrm{PP} 13)$
(iii) D13(1700): This state which has been hinted at in EPSA is definitely present in this analysis (see Fig. 24) decaying strongly into $\pi \Delta$ (DS13) and $\mathrm{N} \epsilon(\mathrm{DP} 13)$. This state has long been required to complete the $\mathrm{S}=0$ members of the $\left[70,1^{-}\right]$supermultiplet of negative baryon states.
(iv) Strong $\mathrm{N}_{\rho}$ couplings are observed for the $\mathrm{P} 13(1860), \mathrm{D} 13(1520)$ and F15(1690) resonances. This is not surprising as the last two
resonances are strongly seen in photoproduction and application of VDM would imply this result.
(v) As can be seen in Figs. 23, 24 and 25 the agreement with the EPSA predictions is excellent.
(b) I=3/2 States. In Fig. 26 are the Argand diagrams of the $\mathrm{I}=3 / 2$ waves.
(i) The following resonances can be clearly seen

| $\mathrm{S} 31(1600)$ | in $\pi \Delta(\mathrm{SD} 31)$ and $\mathrm{N}_{1}(\mathrm{SS} 31)$ |
| :--- | :--- |\(\left\{\begin{array}{l}Although only the upper <br>

part is observed (since <br>
\mathrm{E}_{c.m.}>1650 in our <br>
analysis)\end{array}\right.\)

F35(1890) in $\pi \Delta(\mathrm{FF} 35)$ and $\mathrm{N}_{3}$ (FP35) (see Fig. 27)
F 37 (1930) in $\pi \Delta(\mathrm{FF} 37)$ and $\mathrm{N}_{3}$ (FF37)
(ii) There is no evidence for the existence of a P33 resonant state within the energy range of the analysis (see Fig. 28).
(iii) Again the $N \rho$ couplings are not surprising as these resonances are strongly excited in photoproduction.
(c) Further Comments. There are some problems in the analysis which can be used as a guide to the physics which may be present.
(i) The $\pi^{+} \pi^{+} n$ final state is badly fitted in the analysis. This is not surprising as the only intermediate states which may lead to this channel are $\pi \Delta$ states and from inspection of the $\pi^{+} n$ mass projections it is clear that N* isobars (P11, D13, F15) may be present. The absence of these states in the analysis probably also accounts for the inability to reach the EPSA predictions for the P31 and D35 waves ( $\pi(\mathrm{F} 15)$ is an S-wave and
$\pi(\mathrm{D} 13)$ in a P -wave would be derived from D 35 whereas $\pi(\mathrm{P} 11)$ in an S-wave would come from P31).
(ii) The deterioration of the fits at higher energies is generally associated with being unable to fit the peripheral nucleon. This probably indicates the necessity of including $\pi$-exchange in the production of the $\mathrm{N}_{\rho}$ final state, so that higher partial waves are generated.

## 2. Comparison with Theory

(a) Classifications. The observation of the D13 resonance at 1700 MeV completes the $\left[70,1^{-}\right]$supermultiplet of baryon states. However the absence of any P33 states is embarassing. One such state is required to complete the $\left[56, \mathrm{~L}=2^{+}\right]$while another is required as a partner of the Roper $\operatorname{P11(1470)}$ resonance if this is classified within a radial excitation of the $\left[56, \mathrm{~L}^{2}=^{+}\right]$(Dalitz, 1969; Greenberg, 1969; Feynman et al., 1970). (b) Decays. This area has seen most recent activity. In Fig. 29 I have drawn the arrows which correspond to the sign of the couplings of the resonances observed in solution A of the $\pi \pi \mathrm{N}$ analysis. Should solution B be satisfactory it would imply changes of sign for the lower P11 and D13 states (as indicated by dashed arrows). To emphasize this problem I have also separated the low and high energy parts of the analysis. The predictions for these reactions have been derived from two directions

- Phenomenological analysis of baryon decays assuming a broken form of $\operatorname{SU}(6)_{W}$ Symmetry (Faiman, 1971; Daiman and Plane, 1972; Rosner, 1972; Faiman and Rosner, 1973)
- Application of results related to Melosh transformations (Melosh, 1973) assuming $\operatorname{SU}(6)_{W}$ classification of the baryon states.

But both lead to essentially the same results. However the second approach does give some indication of why the broken form of $\operatorname{SU}(6){ }_{W}$ appears.

These theories can be used to predict numerical results but one expects these to be less satisfactory than the prediction of signs, since they are more susceptible to the details rather than the overall structure. Thus I will concentrate on the prediction and comparison of the signs. (i) Phenomenological analysis. If one assumes $\operatorname{SU}(6)_{W}$ invariance then the decays of all members of one multiplet into the members of another multiplet are related, i.e.,

$$
\begin{align*}
<\mathrm{H}^{\prime} \lambda^{\prime}|\mathrm{T}| \mathrm{H} \lambda_{\mathrm{H}}, \mathrm{M}, \lambda_{\mathrm{M}}> & \sim\left[\mathrm{SU}(6)_{\mathrm{W}} \text { Clebsch-Gordan coefficient }\right] \\
& \times[\mathrm{SU}(3) \text { Clebsch-Gordan coefficient }] \\
& \times[\mathrm{SU}(2) \text { isospin Clebsch-Gordan coefficient }] \\
& \times[\mathrm{SU}(2) \mathrm{W} \text { spin Clebsch-Gordan coefficient }] \\
& \times<\left\|\mathrm{T}_{\mathrm{R}}\right\|> \tag{159}
\end{align*}
$$

where $<\left\|T_{R}\right\|>$ is a reduced helicity matrix element. This is just the Wigner-Eckart theorem. Unfortunately it is already clear that this symmetry must be broken (e.g., for mesons it predicts that in $\mathrm{B} \rightarrow \pi \omega$ the $\omega$ should have $\lambda=0$ whereas in practice it clearly has $\lambda=1$. In photoproduction of $\pi^{\prime} \mathrm{S}$ it predicts the helicity $3 / 2$ couplings of the D13 to be zero and these are the dominant couplings.)

As we see Eq. (159) implies specific relations between different helicity matrix elements. As we have seen in Sections III.A and III.C this means specific relations must exist between the pair of waves allowed in decays to $\pi \Delta$. Thus the manner in which the symmetry is broken is to recast (159) into LS partial wave amplitudes and then assume that these are entirely decoupled, i.e., one uses all the ClebschGordan coefficients of (159) for each partial wave but does not apply the relation between the two waves contained in (159) (Faiman and Plane, 1972; Faiman and Rosner, 1973). This is known as L-broken $\operatorname{SU}(6)_{W}$.

Applying Eq. (159) twice once for the coupling to the initial state and once for the outgoing state allows a calculation of the sign of the amplitude in e.g.,

$$
\begin{equation*}
\pi \mathrm{N} \rightarrow \pi \Delta \tag{160}
\end{equation*}
$$

In Fig. 30 I give clock diagrams (coupling signs) corresponding to the predictions (Faiman and Rosner, 1973) (for comparison with the SLAC/ LBL (Herndon et al., 1972) results all $\mathrm{I}=3 / 2$ predictions should be multiplied by (-1) (Cashmore et 21., 1972)). One further point should be explained. $\operatorname{SU}(6)_{W}$ sign means that the two orbital angular momentum states have the sign predicted by $S U(6)_{W}$ whereas "anti $S U(6)_{W}$ " means that the two amplitudes have exactly the opposite relative sign compared to the $S U(6)_{W}$ prediction.

In certain cases where incoming and outgoing waves are the same the sign of the amplitude is given just by Clebsch-Gordan coefficients. This immediately indicates a suspicious point of solution A clocks. The relative sign of the DD15 and PP11(1470) $\pi \Delta$ decays implies that the

P11 should belong to a [70] representation of $\operatorname{SU}(6)_{W}$ whereas it is usually assigned to a [56]. Furthermore the relative signs of the PP11(1470) and $\operatorname{DD13(1520)} \pi \Delta$ decays would imply that the P1I belongs to a $[56]$ providing the $\operatorname{D} 13(1520)$ belongs to an $\{8\}^{2}$ of $\mathrm{SU}(3) \times \mathrm{SU}(2)_{\mathrm{W}}$. Thus there would appear to be a problem in assigning the P11. However if solution $B$ is valid these problems are immediately removed since the P 11 (1470) and D13(1520) amplitudes are changed in sign. However if solution A is the only tenable result we will immediately have to face the fact that $\operatorname{SU}(6)_{\mathrm{W}}$ symmetry in any of its broken forms can only be saved by configuration mixing. Not only is this unattractive it also implies the presence of other low lying multiplets. It is important to note that one has to consider $\pi \Delta$ decays to test this symmetry as relations amongst $\pi N, \Lambda \pi$, $\Sigma \pi$ only have one possible matrix element and we are thus only testing SU(3) symmetry.

At the present moment it appears that the decays of the $\left[70,1^{-}\right]$ favour anti-SU(6) ${ }_{W}$ signs whereas the $\left[56, L=2^{+}\right]$decays indicate the $\operatorname{SU}(6)_{\mathrm{W}}$ sign. This is summarized in Table VIII .
(ii) Melosh transformations. The whole phenomenological analysis has received important support in recent months from the work of Melosh (Melosh, 1973) and its developements (Gilman et al. , 1973; Gilman, 1973).

Many years ago Gell-Mann (Gell-Mann, 1962) proposed that the 16 vector and axial vector charges, $Q^{\alpha}(\mathrm{t})$ and $\mathrm{Q}_{5}^{\alpha}(\mathrm{t})$ (integrals over space of the weak and electromagnetic current densities) commute at equal
times to form an algebra

$$
\begin{align*}
& {\left[Q^{\alpha}(t), Q^{\beta}(t)\right]=\mathrm{if}^{\alpha \beta \gamma} Q^{\gamma}(\mathrm{t})} \\
& {\left[Q^{\alpha}(\mathrm{t}), \mathrm{Q}_{5}^{\beta}(\mathrm{t})\right]=\mathrm{if}^{\alpha \beta \gamma} \mathrm{Q}_{5}^{\gamma}(\mathrm{t})}  \tag{161}\\
& {\left[\mathrm{Q}_{5}^{\alpha}(\mathrm{t}), \mathrm{Q}_{5}^{\beta}(\mathrm{t})\right]=\mathrm{if} \mathrm{f}^{\alpha \beta \gamma} \mathrm{Q}^{\gamma}(\mathrm{t})}
\end{align*}
$$

From these charges one can define $Q^{\alpha}(t)+Q_{5}^{\alpha}(t)$ and $Q^{\alpha}(t)-Q_{5}^{\alpha}(t)$ and these two charges commute with each other to give chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)$. This group can be cnlarged to form an $\operatorname{SU}(6)_{W}$ algebra whose elements commute like SU(3) and Dirac matrices: $\lambda^{\alpha}, \lambda^{\alpha_{\beta \sigma_{x}}}, \lambda^{\alpha}{ }_{\beta \sigma_{y}}, \lambda^{\alpha_{\sigma}}$ $\left[\right.$ Note $\beta \sigma_{\mathrm{x}}, \beta \sigma_{\mathrm{y}}$ and $\sigma_{\mathrm{z}}$ are the generators of $\operatorname{SU}(2)_{\mathrm{W}}$ (Lipkin and Meshkov, 1965).] This algebra is referred to as the $\underline{S U(6)} \mathrm{W}$ of currents. To label the various representations it is convenient to use the $\operatorname{SU}(3) \times \operatorname{SU}(3)$ subgroup

$$
\begin{equation*}
(\mathrm{A}, \mathrm{~B})_{\mathrm{S}_{\mathrm{z}}} \tag{162}
\end{equation*}
$$

where $A$ is the representation of $Q^{\alpha}+Q_{5}^{\alpha}, B$ the representation of $\mathrm{Q}^{\alpha}-\mathrm{Q}_{5}^{\alpha}$ and $\mathrm{s}_{\mathrm{z}}$ is the value of the (current) quark spin component along the $z$-axis.

From previous comments it appears that the baryon states [and the meson states] lie in irreducible representations of $\operatorname{SU}(6)_{\mathrm{W}} \times 0(3)$. This $\operatorname{SU}(6)_{W}$ algebra acts on constituent quarks, baryons being qqq and mesons $q \bar{q}$. Hadrons are simple in terms of this algebra, referred to as $\xrightarrow{S U(6)}$ W strong.

If these two algebras are identified as the same, many undesirable results follow, e.g., $g_{A}=5 / 3$, zero anomalous magnetic moment for the nucleon, no magnetic $N \rightarrow \Delta$ transition, etc.

It was proposed that there might be a unitary transformation between these two algebras

$$
\begin{equation*}
[\mathrm{SU}(6) \mathrm{W}, \text { strong }]=\mathrm{V}\left[\mathrm{SU}(6)_{\mathrm{W}} \text {, currents }\right] \mathrm{V}^{-1} \tag{163}
\end{equation*}
$$

The achievement of Melosh was to suggest a possible form for $V$ motivated by the free quark model.

Instead of applying this transformation to the operators we can apply it to the states. Then

$$
\begin{equation*}
\text { |hadron }>=\mid I . R . \text { constituents }\rangle=V \mid I . R . \text { currents }\rangle \tag{164}
\end{equation*}
$$

i.e., the I.R. of constituents then becomes a complicated sum of irreducible representations of the $\mathrm{SU}(6)_{\mathrm{W}}$ of currents.

In order to apply these ideas to baryon decays we wish to consider hadron' $\rightarrow$ hadron $+\pi$ which is related through PCAC to <hadron' $\left|Q_{5}^{\alpha}\right|$ hadron $>$. Thus

$$
\begin{align*}
\left.\langle\text { hadron }| Q_{5}^{\alpha} \mid \text { hadron }\right\rangle & \left.=\left\langle I . R .{ }^{\prime} \text { constituents }\right| Q_{5}^{\alpha} \| . R . \text { constituents }\right\rangle \\
& \left.=\left\langle I . R .^{\prime} \text { currents }\right| V^{-1} Q_{5}^{\alpha} \mid I . R . \text { currents }\right\rangle \tag{165}
\end{align*}
$$

All we then have to know is $\mathrm{V}^{-1} \mathrm{Q}_{5}^{\alpha} \mathrm{V}$ and thanks to Melosh we have a possible form for it. As we have seen $Q_{5}^{\alpha}$ itself transforms as $(8,1)_{0}-(1,8)_{0}$ and Melosh found in the free quark model that

$$
\begin{equation*}
\mathrm{V}^{-1} \mathrm{Q}_{5}^{\alpha} \mathrm{V}=\mathrm{R}\left[(8,1)_{0}-(1,8)_{0}\right]+\mathrm{S}\left[(3, \overline{3})_{1}-(\overline{3}, 3)_{-1}\right] \tag{166}
\end{equation*}
$$

This algebraic property can now be extracted and exploited in calculating the decays of (165). Finally in order to complete the calculation we assume that the observed baryon resonances can be identified with I.R. of constituent quarks, and the different quark spin states are related by $\operatorname{SU}(6)_{\mathrm{W}}$ of strong interactions. The two terms of (166) can then be sandwiched between different states of the I.R. representations to give any transition matrix element just as in (159). We can immediately note the two parameters of this approach are then related to the different partial wave couplings of the broken $S U(6)_{W}$ predictions. Indeed the second term corresponds to that breaking.

In Table VIII I reproduce the comparison of signs from Gilman, Kugler and Meshkov (Gilman et al., 1973) with the present solution A. The $\left[56, \mathrm{~L}^{2} 2^{+}\right] \rightarrow\left[56, \mathrm{~L}=0^{+}\right]$decays have consistent signs and indicate the dominance of the $(8,1)_{0}-(1,8)_{0}$ term (in the language of Rosner et al. - the $\operatorname{SU}(6)_{W}$ like sign). However the $\left[70,1^{-}\right] \rightarrow\left[56, L=0^{+}\right]$decays are inconsistent with both sets of relations. However if the D13(1520) signs could be reversed there would be total agreement with the $(3, \overline{3})_{1}-(\overline{3}, 3)_{-1}$ term (the anti $\operatorname{SU}(6)_{W}$ of Rosner et al.). This is just the result that would occur should the solution $B$, at present under study, prove to be tenable. (The changes are recorded in brackets in Table VIII.

I can only stress that these comments would be unnecessary if data in the 'gap' could be analysed. This is just the region which links the 1500 MeV and 1700 MeV groups of resonances.
(iii) $\operatorname{SU}(3)$ comparisons. With improved analyses of the $S=-1$ states it will eventually become important to relate

$$
\left.\begin{array}{lrl}
\mathrm{N}^{*} \rightarrow \pi \Delta(1235) & \mathrm{Y}^{*} & \rightarrow \pi \Sigma(1385) \\
& \rightarrow \overline{\mathrm{K}} \Delta(1235) \tag{168}
\end{array}\right\}
$$

and test the $\operatorname{SU}(3)$ symmetry in these decays.
(iv) Quark model comparisons. At the present time quark model predictions for the photoproduction of $\pi$ mesons from nucleons agree quite well with experiment (Moorhouse and Oberlack, 1973). The same dynamical arguments can be used to predict the $\pi \Delta$ decays of resonant states (Moorhouse and Parsons, 1973) (an essentially parameter free calculation).

Table VIII also contains the results of the comparison of these calculations with the results of the present experimental solution A. The same malady exists in that the $\mathrm{D} 13(1520)$ signs are again wrong compored to the others - of course this problem would be removed if the D13(1520) signs could be reversed as we have noted before. However one further discrepancy is present and that is in the sign of the FP15 $\pi \mathrm{N} \rightarrow \pi \Delta$ transition and this indicates a problem with the detail calculations of the quark model interactions.
(c) Multichannel analyses. The ability to account for all of the $\pi N$ inelasticity in many partial waves means that we are in a position to perform multichannel fits exploiting the constraints of unitarity to the fullest extent in attempting to understand the $\pi \mathrm{N}$ interaction (Cashmore, 1973).

The K-matrix formalism (Dalitz, 1962) is a convenient way of assuming unitarity in these analyses. However the K-matrix parameters obtained often have ridiculous values, i.e., they correspond to resonances having widths of $\sim 500 \mathrm{MeV}$ where one can see that this is not the case from inspection of the Dalitz plot. This problem probably results from the fact that one is not necessarily using the correct parametrizations for resonances and background. However the K-matrix parametrization does give a'good' representation of the Argand diagram, i.e., of the T-matrix elements. In order to identify resonances and properties we now search the T-matrix for poles in the complex energy plane. The motivations for this procedure are
(i) pole positions and residues appear to be unique irrespective of the parametrization of the T-matrix, providing it is good (Lasinski and Barbaro-Galtieri, 1972; Ball et al., 1972; Longacre, 1972)
(ii) we expect the pole position and residues to be closely related to the Breit-Wigner parameters. However the pole position does not equal ( $M_{0}, i / 2 \Gamma_{0}$ ), the parameters of the Breit-Wigner in the physical region, and the residues are not necessarily equivalent to the partial
widths. We expect these equalities to become very poor when we have either large backgrounds or wide resonances.

The results of these investigations are summarized in Table IX and correspond to the present solution $A$ for $\pi N \rightarrow \pi \pi N$. One can observe that many resonance parameters are substantially different from the results of naive estimations and indeed this whole problem of resonance parameter extraction is a thorny one. However any detailed theory must predict T-matrices which should contain this pole structure.

Finally I might add a note of warning - before using any resonance parameters always review their origin. The estimates can be very different.
(d) Dynamical calculations. Attempts to understand the importance of local duality in the region $1500-1700 \mathrm{MeV}$ for $\pi \mathrm{N} \rightarrow \pi \Delta$ have been made (Kernan and Shepherd, 1969) but unfortunately that analysis used preliminary results of

$$
\pi^{-} p \rightarrow \pi^{+} \Delta^{-}
$$

partial wave analysis which has been superceded. It is not clear that these results will still hold. When a final partial wave set is available that exercise will be repeated.

It will also be possible to attempt to correlate the predictions of $\pi$-exchange (a real amplitude) with the present partial waves and also study the relation of photoproduction amplitudes with the vector meson decays.

As one can see the results of such analyses are very rich in valuable data and clearly represent the next step in understanding strong interactions, at least in this energy region.

## B. $\mathrm{K}^{-} \mathrm{p} \rightarrow \Lambda \pi \pi$

The only reliable results on this reaction have been obtained in the region of the $\mathrm{Y}_{0}^{*}(1520)(\mathrm{D} 03)$ in an effort to make detailed measurements of its branching ratios (Mast et al., 1972). This state is predominantly an $\operatorname{SU}(3)$ singlet. If it were entirely such an object it would be forbidden to decay into $\pi \mathrm{Y}_{1}^{*}(1385)$ since an $\underline{8} \times \underline{10}$ coupling does not contain a 1 . This state is then a mixture of singlet and octet (there is a close by octet state, the D03(1690))

$$
\begin{align*}
& |1520\rangle=\cos \theta|1\rangle+\sin \theta|8\rangle  \tag{169}\\
& |1690\rangle=-\sin \theta|1\rangle+\cos \theta|8\rangle \tag{170}
\end{align*}
$$

The value of this mixing angle can be estimated from the Gell-MannOkubo mass relation and from the two body decays $\overline{\mathrm{K}} N$ and $\Sigma \pi$. The result is that $\theta=(21 \pm 5)^{\circ}$ (Plane et al., 1970).

If we now consider the $\pi Y_{1}^{*}(1385)$ decays we see that the decay rates are simply related by

$$
\begin{equation*}
\frac{\Gamma(1520)}{\Gamma(1690)} \frac{\mathrm{PS}(1690)}{\operatorname{PS}(1520)}=\tan ^{2} \theta \tag{171}
\end{equation*}
$$

where PS represents the phase space in the decay. The value of $\theta$ from this relation should be the same as above but unfortunately the result indicates $\theta>50^{\circ}$. We might expect this to be disastrous. However inspection of the $\left[70,1^{-}\right]$supermultiplet indicates yet another $\mathrm{D} 03 \mathrm{Y}_{0}^{*}$
state. If we now require mixing between all three states consistent results can be obtained (Faiman and Plane, 1972). This result then becomes an important factor in the description of the $\left[70,1^{-}\right]$decays, which were mentioned in the previous section.

At present it is important to stress the value of this type of analysis in $\mathrm{K}^{-} \mathrm{p}$ reactions. Just as we are understanding the $\mathrm{N}^{*}$ situation with the advent of this new data, the $\mathrm{Y}^{*}$ region would be similarly helped and indeed our whole view of the symmetries of strong interactions. C. $\mathrm{Kp} \rightarrow \mathrm{K} \pi \mathrm{N}$

The obvious reason for studying this type of reaction is to identify exotic $Z^{* \prime}$. The analyses so far performed have been of the 'selection by cuts' type (see Section II. A). The reaction studied primarily is

$$
\begin{equation*}
\mathrm{K}^{+} \mathrm{p} \rightarrow \mathrm{~K}^{\mathrm{o}} \Delta^{++} \tag{172}
\end{equation*}
$$

and the results have given no indication of such resonant states with I=1 (Bland et al., 1969).

However at this point the elastic phase shift analyses of the $\mathrm{I}=0$ system are much more indicative of a resonant state at $\mathrm{E} \sim 1800 \mathrm{MeV}-$ a $Z_{0}^{*}$. Again it would be desirable to study this possible state in the inelastic channels. Fortunately this is much simpler as the decay into $\mathrm{K} \Delta$ is forbidden. At present this looks the most promising candidate for a $Z_{0}^{*}$.

As I mentioned briefly in the introduction to this section the existence of such states can be predicted in certain dynamical models. These consist of models in which there is strong $\pi$ exchange in the $t$-channel
leading to a strong $K * N$ cross section. Under such circumstances this state can be a major factor in the intermediate states in the KN-KN amplitude (Aaron et al., 1969; Aaron 6t al., 1970) and the isospin crossing matrix then leads to much greater effects in the $I=0 \mathrm{KN}$ system. Thus the observation of such a resonance would not be entirely disastrous to the quark model and moreover it would raise the question of just how frequently this type of situation oceurs. Could many of the resonanoes, we now agsume are elementary particles, have such an origin?

## V. THE RESULTS FROM PRODUCTION REACTIONS

As we have already discussed these isobar methods can be applied to the three body final states obtained in production reactions. The efforts have been almost exclusively limited to the three meson systems obtained in production reactions. These are grouped under the names $A_{1}\left(1^{+}\right), A_{2}\left(2^{+}\right), A_{3}\left(2^{-}\right), Q\left(1^{+}\right), L\left(2^{-}\right)$and are central to the meson classification schemes but confused by the presence of diffraction dissociation. The analyses have been mainly of the type considering just Dalitz plot pepulations or moments $\left(\left\langle\mathrm{Y}_{\mathrm{L}}^{\mathrm{M}}\right\rangle\right)$ of the normal to the three particle plane. The truly complete isobar analysis has only been applied by the group at Illinois (Ascoli et al., 1971; Ascoli, 1972; Hlinois ct al., 1972) and more recently by the SLAC/LBL group (Lasinski et al., 1973).

The Illinois group analysed almost exclusively the reactions

$$
\begin{equation*}
\pi^{-} p \rightarrow \pi^{-} \pi^{-} \pi^{+} p \tag{173}
\end{equation*}
$$

where the $3 \pi$ system is isolated by considering only events in which the four momentum transfer from incident proton to outgoing proton is comparatively small ( $t \lesssim 1.0$ ) and none of the pions resonante with the proton to give a $\Delta$ state. The questions that one would like to answer in such an analysis are
(a) What are the spin parity states present as a function of mass?
(b) What are the production density matrix elements?
(c) What are the $t$-dependences of the processes and the production mechanisms of the states?
(d) What are the s-dependences of the production mechanisms of the states?

I will try to summarize the answers one has to these questions using mainly the results of the Illinois group (Ascoli et al., 1971; Ascoli, 1972; Illinois et al., 1972) (the SLAC/LBL results, although at an early stage in general support their conclusions). I will first tackle the $3 \pi$ system and then comment briefly on the $\mathrm{K} \pi \pi$ and $\mathrm{N} \pi \pi$ situations.

## A. THREE PION FINAL STATES $\pi \pi \pi$

In order to have sufficient statistics to perform the analyses the Illinois group and their collaborators have gathered together essentially three sets of data
(a) $5,7,7.5 \mathrm{GeV} / \mathrm{c}$

$$
\pi^{-} \mathrm{p} \rightarrow \pi^{-} \pi^{+} \pi^{-} \mathrm{p}
$$

$$
\sim 30 \quad 000 \text { events }
$$

(b) 11-25 GeV/c $\pi^{-} p \rightarrow \pi^{-} \pi^{+} \pi^{-} p$ $\sim 15000$ events
(c) $40 \mathrm{GeV} / \mathrm{c}$ $\pi^{-} \mathrm{p} \rightarrow \pi^{-} \pi^{+} \pi^{-} \mathrm{p} \quad \sim 13000$ events (a) and (b) come from bubble chambers and (c) from the CERN boson spectrometer during its stay at Serpukhov. In discussing the results I will not always specify the exact source. An example of the $3 \pi$ invariant mass distribution is given in Fig. 31.

Unfortunately the data are not yet sufficient to allow analysis in both mass and $t$ bins so that the variable of least importance is usually integrated over. Thus for spectroscopic studies one is mainly concerned with the mass dependence and hence $t$ is integrated over, while for discussions of production mechanisms (dynamics) the most important variable is $t$ and larger siices of mass are usually considered.

## 1. Spin Parity States

In Fig. 32 I show the intensities associated with the $0^{-}, 1^{+}, 2^{+}, 2^{-}$ and $3^{+}$waves. Three features are immediately obvious:
(a) the $1^{+}$enhancement around $\sim 1100 \mathrm{MeV}-$ the $\mathrm{A}_{1}$
(b) the $2^{+}$enhancement at $\sim 1300 \mathrm{MeV}$ - the $\mathrm{A}_{2}$
(c) the $2^{-}$enhancement at $\sim 1700 \mathrm{MeV}$ - the $\mathrm{A}_{3}$

Clearly it is important to know the different isobar (two body) states which contribute to this cross section. The $0^{-}, 1^{+}$and $2^{-}$waves are shown in Figs. 33, 34, 35 and have contributions from the $\pi \rho, \pi \in\left(0^{-}, 1^{+}\right)$ and $\pi \rho, \pi f\left(2^{-}\right)$states. The $2^{+}$wave only has contributions from the $\pi \rho$ system. The $1^{+}$wave is dominated by ( $\pi \rho$ ) in an s-wave whereas the $2^{-}$ is predominantly ( $\pi f$ ) in an s-wave. This result has been known for some time as has the fact that Deck diagrams of the type of Fig. 36 give $s$-wave enhancements near threshold. One might hope to identify the $1^{+}, 2^{-}$states as resonant by observing large rapid phase changes of these waves with respect to other partial waves. Unfortunately no such motion is apparent, e.g., in Fig. 37 we see the phase of the $2^{-} \pi$ f wave relative to other waves through the $\mathrm{A}_{3}$ region. No resonant phase variation is observed. Figure 38 demonstrates the same result for the $1^{+}$ $(\pi \rho) s$ wave in the region of the $A_{1}$. These results thus give little encouragement for the identification of these states as resonances. However the presence of small resonant effects on a large background are not ruled out (this could produce little phase variation).

Of course it is always desirable to demonstrate when one can observe a 'good' resonance. This is done for the $A_{2}$ in Fig. 39. There one sees the Breit-Wigner shape in the $2^{+}(\pi \rho)$ d-wave together with correct phase variation across the resonance.

Thus we can conclude that the only resonance positively identified is the $2^{+}(\pi \rho)$ state, the $A_{2}$, and the $1^{+}$and $2^{-}$waves although large and changing in intensity give little support for resonance interpretations.

## 2. Density Matrix Elements

In the last two years the question of s-channel helicity conservation has received a great deal of attention. It appears to be essentially conserved for reactions (Gilman et al. , 1970)

$$
\begin{align*}
& \pi \mathrm{N} \rightarrow \pi \mathrm{~N}  \tag{174}\\
& \gamma \mathrm{~N} \rightarrow \rho^{\mathrm{O}} \mathrm{~N} \tag{175}
\end{align*}
$$

However these reactions do not involve spin changes whereas excitation of the $A_{1}$ and $Q$ enhancements does $\left(0^{-} \rightarrow 1^{+}\right)$. Thus attention has been focussed in the density matrix elements of the $1^{+}$state measured both in the t-channel (Gottfried-Jackson) and s-channel helicity systems (see Section III. D for definition of these). In Fig. 40 one sees $\rho_{00}$ evaluated in the t-channel where it is $\sim 0.9-1.0$. This indicates predominantly t-channel helicity conservation although the existence of $\operatorname{Re} \rho_{01}$ indicates that some $M= \pm 1$ production occurs. However these density matrix elements evaluated in the s-channel system would give values of $\rho_{00} \sim 0.4, \operatorname{Re} \rho_{01} \sim-.3$ at the higher $t$ value. Thus exact helicity conservation does not occur in either system but it is closer to being true in the t-channel.

It should also be noted that the $2^{-}$system is also produced predominantly in the $\mathrm{M}=0$ substate .

In contrast the $A_{2}$ is produced almost entirely in the $\mathrm{M}= \pm 1$ states. In fact $\rho_{11} \sim \rho_{1-1}$ indicating the production of a pure state $|21\rangle+|2-1\rangle$. The near equality of these two density matrix elements indicates that natural spin parity exchange dominates (Ader et al., 1968) since

$$
\begin{align*}
& \sigma_{\text {natural }} \propto \rho_{11}+\rho_{1-1}  \tag{176}\\
& \sigma_{\text {innatural }} \propto \rho_{11}-\rho_{1-1}
\end{align*}
$$

This result is expected if the reaction occurs through $\mathrm{f}^{0}$ or $\rho^{0}$ exchange. 3. Production Mechanisms and t-Dependence

The dominance of natural parity exchange in the production of $\mathrm{A}_{2}$ suggests the presence of $f^{0}$ and/or $\rho^{0}$ exchange terms. However by comparing $A_{2}$ production in the reactions

$$
\begin{align*}
& \pi^{-} \mathrm{p} \rightarrow \mathrm{~A}_{2^{-}}^{\mathrm{p}}  \tag{177}\\
& \pi^{+} \mathrm{n} \rightarrow \mathrm{~A}_{2}^{\mathrm{o} \mathrm{p}}
\end{align*}
$$

it is apparent that $f^{0}$ exchange dominates. Further if the coupling is spin non-flip at the proton vertex (as we guess from elastic scattering) then there will be a net helicity flip leading to zero cross section in the forward direction, i.e., at $\mathrm{t} \mid \sim 0$. This is demonstrated in Fig. 41. 4. s-Dependence of Cross Sections

The ratio of $A_{1}$ to $A_{2}$ to $A_{3}$ cross sections is surprisingly energy independent even up to the $40 \mathrm{GeV} / \mathrm{c}$ data. If these cross sections are
parametrized as

$$
\begin{equation*}
\sigma \propto \mathrm{p}_{\mathrm{lab}}^{-\mathrm{n}}\left(\sim \mathrm{~s}^{-\mathrm{n}}\right) \tag{178}
\end{equation*}
$$

then the value of the exponent is

$$
\begin{array}{ll}
\mathrm{A}_{1}: & \mathrm{n} \sim 0.5 \pm 0.2 \\
\mathrm{~A}_{2}: & \mathrm{n} \sim 0.57 \pm 0.08 \\
\mathrm{~A}_{3}: & \mathrm{n} \sim 0.8 \pm 0.3
\end{array}
$$

Furthermore if one looks at the unnatural and natural parity exchange contributions to the $\mathrm{A}_{2}$ cross section, Fig. 42, one sees that it is the unnatural parity contribution that is falling extremely fast.

The fact that the $A_{2}$ cross section docs not appear to drop any more rapidly than the $A_{1}$ has caused speculation that it may be produced by Pomeron exchange in violation of the empirical rule

$$
\begin{equation*}
\Delta \mathrm{P}=(-1)^{\Delta \mathrm{J}} \tag{179}
\end{equation*}
$$

At this time this can only be speculative and awaits more detailed measurements.

Finally one can attempt to extract information from the relative phases of the $A_{2}$ and $A_{1}$ waves. If we write
then we might expect $\bar{\rho}_{\mathrm{A}_{2}} \mathrm{~A}_{1}$ to be constant in phase and magnitude if the $A_{1}$ phase does not change (independent of $\mathrm{m}_{3 \pi}$ ). This will then represent the relative phase $\delta$ of the $A_{2}$ and $A_{1}$ production amplitudes
$\left(\delta=\phi_{\mathrm{A}_{2}}-\phi_{\mathrm{A}_{1}}=\arg \left(\bar{\rho}_{\mathrm{A}_{2} \mathrm{~A}_{1}}\right)\right)$. The results indicate a slight variation of $\delta$ with c.m. energy, $\delta \sim-46^{\circ}$ at $5 \mathrm{GeV} / \mathrm{c}$ and $\sim-75^{\circ}$ at $40 \mathrm{GeV} / \mathrm{c}$ but it does appear to be independent of $t^{\prime}$. Now if one assumes the $A_{1}$ is produced through the Deck mechanism, Fig. 36, its production phase is that of $\pi p$ diffraction scattering, i.e., pure imaginary, $\phi_{A_{1}}=90^{\circ}$. This then implies

$$
\begin{aligned}
\phi_{\mathrm{A}_{2}} & \sim 44^{\circ} \quad \text { at } 5 \mathrm{GeV} \\
& \sim 15^{\circ} \quad \text { at } 40 \mathrm{GeV} .
\end{aligned}
$$

These phases are clearly inconsistent with Regge model predictions
(a) pure $\mathrm{f}^{\mathrm{O}}$ exchange $\rightarrow \phi_{\mathrm{A}_{2}} \sim 145^{\circ}$ ( $\mathrm{t}^{\mathrm{t}} \sim$. 15)
(b) pure P exchange $\rightarrow \phi_{\mathrm{A}_{2}} \sim 90^{\circ}$
(c) mixture of $f^{0}+$ Pomeron exchange with $\operatorname{Im}\left(f^{0}\right.$ exchange) $>0$
$\rightarrow \phi_{\mathrm{A}_{2}} \sim 120^{\circ}$
However if we have
(d) $\mathrm{f}^{\mathrm{O}}+$ Pomeron with $\operatorname{Im}\left(f^{\mathrm{O}}\right.$ exchange $)<0$
agreement is possible but the energy dependence of $\phi_{\mathrm{A}_{2}}$ on $\mathrm{p}_{\text {lab }}$ in this latter case would be opposite to the observation.

We can summarize these results by the statement that we appear to have problems with our interpretation of the $A_{2}$ production mechanism.

## 5. Summary

This will have given a feeling for the great quantity of information that has been extracted from the analysis of these three pion production reactions, both spectroscopic and dynamical (production mechanisms). However, I think it is true to say that the hoped for goals of proving a
resonance contribution in the $A_{1}$ has not been reached. Clearly at least a large proportion of the $1^{+}$enhancement is dynamic in origin (Deck mechanism) and it is below the sensitivity of these experiments to detect a resonant contribution in this environment. The only hope is to consider reactions in which diffraction dissociation is impossible for the identification of this state, e.g.,

$$
\begin{align*}
& \mathrm{K}^{-} \mathrm{n} \rightarrow \pi^{-} \pi^{+} \pi^{-} \Lambda^{\mathrm{o}} \\
& \pi^{+} \mathrm{n} \rightarrow \pi^{+} \pi^{-} \pi^{\mathrm{o}} \mathrm{p} \tag{181}
\end{align*}
$$

Many of these comments apply equally well to the case of the $2{ }^{-} \mathrm{A}_{3}$ enhancement.

These analyses have increased interest in the production mechanism of the $A_{2}$, the contribution of Pomeron exchange remaining a tantalizing possibility. Measurements of the $K \bar{K}$ decay of the $A_{2}$, if measured with a good absolute normalization, should help resolve this problem.

However it does appear as though further investigation of the $3 \pi$ system in these reactions will not be terribly rewarding and new information on the resonant states will come from reactions such as (181).

## B. $\mathrm{K} \pi \pi$ FINAL STATES

Here the situation is more complicated than for $3 \pi$ 's. We expect (from the $q \bar{q}$ model) two $1^{+}$mesons, the $Q^{\prime} s$, the $K^{*}(1400)$ and a similar $2^{-}$enhancement, the $L$. At this time there are no published results from an isobar model analysis. Analyses have concentrated around the Dalitz plot populations, $\pi \mathrm{K}^{*}$ decays obtained by cuts, and general angular
momentum considerations (see Section II.B). The main results can be summarized as follows
(a) $1^{+}$waves are dominant in the mass region $1.0<\mathrm{M}(\mathrm{K} \pi \pi)<1.4$ with decays into both $\pi \mathrm{K}^{*}$ and $\mathrm{K} \rho$. In fact the $\mathrm{K}_{\rho}$ decay is strongest near threshold.
(b) In this reaction t-channel helicity conservation is ruled out.
(c) Comparison of $Q$ and $\bar{Q}$ states in

$$
\begin{equation*}
\mathrm{K}_{\mathrm{L}}^{\mathrm{o}} \mathrm{p} \rightarrow \mathrm{~K}_{\mathrm{S}^{\pi^{+}} \pi^{-} \mathrm{p}} \tag{182}
\end{equation*}
$$

indieate the probable presence of other contributions to the production mechanism besides Pomeron exchange (diffraction dissociation).
(d) Varying production mechanisms of two interfering $1^{+}$states could explain the changes in $\mathrm{K} \pi \pi$ spectrum that are observed.

The application of isobar model analyses to reactions of the type

$$
\left.\begin{array}{rl}
\mathrm{K}^{ \pm} \mathrm{p} & \rightarrow \mathrm{~K}^{ \pm} \pi^{+} \pi^{-} \mathrm{p} \\
& \rightarrow \overline{\mathrm{~K}}^{\mathrm{o}}  \tag{183}\\
\overline{\mathrm{~K}}^{\mathrm{o}}
\end{array}\right\} \pi^{ \pm} \pi^{\mathrm{o}} \mathrm{p} .
$$

will clearly result in advances similar to those in the $3 \pi$ system. However the problem is more complicated because of the increased number of decay channels which must be considered.
C. $N \pi \pi$ FINAL STATES

There is even less to say about these states, the work being little and the results few. The nucleon having spin $1 / 2$ makes the problem
somewhat more complicated and yet again increases the number of partial waves. Some Dalitz plot analyses have been performed and result in the statements that $J^{P}=1 / 2^{+}$dominates in the region of $1200-1400 \mathrm{MeV}, \mathrm{J}^{\mathrm{P}}=3 / 2^{-}$is present around 1500 MeV and $5 / 2^{+}$large at $\sim 1700 \mathrm{MeV}$. As we have seen in the analysis of $\pi \mathrm{N} \rightarrow \pi \pi \mathrm{N}$ these P11, D13, F15 states are important at these masses and are the very states we expect in diffraction excitation if the $\Delta \mathrm{P}=(-1) \Delta \mathrm{J}$ rule holds.

Application of the isobar model approach is clearly going to bring about a dramatic improvement. Furthermore these would seem to be the obvious states in which to study diffraction dissociation as the formation reactions give us a clear bench mark.

## VI. SUMMARY AND CONCLUSIONS

The isobar model analysis gives an excellent phenomenological description of the data on three body final states both in formation and production reactions.

The major problems in the analysis of

$$
\begin{equation*}
\pi \mathrm{N} \rightarrow \pi \pi \mathrm{~N} \tag{184}
\end{equation*}
$$

is associated with the lack of results from the data that exists in the energy range $1540<\mathrm{E}<1650$. The measurement of the $\pi^{\circ} \pi^{\circ} \mathrm{n}$ final state together with experimental results on single pion production from polarized targets will be sensitive tests of the present partial wave amplitudes. However problems do already exist in the $\pi^{+} \pi^{+} \mathrm{n}$ final state and at the higher energies, and any future analysis will have to introduce both $\mathrm{I}=1 / 2$ isobar intermediate final states and $\pi$-exchange contributions. The wealth of information one has already gained suggests that such an effort will be fruitful. This initial information has already been valuable in discussing higher symmetry schemes for the strong interaction in the resonance region. The existence of the $\mathrm{D} 13(1700)$ is vital to $\mathrm{SU}(6)_{\mathrm{W}}$ schemes but the absence of the P33 states is a nagging embarassment. This is just the beginning of the theoretical activity that should now occur in an effort to reproduce the essentially complete T-matrices that we obtain.

In the future one might expect to see similar analyses of

$$
\begin{align*}
\mathrm{KN} & \rightarrow \Lambda \pi \pi \\
& \rightarrow \Sigma \pi \pi \tag{185}
\end{align*}
$$

where the weak decay of the final baryon gives a good description of its polarization. The $\mathrm{Y}^{*}$ situation needs new information to unravel the exceedingly complicated set of states which should be present (for every octet and decuplet there must be $\mathrm{Y}_{1}^{*}$ states while $\mathrm{Y}_{0}^{*}$ states are singlets and members of octets). The advent of this information will then allow tests of the $\operatorname{SU}(3)$ content of any theory we have for these three body reactions.

The analyses of the positive strangeness KN induced reactions is going to be absolutely vital in testing the validity of quark model schemes. However the preliminary results from analyses of reactions

$$
\begin{equation*}
\mathrm{K}^{+} \mathrm{p} \rightarrow \mathrm{~K}^{\mathrm{o}} \Delta^{++} \tag{186}
\end{equation*}
$$

give little indication of $\mathrm{I}=1$ resonant structure and I do not believe any different conclusions will appear from the more sophisticated isobar analysis. The more intriguing situation is in the $\mathrm{I}=0$ state where there is not only a candidate for a $\mathrm{Z}_{0}^{*}$ at $\sim 1800 \mathrm{MeV}$ but dynamical calculations, considering $\pi$ exchange in the t-channel, do give poles in the $s$-channel. The $Z_{0}^{*}$ system is more likely to contain resonances from this type of mechanism due primarily to the fact that s-t channel isospin crossing matrix predicts the strongest effects in the $I_{S}=0$ state from isospin 1 exchange in the t channel $(\pi)$ to make the $\mathrm{K} * \mathrm{~N}$ intermediate state (Aaron et al., 1969; Aaron et al., 1970). Thus the detailed comparison of observed T-matrix elements with those predicted by theory will be a sensitive test of the validity of such calculations. Indeed this whole question is a somewhat sensitive one - every new resonant state that we
find is automatically regarded as a new elementary particle and ciassified as such. However resonances may occur through an accident of the size of inter-particle forces and the proximity of thresholds, e.g., is the $\mathrm{Y}_{0}^{*}(1405)$ a bound state of the $\overline{\mathrm{K}} N$ system just as the deuteron is a bound state of the ${ }^{3} S_{1}$ pn interaction? Such questions (if they have any meaning) will only be resolved by detail analysis and careful thought.

Let me now turn to production reactions. The question of the existence of the $A_{1}, A_{3}, Q^{\prime} s, L$ is still open and will eventually be resolved by analysis of the hypercharge or charge exchange reactions

$$
\begin{align*}
& \mathrm{K}^{-} \mathrm{p} \rightarrow \Lambda \pi^{+} \pi^{-} \pi^{-}  \tag{187}\\
& \pi^{-} \mathrm{p} \rightarrow \Lambda \mathrm{~K}^{\circ} \pi^{+} \pi^{-}
\end{align*}
$$

There is still no reason to believe that they will be produced as pure $J^{P}$ states (i.e., no other large partial waves present) in such reactions. I expect the isobar model type of analysis will still be required to determine the $J^{P} \mathrm{M}$ states present and their decay channels. As we have seen one automatically obtains an enormous amount of information on the production mechanism of such states and this has already led to a number of surprises, e.g., the energy dependence of the $A_{2}$. In attempts to understand the phenomenon of diffraction excitation both at meson and at nucleon vertices the reactions

$$
\begin{align*}
& \mathrm{K}^{ \pm} \mathrm{p} \rightarrow \mathrm{~K}^{ \pm} \pi^{+-} \pi^{-} \mathrm{p}  \tag{188}\\
& \pi^{ \pm} \mathrm{p} \rightarrow \pi^{ \pm} \pi^{+} \pi^{-} \mathrm{p}
\end{align*}
$$

will continue to receive a lot of attention. Perhaps most results will emerge from the nucleon vertex studies (i.e., diffractive excitation
of $\mathrm{N}^{*}$ 's) where one already has information on the spectrum of states from the analyses of the formation reactions. Finally the increase in the use of polarized targets will lead to an increased understanding of the production mechanisms of these states.

It is clear that we have only begun to extract all of the latent infor mation in these three body states. The indications we have so far are that this information can be very rich indeed and will be a valuable tool in unravelling the mysteries of the strong interaction.

## APPENDIX

In this appendix I gather together the definition of states, the angular momentum projections, normalizations, phase spaces, cross section formulae, etc. The conventions are those of Cashmore et al. (1972) and are essentially taken from Jacob and Wick (JW) (1959).
A. PARTICLE STATES, NORMALIZATIONS, CROSS SECTIONS AND

## ANGULAR MOMENTUM DECOMPOSITION

We use the phase convention of JW but the normalization is different If $\psi_{p \lambda}$ represents a state with momentum $p$ along the $Z$ axis and helicity $\lambda$, then the general state is defined by

$$
\begin{equation*}
|\mathrm{p} \theta \phi \lambda\rangle=\mathrm{R}(\phi, \theta,-\phi) \psi_{\mathrm{p} \lambda} \tag{189}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\mathrm{p}^{\prime} \theta^{\prime} \phi^{\prime} \lambda^{\prime} \mid \mathrm{p} \theta \phi \lambda\right\rangle=2 \mathrm{E} \delta^{3}\left(\overrightarrow{\mathrm{p}^{\prime}}-\overrightarrow{\mathrm{p}}\right) \delta_{\lambda \lambda^{\prime}} \tag{190}
\end{equation*}
$$

we also define states $\chi_{p \lambda}$ by

$$
\begin{equation*}
x_{p \lambda}=(-1)^{S-\lambda} e^{-i \pi J} y_{\psi_{p \lambda}}=(-1)^{S-\lambda} \psi_{-p \lambda} \tag{191}
\end{equation*}
$$

The general $\chi$ state is then given by

$$
\begin{equation*}
1-\mathrm{p} \theta \phi \lambda>=\mathrm{R}(\phi, \theta-\phi) \chi_{\mathrm{p} \lambda} \tag{192}
\end{equation*}
$$

These states are denoted by a minus sign on p. Thus

$$
\begin{equation*}
\left.\left|-\mathrm{p} \theta \phi \lambda>=(-1)^{\mathrm{S}-\lambda} \quad\right| \mathrm{p} \pi-\theta, \phi+\pi, \lambda\right\rangle \tag{193}
\end{equation*}
$$

We also need to know how the states $|\mathrm{p} \theta \phi \lambda\rangle$ transform under Lorentz transformations. Let the Lorentz transformation be $\ell$ where $p^{\prime}=\ell p$ and let $U(\ell)$ be the unitary operator corresponding to $\ell$. Wick
(1962) has shown that

$$
\begin{equation*}
\mathrm{U}(\ell)|\mathrm{p} \theta \phi \lambda\rangle=\sum_{\nu} \mathscr{D}_{\nu \lambda}^{\mathrm{S}}(\Omega \hat{\mathrm{n}})\left|\mathrm{p}^{\prime} \theta^{\prime} \phi^{\prime} \nu\right\rangle \tag{194}
\end{equation*}
$$

where $\hat{\mathrm{n}}$ is a vector along $\overrightarrow{\mathrm{p}} \wedge \overrightarrow{\mathrm{p}}^{\prime}$. We will require this result to transform from an isobar j rest frame to the overall c.m. system (see Fig. 43) $\Omega$ is then an angle of rotation about the normal to the three particle plane and is given (for particle k) by

$$
\begin{equation*}
\cos \Omega=\left(\cosh \rho-\cosh \sigma_{\mathrm{k}} \cosh \sigma_{\mathrm{k}}^{\prime}\right) / \sinh \sigma_{\mathrm{k}} \sinh \sigma_{\mathrm{k}}^{\mathrm{l}} \tag{195}
\end{equation*}
$$

where
$\tanh \rho=\mathrm{v}_{\mathrm{j}}=$ velocity of j in the c.m.s.
$\tanh \sigma_{\mathrm{k}}=\mathrm{v}_{\mathrm{k}}=$ velocity of k in the c.m.s.
$\tanh \sigma_{k}^{\prime}=v_{k}^{\prime}=$ velocity of $k$ in the isobar rest frame.
$\Omega$ can also be calculated as

$$
\begin{equation*}
\Omega=\Theta-\theta-\omega \tag{196}
\end{equation*}
$$

where $\omega$ is the Stapp angle and is given by

$$
\begin{equation*}
\sin \omega=\sin \odot \frac{\beta_{i} \gamma_{i} \beta_{\mathrm{k}} \gamma_{\mathrm{k}}\left[1+\gamma_{\mathrm{i}}+\gamma_{\mathrm{k}}+\gamma_{\mathrm{k}}^{(\mathrm{i})}\right]}{\left(1+\gamma_{\mathrm{i}}\right)\left(1+\gamma_{\mathrm{k}}\right)\left(1+\gamma_{\mathrm{k}}^{(\mathrm{i})}\right)} \tag{197}
\end{equation*}
$$

where
$\beta_{i}-\beta$ for the isobar in the c.m. system
$\beta_{\mathrm{k}}-\beta$ for particle k in the c.m. system
$\beta_{\mathrm{k}}^{(\mathrm{i})}-\beta$ for particle k in the isobar rest frame.

Multiparticle final states are defined as the direct product of one particle states. Thus

$$
\begin{equation*}
\left|\overrightarrow{\mathrm{p}}_{1} \lambda_{1}\right\rangle\left|\overrightarrow{\mathrm{p}}_{2} \lambda_{2}\right\rangle \ldots\left|\overrightarrow{\mathrm{p}}_{\mathrm{n}} \lambda_{\mathrm{n}}\right\rangle=\left|\mathrm{p}_{1} \theta_{1} \phi_{1} \lambda_{1}\right\rangle \ldots\left|\mathrm{p}_{\mathrm{n}} \theta_{\mathrm{n}} \phi_{\mathrm{n}} \lambda_{\mathrm{n}}\right\rangle \tag{198}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\vec{p}_{1}^{\prime} \lambda_{1}^{\prime} \vec{p}_{2}^{\prime} \lambda_{2}^{\prime} \ldots \vec{p}_{n}^{\prime} \lambda_{n}^{\prime}\left|\vec{p}_{1} \lambda_{1} \vec{p}_{2} \lambda_{2} \ldots \vec{p}_{n} \lambda_{n}\right\rangle=\prod_{i=1}^{n}\left(2 E_{i}\right) \delta^{3} \vec{p}_{i}^{\prime}-\vec{p}_{i}\right) \delta_{\lambda_{i}} \lambda_{i}^{\prime} \tag{199}
\end{equation*}
$$

For two body states it is sometimes more convenient to use

$$
\begin{align*}
& \overrightarrow{\mathrm{p}}=\overrightarrow{\mathrm{p}}_{1}+\overrightarrow{\mathrm{p}}_{2} \\
& \left.\overrightarrow{\mathrm{p}}=\overrightarrow{\mathrm{p}}_{1}-\overrightarrow{\mathrm{p}}_{2}\right) / 2 \tag{200}
\end{align*}
$$

Letting ( $p, \theta, \phi$ ) be the polar co-ordinates of $p$, we have

$$
\begin{equation*}
\left.\left|\overrightarrow{\mathrm{P} p} \theta \phi \lambda_{1} \lambda_{2}\right\rangle=\left|\overrightarrow{\mathrm{p}}_{1} \lambda_{1}>\right| \overrightarrow{\mathrm{p}}_{2} \lambda_{2}\right\rangle \tag{201}
\end{equation*}
$$

where the states on the right-hand side are either $\psi$ or $\chi$ states. It can then be shown (Cashmore et al. , 1972) that in the c.m.s., $\overrightarrow{\mathrm{P}}=0$, the normalization is
$\left\langle\overrightarrow{\mathrm{P}}^{\prime}=0, \mathrm{p}^{\prime} \theta^{\prime} \phi^{\prime} \lambda_{1}^{\prime} \lambda_{2}^{\prime} \mid \overrightarrow{\mathrm{P}}=0, \mathrm{p} \theta \phi \lambda_{1} \lambda_{2}\right\rangle=\delta\left(\mathrm{W}^{\prime}-\mathrm{W}\right) \delta^{3}\left(\overrightarrow{\mathrm{P}}^{\prime}-\overrightarrow{\mathrm{P}}\right) \delta^{2}\left(\omega^{\prime}-\omega\right) \frac{4 \mathrm{~W}}{\mathrm{p}}$
where $\omega=(\theta, \phi)$ and $W=E_{1}+E_{2}$.
For the normalizations of (190) the number of particles of type in a volume $V$ is $2 \mathrm{E}_{\mathrm{i}} \mathrm{V} /(2 \pi)^{3}$. In a volume $V$ the total number of states available is $\mathrm{Vd}^{3} \mathrm{p}_{\mathrm{i}} /(2 \pi)^{3}$, so that the density of final states per particle is $d^{3} p_{i} / 2 E_{i}$. Thus the number of three-particle final states available,
$\mathrm{d} \rho_{\mathrm{F}}$ is given by

$$
\begin{equation*}
\mathrm{d}_{\mathrm{F}}=\frac{\mathrm{d}^{3} \mathrm{p}_{1}}{2 \mathrm{E}_{1}} \frac{\mathrm{~d}^{3} \mathrm{p}_{2}}{2 \mathrm{E}_{2}} \frac{\mathrm{~d}^{3} \mathrm{p}_{3}}{2 \mathrm{E}_{3}} \tag{203}
\end{equation*}
$$

With the normalization of (190) the incident flux is

$$
\begin{equation*}
\frac{2 \mathrm{E}_{\mathrm{a}}}{(2 \pi)^{3}} \frac{2 \mathrm{E}_{\mathrm{b}}}{(2 \pi)^{3}}\left(\frac{\mathrm{p}_{\mathrm{a}}}{\mathrm{E}_{\mathrm{a}}}+\frac{\mathrm{p}_{\mathrm{b}}}{\mathrm{E}_{\mathrm{b}}}\right)=\frac{4}{(2 \pi)^{6}} \mathrm{pW} \tag{204}
\end{equation*}
$$

The transition probability/unit volume/unit time is

$$
\begin{equation*}
\frac{\delta^{4}\left(p_{o u t}-p_{i n}\right)}{(2 \pi)^{4}}|M|^{2} d \rho_{F} \tag{205}
\end{equation*}
$$

where $M$ corresponds to the transition matrix element with our normalization of states. Now Berman and Jacob (1965) have shown that

$$
\begin{align*}
\mathrm{d} \rho & =\delta^{4}\left(p_{\text {out }}-p_{\text {in }}\right) \mathrm{d} \rho_{\mathrm{F}} \\
& =\frac{1}{8} \mathrm{dE}_{1} \mathrm{dE}_{2} \mathrm{~d} \cos \Theta \mathrm{~d} \Phi \mathrm{~d} \alpha \tag{206}
\end{align*}
$$

where $\otimes, \Phi$ and $\alpha$ are the Euler angles specifying the orientation of the final three-particle state with respect to the incident system. Equation (206) may also be rewritten in many forms using the kinematical relations that exist

$$
\begin{align*}
\mathrm{d}_{\rho} & =\frac{1}{8} \frac{\mathrm{q}_{1} Q_{1}}{2 \mathrm{~W} \omega_{1}} \mathrm{~d} \omega_{1}^{2} \mathrm{~d} \cos \theta_{1} \mathrm{~d} \cos \Theta \mathrm{~d} \Phi \mathrm{~d} \alpha \\
& =\frac{1}{8} \frac{\mathrm{q}_{1} Q_{1}}{W} \mathrm{~d}_{1} d \cos \theta_{1} d \cos \Theta d \Phi \mathrm{~d} \alpha  \tag{207}\\
& =\frac{1}{8} \frac{1}{4 \mathrm{~W}^{2}} d \omega_{1}^{2} d \omega_{2}^{2} \mathrm{~d} \cos \Theta d \Phi \mathrm{~d} \alpha
\end{align*}
$$

We can summarize by writing

$$
\begin{equation*}
\mathrm{d} v=\frac{\pi^{2}}{\mathrm{~F}}|\mathrm{M}|^{2} \mathrm{~d} \rho \tag{208}
\end{equation*}
$$

where $F=\mathrm{pW}$.
We now turn to the decomposition of the two-particle states into angular momentum states. We work in the two-particle c.m. system and assume particle 2 is in a $\chi$ state, i.e.,

$$
\begin{equation*}
\left|\mathrm{P}=0 \mathrm{p} \theta \phi \lambda_{1} \lambda_{2}\right\rangle=\overrightarrow{\mathrm{p}}_{1} \lambda_{1}>\left|-\overrightarrow{\mathrm{p}}_{1} \lambda_{2}\right\rangle=\mathrm{R}(\phi, \theta,-\phi) \psi_{\mathrm{p}_{1} \lambda_{1}} \chi_{\mathrm{p}_{1} \lambda_{2}} \tag{209}
\end{equation*}
$$

We now define the following state of total angular momentum J and z -component M by

$$
\begin{equation*}
\left|\mathrm{P}=0 \mathrm{pJM} \lambda_{1} \lambda_{2}\right\rangle=\mathrm{N}_{\mathrm{J}} \int \mathscr{D}_{\mathrm{M} \lambda}^{J}(\phi, \theta,-\phi) \mid \mathrm{P}=0, \mathrm{p} \theta \phi \lambda_{1} \lambda_{2}>\mathrm{d}^{2} \omega \tag{210}
\end{equation*}
$$

where $\mathrm{d}^{2} \omega=\mathrm{d} \cos \theta \mathrm{d} \phi$ and $\lambda=\lambda_{1}-\lambda_{2}$. Choosing

$$
\begin{equation*}
N_{J}=\left(\frac{2 J+1}{4 \pi}\right)^{1 / 2}\left(\frac{\mathrm{p}}{4 \mathrm{~W}}\right)^{1 / 2} \tag{211}
\end{equation*}
$$

ensures that

$$
\begin{equation*}
\left\langle\mathrm{P}^{\mathrm{t}}=0, \mathrm{p}^{\prime} \mathrm{J}^{\prime} \mathrm{M}^{\prime} \lambda_{1}^{!} \lambda_{2}^{\prime} \mid \mathrm{P}=0, \mathrm{pJM} \lambda_{1} \lambda_{2}\right\rangle=\delta_{J J} \delta_{M M}, \delta_{\lambda_{1} \lambda_{1}^{\prime}} \delta_{\lambda_{2}} \lambda_{2}^{!} \delta^{3}\left(\mathrm{P}^{\mathrm{t}}-\mathrm{P}\right) \delta\left(\mathrm{W}^{\prime}-\mathrm{W}\right) \tag{212}
\end{equation*}
$$

and we then have

$$
\begin{align*}
& <\mathrm{P}=0, \mathrm{p} \theta \phi \lambda_{1} \lambda_{2}\left|\mathrm{P}^{\mathrm{t}}=0, \mathrm{p}^{\mathrm{J}} \mathrm{JM} \lambda_{1}^{\mathrm{j}} \lambda_{2}^{\prime}\right\rangle \\
&  \tag{213}\\
& \quad=\left(\frac{2 \mathrm{~J}+1}{4 \pi}\right)^{1 / 2}\left(\frac{4 \mathrm{~W}}{\mathrm{p}}\right)^{1 / 2} \mathscr{D}_{\mathrm{M} \lambda}^{\mathrm{J}}(\phi, \theta,-\phi) \delta\left(\mathrm{W}^{\prime}-\mathrm{W}\right) \delta^{3}\left(\mathrm{P}^{\prime}-\mathrm{P}\right) \delta_{\lambda_{1}} \lambda_{1}^{\prime} \delta_{\lambda_{2} \lambda_{2}^{\prime}}
\end{align*}
$$

In terms of the orbital and spin angular momenta, $\cdot \mathrm{L}$ and S , we have $\left.|\mathrm{P}=0 \mathrm{pJMLS}\rangle=\sum_{\mu_{1} \mu_{2}}\left(\frac{2 L+1}{2 J+1}\right)^{1 / 2} \mathrm{C}\left(\operatorname{LSJ} \mid 0 \mu_{1}-\mu_{2}\right) \mathrm{C}\left(\mathrm{S}_{1} \mathrm{~S}_{2} \mathrm{~S} \mid \mu_{1}-\mu_{2}\right) \right\rvert\, \mathrm{P}=0, \operatorname{pJM} \mu_{1}, \mu_{2}>$
with the normalization

$$
\begin{equation*}
<\mathrm{pJ}^{\prime} \mathrm{M}^{\prime} \mathrm{L}^{r} \mathrm{~S}^{\prime}|\mathrm{pJMLS}\rangle=\delta_{\mathrm{JJ}} \delta_{\mathrm{MM}^{\prime}}{ }^{\delta_{\mathrm{LL}}}{ }^{\prime} \delta_{\mathrm{SS}}{ }^{\prime} \tag{215}
\end{equation*}
$$

## B. $\mathscr{D}$ FUNCTIONS

We use the following definition of the rotation operator

$$
\begin{equation*}
R(\alpha, \beta, \gamma)=e^{-i \alpha J} z e^{-i \beta J} y e^{-i \gamma J} z \tag{216}
\end{equation*}
$$

Since the product of two rotations is again a rotation, we have that

$$
\begin{equation*}
\mathrm{R}(\alpha, \beta, \gamma)=\mathrm{R}\left(\alpha^{\prime \prime}, \beta^{\prime \prime}, \gamma^{\prime \prime}\right) \mathrm{R}\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right) \tag{217}
\end{equation*}
$$

The elements of the matrix corresponding to $R$ are given by

$$
\begin{equation*}
\mathscr{D}_{\mathrm{m}_{1} \mathrm{~m}_{2}}^{\mathrm{J}}(\mathrm{R})=\left\langle j \mathrm{~m}_{1}\right| R\left|j \mathrm{~m}_{2}\right\rangle \tag{218}
\end{equation*}
$$

and in the terms of matrices (217) becomes

$$
\begin{equation*}
\mathscr{D}_{\mathrm{m}_{1} \mathrm{~m}_{2}}^{\mathrm{J}}(\alpha, \beta, \gamma)=\sum_{\mathrm{m}} \mathscr{D}_{\mathrm{m}_{1} \mathrm{~m}}^{\mathrm{J}}\left(\alpha^{\prime \prime}, \beta^{\prime \prime}, \gamma^{\prime \prime}\right) \mathscr{D}_{\mathrm{m} \mathrm{~m}_{2}}^{\mathrm{J}}\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right) \tag{219}
\end{equation*}
$$

The matrix elements can be simplified to

$$
\begin{equation*}
\mathscr{D}_{\mathrm{m}_{1} \mathrm{~m}_{2}}^{\mathrm{J}}(\alpha, \beta, \gamma)=\mathrm{e}^{-\mathrm{im} 1^{\alpha}} \mathrm{d}_{\mathrm{m}_{1} \mathrm{~m}_{2}}^{\mathrm{j}}(\beta) \mathrm{e}^{-\mathrm{im} 2_{2} \gamma} \tag{220}
\end{equation*}
$$

where the $d_{m_{1}}^{j} m_{2}^{(\beta)}$ are real. These functions satisfy the general relations

$$
\begin{align*}
& d_{m_{1} m_{2}}^{j}(\beta)=(-1)^{m_{1}-m_{2}} d_{m_{2} m_{1}}^{j}(\beta)=(-1)^{m_{1}-m_{2}} d_{d_{-m_{1}-m_{2}}^{j}}^{(\beta)} \\
& d_{m_{1} m_{2}}^{j}(\pi-\beta)=(-1)^{j-m_{2}} d_{d_{-m_{1} m_{2}}^{j}}^{(\beta)=(-1)^{j+m_{1}} \cdot d_{m_{1}-m_{2}}^{j}(\beta)}  \tag{221}\\
& d_{m_{1} m_{2}}^{j}(-\beta)=d_{m_{2} m_{1}}^{j}(\beta), \quad d_{m_{1} m_{2}}^{(\beta+2 \pi)=(-1)^{2 j} d_{m_{1} m_{2}}^{j}(\beta)}
\end{align*}
$$

The matrix elements have the following normalizations

$$
\begin{equation*}
\int \mathscr{D}_{\mathrm{m}_{1} \mathrm{~m}_{2}}^{\mathrm{j}}(\alpha, \beta, \gamma) \mathscr{D}_{\mathrm{m}_{1}^{\prime} \mathrm{m}_{2}^{\prime}}^{\mathrm{j}^{\prime *}}(\alpha, \beta, \gamma) \mathrm{d} \alpha \mathrm{~d} \cos \beta \mathrm{~d} \gamma=\frac{8 \pi^{2}}{2 \mathrm{j}+1} \delta_{\mathrm{jj}} \delta_{\mathrm{m}_{1} \mathrm{~m}_{1}^{\prime}} \delta_{\mathrm{m}_{2} \mathrm{~m}_{2}^{\prime}} \tag{222}
\end{equation*}
$$

and

$$
\begin{equation*}
\int d_{m_{1} m_{2}}^{j}(\theta) d_{m_{1} m_{2}}^{j^{\prime}}(\theta) d \cos \theta=\frac{2}{2 j+1} \delta_{j j^{\prime}} \tag{223}
\end{equation*}
$$

C. THE REACTION $a+b \rightarrow c+d$

It can be shown (Cashmore et al., 1972) that the cross section for this reaction, with these normalizations of states, is

$$
\begin{equation*}
\sigma=\frac{\pi}{p^{2}} \sum_{J} \frac{(2 J \div 1)}{\left(2 \sigma_{a}+1\right)\left(2 \sigma_{b}+1\right)} \sum_{\text {LSL'S' }} i<Q J M L S^{\prime}|T| p J M L S>\left.\right|^{2} \tag{224}
\end{equation*}
$$

For $\pi \mathrm{N} \rightarrow \pi \mathrm{N}$ this becomes

$$
\begin{equation*}
\sigma\left(J^{\mathrm{P}}\right)=\pi \lambda^{2}(\mathrm{~J}+1 / 2)\left|<\mathrm{J}^{\mathrm{P}}\right| \mathrm{T} \mid \mathrm{J}^{\mathrm{P}}>1^{2} . \tag{225}
\end{equation*}
$$

## REFERENCES

Aaron, R., Teplitz, D. C., Amado, R. D., and Young, J. E. (1969). Phys. Rev. 187, 2047

Aaron, R., Amado, R. D., and Silbar, R. R. (1970). Los Alamos preprint LA-DC-11961

Ader, J. P., Capdeville, M., Cohen-Tannoudji, G., Salin, Ph. (1968). Nuovo Cimento 56A, 952

Ascoli, G., Brockway, D. V., Eisenstein, L., Ioffredo, M. L., Kruse, U. E., Schultz, P. F., Caso, C., Tomasini, G., von Handel, P., Schilling, P., Costa, G., Ratti, S., Daronian, P., Mosca, L., Brenner, A. E., Harrison, W. C., Heyda, D., Johnson, Jr., W. H., Kim, J. K., Law, M. E., Mueller, J. E., Salzberg, B. M., Sisterson, L. K., Johnston, T. F., Prentice, J. D., Steenberg, N. R., Yoon, T. S., Carroll, J. T., Erwin, A. R., Morse, R., Oh, B. Y., Robertson, W., and Walker, W. D. (1971). Phys. Rev. Letters 26, 929

Ascoli, G. (1972). Contributed paper to the 1972 International Conference on High Energy Physics, 16th, National Accelerator Laboratory, Batavia, Mlinois, September 6-13, 1972, No. 443

Ball, J. S., Campbell, R. R., Lee, P. S., and Shaw, G. L. (1972). Phys. Rev. Letters 28, 1143

Berman, S., and Jacob, M. (1965). Phys. Rev. B 139, 1023
Bland, R. W., Bowler, M. G., Brown, J. L., Kadyk, J. A., Goldhaber, G., Goldhaber, S., Seeger, V. H., and Trilling, G. H. (1969). Nucl. Phys. B13, 595

Bowler, M. G., and Cashmore, R. J. (1970). Nucl. Phys. B17, 331
Brody, A. D., Kernan, A. (1969). Phys. Rev. 182, 1785
Brody, A. D., Cashmore, R. J., Kernan, A., Leith, D.W. G. S., Levi, B. G., Shen, B. C., Herndon, D. J., Price, L. R., Rosenfeld, A. H., and Söding, P. (1971). Phys. Letters 34B, 665

Cashmore, R. J., and Hey, A.J. G. (1972). Phys. Rev. D 6, 1303
Cashmore, R. J., Herndon, D. J., and Söding, P. (1972). Lawrence Berkeley Laboratory LBL-543

Cashmore, R. J. (1973). In Proceedings of the International Conference on Baryon Resonances, Purdue, West Lafayette, Indiana, April 20-21, 1973

Chinowsky, W., Mulvey, J. H., and Saxon, D. H. (1970). Phys. Rev. D 2, 1790

Dalitz, R. (1962). In "Strange Particles and Strong Interactions" Oxford University Press

Dalitz, R. (1969). In "Pion-Nucleon Scattering" (G. L. Shaw and D. Y. Wong, eds.) John Wiley and Sons, Inc., New York

DeBeer, M., Deler, B., Dolbeau, J., Neveu, M., Diem, Nguyen Thuc, Smadja, G., and Valladas, G. (1969a). Nucl. Phys. B12, 599

DeBeer, M., Deler, B., Dolbeau, J., Neveu, M., Diem, Nguyen Thuc, Smadja, G., and Valladas, G. (1969b). Nucl. Phys. B12, 617

Deler, B., and Valladas, G., Nuovo Cimento 45A, 559
Faiman, D. (1971). Nucl. Phys. B32, 573
Faiman, D., and Plane, D. E. (1972). Nucl. Phys. B50, 379

Faiman, D., and Rosner, J. (1973). CERN preprint TH. 1636, Phases of resonant amplitudes: $\pi+N \rightarrow \pi+\Delta$ (revised, unpublished)

Feynman, R., Pakvasa, S., and Tuan, S. F. (1970). Phys. Rev. D 2, 1267

Gell-Mann, M. (1962). Phys. Rev. 125, 1067
Gilman, F., Pumplin, J., Schwimmer, A., and Stodolsky, L. (1970). Phys. Letters 31B, 387

Gilman, F., Kugler, M., and Meshkov, S. (1973). Stanford Linear Accelerator Center preprint SLAC-PUB-1235, Pionic transitions as tests of the connection between current and constituent quarks Gilman, F. (1973). In Proceedings of the International Conference on Baryon Resonances, Purdue, West Lafayette, Indiana, April 20-21, 1973

Greenberg, O. (1969). In Proceedings of the International Conference on Elementary Particles, 5th, Lund, Sweden, 1969, Berlingska Boktryckeriet, Lund

Herndon, D. J., Longacre, R., Miller, L. R., Rosenfeld, A. H., Smadja, G., Soding, P., Cashmore, R. J., and Leith, D. W. G.S. (1972). Lawrence Berkeley Laboratory and Stanford Linear Accelerator Center preprint $\mathrm{LBL}-1065 / \mathrm{SLAC}-\mathrm{PUB}-1108$, A partial wave analysis of the reaction $\pi N \rightarrow \pi \pi N$ in the c. m . energy range $1300-$ 2000 MeV

Illinois-GHMS-H-ABBCCH-ND-W Collaboration and CIBS Collaboration (1972). In Proceedings of the International Conference on Experimental Meson Spectroscopy, 3rd, Philadelphia, Pennsylvania, April 28-29, 1972 (Presented by G. Ascoli)

Jacob, M. and Wick, G. C. (1959). Ann. Phys. 7, 404
Kernan, A. and Shepherd, H. K. (1969). Phys. Rev. Letters 23, 1314
Lasinski, T. and Barbaro-Galtieri, L. (1972). Phys. Letters 39B, 1
Lasinski, T., Ronat, E. E., Rosenfeld, A. H., Tabak, M., and
Cashmore, R. J. (1973). Preliminary studies of the $3 \pi$ system in $\pi^{+} p \rightarrow \pi^{+} \pi^{+} \pi^{-} p$ at $7 \mathrm{GeV} / \mathrm{c}$

Levi Setti, R. (1969). In Proceedings of the International Conference on Elementary Particles, 5th, Lund, Sweden, 1969, Berlingska Boktryckeriet, Lund

Lipkin, H. and Meshkov, S. (1965). Phys. Rev. Letters 14, 670
Longacre, R. (1972). Ph.D. Thesis, Lawrence Berkeley Laboratory LBL-948 (unpublished)

Mast, T. S., Alston-Garnjost, M., Bangerter, R. O., Barbaro-Galtieri, A., Solmitz, F. T., and Tripp, R. D. (1972). Phys. Rev. Letters 28, 1220

Mehtani, V., Fung, S. Y., Kernan, A., Schalk, T. L., Williamson, Y., Birge, R. W., Kalmus, G. E., and Michael, W. (1972). Phys. Rev. Letters 29, 1634

Melosh IV, H. J. (1973). Caltech Thesis (unpublished)
Moorhouse, G. and Oberlack, H. (1973). Phys. Letters 43B, 44

Moorhouse, G. and Parsons, N. (1973). Glasgow University preprint Plane, D. E., Baillon, P., Bricman, C., Ferro-Luzzi, M., Meyer, J., Pagiola, E., Schmitz, N., Burkhardt, E., Filfuth, H., Kluge, E., Oberlack, H., Barloutaud, R., Granet, P., Porte, J. P., and Prevost, J. (1970). Nucl. Phys. B22, 93

Rosner, J. (1972). In 1972 International Conference on High Energy Physics, 16th, National Accelerator Laboratory, Batavia, Illinois, September 6-13, 1972, Appendix to Baryon resonances and related phenomenology

Wick, G. C. (1962). Ann. Phys. 18, 65

Table I
The Isospin Decomposition in $\pi \mathrm{N} \rightarrow \pi \pi \mathrm{N}$ Reactions

|  | $\left(\pi_{1} \mathrm{~N}\right)_{\mathrm{I}=3 / 2}$ | $\left(\pi_{2}{ }^{N}\right)_{I=3 / 2}$ | $\left(\pi_{1} N\right)_{1=1 / 2}$ | $\left(\pi_{2} \mathrm{~N}\right)_{\mathrm{I}}=1 / 2$ | $\left(\pi_{1} \pi_{2}\right)_{\mathrm{I}=0}$ | $\left(\pi_{1} \pi_{2}\right)^{1}=1$ | $\left(\pi_{1} \pi_{2}\right)_{\mathrm{I}=2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\pi_{1}^{+} \pi_{2}^{o} p$ | $\frac{3}{\sqrt{15}}$ | $\frac{-2}{\sqrt{15}}$ | 0 | $-\sqrt{\frac{1}{3}}$ | 0 | 1 | $-\sqrt{\frac{1}{5}}$ |
| $\pi_{1}^{+} \pi_{1}^{+} n$ | $\frac{-1}{\sqrt{15}}$ | $\frac{-1}{\sqrt{15}}$. | $\sqrt{\frac{1}{3}}$ | $\sqrt{\frac{1}{3}}$ | 0 | 0 | $\sqrt{\frac{4}{5}}$ |
| $\pi_{1}^{-\pi_{2}^{+}}{ }^{n}$ | $\frac{-3 \sqrt{2}}{\sqrt{135}}$ | $\frac{2 \sqrt{2}}{\sqrt{135}}$ | 0 | $\frac{\sqrt{2}}{\sqrt{27}}$ | 0 | $-\sqrt{\frac{2}{9}}$ | $\sqrt{\frac{2}{45}}$ |
| $\pi_{1}^{-\pi}{ }_{2} p$ | $\frac{-1}{\sqrt{135}}$ | $\frac{4}{\sqrt{135}}$ | $\frac{-2}{\sqrt{27}}$ | $\frac{-1}{\sqrt{27}}$ | 0 | $-\sqrt{\frac{1}{9}}$ | $-\sqrt{\frac{3}{15}}$ |
| $\pi_{1}^{0} \pi_{2}^{0}$ | $\frac{-1}{\sqrt{135}}$ | $\frac{-1}{\sqrt{135}}$ | $\frac{1}{\sqrt{27}}$ | $\frac{1}{\sqrt{27}}$ | 0 | 0 | $\frac{2}{\sqrt{45}}$ |
| $\mathrm{I}=1 / 2$ |  |  |  |  |  |  |  |
|  | $\left(\pi_{1} \mathrm{~N}\right)_{\mathrm{I}=3 / 2}$ | $\left(\pi_{2}{ }^{N}\right)_{\mathrm{I}=3 / 2}$ | $\left(\pi_{1} N\right)^{1}=1 / 2$ | $\left(\pi_{2} \mathrm{~N}\right)_{\mathrm{I}=1 / 2}$ | $\left(\pi_{1} \pi_{2}\right)_{1=0}$ | $\left(\pi_{1} \pi_{2}\right)_{\mathrm{I}=1}$ | $\left(\pi_{1} \pi_{2}\right)_{I=2}$ |
| $\pi_{1}^{-} \pi_{2}^{+} n$ | $\frac{-3}{\sqrt{27}}$ | $\frac{-1}{\sqrt{27}}$ | 0 | $\frac{-2 \sqrt{2}}{\sqrt{27}}$ | $-\frac{2}{3}$ | $\frac{\sqrt{2}}{3}$ | 0 |
| $\pi_{1}^{-} \pi_{2}^{o}$ | $\frac{\sqrt{2}}{\sqrt{27}}$ | $\frac{-\sqrt{2}}{\sqrt{27}}$ | $\frac{-2}{\sqrt{27}}$ | $\frac{+2}{\sqrt{27}}$ | 0 | $-\frac{2}{3}$ | 0 |
| $\pi_{1} \pi_{2}^{0}{ }^{\text {n }}$ | $\frac{\sqrt{2}}{\sqrt{27}}$ | $\frac{\sqrt{2}}{\sqrt{27}}$ | $\frac{+1}{\sqrt{27}}$ | $\frac{+1}{\sqrt{27}}$ | $\frac{\sqrt{2}}{3}$ | 0 | 0 |

Table II
Isospin Decomposition of $\mathrm{I}=0$ and $\mathrm{I}=1$ Decays to Three Pions
$\mathrm{I}=1$

|  | $\left(\pi_{1} \pi_{2}\right)_{\rho}$ | $\left(\pi_{2} \pi_{3}\right)_{\rho}$ | $\left(\pi_{3} \pi_{1}\right)_{\rho}$ | $\left(\pi_{1} \pi_{2}\right)_{\epsilon}$ | $\left(\pi_{2} \pi_{3}\right)_{\epsilon}$ | $\left(\pi_{3} \pi_{1}\right)_{\epsilon}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\pi_{1}^{+} \pi_{2}^{+} \pi_{3}^{-}$ | 0 | $-\frac{1}{2}$ | $+\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ |
| $\pi_{1}^{o} \pi_{2}^{o} \pi_{3}^{+}$ | 0 | $-\frac{1}{2}$ | $+\frac{1}{2}$ | $-\frac{1}{\sqrt{3}}$ | 0 | 0 |
| $\pi_{1}^{+} \pi_{2}^{-} \pi_{3}^{o}$ | 0 | $+\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | $\sqrt{\frac{2}{3}}$ | 0 | 0 |
| $\pi_{1}^{o} \pi_{2}^{o} \pi_{3}^{o}$ | 0 | 0 | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |

$\mathrm{I}=0$

|  | $\left(\pi_{1} \pi_{2}\right)_{\rho}$ | $\left(\pi_{2} \pi_{3}\right)_{\rho}$ | $\left(\pi_{3} \pi_{1}\right)_{\rho}$ | $\left(\pi_{1} \pi_{2}\right)_{\epsilon}$ | $\left(\pi_{1} \pi_{3}\right)_{\epsilon}$ | $\left(\pi_{3} \pi_{1}\right)_{\epsilon}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\pi_{1}^{+} \pi_{2}^{-} \pi_{3}^{o}$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ | 0 | 0 | 0 |

Table III
Isospin Decomposition of $\mathrm{I}=1 / 2$ Decays to $\mathrm{K} \pi \pi$

|  | $\left(\pi_{1} \mathrm{~K}\right)_{\mathrm{I}=1 / 2}$ | $\left(\pi_{2} \mathrm{~K}\right)_{\mathrm{I}=1 / 2}$ | $\left(\pi_{1} \pi_{2}\right)_{\mathrm{I}=1}$ | $\left(\pi_{1} \pi_{2}\right)_{\mathrm{I}=0}$ |
| :--- | :---: | :---: | :---: | :---: |
| $\pi^{+} \pi^{\mathrm{o}} \mathrm{K}^{\mathrm{o}}$ | $+\frac{\sqrt{2}}{3}$ | $-\frac{\sqrt{2}}{3}$ | $+\sqrt{\frac{2}{3}}$ | 0 |
| $\pi_{1}^{+} \pi_{2}^{-} \mathrm{K}^{+}$ | 0 | $+\frac{2}{3}$ | $-\sqrt{\frac{1}{3}}$ | $\sqrt{\frac{2}{3}}$ |
| $\pi_{1}^{\mathrm{o} \pi_{2}^{o} \mathrm{~K}^{+}}$ | $-\frac{1}{3 \sqrt{2}}$ | $-\frac{1}{3 \sqrt{2}}$ | 0 | $-\sqrt{\frac{1}{3}}$ |
| $\pi_{1}^{+} \pi_{2}^{-} \mathrm{K}^{o}$ | $\frac{2}{3}$ | 0 | $\sqrt{\frac{1}{3}}$ | $\sqrt{\frac{2}{3}}$ |
| $\pi_{1}^{-} \pi_{2}^{\mathrm{o}} \mathrm{K}^{+}$ | $+\frac{\sqrt{2}}{3}$ | $-\frac{\sqrt{2}}{3}$ | $\sqrt{\frac{2}{3}}$ | 0 |

Table IV
The Observable Quantities in $\mathrm{MB} \rightarrow \mathrm{BMM}$

$$
\begin{aligned}
& I_{0}=\frac{1}{2}\left\{\left|A_{++}\right|^{2}+\left|A_{+-}\right|^{2}+\left|A_{-+}\right|^{2}+\left|A_{--}\right|^{2}\right\} \\
& I_{0} A_{x}=\operatorname{Re}\left[A_{++} A_{+-}^{*} e^{i \alpha}\right]+\operatorname{Re}\left[A_{++} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} A_{y}=\operatorname{Im}\left[A_{++} A_{+-}^{*} e^{i \alpha}\right]+\operatorname{Im}\left[A_{-+} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} A_{z}=\frac{1}{2}\left\{\left|A_{++}\right|^{2}+\left|A_{-+}\right|^{2}-\left|A_{+-}\right|^{2}-\left|A_{--}\right|^{2}\right\} \\
& \mathrm{I}_{0} \mathrm{P}_{\mathrm{X}}^{(0)}=\operatorname{Re}\left[\mathrm{A}_{++} \mathrm{A}_{-+}^{*}\right]+\operatorname{Re}\left[\mathrm{A}_{+-} \mathrm{A}_{--}^{*}\right] \\
& I_{0} P_{Y}^{(0)}=-\operatorname{Im}\left[A_{++} A_{-+}^{*}\right]-\operatorname{Im}\left[A_{+-} A_{--}^{*}\right] \\
& I_{0} P_{Z}^{(0)}=\frac{1}{2}\left\{\left|A_{++}\right|^{2}+\left|A_{+-}\right|^{2}-\left|A_{-+}\right|^{2}-\left|A_{--}\right|^{2}\right\} \\
& I_{0} D_{X X}=\operatorname{Re}\left[A_{+-} A_{-+}^{*} e^{-i \alpha}\right]+\operatorname{Re}\left[A_{++} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} D_{X Y}=-\operatorname{Im}\left[A_{+-} A_{++}^{*} e^{-i \alpha}\right]-\operatorname{Im}\left[A_{++} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} D_{x Z}=\operatorname{Re}\left[A_{++} A_{+-}^{*} e^{i \alpha}\right]-\operatorname{Re}\left[A_{-+} A_{--}^{*} e^{i \alpha}\right] \\
& \left.I_{0} D_{y X}=-\operatorname{Im}\left[A_{+-} A_{-+}^{*} e^{-i \alpha}\right]+\operatorname{Im} A_{++} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} D_{y Y}=\operatorname{Re}\left[A_{++-} A_{-+}^{*} e^{i \alpha}\right]-\operatorname{Re}\left[A_{+-} A_{-+}^{*} e^{-i \alpha}\right] \\
& I_{0} D_{y Z}=\operatorname{Im}\left[A_{++} A_{+-}^{*} e^{i \alpha}\right]-\operatorname{Im}\left[A_{++} A_{--}^{*} e^{i \alpha}\right] \\
& I_{0} D_{z X}=\operatorname{Re}\left[A_{++} A_{-+}^{*}\right]-\operatorname{Re}\left[A_{+-} A_{--}^{*}\right] \\
& I_{0} D_{Z Y}=-\operatorname{Im}\left[A_{++} A_{-+}^{*}\right]+\operatorname{Im}\left[A_{+-} A_{--}^{*}\right] \\
& I_{0} D_{z Z}=\frac{1}{2}\left\{\left|A_{++}\right|^{2}+\left|A_{--}\right|^{2}-\left|A_{++}\right|^{2}-\left|A_{+-}\right|^{2}\right\}
\end{aligned}
$$

The amplitudes $A_{\mu_{f}} \mu_{i}$ are written with $\mu_{i}, \mu_{f}= \pm 1 / 2$. Also note that the order of the subscripts of $A$ has been reversed.

Table V
The Partial Waves Used in $\pi N \rightarrow \pi \pi N$

| Isospin | Incident Wave | $\pi \Delta$ | ${ }^{\mathrm{N}}{ }_{3}$ | $\mathrm{N}_{1}$ | N $\sigma$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{I}=1 / 2$ | S11 | SD11 | SD11 | SS11 | SP11 |
|  | P11 | PP11 | PP11 | PP11 | PS11 |
|  | D13 | $\begin{aligned} & \text { DS13 } \\ & \text { DD13 } \end{aligned}$ | $\begin{aligned} & \text { DS13 } \\ & \text { DD13 } \end{aligned}$ | DD13 | DP13 |
|  | P13 | $\begin{aligned} & \text { PP13 } \\ & \text { PF13 } \end{aligned}$ | PP13 <br> PF13 | PP13 | PD13 |
|  | D15 | DD15 | DD15 | DD15 | DF15 |
|  | F15 | FP15 <br> FF15 | FP15 <br> FF15 | FF15 | FD15 |
|  | F17 | FF17 | FF17 | FF17 |  |
| $\mathrm{I}=3 / 2$ | S31 | SD31 | SD31 | SS31 |  |
|  | P31 | PP31 | PP31 | PP31 |  |
|  | D33 | $\begin{aligned} & \text { DS33 } \\ & \text { DD33 } \end{aligned}$ | $\begin{aligned} & \text { DS33 } \\ & \text { DD33 } \end{aligned}$ | DD33 |  |
|  | P33 | $\begin{aligned} & \text { PP33 } \\ & \text { PF33 } \end{aligned}$ | $\begin{aligned} & \text { PP33 } \\ & \text { PF33 } \end{aligned}$ | PP33 |  |
|  | D35 | DD35 | DD35 | DD35 |  |
|  | F35 | FP35 FF35 | FP35 FF35 | FF35 |  |
|  | F37 | FF37 | FF37 | FF37 |  |

The 60 waves with angular momenta $L, L^{\prime}, \ell$ each $\leq 3$. There are two nucleon-rho terms in the isobar model, indicated by $\rho_{3}$ and $\rho_{1}$, where the subscript indicates the coupling between the spin of the $\rho(\ell=1)$ and the spin of the outgoing nucleon.

## Table VI

Partial Waves in Decay to Three Pseudoscalar Mesons for $\ell \leq 2(\epsilon, \rho, f)$ and $L \leq 3$

| $J^{P}$ | $2 \mathrm{~J}+1$ | L | $\ell$ | Number of Amperes |
| :---: | :---: | :---: | :---: | :---: |
| $0^{-}$ | 1 | 0 | 0 | 1 |
|  |  | 1 | 1 | 1 |
|  | - | 2 | 2 | 1 |
| $1^{-}$ | 3 | 1 | 1 | 3 |
|  |  | 2 | 2 | 3 |
| ${ }^{1}+$ | 3 | 1 | 0 | 3 |
|  |  | 0 | 1 | 3 |
|  |  | 2 | 1 | 3 |
|  |  | 1 | 2 | 3 |
|  |  | 3 | 2 | 3 |
| $2^{+}$ | 5 | 2 | 1 | 5 |
|  |  | 1 | 2 | 5 |
|  |  | 3 | 2 | 5 |
| $2^{-}$ | 5 | 2 | 0 | 5 |
|  |  | 1 | 1 | 5 |
|  |  | 3 | 1 | 5 |
|  |  | 0 | 2 | 5 |
|  |  | 2 | 2 | 5 |
| $3^{-}$ | 7 | 3 | 1 | 7 |
|  |  | 2 | 2 | 7 |
| $3^{+}$ | 7 | 3 | 0 | 7 |
|  |  | 2 | 1 | 7 |
|  |  | 1 | 2 | 7 |
|  |  | 3 | 2 | 7 |
| Total ( $J \leq 2, \quad \ell \leq 1)$ |  |  |  | 29 |

## Table VII

Number of Events for the Energy Bins Used in the Fits

| C. M. Energy | Range <br> $(\mathrm{MeV})$ | $\pi^{-} p \rightarrow \pi^{+} \pi^{-} \mathrm{n}$ | $\pi^{-} \mathrm{p} \rightarrow \pi^{-} \pi^{0} p$ | $\pi^{+} p \rightarrow \pi^{+} \pi^{0} p$ |
| :---: | :---: | :---: | :---: | :---: |
| 1310 | $1300-1330$ | 1069 | 151 |  |
| 1340 | $1330-1360$ | 1664 | 11 |  |
| 1370 | $1360-1380$ | 2471 | 2 |  |
| 1400 | $1380-1410$ | 5049 | 964 | 78 |
| 1440 | $1430-1460$ | 4918 | 1802 | 359 |
| 1470 | $1460-1480$ | 3252 | 1629 | 175 |
| 1490 | $1480-1510$ | 5555 | 3197 | 1523 |
| 1520 | $1510-1530$ | 3241 | 2588 | 795 |
| 1540 | $1530-1560$ | 3905 | 3285 | 1114 |
| 1650 | $1630-1670$ | 6061 | 3757 | 2467 |
| 1690 | $1670-1710$ | 5901 | 3689 | 1139 |
| 1730 | $1710-1750$ | 3455 | 2630 | 4061 |
| 1770 | $1750-1790$ | 3214 | 2352 | 2853 |
| 1810 | $1790-1830$ | 2447 | 1541 | 3855 |
| 1850 | $1380-1870$ | 3931 | 3183 | 6372 |
| 1890 | $1870-1910$ | 5072 | 3170 | 12690 |
| 1930 | $1910-1950$ | 5817 | 4080 | 4298 |
| 1970 | $1950-1990$ | 5277 | 3544 | 7744 |
| $707 a 1$ | $1300-1990$ | 72299 | 41575 | 49523 |

Table VIII
Signs of the Amplitudes for $\pi N \rightarrow N^{*} \rightarrow \pi \Delta$ for $\mathrm{N}^{*!} \mathrm{S}$ in the $[70, \mathrm{~L}=1]$ and $[56, \mathrm{~L}=2]$

|  |  | $\begin{aligned} & \operatorname{SU}(6)_{W} \operatorname{Sign} \\ & (8,1)_{0}-(1,8)_{0} \end{aligned}$ | $\begin{gathered} \text { Anti-SU }(6){ }_{\mathrm{W}} \text { Sign } \\ (3, \overline{3})_{1}-(\overline{3}, 3) \\ -1 \end{gathered}$ | Quark Model |
| :---: | :---: | :---: | :---: | :---: |
|  | $\left(\begin{array}{l}\text { DD13(1520) } \\ \text { DS13(1520) }\end{array}\right.$ | $\begin{array}{ll}+* & (-) \\ - & (+)\end{array}$ | $\begin{array}{ll}+* & (-) \\ + & (-)\end{array}$ | $\begin{array}{ll}+ & (-) \\ + & (-)\end{array}$ |
| $\begin{aligned} & {[70, \mathrm{~L}=1]} \\ & {[56, \overrightarrow{\mathrm{~L}}=0]} \end{aligned}$ | $\left\{\begin{array}{l}\text { SD31(1640) } \\ \text { DS33(1690) } \\ \text { DS13(1700) } \\ \text { DD15 (1670) }\end{array}\right.$ | $\begin{aligned} & + \\ & + \\ & + \\ & +_{*}^{*} \end{aligned}$ | _ * |  |
| $\begin{aligned} & {[56, \mathrm{~L}=2]} \\ & {[56, \overrightarrow{\mathrm{~L}}=0]} \end{aligned}$ | $\left\{\begin{array}{l}\text { FP15(1688) } \\ \text { FF35(1880) } \\ \text { FF37(1950) }\end{array}\right.$ | - $-*$ $\_^{*}$ | $+$ _ * _ * | $+$ <br> _ * $-*$ |

Products of the experimental and theoretical signs (in various models) are presented. Signs which are independent of the model are denoted by ${ }^{1 *}$. Exponent and theory agree within the $[70, \mathrm{~L}=1]$ or within the $[56, \mathrm{~L}=2]$ if all the signs in any column are the same.

Table IX
Pole Parameters and $\Gamma_{\text {partial }}$ from Pole Residues

| Wave | Pole | $\Gamma_{\pi N}$ | $\Gamma^{\pi \Delta_{\mathrm{L}}}$ | $\Gamma_{\pi \Delta_{L}}$ | $\Gamma_{\mathrm{N}_{\rho}}$ | $\Gamma^{\mathrm{N}} \rho_{1}$ | $\Gamma_{N \epsilon}$ | Other <br> Channel | $\Gamma_{\text {tot }}=\Sigma \Gamma_{i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| S11 | $\begin{aligned} & 1 \quad 1498-\mathrm{i} \frac{66}{2} \\ & 2 \quad 1648-\mathrm{i} \frac{103}{2} \end{aligned}$ | 11 $46$ |  |  |  | 11 $12$ | $34$ $4$ | $14(\eta N)$ $32(\eta \mathrm{~N})$ | $\begin{aligned} & 70 \\ & 94 \end{aligned}$ |
| P11 | $\begin{aligned} & 1 \quad 1383-\mathrm{i} \frac{200}{2} \\ & 2 \quad 1724-\mathrm{i} \frac{290}{2} \end{aligned}$ | $80$ $120$ | 40 $37$ |  |  | - | $5$ $75$ |  | $\begin{aligned} & 125 \\ & 232 \end{aligned}$ |
| P13 | $\begin{array}{lll}1 & 1728-\mathrm{i} & \frac{159}{2}\end{array}$ | 25 |  |  |  | 84 |  |  | 109 |
| D13 | $\begin{aligned} & 1 \quad 1515-\mathrm{i} \frac{143}{2} \\ & 2 \\ & 2 \end{aligned} 1646-\mathrm{i} \frac{114}{2} .$ | $90$ $16$ | $38$ $22$ | 15 $3$ | $34$ $3$ |  | 3 $62$ |  | $\begin{aligned} & 180 \\ & 108 \end{aligned}$ |
| D15 | $1666-\mathrm{i} \frac{159}{2}$ | 69 | 92 |  |  |  |  |  | 161 |
| F15 | $1672-\mathrm{i} \frac{155}{2}$ | 101 | 12 |  | 42 |  | 21 |  | 182 |
| S31 | 1605-i $\frac{59}{2}$ | 17 | 11 |  |  | 35 |  |  | 63 |
| D33 | $1650-\mathrm{i} \frac{150}{2}$ | 13 | 56 |  | 56 |  |  |  | 125 |
| F35 | 1824 -i $\frac{282}{2}$ | 44 |  | 26 | 107 |  |  |  | 177 |
| F37 | 1866 -i $\frac{255}{2}$ | 111 | 76 |  | 57 |  |  | 132 (junk) | 380 |

## FIGURE CAPTIONS

1. The three body final states: (a) formation reactions; (b) production reactions.
2. Dalitz plot for $\pi^{-} p \rightarrow \pi^{-} \pi^{+} n$.
3. Diffraction excitation.
4. Hypercharge and charge exchange reactions.
5. Diffractive excitation of the nucleon.
6. The moments $W_{L}^{M}$ as a function of energy in the final state $\pi^{-} \pi^{+} n$ normalized such that $W_{0}^{0}=1$. The x axis is defined as $\overrightarrow{\mathrm{p}}_{\mathrm{N}}$ and the z-axis as $\overrightarrow{\mathrm{p}}_{\pi^{-}} \wedge \overrightarrow{\mathrm{p}}_{\pi^{+}}$.
7. $\pi^{+} \pi^{-} \pi^{-}$Dalitz plot $\mathrm{M}^{2}\left(\pi^{+} \pi_{1}^{-}\right)$vs $\mathrm{M}^{2}\left(\pi^{+} \pi_{2}^{-}\right)$for events having $3 \pi$ masses in the interval $1.0-1.4 \mathrm{GeV}$. The curves define the Dalitz plot boundaries for $\mathrm{M}_{3 \pi}=1.20$ and 1.40 GeV . The data is from 5 and $7.5 \mathrm{GeV} / \mathrm{c} \pi{ }^{-} \mathrm{p}$ collisions.
8. The isobar model.
9. Definitions in the three particle decay.
10. Definitions in the incident two particle state in formation reactions.
11. Definitions in production reactions.
12. The co-ordinate system S .
13. The isobar model.
14. The decay angles of the isobar.
15. The helicity frame axes $0 x^{\prime} y^{\prime} z^{\prime}$ for particle $j$. $0 x y z$ are the axes of frame S .
16. Production reactions.
17. The Gottfried-Jackson system.
18. Production angles of $x$ in $S^{\prime \prime}$, a system defined by a polarized target.
19. Relative signs of $\mathrm{Y}^{*}$ couplings. The arrow is the prediction from SU(3) and the X represents the experimentally observed coupling.
20. Fits to the reaction $\pi^{-} \mathrm{p} \rightarrow \pi^{+} \pi^{-} \mathrm{n}$ at a c.m. energy of 1690 MeV . The figure contains $\cos \theta$ vs $\phi$ plots for individual regions of the Dalitz plot where $\cos \theta$ and $\phi$ are the polar angles of the incident pion in a co-ordinate system defined by the final state. The $z$ axis lies along $\overrightarrow{\mathrm{p}}_{\mathrm{N}}$ and the y axis lies along $\overrightarrow{\mathrm{p}}_{\pi^{-}} \times \overrightarrow{\mathrm{p}}_{\pi^{+}}$. The plots outside the Dalitz plot are the sums of the corresponding plots within the boundary.
21. Single pion production cross sections. Data points are indicated by I and the predictions from our partial wave amplitudes by x .
22. $\mathrm{I}=1 / 2$ partial wave amplitudes. Arrows are spaced every 20 MeV , with wide arrows every 100 MeV : base of wide arrows mark integral hundreds of MeV . Lower $-\ell$ waves are plotted starting at $\mathrm{s}=1400 \mathrm{MeV}$; higher - $\ell$ waves only where they were first needed. Last arrowhead is always at 1940 MeV .
23. Partial waves derived from the incident $P 11$ wave.
24. Partial waves derived from the incident D13 wave.
25. The $F_{15}$ partial wave.
26. I=3/2 partial wave amplitudes. Arrows are spaced every 20 MeV , with wide arrows every 100 MeV : base of wide arrows mark integral hundreds of MeV . Lower $-\ell$ waves are plotted starting at $s=1400 \mathrm{MeV}$; higher $-\ell$ waves only where they were first needed. Last arrowhead is always at 1940 MeV .
27. Partial waves derived from the incident F35 wave.
28. Partial waves derived from the incident P33 wave.
29. Relative coupling signs of the resonances in all inelastic channels. The vertical double line separates the two regions, since the continuity is not unambiguously ascertained. The heavy arrows correspond to the present solution A. The dashed arrows will be the orientation if the second solution, B, is satisfactory.
30. Prediction of resonant phases in $\pi N \rightarrow N^{*} \rightarrow \pi \Delta$ corresponding to a choice of the 'anti $\mathrm{SU}(6)_{\mathrm{W}}{ }^{\prime}$ ' relative sign for the two partial waves in the decay. To obtain $S U(6)_{W}$ results, reverse all double handed clocks. Finally to compare with experimental results all $\mathrm{I}=3 / 2$ waves must be multiplied by ( -1 ) to give the same phase conventions.
31. The $\pi^{+} \pi^{-} \pi^{-}$mass spectrum for the data at $5.0-7.5 \mathrm{GeV} / \mathrm{c}$. The shaded histogram is for $\mathrm{t}^{\prime}<0.7 \mathrm{GeV}^{2}, \mathrm{M}_{\mathrm{p} \pi^{+}}<1.4$ out, $1.16<\mathrm{M}_{\mathrm{p} \pi^{-}}<1.32$ out.
32. Contributions of various $\mathrm{J}^{\mathrm{P}}$ states to the $\mathrm{M}_{3 \pi}$ distribution.
33. Individual partial wave contributions to the $0^{-}$state.
34. Individual partial wave contributions to the $1^{+}$state.
35. Individual partial wave contributions to the $2^{-}$state.
36. The Deck mechanism for the A1 and A3.
37. Phase of $2^{-} \pi f$ (s-wave) production amplitude relative to other partial waves as a function of $3 \pi$ mass.
38. The $1^{+} \pi \rho$ (s-wave). Its intensity and phase relative to other partial waves as a function of $3 \pi$ mass.
39. The $2^{+} \pi \rho$ (d-wave). Its intensity and phase variation relative to the $1^{+}$waves as a function of $3 \pi$ mass.
40. The density matrix of the $1^{+}$state.
41. Production angular distribution of spin parity $2^{+}$events. Curves are fits of the form $t^{\prime} e^{-B t^{\prime}}$ with $B=6.41$ and $B=9.04$ for low (5-7.5) and high (11-25) incident $\pi$ momenta.
42. Cross sections as a function of incident momentum for production of $2^{+}$by 'natural parity exchange' and 'unnatural parity exchange'.
43. Wigner angle - the rotations due to Lorentz transformations.

(a) Formation

(b) Production

2345A1

FIG. 1


FIG. 2


FIG. 3


FIG. 4


234545

FIG. 5

MOMENTS $\mathbb{I N}$ THE FINAL STATE $\pi^{-} \pi^{+} n$


FIG. $6 a$

MOMENTS IN THE FINAL STATE $\pi^{-} \pi^{+} n$


FIG. 6b


FIG. 7


2345A9
FIG. 8


FIG. 9


FIG. 10


FIG. 11


FIG. 12


FIG. 13


FIG. 14


FIG. 15


FIG. 16


$$
\begin{aligned}
& O Z^{\prime}=\vec{\pi} \\
& O Y^{\prime}=\vec{p}_{\text {in }} \times \vec{p}_{\text {out }}
\end{aligned}
$$

FIG. 17


$\mathrm{SU}_{3}$ RELATIVE SIGN OF RESONANT AMPLITUDES $T_{\text {RES }} \sim \phi\left(g_{N \bar{K} Y^{*}} \cdot g_{Y \pi Y^{*}}\right) /\left(E_{R}-E-i \Gamma / 2\right)$


FIG. 19


FIG. 20


FIG. 21


FIG. 22


FIG. 23


FIG. 24


FIG. 25


FIG. 26


FIG. 27


FIG. 28


FIG. 29


FIG. 30


FIG. 31


FIG. 32


FIG. 33


FIG. 34


FIG. 35


FIG. 36


FIG. 37

$$
\pi^{-} p \rightarrow \pi^{-} \pi^{-} \pi^{+} p
$$

CUBS $197240 \mathrm{GeV} / \mathrm{c}$
U. of Illinois $5+7.5 \mathrm{GeV} / \mathrm{c}$ $0.04<\mathrm{t}<0.33(\mathrm{GeV} / \mathrm{c})^{2}$ $\mathrm{t}<0.7(\mathrm{GeV} / \mathrm{c})^{2}$
INTENSITY OF $\left.\right|^{+} \mathrm{S}$-WAVE



FIG. 38

$$
\begin{gathered}
\pi^{-} p \rightarrow \pi^{-} \pi^{-} \pi^{+} p \quad 40 \mathrm{GeV} / \mathrm{c} \\
0.17<t<0.33(\mathrm{GeV} / \mathrm{c})^{2}
\end{gathered}
$$




FIG. 39


FIG. 40


FIG. 42


FIG. 43

