# Alternating Euler Paths for Packings and Covers 

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1. Introduction. An interesting combinatorial problem known as the "schoolgirls' walk" asks if the girls in an all-girl school can take a walk in two-by-two fashion so that each pair walking side by side are on friendly terms, it being known which pairs are friendly among all possible pairings. If such a utopian arrangement is not possible, then what is the largest number of friendly pairings that can be achieved simultaneously and how can such an optimal set of pairings be found? This problem is abstractly equivalent to a problem in graph theory which is as follows: Let $G$ be a finite graph with vertex set $V$ and edge set $E$; a matching $M$ of graph $G$ is a subset of $E$ such that no two edges in $M$ have a vertex in common. A matching $M^{*}$ is maximum if no other matching has more edges than $M^{*}$. A matching $P$ is perfect if each vertex in $V$ belongs to an edge of $P$. In this abstract version the vertices represent girls and edges represent pairs of friendly girls. A matching is a pairing of friendly girls with no girl appearing twice, an obviously necessary requirement. A perfect matching represents the utopian arrangement and a maximum matching achieves the largest number of friendly pairings.

A related problem concerns a minimum cover for a graph $G$ where a cover $C$ is a subset of $E$ such that every vertex belongs to at least one edge in $C$. A perfect matching is then a matching which is also a cover and it is easy to see that any subset of edges which is both a matching and a cover is necessarily a maximum matching and a minimum cover.

A good algorithm for finding a maximum matching requires a reasonably simple condition which, when true, assures that a matching $M$ is maximum and, when false, implies $M$ is not maximum, and further indicates how to modify $M$ to obtain a larger matching $M^{\prime}$. Such a condition is afforded by augmenting paths. A path in $G$ is a sequence of edges such that two edges adjacent in the sequence share a vertex in $G$. For our purposes, no edge can appear more than once in a path. If $S$ is a subset of edges in $G$, an $S$-alternating path is a path whose edges are alternately in $S$ and in $\overline{\boldsymbol{S}}=E-S$. A vertex $\mathbf{v}$ is $S$-exposed if $\mathbf{v}$ belongs to no edge in $S$. An $M$-augmenting path for matching $M$ is an $M$-alternating path whose end vertices are $M$-exposed. Notice this implies the end edges are in $\bar{M}$ and the path has odd length. Such a path is called augmenting because by interchanging the $M$ and $\bar{M}$ status of the edges of the path, a new matching $M^{\prime}$ results with $\left|\dot{M}^{\prime}\right|=|M|+1$. Hence, the existence

[^0]of an $M$-augmenting path implies $M$ is not maximum. What is not so obvious is that the non-existence of an $M$-augmenting path implies $M$ is maximum. This result, first obtained by Berge [1] means that the non-existence of an augmenting path is a condition which can be used to find a maximum matching. Interest in an efficient algorithm stems from several interesting practical problems which can be formulated as optimum matching or cover problems (see [2], [3], [4, p. 177] for details).

An analogous situation holds for minimum covers. A vertex $\mathbf{v}$ is $\boldsymbol{S}$-doubled for a subset $S$ of edges if v belongs to at least two edges of $S$. A C-reducing path for a cover $C$ is a $C$-alternating path whose end edges are in $C$ and whose end vertices are $C$-doubled. Once again a $C$-reducing path leads to a new cover $C^{\prime}$ with $\left|C^{\prime}\right|=$ $|C|-1$. Furthermore, Norman and Rabin [2] have shown that the non-existence of a $C$-reducing path implies $C$ is minimum, leading to an algorithm for finding a minimum cover. They also show that a maximum matching $M^{*}$ can be obtained from a minimum cover $C^{*}$ by dzleting all but one $C^{*}$-edge from each $C^{*}$-doubled vertex. Adding an edge to $M^{*}$ to cover each $M^{*}$-exposed vertex of a maximum matching $M^{*}$ produces a minimum cover $C^{*}$. Edmonds [3] has generalized the theorems of Berge [1] and Norman-Rabin [2] by replacing edges (2-element subsets) with more general subsets of vertices whereupon the appropriate improvement structures become trees in a certain graph.
2. Packings and covers. Here we are concerned with a different generalization of the matching and cover problems. Let $\delta$ be a function which assigns a nonnegative integer to each vertex $\mathbf{v}$ of $G$. If $d(v)$, the local degree of $\mathbf{v}$, is the number of edges to which $\mathbf{v}$ belongs and $\delta(\mathbf{v}) \leqq d(\mathbf{v})$ for all $\mathbf{v}$ in $V$ then $\delta$ is called a local degree constraint on $G$. A $\delta$-packing in $G$ is a subset $P$ of edges such that each vertex $\mathbf{v}$ in $V$ belongs to, at most, $\delta(\mathbf{v})$ edges in $P$. A $\delta$-cover $C$ is a subset of edges, such that each vertex $\mathbf{v}$ belongs to at least $\delta(\mathbf{v})$ edges in $C$. In this terminology, a matching is a 1 -packing (i.e., a $\delta$-packing with $\delta \equiv 1$ ) and a cover is a 1 -cover. Optimum $\delta$-packings and $\delta$-covers are defined in the obvious way.

There is a strong duality between $\delta$-packings and $\delta$-covers which does not exist between matchings and covers. If $\delta$ is a local degree constraint on $G$, then $\delta=d-\delta$ is also a local degree constraint on $G$ and we say $\delta$ and $\delta$ are complementary. It is not hard to see that a subset $S$ of edges of $G$ is a $\delta$-packing if and only if $S$ is a $\delta$-c $J$ ver. Let $\mathbf{v}$ be a vertex and $S$ be a subset of edges; then $\mathbf{v}$ is $(S, \delta)$-deficient; if $\mathbf{v}$ belongs to less than $\delta(\mathbf{v})$ edges in $S$ and is $(S, \delta)$-surfeited if $\mathbf{v}$ belongs to more than $\delta(\mathbf{v})$ edges in $S$. A $(P, \delta)$-augmenting path for $\delta$-packing $P$ is a $P$-alternating path whose end edges are in $\bar{P}$ and whose end vertices are ( $P, \delta$ )-deficient a ( $C, \delta$ )- reducing path for $\delta$-cover $C$ is a $C$-alternating path whose end edges are in $C$ and whose end vertices are ( $C, \delta$ )-surfeited. In case the end vertices of an augmenting (reducing) path are identical, the deficiency (surfeit) is required to be at least two. Now the duality means that path $\pi$ in $G$ is $(P, \delta)$-augmenting for $\delta$-packing
$P$ if and only if $\pi$ is $(\bar{P}, \delta)$-reducing for $\bar{\delta}$-cover $\bar{P}$. It is also clear that $P$ is a maximum $\delta$-packing if and only if $C=\bar{P}$ is a minimum $\bar{\delta}$-cover.

The theorem of Berge-Norman-Rabin proved by Berge [4, p. 175] asserts that the non-existence of a ( $P, \delta$ )-augmenting path for $\delta$-packing $P$ implies $P$ is maximum. Using duality, we see immediately that non-existence of a ( $C, \delta$ )-reducing path for $\delta$-cover $C$ insures $C$ is minimum. Goldman [5] has proved the B-N-R theorem on augmentable $\delta$-packings by a direct reduction to the theorem of Berge [1] on augmentable matchings (1-packings). The main result of this paper is a simple direct proof of the B-N-R theorem using ideas from Edmonds' simple proof [6] of the original augmenting path theorem of Berge [1] and the notion of Euler path [7] which dates back to 1736.

As regards design of algorithms, Edmonds [6] has found an exceptionally efficient algorithm for determining a maximum matching by growing alternating-path trees and occasionally shrinking odd-length cyclic paths until an augmenting path is discovered or the edges of the graph have been depleted. Witzgall and Zahn [8] have devised a modified version of the Edmonds algorithm which does not shrink and Edmonds [9] has extended his algorithm to the case where edges have realvalued weights and maximum is defined accordingly.

The reader is referred to Berge [4] and Ore [10] for more leisurely discussions of matchings and coverings in graphs. Alternating paths were invented by Petersen [11] in the last century and augmenting paths for $\delta$-packings occur in Tutte's [12] paper on $f$-factors (perfect $f$-packings) in a graph.
3. Euler paths. One of the earliest problems in graph theory was posed and solved by Euler [7] in the year 1736. The problem is known as the"Königsberg bridge problem" and intrigued the inhabitants of this Prussian town until solved by Euler. We quote Euler's [7] statement of the problem:

In the town of Königsberg in Prussia there is an island A, called "Kneiphof", with the two branches of the river (Pregel) flowing around it, as shown in Figure 1 . There are seven bridges, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ and $\mathbf{g}$, crossing the two branches. The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once.
Euler recognized the combinatorial nature of the problem and his solution can be phrased in terms of the graph in Figure 1. An Euler path in a graph $G$ is a path containing each edge of $G$ exactly once. Euler showed that such a path is possible if and only if no more than 2 vertices of $G$ have odd local degree and $G$ is connected (an obviously necessary condition). Hence, the answer to the Königsberg bridge problem is negative, there being 4 odd vertices. If the graph has exactly two odd vertices $\mathbf{a}$ and $\mathbf{b}$, then an Euler path must have $\mathbf{a}$ and $\mathbf{b}$ as end vertices. If no vertices are odd, the graph contains a closed Euler path.

A constructive, algorithmic demonstration of the Euler path result borrowing from [4, p. 165] and [10, p. 39] is as follows: Suppose graph $G$ has exactly two


Fig. 1
odd vertices a and $\mathbf{b}$. Start growing a path $\pi_{1}$ at vertex $\mathbf{a}$ and continue it as far as possible without repeating any edges. This path cannot get stuck at an even vertex because each time a vertex is crossed by the path two of its edges are used; furthermore, the path cannot stop at a because the first edge of the path is adjacent to vertex a leaving an even number of unused edges for subsequent crossings through a. Therefore, path $\pi_{1}$ stops at the only other odd vertex, b. If $\pi_{1}$ includes all edges of $G$, then it is the required Euler path. If not, we delete the edges of $\pi_{1}$ from $G$ and obtain a new graph $G^{\prime}$ whose local degrees are all even since $\pi_{1}$ meets each vertex through an even number (possibly zero) of edges with the exception of $\mathbf{a}$ and $\mathbf{b}$. The connectedness of $G$ implies there is a vertex $\mathbf{c}$ on $\pi_{1}$ which is contained in an edge of $G^{\prime}$. We construct a path $\pi_{2}$ in $G^{\prime}$ starting at $\mathbf{c}$, which must return to $\mathbf{c}$ because all local degrees of $G^{\prime}$ are even. Enlarge $\pi_{1}$ to the "spliced" path $\pi_{1}(\mathbf{a}, \mathbf{c}) / \pi_{2} / \pi_{1}(\mathbf{c}, \mathbf{b})$ and repeat the process until $\pi_{1}$ contains all edges of $G$. It can be shown by a similar argument [10, p. 40] that a graph with $2 N>0$ odd vertices can be covered by exactly $N$ paths. More detailed discussions on Euler paths can be found in Ore [10] and Berge [4].
4. Edmonds' lemma. Edmonds [6, p. 453] gave a one-sentence proof of the theorem of Berge [1] based on the following lemma:

Lemma E. Let $M_{1}$ and $M_{2}$ be matchings in graph $G$ and let $M_{1}+M_{2}$ denote the set of edges in $M_{1}$ or $M_{2}$ but not both. Then the subgraph $G_{12}$ formed by $M_{1}+M_{2}$ has connected components which are paths and circuits, each of which is $M_{1^{-}}$ alternating as well as $M_{2}$-alternating. Each end vertex of these paths is either $M_{1}$-exposed or $M_{2}$-exposed.

Proof. No vertex of $G_{12}$ has local degree greater than two since a vertex can meet at most one $M_{1}$ edge and one $M_{2}$ edge. Hence, the graph $G_{12}$ consists entirely of paths and circuits. Let $a$ be an end vertex of one of the paths in $G_{12}$ and suppose the adjacent end edge belongs to $M_{1} \cap \bar{M}_{2}$. Since $M_{1}$ is a matching, $\boldsymbol{a}$ is not adjacent
to any other $M_{1}$ edge. Any $M_{2}$ edge adjacent to $a$ would therefore be in $M_{1}+M_{2}$ contradicting the fact that $a$ is an end vertex of $G_{12}$. We conclude that $a$ is $M_{2^{-}}$ exposed.

To prove a non-maximum matching $M$ contains an $M$-augmenting path is now a matter of simple arithmetic! If $M^{\prime}$ is a matching larger than $M$, then some component of the subgraph $M+M^{\prime}$ must contain more $M^{\prime}$ edges than $M$ edges implying the end edges are in $\bar{M}$ and $M$-exposed.

The natural decomposition of $M_{1}+M_{2}$ into alternating paths depends heavily on the special "oneness' of matchings (1-packings). An analogous result for general $\delta$-packings requires some extra device for generating the alternating paths. The Euler paths of Section 3 supply an adequate mechanism for this purpose.
5. Alternating Euler paths. We begin this section by applying the Euler path idea to prove a generalization of Edmonds' Lemma E. We have used the notation $d(\mathbf{v})$ to represent the local degree of vertex $\mathbf{v}$ in graph $G$. In what follows, we shall be concarned with the various local degrees of a single vertex $v$ in various subgraphs $\mathrm{H}_{i}$ of $G$ and we use $d\left(\mathbf{v}, \mathrm{H}_{i}\right)$ to denote the number of edges of $\mathrm{H}_{i}$ which contain $\mathbf{v}$.

Lemma EEZ. Let $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$ be subgraphs of $G$ which have no common edges. Then the subgraph $G_{12}$ generated by $\mathrm{H}_{1} \cup \mathrm{H}_{2}$ can be decomposed into an edgewise disjoint family of paths whose edges alternate between $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$, such that each path is one of the following types:
(1) Closed paths of even length.
(2) Closed paths of odd length such that the unique vertex $\mathbf{v}$ incident to two adjacent edges of the same $\mathrm{H}_{i}$ satisfies $d\left(\mathbf{v}, \mathrm{H}_{i}\right)-d\left(\mathbf{v}, \mathrm{H}_{j}\right) \geqq 2$ where $j \neq i$.
(3) Non-closed paths such that if $e$ is an end edge in $\mathrm{H}_{i}$ containing end vertex $\mathbf{v}$, then $d\left(\mathbf{v}, \mathrm{H}_{i}\right)-d\left(\mathbf{v}, \mathrm{H}_{j}\right) \geqq 1$, where $j \neq i$.

Proof. Let $\Delta_{12}(\mathbf{v})=d\left(\mathbf{v}, \mathrm{H}_{1}\right)-d\left(\mathbf{v}, \mathrm{H}_{2}\right)$ for all vertices in $G_{12}$ and call vertex $\mathbf{v}$ batanced, positive or negative ascording as $\Delta_{12}(v)$ is zero, positive or negative. If all vertices of $G_{12}$ are balanced, then each connected component of $G_{12}$ enjoys the same property and hence contains an Euler circuit (à closed Euler path). Furthermore, since $d\left(\mathbf{v}, \mathrm{H}_{1}\right)=d\left(\mathbf{v}, \mathrm{H}_{2}\right)$ for each vertex, the Euler path can be chosen to alternate between edges of $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$, and so becomes a path of type 1 .

If $G_{12}$ contains unbalanced vertices, let $\mathbf{v}_{1}$ be one such and, for convenience, suppose it to be positive. The argument is similar for negative vertices.

Since $\mathbf{v}_{1}$ is positive $d\left(\mathbf{v}_{1}, \mathrm{H}_{1}\right)-d\left(\mathbf{v}_{1}, \mathrm{H}_{2}\right) \geqq 1$ and, therefore, $d\left(\mathbf{v}_{1}, \mathrm{H}_{1}\right) \geqq 1$. Let $\mathbf{e}_{1}$ be one of the edges of $H_{1}$ adjacent to $\mathbf{v}_{1}$ and let $\mathbf{v}_{2}$ be the other vertex of $\mathbf{e}_{1}$. We select an eige $e_{2}$ among the $H_{2}$ elges at $v_{2}$ and add it to the path (if there exists such an eige). This prosess is continued as long as the path alternates between $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$ and does not use the same edge twice. The finiteness of the graph ensures termination; this can happen in several ways.

If the path terminates at $\mathbf{v}_{t}$ via edge $\mathrm{e}_{2 p}$ in $\mathrm{H}_{2}$, then $\mathbf{v}_{t} \neq \mathbf{v}_{1}$ since the positiveness
of $\mathbf{v}_{1}$ ensures that every time we enter that vertex via an $\mathrm{H}_{2}$ edge, there will be an unused $H_{1}$ edge available for exit. Since the path uses up equal numbers of $H_{1}$ and $\mathrm{H}_{2}$ edges at each vertex it crosses (except $\mathbf{v}_{1}$ and $\mathbf{v}_{t}$ ), we can be forced to stop at $\mathbf{v}_{t}$ after edge $\mathbf{e}_{2 p}$ in $\mathrm{H}_{2}$ only if $\mathbf{v}_{t}$ is negative. In this case, the path $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}, \cdots, \mathbf{e}_{2 p}\right\}$ is of type 3. Deleting this path from $G_{12}$ produces a new graph $G_{12}^{\prime}$ in which the path tracing can be resumed. Had $\mathbf{v}_{1}$ been a negative vertex, the even length path would have ended at a positive vertex $\mathbf{v}_{t}$. In either case, the deletion of such a path may create some balanced vertices but never any positive or negative ones.

If the path from positive $\mathbf{v}_{1}$ ends at $\mathbf{v}_{t}$ via edge $e_{2 p+1}$ in $H_{1}$, then either $\mathbf{v}_{t}=\mathbf{v}_{1}$ and $\Delta_{12}\left(\mathbf{v}_{1}\right) \geqq 2$ or else $\boldsymbol{v}_{t} \neq \mathbf{v}_{1}$ and $\mathbf{v}_{t}$ is positive. The first case gives a path of type 2 and the second case one of type 3 . Similar results hold if $\mathbf{v}_{1}$ is negative and once again the paths can be deleted and the path tracing resumed in the reduced graph. When we arrive at a reduced graph with no unbalanced vertices, we decompose each connected component into a path of type 2, as indicated earlier in the proof. The path tracing must terminate for lack of edges or unbalanced vertices so the lemma is proved.

We call the paths of Lemma EEZ alternating Euler paths.
Corollary 1. If $\alpha_{1}$ is the number of odd-length paths (possibly closed) with more $\mathrm{H}_{1}$ edges than $\mathrm{H}_{2}$, and similarly for $\alpha_{2}$, then

$$
\alpha_{1}-\alpha_{2}=\left|H_{1}\right|-\left|H_{2}\right|=\frac{1}{2} \sum_{v \in G_{12}} \Delta_{12}(v) .
$$

Proof. Even-length alternating paths have equal numbers of edges from $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$ and so contribute nothing to the expression $\left|\mathrm{H}_{1}\right|-\left|\mathrm{H}_{2}\right|$. Each odd-length alternating path has exactly one more $\mathrm{H}_{1}$ edge than it has $\mathrm{H}_{2}$ edges or vice versa thereby contributing $\pm 1$ to $\left|\mathrm{H}_{1}\right|-\left|\mathrm{H}_{2}\right|$. Because each edge is counted twice when local vertex degrees are summed over all vertices

$$
\sum_{\mathbf{v} \in G_{12}} \Delta_{12}(\mathbf{v})=\sum_{\mathbf{v} \in G_{12}} d\left(\mathbf{v}, \mathrm{H}_{1}\right)-\sum_{\mathbf{v} \in G_{12}} d\left(\mathbf{v}, \mathrm{H}_{2}\right)=2\left|\mathrm{H}_{1}\right|-2\left|\mathrm{H}_{2}\right| .
$$

Corollary 2. Let $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$ be as in Lemma EEZ and let $\left|\mathrm{H}_{1}\right|>\left|\mathrm{H}_{2}\right|$. Then $G_{12}$ contains a path $\pi$ which alternates between $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$ has end edges in $\mathrm{H}_{1}$ and end vertices $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ satisfying one of the following conditions:
(1) $\Delta_{12}\left(\mathbf{v}_{i}\right) \geqq 1$ for $i=1,2$ if $\mathbf{v}_{1} \neq \mathbf{v}_{2}$.
(2) $\Delta_{12}\left(\mathbf{v}_{i}\right) \geqq 2$ if $\mathbf{v}_{1}=\mathbf{v}_{2}$.

Proof. By Corollary 1 we get $\alpha_{1}-\alpha_{2}=\left|\mathrm{H}_{1}\right|-\left|\mathrm{H}_{2}\right| \geqq 1$ and hence $\alpha_{1} \geqq 1$. This assures the existence of an odd length path of type 2 or 3 with end edges in $\mathrm{H}_{1}$.
6. The theorem of Berge-Norman-Rabin. We can now give a simple proof of the Berge-Norman-Rabin theorem [4, p.175] using Lemma EEZ and Corollary 2.

Theorem BNR. If P is a non-maximum $\delta$-packing in graph $G$, then $G$ contains a ( $P, \delta$ )-augmenting path.

Proof. Let $P^{*}$ be a larger $\delta$-packing and put $\mathrm{H}_{1}=P^{*}-P$ and $\mathrm{H}_{2}=P-P^{*}$. Applying Lemma EEZ and Corollary 2 (since $\left|\mathrm{H}_{1}\right|>\left|\mathrm{H}_{2}\right|$ ), we get a path which alternates edges of $P$ and $\bar{P}$ (i.e., $P$-alternating), has end edges in $\bar{P}$ and end vertices $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ each satisfying condition (1) or (2) of Corollary 2 . Since the edges in $P \cap P^{*}$ contribute to both terms of the expression $d\left(\mathbf{v}_{i}, P^{*}\right)-d\left(\mathbf{v}_{i}, P\right)$, we see easily that

$$
d\left(\mathbf{v}_{i}, P^{*}\right)-d\left(\mathbf{v}_{i}, P\right)=d\left(\mathbf{v}_{i}, \mathrm{H}_{1}\right)-d\left(\mathbf{v}_{i}, \mathrm{H}_{2}\right)
$$

Combining this with conditions (1) and (2) of Corollary 2 and the inequality $d\left(\mathbf{v}_{i}, P^{*}\right) \leqq \delta\left(\mathbf{v}_{i}\right)$, we find that

$$
\begin{aligned}
& d\left(\mathbf{v}_{i}, P\right) \leqq \delta\left(\mathbf{v}_{i}\right)-1 \quad \text { for } i=1,2 \text { if } \mathbf{v}_{1} \neq \mathbf{v}_{2} \\
& d\left(\mathbf{v}_{1}, P\right) \leqq \delta\left(\mathbf{v}_{1}\right)-2 \quad \text { if } \mathbf{v}_{1}=\mathbf{v}_{2}
\end{aligned}
$$

Hence, the path is ( $P, \delta$ )-augmenting.
7. Graphs with edge dichotomies. Any graph $G$ whose edge set $E$ has been dichotomized (i.e., partioned into two subsets) can be decomposed by Lemma EEZ into a family of edge-disjoint alternating Euler paths with fairly natural conditions on the end vertices. If $E=E_{1} \cup E_{2}$ is the dichotomy, let $H_{i}$ for $i=1,2$ be the subgraph of $G$ generated by the edge set $E_{i}$, and apply Lemma EEZ (in this case $G_{12}=G$ ). We separated Lemma EEZ from the proof of Theorem BNR because the alternating path decomposition is a general phenomenon not dependent on packings or covers or local degree constraints. The following corollaries strengthen Lemma EEZ somewhat:

Corollary 3. If graph $G_{12}$ in Lemma EEZ is connected, then it can be decomposed so that there is, at most, one path of type 1, and that only if it is the sole path covering all of $G_{12}$.

Proof. Because $G_{12}$ is connected, the even-length closed (type 1) paths can be "spliced" together or into other paths of types 2 or 3 . If at least one path of type 2 or 3 exists, then all the paths of type 1 can be made to disappear into one or more of the paths of types 2 or 3 . The splicing is similar to that used in section 2 for Euler paths. Clearly, at least one path is required, so a single type 1 path is possible.

To characterize further the alternating path decompositions, we need some additional terminology. Let $\alpha_{0}$ be the number of even-length paths of type 3 and let $\alpha_{i}$ for $i=1$ and 2 be, as before, the number of odd-length paths of types 2 and 3 with more $H_{i}$ edges. It is then obvious that the number of paths of type 2 or 3 is exactly $\left(\alpha_{0}+\alpha_{1}+\alpha_{2}\right)$. We call $\Delta^{T}=\Sigma\left|\Delta_{12}(\mathrm{v})\right|$ the total vertex imbalance for dichotomy $\left(\mathrm{H}_{1}, \mathrm{H}_{2}\right)$.

COROLLARy 4. The path decomposition of a connected $G_{12}$ as presented in Corollary 3, is minimum in the sense that no other representation of $G_{12}$ as a family of alternating paths has fewer paths. Furthermore, the number of paths of type 2 or

3 is related directly to the vertex differentials $\Delta_{12}(\mathrm{v})$ by

$$
\alpha_{0}+\alpha_{1}+\alpha_{2}=\frac{1}{2} \sum_{v \in G_{12}}\left|\Delta_{12}(v)\right|=\Delta^{T} / 2
$$

Proof. First we show that $\Delta^{T} / 2$ is a lower bound for the number of paths in an alternating family. Let $\mathscr{F}$ be a family of alternating paths for $G_{12}$ and consider a single vertex $v$ with differential $\Delta_{12}(v)>0$. The edges of $H_{1}$ and $H_{2}$ incident to $v$ can be paired off except for exactly $\left|\Delta_{12}(v)\right|$ extra $\mathrm{H}_{1}$ edges. Each pair of edges corresponds to the occurrence of $\mathbf{v}$ as an internal vertex of an alternating path of $\mathscr{F}$. Each extra edge must represent the occurrence of $v$ as an end vertex. An identical argument holds for $\Delta_{12}(v)<0$. The paths of any alternating path decomposition must hence account for at least $\Delta^{T}$ end vertex occurrences, but each path can handle, at most 2 so $\Delta^{T} / 2$ is indeed a lower bound. In the proof of Lemma EEZ, paths of type 2 or 3 are constructed only between end vertices which are currently unbalanced and each path deletion decreases the total vertex imbalance by 2 units. The construction of type 2 and type 3 paths terminates when the total vertex imbalance is reduced to zero so the total number of such paths is precisely $\Delta^{T} / 2$. This establishes the formula and the minimality follows because our particular decomposition achieves the lower bound.

Let us call $\Delta^{\Sigma}=\Sigma \Delta_{12}(v)$ the net vertex imbulance. It is then tempting to ask if a decomposition of $G_{12}$ into alternating paths can be accomplished with $\alpha_{1}+\alpha_{2}=\left|\Delta^{\Sigma}\right| / 2$, it being clear that $\alpha_{1}+\alpha_{2} \geqq\left|\Delta^{\Sigma}\right| / 2$. If equality does hold, then either $\alpha_{1}=\Delta^{\Sigma} / 2$ while $\alpha_{2}=0$, or else $\dot{x}_{2}=-\Delta^{\Sigma} / 2$ while $\alpha_{1}=0$. Figure 2 depicts a simple dichotomized graph with $\Delta^{\Sigma}=0$ which requires $\alpha_{1}+\alpha_{2} \geqq 2$.


Fig. 2
On the other hand, this seems to result from the lack of connectedness between positive and negative vertices so equality may be achievable under some sort of multiple connectedness assumption. In any case, it would be interesting to find a decomposition with minimum value of $\alpha_{1}+\alpha_{2}$ and know how the minimum relates to the structure of graph $G_{12}$.
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