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AN ANGULAR MOMENTUM REDUCTION FOR PHYSICAL AMPLITUDES IN THE THREE BODY PROBLEM*

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ABSTRACT

We give here an approach to the angular momentum reduction that is tailored to be appropriate for the multichannel structure of the three-body problem. Where ever possible we work directly with the physical scattering amplitudes. We obtain concrete partial wave expansions of elastic, rearrangement and breakup amplitudes. For these amplitudes we obtain a coupled two-variable integral equation. The effects of parity, time reversal and rotational invariance are fully discussed. Finally, we provide expressions for the multichannel partial-wave cross sections, asymptotic coordinate-space wave functions, off-shell unitarity and the partial-wave version of the optical theorem as well as phase-shift parametrizations of the amplitudes.

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1. INTRODUCTION

We present a new angular momentum analysis of Faddeev's[1] three-body equations. This study is tailored to be appropriate for the multichannel scattering aspects of the three-body problem. The underlying philosophy of our approach is to always work directly with the various observable three-body amplitudes. Our aim is to obtain simple and physically transparent angular momentum representations for these observable amplitudes.

The physical problem assumed in Faddeev's work is that of three spinless nonrelativistic particles interacting via potentials. In the following analysis we shall restrict this problem by adding two further assumptions. First, we assume that the pairwise two-body potentials are spherically symmetric. Secondly, we assume that each two-body channel has only one bound state, which is an s-wave state. However, our general method is not limited to the case where the above restrictions apply. We have assumed spherical symmetry because this condition is both necessary and sufficient for the conservation of angular momentum. The second additional assumption, of s-wave two-body bound states, is chosen primarily in order to simplify the presentation of our results.

Let us now pose in a detailed manner the problem we will solve. To understand the difficulties of an angular momentum analysis in the scattering region we recount the salient features of Faddeev's treatment. Faddeev's time independent account begins with the definition of a partial transition amplitude given by

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}V_\alpha - V_\alpha G(z) V_\beta \qquad \alpha, \beta = 1, 2, 3 \tag{1.1}$$

The quantities in this equation are operators in the three-body center-of-mass momentum space. The pair-wise potential between β and γ is represented by

V_α and $G(z) = (H-z)^{-1}$ is the Green's function at a complex energy z for the fully interacting Hamiltonian $H = H_0 + \sum_\alpha V_\alpha$. The 3-to-3 transition amplitude would be just the sum over α and β of the partial transition amplitudes $M_{\alpha\beta}$. As is well known[2] the $M_{\alpha\beta}$ satisfy coupled linear equations of the form

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}T_\alpha(z) - \sum_{\gamma\neq\alpha} T_\alpha(z)G_0(z)M_{\gamma\beta}(z)$$

$$= \delta_{\alpha\beta}T_\alpha(z) - \sum_{\gamma\neq\beta} M_{\alpha\gamma}(z)G_0(z)T_\beta(z) \qquad (1.2)$$

The Green's function appearing here, $G_0(z)$, is the resolvent for the free Hamiltonian, H_0. The operator, $T_\alpha(z)$, is the two-body t-matrix realized in the three-body Hilbert space and is determined by its kernel representation,

$$T_\alpha(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\alpha, \vec{q}'_\alpha; z) = \delta^3(\vec{p}_\alpha - \vec{p}'_\alpha)\, t_\alpha(\vec{q}_\alpha; \vec{q}'_\alpha; z - \vec{p}_\alpha^{\,2}/2n_\alpha). \qquad (1.3)$$

In this expression the, $\vec{p}_\alpha$, is the individual particle momentum of particle α in the three-particle center of mass system. The momentum, $\vec{q}_\alpha$, is the relative momentum of the constituents of the cluster $\beta\gamma$ i.e., $\vec{q}_\alpha = (m_\gamma\vec{p}_\beta - m_\beta\vec{p}_\gamma)/(m_\beta + m_\gamma)$. These momenta $\vec{p}_\alpha$, $\vec{q}_\alpha$ are the conjugates to the coordinate-space Jacobi variables. (See the appendix of Ref. 4 for a detailed account of these coordinates.) The kernel t_α is the off-shell two-body transition amplitude with an energy argument $z - \vec{p}_\alpha^{\,2}/2n_\alpha$. The factor $\vec{p}_\alpha^{\,2}/2n_\alpha$ is the kinetic energy of the relative motion of the particle α and the cluster $\beta\gamma$, and $n_\alpha = m_\alpha(m_\beta + m_\gamma)/(m_\alpha + m_\beta + m_\gamma)$ is the appropriate reduced mass.

A central feature in Faddeev's method is to describe the amplitudes, $M_{\alpha\beta}$, in terms of a primary pole decomposition. The t-matrices, $T_\alpha(z)$, in Eq. (1.2) have poles in the energy arguments which are caused by the two-body bound state in channel α. By iterating, Eq. (1.2), one may find all the singularities

in $M_{\alpha\beta}$. The primary pole representation so obtained for $M_{\alpha\beta}$ can be written[1,3]

$$M_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta, \vec{q}'_\beta; z) = \delta_{\alpha\beta}\delta(\vec{p}_\alpha - \vec{p}'_\alpha)\, t_\alpha(\vec{q}_\alpha, \vec{q}'_\beta; z - \tilde{p}_\alpha^2)$$

$$+ \mathscr{F}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}_\beta, \vec{q}'_\beta; z) + \frac{\mathscr{G}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}_\beta; z)\, \phi^*_\beta(\vec{q}'_\beta)}{z + \chi_\beta^2 - \tilde{p}'^2_\beta}$$

$$+ \frac{\phi_\alpha(\vec{q}_\alpha)\tilde{\mathscr{G}}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta, \vec{q}'_\beta; z)}{z + \chi_\alpha^2 - \tilde{p}_\alpha^2} + \frac{\phi_\alpha(\vec{q}_\alpha)\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; z)\, \phi^*_\beta(\vec{q}'_\beta)}{(z + \chi_\alpha^2 - \tilde{p}_\alpha^2)(z + \chi_\beta^2 - \tilde{p}'^2_\beta)} \tag{1.4}$$

On the right hand side of expression (1.4), the function ϕ_α is the "vertex function" related to the normalized two-body bound state wave function, ψ_α, of binding energy $-\chi_\alpha^2$, in the following way:

$$\phi_\alpha(\vec{q}_\alpha) = (\tilde{q}_\alpha^2 + \chi_\alpha^2)\, \psi_\alpha(\vec{q}_\alpha) \tag{1.5}$$

In both Eqs. (1.4) and (1.5) we introduce an abbreviated notation for the kinetic energies, viz.

$$\tilde{p}_\alpha^2 = \vec{p}_\alpha^{\,2}/2n_\alpha, \qquad \tilde{q}_\alpha^2 = \vec{q}_\alpha^{\,2}/2\mu_\alpha, \tag{1.6}$$

where

$$\mu_\alpha = m_\beta m_\gamma/(m_\beta + m_\gamma)\,.$$

For scattering from a bound state the residue functions $\mathscr{F}_{\alpha\beta}$, $\mathscr{G}_{\alpha\beta}$, $\tilde{\mathscr{G}}_{\alpha\beta}$, $\mathscr{H}_{\alpha\beta}$ are all singularity free kernels. Furthermore, $\mathscr{H}_{\alpha\beta}$ has the interpretation of the transition amplitude for the two-body like scattering of channel β into channel α.[3] In fact in Ref. 4 it is proved that the cross-section for elastic or

rearrangement scattering is,

$$\frac{d\sigma_{\alpha\beta}(\hat{x}_\alpha)}{d\Omega_{\hat{x}_\alpha}} = (2\pi)^4 n_\alpha n_\beta \frac{p^f_\alpha}{p'_\beta} \left|\mathscr{H}_{\alpha\beta}(p^f_\alpha \hat{x}_\alpha; \vec{p}'_\beta; E + i0)\right|^2 \tag{1.7}$$

This cross-section is for the physical process where particle β is incident with momentum $\vec{p}'_\beta$ onto the cluster $\alpha\gamma$ and the final state is characterized by the cluster $\beta\gamma$ and the free particle α. The momentum available to α, p_α, in the final state is just

$$p^f_\alpha = \left[2n_\alpha(\tilde{p}'^2_\beta - \chi^2_\beta + \chi^2_\alpha)\right]^{1/2} . \tag{1.8}$$

The spatial coordinate, $\vec{x}_\alpha$, in our formula is the vector separation of the position of the center-of-mass of the $\beta\gamma$ cluster and the position of particle α. The variable E denotes the total three-body energy available for this scattering.

If we examine breakup scattering, then a linear combination of $\mathscr{G}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ determine the breakup cross section. Define $\mathscr{K}_{\alpha\beta}$ to be

$$\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) = \mathscr{G}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) - \frac{\phi_\alpha(\vec{q}_\alpha)\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; z)}{\tilde{p}^2_\alpha - \chi^2_\alpha - z} \tag{1.9}$$

Then the cross section for scattering from the incident channel β with momentum $\vec{p}'_\beta$ into a final state of three free particles is[4]

$$\frac{d\sigma_{0\beta}(\hat{x}_\beta, \hat{y}_\beta, q_\beta)}{d\Omega_{\hat{x}_\beta}\, d\Omega_{\hat{y}_\beta}\, q^2_\beta dq_\beta} = (2\pi)^4 n^2_\beta \frac{p^f_\beta}{p'_\beta} \left|\mathscr{B}_{0\beta}(p^f_\beta \hat{x}_\beta, q_\beta \hat{y}_\beta; \vec{p}'_\beta; E + i0)\right|^2 \tag{1.10}$$

where,

$$\mathscr{B}_{0\beta}(\vec{p}, \vec{q}; \vec{p}'_\beta; z) = -\sum_{\alpha=1}^{3} \mathscr{K}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) , \tag{1.11}$$

and,

$$p^f_\beta = \left[2n_\beta(\tilde{p}'^2_\beta - \chi^2_\beta - \tilde{q}^2_\beta)\right]^{1/2} . \tag{1.12}$$

The variables $\hat{x}_\beta$, $\hat{y}_\beta$, q_β are five independent variables which completely specify the final state. The spatial separation of α and γ is given by $\vec{y}_\beta$, and $\vec{q}_\beta$ is the internal momentum of the pair $\alpha\gamma$. Once q_β is given then p^f_β represents the momentum that energy conservation permits particle β to have.

The cross section formula (1.7), (1.9) and (1.10) indicate that the functions $\mathscr{H}_{\alpha\beta}$ and $\mathscr{G}_{\alpha\beta}$ may be thought of as physical observables in the three-body problem. This point is reinforced when one examines the multichannel S matrices. There the functions which appear are just $\mathscr{H}_{\alpha\beta}$ and $\mathscr{K}_{\alpha\beta}$.[1,3] Thus it is desirable to work directly with the functions $\mathscr{H}_{\alpha\beta}$ and $\mathscr{G}_{\alpha\beta}$, because their behavior immediately affects the behavior of the cross sections. Furthermore, given $\mathscr{H}_{\alpha\beta}$ and $\mathscr{G}_{\alpha\beta}$ the exact scattering wave function is completely specified.[4]

Faddeev completes the description of the time independent scattering problem by giving an integral equation for $\mathscr{H}_{\alpha\beta}$ and $\mathscr{G}_{\alpha\beta}$. This equation is a consequence of the relations (1.2) and has the form[3]

$$\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; z) = \mathscr{H}^0_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; z) - \sum_{\delta\neq\alpha}\int \frac{\phi^*_\alpha(\vec{q}''_\alpha)\,\delta(\vec{p}_\alpha - \vec{p}''_\alpha)}{\tilde{p}''^2_\alpha + \tilde{q}''^2_\alpha - z}\,\mathscr{K}_{\delta\beta}(\vec{p}''_\delta, \vec{q}''_\delta; \vec{p}'_\beta; z)\, d\vec{p}''_\delta\, d\vec{q}''_\delta$$

$$\mathscr{G}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) = \mathscr{G}^0_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) - \sum_{\delta\neq\alpha}\int \frac{\hat{t}_\alpha(\vec{q}_\alpha, \vec{q}''_\alpha; z - \tilde{p}''^2_\alpha)\,\delta(\vec{p}_\alpha - \vec{p}''_\alpha)}{\tilde{p}''^2_\alpha + \tilde{q}''^2_\alpha - z}\,\mathscr{K}_{\delta\beta}(\vec{p}''_\delta, \vec{q}''_\delta; \vec{p}'_\beta; z)\, d\vec{p}''_\delta\, d\vec{q}''_\delta \tag{1.13}$$

The function $\hat{t}_\alpha$ in the kernel of the second Eq. (1.13) is the non-pole portion of the two-body t matrix defined as,

$$t_\alpha(\vec{q}_\alpha, \vec{q}'_\alpha; z) = \frac{\phi_\alpha(\vec{q}_\alpha)\,\phi^*_\alpha(\vec{q}'_\alpha)}{z + \chi^2_\alpha} + \hat{t}_\alpha(\vec{q}_\alpha, \vec{q}'_\alpha; z) \tag{1.14}$$

The terms $\mathscr{H}^{o}_{\alpha\beta}$, $\mathscr{G}^{o}_{\alpha\beta}$ denote the driving terms of the integral equation. Specifically they are,

$$\mathscr{H}^{o}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; z) = \frac{-\delta_{\alpha\beta}\,\phi^*_\alpha(\vec{p}_\alpha, \vec{p}'_\beta)\,\phi_\beta(\vec{p}_\alpha, \vec{p}'_\beta)}{\dfrac{p_\alpha^2}{2\mu_\beta} + \dfrac{\vec{p}_\alpha \cdot \vec{p}'_\beta}{m_\gamma} + \dfrac{p'^2_\beta}{2\mu_\alpha} - z}\,, \tag{1.15}$$

and,

$$\mathscr{G}^{o}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta; z) = \frac{-\bar{\delta}_{\alpha\beta}\,\hat{t}_\alpha(\vec{q}_\alpha, \vec{q}_\alpha(\vec{p}_\alpha, \vec{p}'_\beta); z - \tilde{p}^2_\alpha)\,\phi_\beta(\vec{p}_\alpha, \vec{p}'_\beta)}{\dfrac{p_\alpha^2}{2\mu_\beta} + \dfrac{\vec{p}_\alpha \cdot \vec{p}'_\beta}{m_\gamma} + \dfrac{\vec{p}'^2_\beta}{2\mu_\alpha} - z} \tag{1.16}$$

In these expressions, the $\vec{p}_\alpha$, $\vec{p}'_\beta$ argument of ϕ_α or ϕ_β denotes the vector $\vec{q}_\alpha$ or $\vec{q}_\beta$ determined by the vectors $\vec{p}_\alpha$, $\vec{p}'_\beta$ $(\alpha \neq \beta)$. The symbol $\bar{\delta}_{\alpha\beta}$ denotes $(1 - \delta_{\alpha\beta})$. We note in passing, that Faddeev's demonstration of the uniqueness of three-body wave functions utilized Eq. (1.13). When set in an appropriately structured Banach space the kernel of Eq. (1.13) can be shown to be compact. This ensures that Eq. (1.13) is Fredholm. Employing the Fredholm alternative then leads to uniqueness.

With the fundamental structure of the three-body scattering problem set forth in Eqs. (1.4 — 1.13) one can list the features a physically interesting angular momentum analysis should possess. First, one should obtain simple expansions for the observable quantities $\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta)$, $\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha\, \vec{q}_\alpha; \vec{p}'_\beta)$. The expansions for the two-body like amplitude $\mathscr{H}_{\alpha\beta}$ should have the form of an expansion in $P_\ell(\hat{p}_\alpha \cdot \hat{p}'_\beta)$ and its coefficients should be defined in terms of multichannel phase shifts. Second, one needs to derive a well-behaved integral equation for the expansion coefficients of $\mathscr{H}_{\alpha\beta}$, $\mathscr{K}_{\alpha\beta}$. This equation should possess some of the features of (1.13). It needs to be written for the angular momentum reduced observables one obtains from $\mathscr{H}_{\alpha\beta}$ and $\mathscr{K}_{\alpha\beta}$ and it should

retain the compactness virtue that Eq. (1.13) has. Hopefully the resulting integral equation will be simple enough to be solved numerically. Finally, one should give formulae for all the partial-wave cross-sections as well as the relevant form of the optical theorem. All this is accomplished in the following sections.

Our general method is to introduce the angular momenta via the Omnes[5] approach. However we find that in order to meet the first requirement cited above we are forced to alter Omnes approach. Our modified Omnes equations are given in Section II. In Section III we obtain coupled two-variable equations, using a method similar to Ref. 6. These equations are the analog of Eq. (1.13). Section III also gives a complete account of the invariances that are caused by parity, rotation and time reversal. The last Section gives the partial-wave representations for cross sections, wave functions, the optical theorem, as well as the phase-shift parameterization of the amplitudes.

In closing we mention extant literature concerning angular momentum reductions of the three-body problem. A helpfull guide to the literature is the review article of El Baz et al.[7] This paper gives a comparative summary of most of the different conceptual approaches used. In Ref. 8 we have compiled all the angular reductions known to us. The method and results we present are in marked contrast to those given in Ref. 8. In particular our approach is distinctive in that we work directly with the observed multichannel amplitudes.

2. AN EULER ANGLE REDUCTION

The fundamental structure of any angular momentum reduction of a physical problem stems from the commutators of the angular momentum operators with the Hamiltonians which define the problem. It is assumed in the Introduction that the two-body potentials, v_α, are such that they commute with the α-channel

two-body angular momentum operator $\vec{\ell}_\alpha = \vec{y}_\alpha \times \vec{q}_\alpha$ — namely $[\vec{\ell}_\alpha, v_\alpha] = 0$. If we let $\vec{L}_\alpha = \vec{x}_\alpha \times \vec{p}_\alpha$, be the angular momentum of particle α relative to the cluster $\beta\gamma$ then the total angular momentum operator can be written as $\vec{J} = \vec{L}_\alpha + \vec{\ell}_\alpha$, $(\alpha = 1, 2, 3)$. Since the three-body pair wise potential, V_α, is

$$V_\alpha(\vec{x}_\alpha, \vec{y}_\alpha) = v_\alpha(\vec{y}_\alpha), \tag{2.1}$$

it follows from the above definitions that $[\vec{L}_\alpha, v_\alpha] = 0$, so $[\vec{J}, V_\alpha] = 0$. We may now conclude that $[\vec{J}, M_{\alpha\beta}(z)] = 0$, since $M_{\alpha\beta}(z)$ is a function of V_α and H_0. So all our representations of $M_{\alpha\beta}(z)$ will be diagonal in the eigen functions corresponding to $\vec{J}^2$ and J_z. We shall use this fact to simplify the notation in our representations.

Let us now define the manner in which we shall attach the Euler rotation to the vectors $\vec{p}_\alpha$ and $\vec{q}_\alpha$ which describe the three-body system. We have three equivalent but different momentum space coordinate systems: $(\vec{p}_1, \vec{q}_1)$, $(\vec{p}_2, \vec{q}_2)$, $(\vec{p}_3, \vec{q}_3)$. The center-of-mass condition implies that

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0. \tag{2.2}$$

Thus the three individual particle momentum lie in a plane. We use this fact to introduce the following angular convention depicted in Fig. 1. Our definition of the angle $\theta_{\beta\alpha}$ is the angle needed to take the direction of $\vec{p}_\alpha$ into the direction of $\vec{p}_\beta$ with a right hand positive rotation. For $\alpha\beta\gamma$ chosen to be ordered in a counter-clockwise fashion then $\theta_{\beta\alpha}$, $\theta_{\alpha\gamma}$, $\theta_{\gamma\beta}$ vary between 0 and π. The angle between $\vec{p}_\alpha$ and $\vec{q}_\alpha$ we will call Γ_α. This will be defined as the right hand rotation of $\hat{p}_\alpha$ into the direction of $\hat{q}_\alpha$. The angle Γ_α varies between 0 and 2π.

We shall denote by R a 3×3 Euler rotation matrix. Specifically

$$R = R(\psi, \theta, \phi) = R_z(\psi) R_y(\theta) R_z(\phi) \tag{2.3}$$

where R_z, R_y represent rotations about a spatially fixed z and y axis respectively. The angles ψ, θ, ϕ are the three independent Euler angles that uniquely determine a rotation. The convention we use for R_y is

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

with a similar form obtaining for $R_z(\psi)$. This representation is consistent with Rose's treatment[9] of the $\mathscr{D}$ functions that describe rotations of functions. In the following we shall employ Rose's conventions.

Now consider the basic amplitude $M_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta, \vec{q}'_\beta)$. Note that, we shall suppress the energy variable when it is of no importance to our arguments. The channel indices α and β on $M_{\alpha\beta}$ have a simple physical interpretation. The β indicates that the pair $\alpha\gamma$ interact through a potential V_β before anything else occurs. Similarly, α indicates that the last interaction is V_α. We shall define an angular representation so that the vector $\vec{p}_\alpha$, in the left set of variables for $M_{\alpha\beta}$, has a preferred simple description. For the right-hand set of variables we arrange our definitions so that $\vec{p}'_\beta$ has the simple description. So, we define

$$\hat{p}_\alpha = R_\alpha \hat{z} = R(\psi_\alpha, \theta_\alpha, \phi_\alpha)\,\hat{z}$$

$$\hat{q}_\alpha = R_\alpha R_y(\Gamma_\alpha)\,\hat{z} \tag{2.4}$$

where $\hat{z}$ is the unit vector pointing in the z direction of a spatially fixed coordinate system. Likewise

$$\hat{p}'_\beta = R'_\beta \hat{z} = R(\psi'_\beta, \theta'_\beta, \phi'_\beta)\,\hat{z}$$

$$\hat{q}'_\beta = R'_\beta R_y(\Gamma'_\beta)\,\hat{z} \tag{2.5}$$

Furthermore, let us denote the triplet $(p_\alpha, q_\alpha, \Gamma_\alpha)$ by P_α. This triplet of variables is the orthogonal complement of $\psi_\alpha, \theta_\alpha, \phi_\alpha$. Together these two sets span the six dimensional space of $(\vec{p}_\alpha, \vec{q}_\alpha)$. With this choice of variables we may now define

$$M_{\alpha\beta}(P_\alpha, \psi_\alpha, \theta_\alpha, \phi_\alpha; P'_\beta, \psi'_\beta, \theta'_\beta, \phi'_\beta)$$

$$\equiv M_{\alpha\beta}(p_\alpha R_\alpha \hat{z},\ q_\alpha R_\alpha R_y(\Gamma_\alpha)\hat{z};\ p'_\beta R'_\beta \hat{z};\ q'_\beta R'_\beta R_y(\Gamma'_\beta)\hat{z}) \tag{2.6}$$

As the index α and β change then so does our choice of the representing coordinate system. Clearly $\vec{p}_\alpha$ and $\vec{p}'_\beta$ are preferred variables because the direction of these two variables are simply given by $(\theta_\alpha, \psi_\alpha)$ and $(\theta'_\beta, \psi'_\beta)$. This varying choice of coordinate representation differs from Omnes' approach. The Omnes method uses a common rotation to construct the positions of all three equivalent vector sets $(\vec{p}_\alpha, \vec{q}_\alpha)$.

Now that $M_{\alpha\beta}(P_\alpha, \psi_\alpha, \theta_\alpha, \phi_\alpha; P'_\beta, \psi'_\beta, \theta'_\beta, \phi'_\beta)$ is defined, let us introduce an orthogonal expansion of this kernel. We introduce the functions $\tilde{\mathscr{D}}^J_{M\lambda}(\psi, \theta, \phi)$.

$$\tilde{\mathscr{D}}^J_{M\lambda}(\psi, \theta, \phi) \equiv \left(\frac{2J+1}{8\pi^2}\right)^{1/2} \mathscr{D}^{J*}_{M\lambda}(\psi, \theta, \phi) \tag{2.7}$$

where the standard $\mathscr{D}^J_{M\lambda}$ is defined as

$$\mathscr{D}^J_{M\lambda}(\psi, \theta, \phi) = \langle JM | D(R(\psi, \theta, \phi)) | J\lambda \rangle \tag{2.8}$$

and

$$D(R(\psi, \theta, \phi)) = e^{-i\psi J_z} e^{-i\theta J_y} e^{-i\phi J_z}. \tag{2.9}$$

The $|J\lambda\rangle$ are the normalized eigen functions of J^2 and J_z. The rotation operator, $D(R)$, acting on functions of $\vec{x}_\alpha$ and $\vec{y}_\alpha$ or $\vec{p}_\alpha$ and $\vec{q}_\alpha$ is that induced by the Euler matrix rotation R. The modified form of the $\mathscr{D}^J_{M\lambda}$ function we introduce in Eq. (2.7) is a complete orthonormal set with respect to the rotation measure

$dR = \sin\theta\, d\theta\, d\psi d\phi$. The functions $\widetilde{\mathscr{D}}^{J}_{M\lambda}$ are eigen functions of the operators $\vec{J}^2$ and J_z with eigenvalues $2J + 1$ and M respectively.

Employing the $\widetilde{\mathscr{D}}^{J}_{M\lambda}$ we may expand $M_{\alpha\beta}$ in the form,

$$M_{\alpha\beta}(P_\alpha, \psi_\alpha \theta_\alpha \phi_\alpha; P'_\beta, \psi'_\beta \theta'_\beta \phi'_\beta)$$

$$= \sum_{JM\lambda\lambda'} \widetilde{\mathscr{D}}^{J}_{M\lambda}(\psi_\alpha, \theta_\alpha, \phi_\alpha)\, M^{JM}_{\alpha\beta;\lambda\lambda'}(P_\alpha; P'_\beta)\, \widetilde{\mathscr{D}}^{J*}_{M\lambda'}(\psi'_\beta, \theta'_\beta, \phi'_\beta) \quad (2.10)$$

The coefficients $M^{JM}_{\alpha\beta;\lambda\lambda'}$ are determined by

$$M^{JM}_{\alpha\beta;\lambda\lambda'}(P_\alpha; P'_\beta)\, \delta_{JJ'}\, \delta_{MM'}$$

$$= \int \widetilde{\mathscr{D}}^{J*}_{M\lambda}(R_\alpha)\, M_{\alpha\beta}(\vec{p}_\alpha\, \vec{q}_\alpha; \vec{p}'_\beta \vec{q}'_\beta)\, \widetilde{\mathscr{D}}^{J'}_{M'\lambda'}(R'_\beta)\, dR_\alpha\, dR'_\beta \quad (2.11)$$

The diagonality in J and M comes from the commutation relation, $[\vec{J}, M_{\alpha\beta}] = 0$, mentioned earlier. In the integral in Eq. (2.11), the direction of the vectors are those given by the relations (2.4) and (2.5).

Once the $M^{JM}_{\alpha\beta;\lambda\lambda'}$ are defined we can now derive an Omnes-like[5] equation for these coefficients. To effect this derivation one only need to know how two $M_{\alpha\beta}$ operators multiply in the $JM\lambda\lambda'$ representation that we have introduced in Eq. (2.10). Consider a second operator of the M type, $N_{\gamma\delta}$. This operator will have the representation,

$$N_{\gamma\delta}(\vec{p}''_\gamma \vec{q}''_\gamma; \vec{p}'_\delta \vec{q}'_\delta) = N_{\gamma\delta}(P''_\gamma\, \psi''_\gamma \theta''_\gamma \phi''_\gamma; P'_\delta\, \psi'_\delta \theta'_\delta \phi'_\delta)$$

$$= \sum_{JM\lambda''\lambda'} \widetilde{\mathscr{D}}^{J}_{M\lambda''}(R''_\gamma)\, N^{JM}_{\gamma\delta;\lambda''\lambda'}(P''_\gamma; P'_\delta)\, \widetilde{\mathscr{D}}^{J*}_{M\lambda'}(R'_\delta)\,. \quad (2.12)$$

The operator product is defined as

$$[M_{\alpha\beta}N_{\gamma\delta}](\vec{p}_\alpha\vec{q}_\alpha;\vec{p}'_\delta\vec{q}'_\delta)$$

$$= \int M_{\alpha\beta}(\vec{p}_\alpha\vec{q}_\alpha;\vec{p}''_\beta\vec{q}''_\beta)\, N_{\gamma\delta}(\vec{p}''_\gamma\vec{q}''_\gamma;\vec{p}'_\delta\vec{q}'_\delta)\, d^3p''\, d^3q''$$

$$= \sum_{JM\lambda\lambda'} \tilde{\mathscr{D}}^J_{M\lambda}(R_\alpha)\,[M_{\alpha\beta}N_{\gamma\delta}]^{JM}_{\lambda\lambda'}(P_\alpha;P'_\delta)\,\tilde{\mathscr{D}}^{J*}_{M\lambda'}(R'_\delta). \tag{2.13}$$

We wish to find an expression for $[M_{\alpha\beta}N_{\gamma\delta}]^{JM}_{\lambda\lambda'}$. This can be done by substituting the representations (2.10) and (2.12) into the integral appearing in (2.13). To carry out the integration however one needs to have both $\vec{p}''_\beta\vec{q}''_\beta$ and $\vec{p}''_\gamma\vec{q}''_\gamma$ expressed in the same angular coordinate system. By reference to Fig. 1 and the definitions we have employed it is easy to see that

$$R''_\gamma = R''_\beta R_y(\theta''_{\gamma\beta}) \tag{2.14}$$

Thus the $\tilde{\mathscr{D}}^J_{M\lambda}(R''_\gamma)$ appearing in (2.12) may be represented as

$$\tilde{\mathscr{D}}^J_{M\lambda}(R''_\gamma) = \sum_{\lambda_\beta} \tilde{\mathscr{D}}^J_{M\lambda_\beta}(R''_\beta)\, d^J_{\lambda_\beta\lambda}(\theta''_{\gamma\beta}). \tag{2.15}$$

Eq. (2.15) is a consequence of the group property that the $\mathscr{D}^J_{M\lambda}$ functions possess. The function $d^J_{\lambda\lambda'}(\theta)$ is just $\mathscr{D}^J_{\lambda\lambda'}(\psi,\theta,\phi)$ with ψ and ϕ set equal to 0. Using the differential relation (A.6) given in Appendix A, we carry out the integration on the right of Eq. (2.13). We obtain,

$$[M_{\alpha\beta}N_{\gamma\delta}]^{JM}_{\lambda\lambda'}(P_\alpha;P'_\delta)$$

$$= \sum_{\lambda_\beta\lambda_\gamma} \int M^{JM}_{\alpha\beta;\lambda\lambda_\beta}(P_\alpha;P''_\beta)\, d^J_{\lambda_\beta\lambda_\gamma}(\theta''_{\gamma\beta})\quad N^{JM}_{\gamma\delta;\lambda_\gamma\lambda'}(P''_\gamma;P'_\delta)\, dP'' \tag{2.16}$$

The angle $\theta''_{\gamma\beta}$ appearing in the integral can be written in terms of the variables P''_β or P''_γ. Eq. (2.16) provides us with the relationship governing the multiplication of two operators represented in the $JM\lambda\lambda'$ basis. It is important to keep

in mind that the angular coordinate system used in the representation of $M_{\alpha\beta}$ and $N_{\gamma\delta}$ depends on the indices $\alpha\beta$ and $\gamma\delta$.

Let us proceed to obtain our equivalent of the Omnes equations. Once we have (2.16) we may write down by inspection the forms (1.2) take. Thus,

$$M^{JM}_{\alpha\beta;\,\lambda_\alpha\lambda'_\beta}(P_\alpha;\,P'_\beta;\,z) - \delta_{\alpha\beta}\,T^{JM}_{\alpha;\lambda_\alpha\lambda'_\beta}(P_\alpha;\,P'_\beta;\,z)$$

$$= -\sum_{\gamma\neq\alpha}\;\sum_{\lambda''_\alpha\lambda''_\gamma}\int T^{JM}_{\alpha;\lambda_\alpha\lambda''_\alpha}(P_\alpha;\,P''_\alpha;\,z)\;d^J_{\lambda''_\alpha\lambda''_\gamma}(\theta''_{\gamma\alpha})\;M^{JM}_{\gamma\beta;\,\lambda''_\gamma\lambda'_\beta}(P''_\gamma;\,P'_\beta;\,z)$$

$$\times\;(\tilde{p}''^2_\gamma + \tilde{q}''^2_\gamma - z)^{-1}\,dP''_\gamma\;. \tag{2.17}$$

A similar equation holds for the second form of (1.2).

The matrix element $T^{JM}_{\alpha;\lambda_\alpha\lambda'_\alpha}$ is defined by an equation analogous to Eq. (2.11),

$$T^{JM}_{\alpha;\lambda_\alpha\lambda'_\alpha}(P_\alpha;\,P'_\alpha;\,z)$$

$$= \int \tilde{\mathscr{D}}^{J*}_{M\lambda_\alpha}(R_\alpha)\;T_\alpha(\vec{p}_\alpha,\vec{q}_\alpha;\,\vec{p}'_\alpha,\,\vec{q}'_\alpha;\,z)\;\tilde{\mathscr{D}}^{J}_{M\lambda'_\alpha}(R'_\alpha)\;dR_\alpha\,dR'_\alpha \tag{2.18}$$

where T_α appearing in the inner product is given by Eq. (1.3). Because the two-body potential $v_\alpha(\vec{y}_\alpha)$ is spherically symmetric, the related t-matrix, t_α, has the expansion,

$$t_\alpha(\vec{q}_\alpha;\,\vec{q}'_\alpha;\,z-\tilde{p}^2_\alpha) = \sum_{\ell_\alpha}\frac{2\ell_\alpha+1}{4\pi}\;t^\alpha_{\ell_\alpha}(q_\alpha;\,q'_\alpha;\,z-\tilde{p}^2_\alpha)\;P_{\ell_\alpha}(\hat{q}_\alpha\cdot\hat{q}'_\alpha) \tag{2.19}$$

One may exploit this rotational invariance of the t_α to calculate[5] the integral in (2.18). The result is

$$T^{JM}_{\alpha;\lambda_\alpha\lambda'_\alpha}(P_\alpha;\,P'_\alpha;\,z) \equiv \delta_{\lambda_\alpha\lambda'_\alpha}\;T^J_{\alpha;\,\lambda_\alpha}(P_\alpha;\,P'_\alpha;\,z), \tag{2.20}$$

where the $T^J_{\alpha;\lambda_\alpha}$ is given by,

$$T^J_{\alpha;\lambda_\alpha}(P_\alpha;P'_\alpha;z) = \frac{2\pi\delta(p_\alpha - p'_\alpha)}{p^2_\alpha}$$

$$\times \sum_{\ell_\alpha} t^\alpha_{\ell_\alpha}(q_\alpha;q'_\alpha;z-\tilde{p}^2_\alpha)\, Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha,0)\, Y^*_{\ell_\alpha\lambda_\alpha}(\Gamma'_\alpha,0) \tag{2.21}$$

Some simplification of the integral equation for $M^{JM}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}$ takes place when we substitute the relation (2.20) into (2.17). We have then

$$M^J_{\alpha\beta;\lambda_\alpha\lambda'_\beta}(P_\alpha;P'_\beta;z) - \delta_{\alpha\beta}T^J_{\alpha;\lambda_\alpha}(P_\alpha;P'_\beta;z)$$

$$= -\sum_{\gamma\neq\alpha}\sum_{\lambda''_\gamma}\int T^J_{\alpha;\lambda_\alpha}(P_\alpha;P''_\alpha;z)\, d^J_{\lambda_\alpha\lambda''_\gamma}(\theta''_{\gamma\alpha})\, M^J_{\gamma\beta;\lambda''_\gamma\lambda'_\beta}(P''_\gamma;P'_\beta;z)$$

$$(\tilde{p}''^2_\gamma + \tilde{q}''^2_\gamma - z)^{-1}\, dP''$$

$$= -\sum_{\delta\neq\beta}\sum_{\lambda''_\delta}\int M^J_{\alpha\delta;\lambda_\alpha\lambda''_\delta}(P_\alpha;P''_\delta;z)\, d^J_{\lambda''_\delta\lambda'_\beta}(\theta''_{\beta\delta})\, T^J_{\beta;\lambda'_\beta}(P''_\beta;P'_\beta;z)$$

$$(\tilde{p}''^2_\delta + \tilde{q}''^2_\delta - z)^{-1}\, dP'' \tag{2.22}$$

Equations (2.22) represent our final form for the Omnes equations. These differ in structure from the original Omnes equations by the presence of the $d^J_{\lambda\lambda'}$ functions occurring in the kernels. Also, of course, the meaning of the $JM\lambda$ amplitudes differ here. It is seen from Eqs. (2.19) and (2.20) that spherical symmetry of v_α has led to an invariance of the amplitudes $T^{JM}_{\alpha;\lambda\lambda'}$ in the M index. This in turn means that $M^{JM}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}$ will be M independent. We have accordingly simplified our notation for Eq. (2.22). This M independence is due to the rotational invariance of the entire three-body problem.

We note in conclusion of this section that our equations (2.22) in the JMλ basis have transformed the six-dimensional momentum space integral equations into coupled integral equations in 3 continuous variables. In the next section we will further simplify these equations so that they are reduced to integral equations in 2 continuous variables.

3. TWO VARIABLE EQUATIONS

In this section we study several inter-related problems. We first introduce a set of basis functions which permit a reduction of our integral equations to two variables. In this basis set we investigate the behavior of the primary pole representation (1.4). From this we can extract definitions of the partial-wave forms of $\mathscr{K}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$. We then can obtain the integral equations that these forms satisfy.

We now consider the appropriate basis of functions for a two variable reduction. If one substitutes one form of (2.22) into the other and uses the representation (2.21), it follows that the amplitude $M^{J}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}$ has an explicit dependence on the variables Γ_α and Γ'_β which may be summarized by

$$M^{JM}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}(P_\alpha;\, P'_\beta;\, z) = 2\pi \sum_{\ell_\alpha\ell'_\beta} \mathscr{M}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha;\ell'_\beta\lambda'_\beta}\,(p_\alpha q_\alpha;\, p'_\beta q'_\beta;\, z)$$

$$Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\; Y^{*}_{\ell'_\beta\lambda'_\beta}(\Gamma'_\beta, 0)\;. \tag{3.1}$$

If this expansion is considered together with that of Eq. (2.10) it follows that $\mathscr{M}^{J}_{\alpha\beta;\ell_\alpha\lambda_\alpha;\ell'_\beta\lambda'_\beta}$ is the matrix element of $M_{\alpha\beta}$ with respect to the functions $Q^{JM}_{\ell_\alpha\lambda_\alpha}$ defined as,

$$Q^{JM}_{\ell_\alpha\lambda_\alpha}(\psi_\alpha, \theta_\alpha, \phi_\alpha, \Gamma_\alpha) = \sqrt{2\pi}\; Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\; \widetilde{\mathscr{D}}^{J}_{M\lambda_\alpha}(\psi_\alpha, \theta_\alpha, \phi_\alpha). \tag{3.2}$$

These Q are orthonormal relative to the differential $d^4Q_\alpha \equiv d\cos\Gamma_\alpha d\psi_\alpha$ $d\cos\theta_\alpha d\phi_\alpha$. So we have

$$\mathcal{M}^{JM}_{\alpha\beta;\ell_\alpha\lambda_\alpha;\ell'_\beta\lambda'_\beta}(p_\alpha q_\alpha;\ p'_\beta q'_\beta;\ z)$$

$$= \int Q^{JM*}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{q}_\alpha)\, M_{\alpha\beta}(\vec{p}_\alpha\ \vec{q}_\alpha;\ \vec{p}'_\beta\vec{q}'_\beta;\ z)\ Q^{JM}_{\ell'_\beta\lambda'_\beta}(\hat{p}'_\beta,\ \hat{q}'_\beta)\, d^4Q_\alpha\, d^4Q'_\beta \tag{3.3}$$

In this formula the vectors $\vec{p}_\alpha, \vec{q}_\alpha$ and $\vec{p}'_\beta, \vec{q}'_\beta$ are given in terms of (2.4) and (2.5). In writing the argument of Q we have used the fact that knowledge of $\psi_\alpha, \theta_\alpha, \phi_\alpha, \Gamma_\alpha$ implies the orientation of $\hat{p}_\alpha, \hat{q}_\alpha$.

Furthermore the set of functions $Q^{JM}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha\hat{q}_\alpha)$ are complete on the Hilbert space related to $\hat{p}_\alpha$ and $\hat{q}_\alpha$, i.e., that space associated with the inner product appearing in (3.3). The only open problem here is showing that $Q^{JM}_{\ell_\alpha\lambda_\alpha}$ span the space. This is verified by noting a simple connection between $Y_{L_\alpha M_\alpha}(\hat{p}_\alpha)\, Y_{\ell_\alpha m_\alpha}(\hat{q}_\alpha)$ and our Q's. Using the relation (2.4) one may show,

$$Y_{L_\alpha M_\alpha}(\hat{p}_\alpha)\, Y_{\ell_\alpha m_\alpha}(\hat{q}_\alpha)$$

$$= \sum_{J\lambda_\alpha} \left(\frac{2L_\alpha + 1}{2J + 1}\right)^{1/2} C(L_\alpha\ell_\alpha J; M_\alpha m_\alpha)\, C(L_\alpha\ell_\alpha J;\ 0,\ \lambda_\alpha)\, Q^{J,\, M_\alpha + m_\alpha}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{q}_\alpha). \tag{3.4}$$

The C are the Clebsch-Gordon coefficients in Rose's notation.[9] Equation (3.4), tells us that each $Y_{L_\alpha M_\alpha}(\hat{p}_\alpha)\, Y_{\ell_\alpha m_\alpha}(\hat{q}_\alpha)$ is a linear combination of the Q's. Since the pair of Y_{LM} functions are complete, then so must be the Q's.

The indices that Q possess all have a simple meaning. Of course, J and M represent the total angular momentum and its projection along a spatially fixed z axis. These two quantum numbers are eigenvalues of both the exact Hamiltonian, H, and the channel Hamiltonian, H_α. The internal angular momentum of the cluster α is ℓ_α. Finally, it follows from the definition of the rotation operators that λ_α is the projection of the angular momentum ℓ_α on the fixed-body axis. In this case the fixed body axis points in the $\hat{p}_\alpha$ direction. So λ_α is the projection of ℓ_α onto the $\hat{p}_\alpha$ direction. It is easily seen that both ℓ_α and λ_α are eigenvalues of H_α. The operator H_α leaves the direction of $\vec{p}_\alpha$ unchanged, so $\vec{p}_\alpha$ may be regarded as a fixed vector in the α channel. Then λ_α is just the component of ℓ_α in this fixed direction.

Finally, we note that we have three distinct $Q^{JM}_{\ell_\alpha\lambda_\alpha}$, one set for each asymptotic Hamiltonian, H_α. It is also clear that these three sets of functions of four angular variables do not span a common four dimensional subspace of $\vec{p},\vec{q}$. This is in contrast to our three sets of $\widetilde{\mathscr{D}}^J_{M\lambda_\alpha}(R_\alpha)$, all of which do span the same space. Thus no unitary transformation between the three Q's exists. This is one of the reasons why the derivations in this section must be somewhat more lengthy than in the previous section. Specifically, for the Q's there is no multiplication law similar to (2.16).

Now let us obtain the two-variable integral equation for $\mathscr{M}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha;\ell'_\beta\lambda'_\beta}$. If we substitute (3.1) into the Omnes Eq. (2.22) and equate the coefficient of $Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha,0)\,Y^*_{\ell'_\beta\lambda'_\beta}(\Gamma'_\beta,0)$ we have,

$$\mathcal{M}^{J}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha;\,\ell'_\beta\lambda'_\beta}(p_\alpha q_\alpha;\, p'_\beta\ q'_\beta;\, z) = \delta_{\alpha\beta}\delta_{\lambda_\alpha\lambda'_\beta}\ \frac{\delta(p_\alpha - p'_\alpha)}{p_\alpha^2}\ t^{\alpha}_{\ell_\alpha}(q_\alpha;\, q'_\alpha;\, z - \tilde{p}^2_\alpha)$$

$$- \sum_{\gamma\neq\alpha}\ \sum_{\ell''_\gamma\lambda''_\gamma} \int \frac{\delta(p_\alpha - p''_\alpha)\, t^{\alpha}_{\ell_\alpha}(q_\alpha;\, q''_\alpha;\, z - \tilde{p}^2_\alpha)}{p_\alpha^2(p''^2_\gamma + q''^2_\gamma - z)} \left\{ 2\pi\, Y^*_{\ell_\alpha\lambda_\alpha}(\Gamma''_\alpha, 0)\, d^J_{\lambda_\alpha\lambda''_\gamma}(\theta''_{\gamma\alpha})\, Y_{\ell''_\gamma\lambda''_\gamma}(\Gamma''_\gamma, 0) \right\}$$

$$\mathcal{M}^{J}_{\gamma\beta;\,\ell''_\gamma\lambda''_\gamma;\,\ell'_\beta\lambda'_\beta}(p''_\gamma q''_\gamma;\, p'_\beta q'_\beta;\, z)\ dP'' \tag{3.5}$$

This is a coupled two-variable integral equation for $\mathcal{M}$. By using, $dP'' = p''^2_\gamma\, dp''_\gamma\, q''^2_\gamma\, dq''_\gamma\, d\cos\Gamma''_\gamma$, one may integrate out the $\cos\Gamma''_\gamma$ dependence in the kernel of (3.5) and further simplify the equation. We define

$$V^J(p_\alpha;\, p''_\gamma q''_\gamma)_{\ell_\alpha\lambda_\alpha;\,\ell''_\gamma\lambda''_\gamma}$$

$$= (2\pi)\ \frac{p''_\gamma q''_\gamma}{p_\alpha^2} \int_{-1}^{+1} \delta(p_\alpha - p''_\alpha)\, Y^*_{\ell_\alpha\lambda_\alpha}(\Gamma''_\alpha, 0)\ \ d^J_{\lambda_\alpha\lambda''_\gamma}(\theta''_{\gamma\alpha})\, Y_{\ell''_\gamma\lambda''_\gamma}(\Gamma''_\gamma, 0)\, d\cos\Gamma''_\gamma . \quad \gamma\neq\alpha . \tag{3.6}$$

In the Appendix B we evaluate this integral and provide an explicit algebraic form for it.

With this definition, now Eq. (3.5) becomes

$$\mathcal{M}^{J}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha;\,\ell'_\beta\lambda'_\beta}(p_\alpha, q_\alpha;\, p'_\beta, q'_\beta;\, z) = \delta_{\alpha\beta}\delta_{\lambda_\alpha\lambda'_\beta}\ \frac{\delta(p_\alpha - p'_\alpha)}{p_\alpha^2}\ t^{\alpha}_{\ell_\alpha}(q_\alpha;\, q'_\alpha;\, z - \tilde{p}^2_\alpha)$$

$$- \sum_{\gamma\neq\alpha}\ \sum_{\ell''_\gamma\lambda''_\gamma} \int \frac{t^{\alpha}_{\ell_\alpha}(q_\alpha;\, q''_\alpha;\, z - \tilde{p}^2_\alpha)}{(\tilde{p}''^2_\gamma + \tilde{q}''^2_\gamma - z)}\ V^J(p_\alpha;\, p''_\gamma q''_\gamma)_{\ell_\alpha\lambda_\alpha;\,\ell''_\gamma\lambda''_\gamma}$$

$$\mathcal{M}^{J}_{\gamma\beta;\,\ell''_\gamma\lambda''_\gamma;\,\ell'_\beta\lambda'_\beta}(p''_\gamma, q''_\gamma;\, p'_\beta, q'_\beta;\, z)\ p''_\gamma\, dp''_\gamma\, q''_\gamma\, dq''_\gamma \tag{3.7}$$

The term V^J occurring in the kernel of (3.7) is purely a universal kinematic factor common to all three-body scattering problems. Such a factor is endemic to all the angular momentum reductions in the literature.[8]

As far as obtaining a two-variable integral equation is concerned Eq. (3.7) provides the desired form. However, as emphasized in the Introduction, (3.7) has several drawbacks. First, it contains the primary-pole structure implicitly; nor, it is an equation for observable multichannel amplitudes. We proceed now to develop such physical integral equations. The first step is to analyze the primary-pole decomposition when represented in the $Q^{JM}_{\ell_\alpha \lambda_\alpha}$ basis. This will give us the definitions of the partial-wave observable amplitudes. Once we have these amplitudes it is a simple matter to obtain the integral equation they satisfy.

Let us take an integral of the primary-pole decomposition for $M_{\alpha\beta}$, Eq. (1.4), with respect to

$$Q^{JM*}_{\ell_\alpha \lambda_\alpha}(\hat{p}_\alpha, \hat{q}_\alpha)\, Q^{JM}_{\ell'_\beta \lambda'_\beta}(\hat{p}'_\beta, \hat{q}'_\beta)\, d^4Q_\alpha\, d^4Q'_\beta \, .$$

Our s-wave bound-state assumption means the vertex function ϕ_β may be written,

$$\phi_\beta(\vec{q}_\beta) = \frac{1}{\sqrt{4\pi}}\ \phi^o_\beta(q_\beta) \, . \tag{3.8}$$

The $(4\pi)^{-1/2}$ is the value of $Y_{00}(\hat{q}_\beta)$. All the integrals are trival and one obtains,

$$\begin{aligned}
\mathscr{M}^{JM}_{\alpha\beta;\, \ell_\alpha\lambda_\alpha;\, \ell'_\beta\lambda'_\beta}(p_\alpha q_\alpha;\, p'_\beta q'_\beta) &= \mathscr{F}^{JM}_{\alpha\beta;\, \ell_\alpha\lambda_\alpha;\, \ell'_\beta\lambda'_\beta}(p_\alpha q_\alpha;\, p'_\beta q'_\beta) \\
&+ \mathscr{G}^{JM}_{\alpha\beta;\, \ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha;\, p'_\beta)\ \frac{\delta_{\ell'_\beta 0}\,\delta_{\lambda'_\beta 0}\,\phi^o_\beta(q'_\beta)^*}{z + \chi^2_\beta - \widetilde{p}'^2_\beta} + \frac{\delta_{\ell_\alpha 0}\,\delta_{\lambda_\alpha 0}\,\phi^o_\alpha(q_\alpha)}{z + \chi^2_\alpha - \widetilde{p}^2_\alpha}\ \widetilde{\mathscr{G}}^{JM}_{\alpha\beta;\, \ell'_\beta\lambda'_\beta}(p_\alpha;\, p'_\beta q'_\beta) \\
&+ \frac{\delta_{\ell_\alpha 0}\,\delta_{\lambda_\alpha 0}\,\phi^o_\alpha(q_\alpha)}{z + \chi^2_\alpha - \widetilde{p}^2_\alpha}\ \mathscr{H}^{JM}_{\alpha\beta}(p_\alpha;\, p'_\beta)\ \frac{\delta_{\ell'_\beta 0}\,\delta_{\lambda'_\beta 0}\,\phi^o_\beta(q'_\beta)^*}{z + \chi^2_\beta - \widetilde{p}'^2_\beta} \, .
\end{aligned} \tag{3.9}$$

The partial-wave amplitudes in this expansion are:

$$\mathscr{H}^{JM}_{\alpha\beta}(p_\alpha; p'_\beta) = \int Y^*_{JM}(\hat{p}_\alpha)\ \mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta)\ Y_{JM}(\hat{p}'_\beta)\, d\hat{p}_\alpha\, d\hat{p}'_\beta \tag{3.10}$$

$$\mathscr{G}^{JM}_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta) = \int Q^{JM*}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{q}_\alpha)\ \mathscr{G}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta)\ Y_{JM}(\hat{p}'_\beta)\, d^4\hat{Q}_\alpha\, d\hat{p}'_\beta \tag{3.11}$$

$$\tilde{\mathscr{G}}^{JM}_{\alpha\beta;\ \ell'_\beta\lambda'_\beta}(p_\alpha; p'_\beta q'_\beta) = \int Y^*_{JM}(\hat{p}_\alpha)\ \tilde{\mathscr{G}}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta, \vec{q}'_\beta)\ Q^{JM}_{\ell'_\beta\lambda'_\beta}(\hat{p}'_\beta, \hat{q}'_\beta)\, d\hat{p}_\alpha\, d^4\hat{Q}'_\beta \tag{3.12}$$

and $\mathscr{F}^{JM}_{\alpha\beta;\ \ell_\alpha\lambda_\alpha;\ \ell'_\beta\lambda'_\beta}$ is defined as $\mathscr{M}^{JM}_{\alpha\beta;\ \ell_\alpha\lambda_\alpha;\ \ell'_\beta\lambda'_\beta}$ in (3.3) but with respect to the amplitude $\mathscr{F}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta, \vec{q}'_\beta)$.

Let us examine $\mathscr{H}^{JM}_{\alpha\beta}$ in detail. First we know that the amplitude, $\mathscr{H}^{JM}_{\alpha\beta}$, is independent of M since $\mathscr{M}^{JM}_{\alpha\beta;\ \ell_\alpha\lambda_\alpha;\ \ell'_\beta\lambda'_\beta}$ is so. This fact, combined with definition (3.10) means $\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha, \vec{p}'_\beta)$ is a function only of the variables p_α, p'_β and $\hat{p}_\alpha \cdot \hat{p}'_\beta$. Or mathematically,

$$\mathscr{H}^{J}_{\alpha\beta}(p_\alpha; p'_\beta) = \sum_{M=-J}^{J} \frac{1}{2J+1} \int Y^*_{JM}(\hat{p}_\alpha)\ \mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta)\ Y_{JM}(\hat{p}'_\beta)\, d\hat{p}_\alpha d\hat{p}'_\beta$$

$$= 2\pi \int_{-1}^{+1} \mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; \cos\theta_{\alpha\beta})\ P_J(\cos\theta_{\alpha\beta})\, d\cos\theta_{\alpha\beta} \tag{3.13}$$

and correspondingly,

$$\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta) = \sum_{JM} Y_{JM}(\hat{p}_\alpha)\ \mathscr{H}^{J}_{\alpha\beta}(p_\alpha, p'_\beta)\ Y^*_{JM}(\hat{p}'_\beta)$$

$$= \sum_{J} \frac{2J+1}{4\pi}\ \mathscr{H}^{J}_{\alpha\beta}(p_\alpha, p'_\beta)\ P_J(\hat{p}_\alpha \cdot \hat{p}'_\beta) \tag{3.14}$$

Thus we shall be able to parametrize the on-shell $\mathscr{H}^{J}_{\alpha\beta}$ by a phase-shift representation. This simple expansion of $\mathscr{H}_{\alpha\beta}$ owes its existence to two circumstances. The first is the M independence of the amplitudes. The second is the definitions of the angular rotations, (2.4) and (2.5), introduced in Section II.

The remaining amplitude of physical interest for us is the partial-wave breakup amplitude. The breakup amplitude is a linear combination of $\mathscr{H}^{JM}_{\alpha\beta}$ and $\mathscr{G}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}$, viz.

$$\mathscr{K}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta) = \mathscr{G}^{JM}_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta) + \frac{\delta_{\ell_\alpha 0}\delta_{\lambda_\alpha 0}\,\phi^{0}_{\alpha}(q_\alpha)}{z + \chi^{2}_{\alpha} - \tilde{p}^{2}_{\alpha}}\,\mathscr{H}^{JM}_{\alpha\beta}(p_\alpha; p'_\beta), \tag{3.15}$$

where $\mathscr{K}^{JM}_{\alpha\beta;\ell_\alpha\lambda_\alpha}$ is defined like $\mathscr{G}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}$ in Eq. (2.11) but with $\mathscr{K}_{\alpha\beta}$ replacing $\mathscr{G}_{\alpha\beta}$. From the definition of $\mathscr{K}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}$ it follows that $\mathscr{K}_{\alpha\beta}$ may be represented

$$\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha \vec{q}_\alpha; \vec{p}'_\beta) = \sum_{JM\ell_\alpha\lambda_\alpha} Q^{JM}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{q}_\alpha)\,\mathscr{K}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta)\,Y^{*}_{JM}(\hat{p}'_\beta)\,. \tag{3.16}$$

As with our other amplitudes, the breakup amplitudes will be M independent. We would like to find the geometrical invariance $\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha \vec{q}_\alpha; \vec{p}'_\beta)$ will possess as a consequence of this independence. Using Eq. (3.2) we have

$$\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha \vec{q}_\alpha; \vec{p}'_\beta) = \sum_{J\ell_\alpha\lambda_\alpha} (2\pi)^{1/2}\,Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\,\mathscr{K}^{J}_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta)$$

$$\times\left\{\sum_{M}\left(\frac{2J+1}{8\pi^2}\right)^{1/2} \mathscr{D}^{J*}_{M\lambda_\alpha}(R_\alpha)\,Y^{*}_{JM}(\hat{p}'_\beta)\right\}$$

$$= \sum_{J\ell_\alpha\lambda_\alpha}\left(\frac{2J+1}{4\pi}\right)^{1/2} Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\,\mathscr{K}^{J}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta)\,Y^{*}_{J\lambda_\alpha}(R_\alpha \hat{p}'_\beta)$$

This can be further simplified by choosing the z axis along the p'_β direction. Then $R_\alpha \hat{p}'_\beta = \hat{p}_\alpha$, so

$$\mathcal{K}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta) = \sum_{J\ell_\alpha\lambda_\alpha} \left(\frac{2J+1}{4\pi}\right)^{1/2} Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\, Y^*_{J\lambda_\alpha}(\hat{p}_\alpha)\, \mathcal{K}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta) \tag{3.17}$$

From examination of the right hand side of Eq. (3.17) we see that $\mathcal{K}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta)$ has no dependence on the azimuthal angle of $\hat{q}_\alpha$ about the $\hat{p}_\alpha$ direction. Equivalently, any rotation about the direction $\hat{p}_\alpha$ leaves the amplitude invarient.

We proceed by giving the integral equations for $\mathcal{K}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}$ and $\mathcal{H}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}$. We desire these equations because their amplitudes are the multi-channel partial-wave observable amplitudes. The equation for $\mathcal{K}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}$ is obtained from that for $\mathcal{M}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha;\ell'_\beta\lambda'_\beta}$, Eq. (3.7), by equating the coefficient of the singular factor $\delta_{\ell'_\beta 0}\delta_{\lambda'_\beta 0}\phi^o_\beta(q'_\beta)^*(z + \chi^2_\beta - \tilde{p}'^2_\beta)^{-1}$ which appears in the primary-pole expansion (3.9). By inspection we deduce from Eq. (3.7) that

$$\mathcal{K}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta; z) = \mathcal{K}^{oJ}_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha\ q_\alpha; p'_\beta; z)$$

$$- \sum_{\gamma\neq\alpha} \sum_{\ell''_\gamma\lambda''_\gamma} \int \frac{t^\alpha_{\ell_\alpha}(q_\alpha; q''_\alpha; z - \tilde{p}^2_\alpha)}{\tilde{p}''^2_\gamma + \tilde{q}''^2_\gamma - z}\, V^J(p_\alpha; p''_\gamma\ q''_\gamma)_{\ell_\alpha\lambda_\alpha;\ell''_\gamma\lambda''_\gamma}\, \mathcal{K}^J_{\gamma\beta;\ell''_\gamma\lambda''_\gamma}(p''_\gamma\ q''_\gamma; p'_\beta)\, p''_\gamma dp''_\gamma q''_\gamma dq''_\gamma \tag{3.18}$$

This above equation still is characterized by singular solutions as can be seen from examining representation (3.15) for $\mathcal{K}^J_{\alpha\beta;\ell_\alpha\lambda_\alpha}$. If we introduce Eq. (3.15) into (3.18) and equate the residues of the singular term $(z + \chi^2_\alpha - \tilde{p}^2_\alpha)^{-1}\delta_{\ell_\alpha 0}\delta_{\lambda_\alpha 0}\phi^o_\alpha(q_\alpha)$ then we derive,

$$\mathscr{H}^{J}_{\alpha\beta}(p_\alpha;\, p'_\beta;\, z) = \mathscr{H}^{oJ}_{\alpha\beta}(p_\alpha;\, p'_\beta;\, z)$$

$$- \sum_{\gamma\neq\alpha} \sum_{\ell''_\gamma \lambda''_\gamma} \int \frac{\phi^{o}_{\alpha}(q''_\alpha)^* \, V^{J}(p_\alpha;\, p''_\gamma q''_\gamma)_{0,0;\ell''_\gamma \lambda''_\gamma}}{\widetilde{p''}_\gamma^{\,2} + \widetilde{q''}_\gamma^{\,2} - z} \; \mathscr{K}^{J}_{\gamma\beta;\, \ell''_\gamma \lambda''_\gamma}(p''_\gamma\; q''_\gamma;\, p'_\beta;\, z) p''_\gamma dp''_\gamma q''_\gamma dq''_\gamma \tag{3.19}$$

and

$$\mathscr{G}^{J}_{\alpha\beta;\, \ell_\alpha \lambda_\alpha}(p_\alpha q_\alpha;\, p'_\beta;\, z) = \mathscr{G}^{oJ}_{\alpha\beta;\, \ell_\alpha \lambda_\alpha}(p_\alpha\; q_\alpha;\, p'_\beta;\, z)$$

$$- \sum_{\gamma\neq\alpha} \sum_{\ell''_\gamma \lambda''_\gamma} \int \frac{\hat{t}^{\alpha}_{\ell_\alpha}(q_\alpha;\, q''_\alpha;\, z)\; V^{J}(p_\alpha;\, p''_\gamma q''_\gamma)_{\ell_\alpha \lambda_\alpha;\, \ell''_\gamma \lambda''_\gamma}}{\widetilde{p''}_\gamma^{\,2} + \widetilde{q''}_\gamma^{\,2} - z}$$

$$\mathscr{K}^{J}_{\gamma\beta;\, \ell''_\gamma \lambda''_\gamma}(p''_\gamma q''_\gamma;\, p'_\beta;\, z)\; p''_\gamma dp''_\gamma q''_\gamma dq''_\gamma \tag{3.20}$$

where we have used the pole-decomposition of $t^{\alpha}_{\ell_\alpha}$

$$t^{\alpha}_{\ell_\alpha}(q_\alpha,\, q'_\alpha\;;\, z) = \delta_{\ell_\alpha 0} \; \frac{\phi^{o}_{\alpha}(q_\alpha)\; \phi^{o}_{\alpha}(q'_\alpha)^*}{z + \chi^{2}_{\alpha}} + \hat{t}^{\alpha}_{\ell_\alpha}(q_\alpha, q'_\alpha;\, z)\;. \tag{3.21}$$

In solving these equations one must employ Eq. (3.15) to insert $\mathscr{H}^{J}_{\alpha\beta}$ and $\mathscr{G}^{J}_{\alpha\beta;\, \ell_\alpha \lambda_\alpha}$ on the right hand sides of (3.19) and (3.20). These equations will share nice mathematical properties of their momentum-space analogs, Eq. (1.13). These essential properties are that $\mathscr{H}_{\alpha\beta}$, $\mathscr{G}^{J}_{\alpha\beta;\, \ell_\alpha \lambda_\alpha}$ are singularity free Holder continuous functions. Although, we have not proved it, we think it is clear that the operator associated with the kernel of (3.19) and (3.20) will be compact in an appropriate reduced Banach space, so that these equations will then have unique solutions. Finally, these equations should be attractive to physicists since the unknown functions $\mathscr{H}^{J}_{\alpha\beta}$, and $\mathscr{K}^{J}_{\alpha\beta;\, \ell_\alpha \lambda_\alpha}$ are observable partial-wave amplitudes.

Our description of these equations will be complete when we have given formulae for the driving terms. These terms are computed just by taking matrix elements of (1.15) and (1.16) with respect to the correct angular functions. We shall just quote the result of these straight forward evaluations,

$$\mathscr{H}^{oJ}_{\alpha\beta}(p_\alpha, p'_\beta; z) = -\frac{\bar{\delta}_{\alpha\beta}}{2} \int_{-1}^{1} P_J(\cos\theta''_{\beta\alpha})$$

$$\frac{\phi^{o*}_\alpha\left(p'^2_\beta + \left(\frac{m_\beta}{m_\beta+m_\gamma}\right)^2 p^2_\alpha + \frac{2m_\beta}{m_\beta+m_\gamma} p_\alpha p'_\beta \cos\theta''_{\beta\alpha}\right)\phi^o_\beta\left(p^2_\alpha + \left(\frac{m_\alpha}{m_\beta+m_\gamma}\right)^2 p'^2_\beta + \frac{2m_\alpha}{m_\alpha+m_\gamma} p_\alpha p'_\beta \cos\theta''_{\beta\alpha}\right)}{\frac{p^2_\alpha}{2\mu_\beta} + \frac{p_\alpha p'_\beta}{m_\gamma}\cos\theta''_{\beta\alpha} + \frac{p'^2_\beta}{2\mu_\alpha} - z} \, d\cos\theta''_{\beta\alpha}$$

(3.22)

The term $\mathscr{G}^{oJ}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}$ is,

$$\mathscr{G}^{oJ}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta)$$

$$= -\bar{\delta}_{\alpha\beta}\sqrt{\pi}\int_{-1}^{1} \hat{t}^{\alpha}_{\ell_\alpha}(q_\alpha, q''_\alpha; z - \tilde{p}^2_\alpha) \frac{Y^*_{\ell_\alpha\lambda_\alpha}(\Gamma''_\alpha, 0)\, d^J_{\lambda_\alpha 0}(\theta''_{\beta\alpha})\, \phi^o_\beta(q''_\beta)}{\frac{p^2_\alpha}{2\mu_\beta} + \frac{p_\alpha p'_\beta}{m_\gamma}\cos\theta''_{\beta\alpha} + \frac{p'^2_\beta}{2\mu_\alpha} - z} \, d\cos\theta''_{\beta\alpha}$$

(3.23)

Earlier in this section we have studied the properties of our amplitudes that stem from the rotational invarience inherent in this problem. We would like to consider here the invarience properties associated with parity and time reversal. Let us consider parity first.

We shall denote by $\mathscr{P}$ the parity operator. The parity operator effects the transformation $\vec{x}_\alpha \to -\vec{x}_\alpha$; $\vec{y}_\alpha \to -\vec{y}_\alpha$ and $\vec{p}_\alpha \to -\vec{p}_\alpha$; $\vec{q}_\alpha \to -\vec{q}_\alpha$. Parity commutes

with the three-body potentials, since they are spherically symmetric, viz.

$$\mathscr{P}V_\alpha(\vec{x}_\alpha,\vec{y}_\alpha) = \mathscr{P}v_\alpha(\vec{y}_\alpha) = v_\alpha(-\vec{y}_\alpha)\mathscr{P} = V_\alpha(\vec{x}_\alpha,\vec{y}_\alpha)\mathscr{P} \tag{3.24}$$

Parity also commutes with H_0, so we may conclude that $[M_{\alpha\beta}, \mathscr{P}] = 0$. Invariance due to parity comes from evaluating this last commutator in the $JM\lambda$ basis. For example the $\mathscr{P}M_{\alpha\beta}$ leads to the integral,

$$\int \tilde{\mathscr{D}}^{J*}_{M\lambda_\alpha}(R_\alpha)\, M_{\alpha\beta}(p_\alpha \mathscr{P} R_\alpha \hat{z},\, q_\alpha \mathscr{P} R_\alpha R_y(\Gamma_\alpha)\hat{z};\, \vec{p}'_\beta, \vec{q}'_\beta)\, \tilde{\mathscr{D}}^{J}_{M\lambda'_\beta}(R'_\beta)\, dR_\alpha\, dR'_\beta\,. \tag{3.25}$$

One computes this integral by using the fact that $\mathscr{P}$ commutes with all rotations to write (say for the second argument of $M_{\alpha\beta}$),

$$\mathscr{P}R_\alpha R_y(\Gamma_\alpha)\hat{z} = R_\alpha R_y(\Gamma_\alpha)(-\hat{z}) = R_\alpha R_y(\Gamma_\alpha)R_y(\pi)\hat{z} = R_\alpha R_y(\pi)R_y(\Gamma_\alpha)\hat{z} \tag{3.26}$$

Thus we define a new rotation, R_T, by $R_T = R_\alpha R_y(\pi)$ and use $dR_\alpha = dR_T$. Now, the integral (3.25) becomes, after using the group property (2.15), equal to

$$\sum_{\lambda_T} M^{JM}_{\alpha\beta;\lambda_T\lambda'_\beta}(P_\alpha; P'_\beta; z)\, d^J_{\lambda_T\lambda_\alpha}(-\pi) = (-1)^{J+\lambda_\alpha} M^{JM}_{\alpha\beta;-\lambda_\alpha,\lambda'_\beta}(P_\alpha; P'_\beta; z) \tag{3.27}$$

In a similar fashion the matrix element of $M_{\alpha\beta}\mathscr{P}$ is determined to be,

$$(-1)^{J+\lambda'_\beta} M^{JM}_{\alpha\beta;\lambda_\alpha,-\lambda'_\beta}(P_\alpha, P'_\beta; z) \tag{3.28}$$

Because $[M_{\alpha\beta}, \mathscr{P}] = 0$ these two expressions, (3.27) and (3.28), must be equal. So we find

$$M^{JM}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}(P_\alpha; P'_\beta; z) = (-1)^{\lambda'_\beta+\lambda_\alpha} M^{JM}_{\alpha\beta;-\lambda_\alpha,-\lambda'_\beta}(P_\alpha; P'_\beta; z) \tag{3.29}$$

We can easily translate this result into a relation which applies to $\mathscr{K}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$. For $\mathscr{K}_{\alpha\beta}$ we have

$$\mathscr{K}^{JM}_{\alpha\beta;\,\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha;\, p'_\beta;\, z) = (-1)^{\lambda_\alpha}\,\mathscr{K}^{JM}_{\alpha\beta;\,\ell_\alpha -\lambda_\alpha}(p_\alpha q_\alpha;\, p'_\beta; z) \tag{3.30}$$

For $\mathscr{H}_{\alpha\beta}$ there is no restriction on the amplitude. This is because our bound states are s-waves.

Finally, we turn to the effect of time-reversal invariance. The time-reversal operator as defined by Wigner[10] is denoted by Ⓗ. Its effect on any element in our momentum Hilbert space is,

$$\text{Ⓗ}\, f(\vec{p}, \vec{q}) = f^*(-\vec{p}, -\vec{q}) \tag{3.31}$$

According to assumption (R) of Faddeev[1] the two-body potentials satisfy $v_\alpha(\vec{q}_\alpha) = v^*_\alpha(-\vec{q}_\alpha)$. This immediately ensures that $[\text{Ⓗ}, V_\alpha] = 0$. Now let us examine the effect of Ⓗ on $M_{\alpha\beta}(z)$ as defined by (1.1). Because Ⓗ is anti-linear, $\text{Ⓗ}\, G(z) = G(\bar{z})\, \text{Ⓗ}$. So we have

$$\text{Ⓗ}\, M_{\alpha\beta}(z) = M_{\alpha\beta}(\bar{z})\, \text{Ⓗ} = M^\dagger_{\beta\alpha}(z)\, \text{Ⓗ}\,. \tag{3.32}$$

This relation is the fundamental operator consequence of time-reversal invariance. We shall now compute the result Eq. (3.32) implies for our matrix elements. Let f, q be arbitrary elements of our Hilbert space, then

$$\begin{aligned}(f, M_{\alpha\beta}(z)g) &= (\text{Ⓗ}\, f, \text{Ⓗ}\, M_{\alpha\beta}(z)g)^* = (\text{Ⓗ}\, f, M^\dagger_{\beta\alpha}(z)\, \text{Ⓗ}\, g)^* \\ &= (M_{\beta\alpha}(z)\, \text{Ⓗ}\, f, \text{Ⓗ}\, g)^* = (\text{Ⓗ}\, g, M_{\beta\alpha}(z)\, \text{Ⓗ}\, f).\end{aligned} \tag{3.33}$$

The above equation is also valid for the inner products associated with $\widetilde{\mathscr{D}}^J_{M\lambda}$ functions. Let f be $\widetilde{\mathscr{D}}^J_{M\lambda_\alpha}(R_\alpha)$ and g be $\widetilde{\mathscr{D}}^J_{M\lambda'_\beta}(R'_\beta)$. We need the effect of Ⓗ on $\widetilde{\mathscr{D}}^J_{M\lambda}$. Recall that Ⓗ reverses the direction of all the momenta. With the

convention given by (2.4) we have, e.g.

$$\text{Ⓗ}\ \vec{q}_\alpha = \vec{-q}_\alpha = -q_\alpha R_\alpha R_y(\Gamma_\alpha)\hat{z} = q_\alpha R_\alpha R_y(\Gamma_\alpha) R_y(\pi)\hat{z} = q_\alpha R_\alpha R_y(\pi) R_y(\Gamma_\alpha)\hat{z} \ . \tag{3.34}$$

Thus,

$$\text{Ⓗ}\ \tilde{\mathscr{D}}^J_{M\lambda}(R) = \tilde{\mathscr{D}}^{J*}_{M\lambda}(R\, R_y(-\pi)) = \sum_{\lambda'} \tilde{\mathscr{D}}^{J*}_{M\lambda'}(R)\, d^J_{\lambda'\lambda}(-\pi) \ . \tag{3.35}$$

Using $d^J_{\lambda'\lambda}(-\pi) = (-1)^{J+\lambda}\, \delta_{\lambda,\, -\lambda'}$ and $\tilde{\mathscr{D}}^{J*}_{M\lambda}(R) = (-1)^{-M+\lambda}\, \tilde{\mathscr{D}}^J_{-M-\lambda}(R)$, Eq. (3.35) then becomes

$$\text{Ⓗ}\ \tilde{\mathscr{D}}^J_{M\lambda}(R) = (-1)^{J-M}\ \ \tilde{\mathscr{D}}^J_{-M\lambda}(R) \ . \tag{3.36}$$

Substituting this result into (3.33) gives us

$$(\ \tilde{\mathscr{D}}^J_{M\lambda_\alpha},\ M_{\alpha\beta}(P_\alpha; P'_\beta; z)\ \ \tilde{\mathscr{D}}^J_{M\lambda'_\beta}) = (\ \tilde{\mathscr{D}}^J_{-M\lambda'_\beta},\ M_{\beta\alpha}(P'_\beta; P_\alpha; z)\ \tilde{\mathscr{D}}^J_{-M\lambda_\alpha}\ ) \ ,$$

or

$$M^{JM}_{\alpha\beta;\lambda_\alpha\lambda'_\beta}(P_\alpha;\ P'_\beta;\ z) = M^{JM}_{\beta\alpha;\lambda'_\beta\lambda_\alpha}(P'_\beta;\ P_\alpha;\ z), \tag{3.37}$$

where we have used rotational invariance to replace -M by M. In general this result implies relations between $\mathscr{K}_{\alpha\beta}$ and $\tilde{\mathscr{K}}_{\beta\alpha}$. However, it has a simple form for $\mathscr{H}_{\alpha\beta}$, namely

$$\mathscr{H}^J_{\alpha\beta}(p_\alpha;\ p'_\beta;\ z) = \mathscr{H}^J_{\beta\alpha}(p'_\beta;\ p_\alpha;\ z) \ . \tag{3.38}$$

This is useful in deriving the off-shell unitarity relation $\mathscr{H}_{\alpha\beta}$ satisfy.

4. WAVE FUNCTIONS, CROSS SECTIONS AND PHASE SHIFTS

Throughout this paper we have repeatedly stressed that $\mathscr{H}^{J}_{\alpha\beta}$ and $\mathscr{K}^{J}_{\alpha\beta;\ell_\alpha\lambda_\alpha}$ possess interpretations as observable partial-wave amplitudes. In this section we give the various representations which support this view. We obtain explicit forms for the asymptotic behavior of the coordinate-space wave functions, written in our partial-wave form. We also give the formulae for the various partial-wave cross sections, find the related form of the optical theorem, and obtain a phase-shift parametrization of the observable amplitudes.

We consider the wave-function properties first. In Ref. (4) it is proved that the form the exact wave function, $\psi^{(+)}(\vec{x}_\alpha, \vec{y}_\alpha; \vec{p}'_\beta)$, has in the limit $|\vec{x}_\alpha| \to \infty$, $\vec{y}_\alpha = \text{const}$ is given by,

$$\psi^{(+)}(\vec{x}_\alpha, \vec{y}_\alpha; \vec{p}'_\beta, E + i0) \underset{|x_\alpha| \to \infty}{\sim} \frac{\psi_\alpha(\vec{y}_\alpha)}{(2\pi)^{3/2}} \left\{ \delta_{\alpha\beta} e^{+i\vec{p}'_\beta \cdot \vec{x}_\beta} - n_\alpha (2\pi)^2 \frac{e^{ip^f_\alpha |\vec{x}_\alpha|}}{|\vec{x}_\alpha|} \mathscr{H}_{\alpha\beta}(p^f_\alpha \hat{x}_\alpha; p'_\beta; E + i0) \right\} \tag{4.1}$$

where p^f_α is defined by (1.8), the coordinates $\vec{x}_\alpha$, $\vec{y}_\alpha$ are defined in Section 1, and E is the incident scattering energy. ψ_α is the unit-normalized bound state of the Hamiltonian $q^2_\alpha/2\mu_\alpha + v_\alpha$. The terms on the right are the leading terms in $|x_\alpha|^{-1}$ and higher order terms are omitted.

The partial-wave version of Eq. (4.1) is given by substituting expansion (3.14) for $\mathscr{H}_{\alpha\beta}$ and using the spherical-harmonic expansion of $e^{i\vec{p}'_\beta \cdot \vec{x}_\beta}$. One then takes the inner product of Eq. (4.1) with respect to $Q^{JM*}_{\ell_\alpha\lambda_\alpha}(\hat{x}_\alpha, \hat{y}_\alpha)$ and $Y_{JM}(\hat{p}'_\beta)$ to obtain,

$$\psi^{(+)J}_{\ell_\alpha\lambda_\alpha}(x_\alpha y_\alpha; p'_\beta) = \int Q^{JM*}_{\ell_\alpha\lambda_\alpha}(\hat{x}_\alpha \hat{y}_\alpha) \; \psi^{(+)}(\vec{x}_\alpha, \vec{y}_\alpha; \vec{p}'_\beta; E + i0) \; Y_{JM}(\hat{p}'_\beta) \, d\hat{x}_\alpha d\hat{y}_\alpha d\hat{p}'_\beta ,$$

$$\psi^{(+)J}_{\ell_\alpha \lambda_\alpha}(x_\alpha y_\alpha ; p'_\beta) \underset{x_\alpha \to \infty}{\sim}$$

$$\frac{\delta_{\ell_\alpha 0}\delta_{\lambda_\alpha 0}}{(2\pi)^{3/2}} \psi^{o}_{\alpha}(y) \left\{ \delta_{\alpha\beta} 4\pi i^J j_J(p'_\beta x_\beta) - n_\alpha (2\pi)^2 \frac{e^{ip^f_\alpha x_\alpha}}{x_\alpha} \mathscr{H}^J_{\alpha\beta}(p^f_\alpha ; p'_\beta ; E + i0). \right\} \tag{4.2}$$

Here, $\psi_\alpha(\vec{y}) = (4\pi)^{-1/2} \psi^o_\alpha(y)$ defines a normalized $\ell_\alpha = 0$ wavefunction ψ^o_α. The j_J are the customary spherical-Bessel functions. As is expected the x_α, y_α dependence on the right-hand side of Eq. (4.2) occurs only in the Bessel function and the radiation term $\frac{e^{ip^f_\alpha x_\alpha}}{x_\alpha}$. Clearly $\mathscr{H}^J_{\alpha\beta}$ has the interpretation of the partial-wave amplitude of total angular momentum J.

Now let us discuss the breakup asymptotic limit. As explained in Ref. (4) the contribution of the exact wave function to the breakup is given by the limit $|\vec{x}_\beta| \to \infty$, $|\vec{y}_\beta| \to \infty$ with the condition that $|\vec{x}_\beta||\vec{y}_\beta|^{-1}$ remain constant. The leading contribution in this limit is,

$$\psi^{(+)}(\vec{x}_\beta, \vec{y}_\beta ; \vec{p}'_\beta) \underset{\substack{|x_\beta| \to \infty \\ |y_\beta||y_\beta|^{-1} = \text{const}}}{\sim}$$

$$\left\{ e^{\frac{i\pi}{4}} 4(2\pi)^{1/2} n_\beta^{3/2} \mu_\beta^{3/2} E^{3/4} \frac{e^{i\sqrt{E}\tilde{\rho}}}{\tilde{\rho}^{5/2}} \right\} \mathscr{B}_{0\beta}(p^f_\beta \hat{x}_\beta, q_\beta \hat{y}_\beta ; \vec{p}'_\beta ; E + i0) \tag{4.3}$$

where $\mathscr{B}_{0\beta}$ is given by (1.11). The invariant metric, $\tilde{\rho}$, is defined as

$$\tilde{\rho} = (2n_\beta x_\beta^2 + 2\mu_\beta y_\beta^2)^{1/2}, \qquad \beta = 1, 2, 3 \tag{4.4}$$

In this case we want to take matrix elements of Eq. (4.3) with respect to $\tilde{\mathscr{D}}^J_{M\lambda_\beta}(R_\beta)$ and $Y_{JM}(\hat{p}'_\beta)$. We note that in order to carry out the evaluation of this inner product that all the terms $\mathscr{K}_{\alpha\beta}$ appearing in $\mathscr{B}_{0\beta}$ must be expressed in the same coordinate system. Here we have selected the coordinate system

with unrotated $\vec{p}_\beta$ oriented along z to be the common coordinate system. Our Euler angle representation of $\mathscr{K}_{\alpha\beta}$ is

$$\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha \vec{q}_\alpha; \vec{p}'_\beta) = \sum_{\lambda_\alpha} \widetilde{\mathscr{D}}^J_{M\lambda_\alpha}(\psi_\alpha, \theta_\alpha, \phi_\alpha)\ \mathscr{K}^{JM}_{\alpha\beta;\lambda_\alpha}(P_\alpha; p'_\beta)\ Y^*_{JM}(p'_\beta) \qquad (4.5)$$

where,

$$\mathscr{K}^{JM}_{\alpha\beta;\lambda_\alpha}(P_\alpha; p'_\beta) = \sum_{\ell_\alpha} \sqrt{2\pi}\, Y_{\ell_\alpha\lambda_\alpha}(\Gamma_\alpha, 0)\ \mathscr{K}^{JM}_{\alpha\beta;\ell_\alpha\lambda_\alpha}(p_\alpha q_\alpha; p'_\beta) \qquad (4.6)$$

To obtain the representation in the R_β basis for all α we use $R_\alpha = R_\beta R_y(\theta_{\alpha\beta})$ and the group property of the $\mathscr{D}$ functions.

$$\mathscr{K}_{\alpha\beta}(\vec{p}_\alpha, \vec{q}_\alpha; \vec{p}'_\beta) = \sum_{\lambda'_\beta \lambda_\alpha} \widetilde{\mathscr{D}}^J_{M\lambda'_\beta}(\psi_\beta, \theta_\beta, \phi_\beta)\ d^J_{\lambda'_\beta\lambda_\alpha}(\theta_{\alpha\beta})\ \mathscr{K}^{JM}_{\alpha\beta;\lambda_\alpha}(P_\alpha; p'_\beta)\ Y^*_{JM}(\hat{p}'_\beta) \qquad (4.7)$$

We may now substitute Eq. (4.7) into (4.3) and form the inner product with respect to $\widetilde{\mathscr{D}}^{J*}_{M\lambda_\beta}(R_\beta)$ and $Y_{JM}(\hat{p}'_\beta)$. We obtain,

$$\psi^{(+)J}_{\lambda_\beta}(x_\beta, y_\beta, \Gamma_\beta; p'_\beta) = \int \widetilde{\mathscr{D}}^{J*}_{M\lambda_\beta}(R_\beta)\ \psi^{(+)}(\vec{x}_\beta, \vec{y}_\beta; \vec{p}'_\beta)\ Y_{JM}(\hat{p}'_\beta)\ dR_\beta d\hat{p}'_\beta \qquad (4.8)$$

$$\psi^{(+)J}_{\lambda_\beta}(x_\beta, y_\beta, \Gamma_\beta; p'_\beta) \underset{\substack{|x_\beta| \to \infty \\ |x_\beta|\,|y_\beta|^{-1} = \text{const}}}{\sim} \left\{ e^{\frac{i\pi}{4}}\, 4(2\pi)^{1/2}\, n_\beta^{3/2}\, \mu_\beta^{3/2}\, E^{3/4}\, \frac{e^{i\sqrt{E}\,\widetilde{\rho}}}{\widetilde{\rho}^{5/2}} \right\}$$

$$\times \left\{ - \sum_{\alpha\lambda_\alpha} d^J_{\lambda_\beta\lambda_\alpha}(\theta_{\alpha\beta})\ \mathscr{K}^{JM}_{\alpha\beta;\lambda_\alpha}(p^f_\beta, q_\beta, \Gamma_\beta; p'_\beta; E + i0) \right\} \qquad (4.9)$$

The physical meaning of our integration over R_β is to average over all orientations of the plane formed by $\vec{p}_1, \vec{p}_2, \vec{p}_3$ with a weigthing factor $\widetilde{\mathscr{D}}^J_{M\lambda_\beta}$. After this

average is performed then the only independent coordinate variables left in the mixed representation above are the distances x_β, y_β and Γ_β — the angle between $\vec{x}_\beta$ and $\vec{y}_\beta$. We note the term in the first curly brackets, which in this breakup limit represents the radiation condition, is independent of all our integration variables. If we want to display the dependence on the Γ's we can introduce (4.6) into the right-hand side of (4.9).

We shall now consider the various partial-wave cross sections that exist in this problem. Let us take up the elastic and rearrangement cross sections first. By combining the general cross section given by (1.7) with the representation of $\mathscr{H}_{\alpha\beta}$ found in (3.14) we find

$$\frac{d\sigma_{\alpha\beta}(\hat{x}_\alpha)}{d\Omega_{\hat{x}_\alpha}} = (2\pi)^4 \, n_\alpha n_\beta \, \frac{p^f_\alpha}{p'_\beta} \left| \sum_J \frac{2J+1}{4\pi} \, \mathscr{H}^J_{\alpha\beta}(p^f_\alpha ; p'_\beta) \, P_J(\hat{x}_\alpha \cdot \hat{p}'_\beta) \right|^2 \tag{4.10}$$

In the above and following formula we shall omit the common energy argument, $E = \tilde{p}'^2_\beta - \chi^2_\beta + i0$. If we construct the total cross section by integrating the differential cross section relative to $d\Omega_{\hat{x}_\alpha}$, we have

$$\sigma_{\alpha\beta} = \int \frac{d\sigma_{\alpha\beta}(\hat{x}_\alpha)}{d\Omega_{\hat{x}_\alpha}} \, d\Omega_{\hat{x}_\alpha} = (2\pi)^4 \, n_\alpha n_\beta \, \frac{p^f_\alpha}{p'_\beta} \sum_J \frac{(2J+1)}{4\pi} \left| \mathscr{H}^J_{\alpha\beta}(p^f_\alpha ; p'_\beta) \right|^2 \tag{4.11}$$

From Eq. (4.11) we can read off the partial-wave cross sections, $\sigma^J_{\alpha\beta}$,

$$\sigma^J_{\alpha\beta} = \frac{(2J+1)}{4\pi} (2\pi)^4 \, n_\alpha n_\beta \, \frac{p^f_\alpha}{p'_\beta} \left| \mathscr{H}^J_{\alpha\beta}(p^f_\alpha ; p'_\beta) \right|^2 \tag{4.12}$$

where,

$$\sigma_{\alpha\beta} = \sum_J \sigma^J_{\alpha\beta} \, . \tag{4.13}$$

Equation (4.12) is an explicit verification that $\mathscr{H}^J_{\alpha\beta}$ is the physical partial-wave amplitude.

Next we must determine the partial wave forms of the breakup cross sections. We start with the general breakup cross section given by (1.10). Into (1.10) we insert the form of $\mathscr{K}_{\alpha\beta}$ given by (4.7). We have then

$$\frac{d\sigma_{0\beta}(\hat{x}_\beta, \hat{y}_\beta, q_\beta)}{d\Omega_{\hat{x}_\beta}\, d^3\vec{q}_\beta} = (2\pi)^4\, n_\beta^2\, \frac{p_\beta^f}{p'_\beta} \left| \sum_{\substack{\alpha\lambda'_\beta\lambda_\alpha \\ JM}} \tilde{\mathscr{D}}^J_{M\lambda'_\beta}(R_\beta)\, d^J_{\lambda'_\beta\lambda_\alpha}(\theta_{\alpha\beta})\, \mathscr{K}^J_{\alpha\beta;\lambda_\alpha}(P_\alpha; p'_\beta)\, Y^*_{JM}(\hat{p}'_\beta) \right|^2 \tag{4.14}$$

One can show using the results of Appendix A that

$$d\Omega_{\hat{x}_\beta}\, d\Omega_{\hat{y}_\beta}\, q_\beta^2\, dq_\beta = \frac{p_\alpha p_\gamma}{p_\beta}\, dp_\alpha\, dp_\gamma\, dR_\beta \tag{4.15}$$

Before integrating (4.14) with respect to dR_β, we simplify it by choosing the z axis to lie along $\vec{p}'_\beta$ so, $Y^*_{JM}(\vec{p}'_\beta) = \left(\frac{2J+1}{4\pi}\right)^{1/2} \delta_{M0}$. The integration gives

$$\frac{d\sigma_{0\beta}}{dp_\alpha dp_\gamma} = (2\pi)^4\, n_\beta^2\, \frac{p_\alpha p_\gamma}{p'_\beta} \sum_{J\lambda'_\beta} \frac{2J+1}{4\pi} \left| \sum_{\alpha\lambda_\alpha} d^J_{\lambda'_\beta\lambda_\alpha}(\theta_{\alpha\beta})\, \mathscr{K}^J_{\alpha\beta;\lambda_\alpha}(p_1, p_2, p_3; p'_\beta) \right|^2 \tag{4.16}$$

The partial-wave cross section is clearly

$$\frac{d\sigma^J_{0\beta}}{dp_\alpha dp_\gamma} = (2J+1)\frac{(2\pi)^3}{2}\, n_\beta^2\, \frac{p_\alpha p_\gamma}{p'_\beta} \sum_{\lambda'_\beta} \left| \sum_{\alpha\lambda_\alpha} d^J_{\lambda'_\beta\lambda_\alpha}(\theta_{\alpha\beta})\, \mathscr{K}^J_{\alpha\beta;\lambda_\alpha}(p_1, p_2, p_3; p'_\beta) \right|^2 \tag{4.17}$$

and

$$\frac{d\sigma_{0\beta}}{dp_\alpha dp_\gamma} = \sum_J \frac{d\sigma^J_{0\beta}}{dp_\alpha dp_\gamma}$$

We will now employ our partial-wave cross sections in the unitarity relation to obtain a partial-wave version of the optical theorem. The off-shell unitarity relation is given in Ref. (4) and is

$$\begin{aligned}
&\mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; E'_\beta + i0) - \mathscr{H}_{\alpha\beta}(\vec{p}_\alpha; \vec{p}'_\beta; E'_\beta - i0) \\
&= -2i\pi \int \mathscr{B}^*_{0\alpha}(\vec{p}''\vec{q}''; \vec{p}_\alpha; E'_\beta + i0)\, \delta(\tilde{p}''^2 + \tilde{q}''^2 - E'_\beta)\, \mathscr{B}_{0\beta}(\vec{p}''\vec{q}''; \vec{p}'_\beta; E'_\beta + i0)\, d\vec{p}''\, d\vec{q}'' \\
&- 2i\pi \sum_\gamma \int \mathscr{H}^*_{\gamma\alpha}(\vec{p}''_\gamma; \vec{p}_\alpha; E'_\beta + i0)\, \delta(\tilde{p}''^2_\gamma - \chi^2_\gamma - E'_\beta)\, \mathscr{H}_{\gamma\beta}(\vec{p}''_\gamma; \vec{p}'_\beta; E'_\beta + i0)\, d\vec{p}''_\gamma
\end{aligned} \tag{4.18}$$

Let us now integrate this with respect to $Y^*_{JM}(\hat{p}_\alpha)\, Y_{JM}(\hat{p}'_\beta)\, d\hat{p}_\alpha\, d\hat{p}_\beta$. The left-hand side is

$$\mathscr{H}^J_{\alpha\beta}(p_\alpha; p'_\beta; E'_\beta + i0) - \mathscr{H}^J_{\alpha\beta}(p_\alpha; p'_\beta; E'_\beta - i0) \tag{4.19}$$

Let us compute the second term on the right of (4.18). Using the expansion (3.14) we find at once that the term is

$$-2i\pi \sum_\gamma \int \mathscr{H}^{J*}_{\gamma\alpha}(p''_\gamma; p_\alpha; E'_\beta + i0)\, \delta(\tilde{p}''^2_\gamma - \chi^2_\gamma - E'_\beta)\, \mathscr{H}^J_{\gamma\beta}(p''_\gamma, p'_\beta; E'_\beta + i0)\, p''^2_\gamma dp''_\gamma \tag{4.20}$$

Finally we must evaluate the first term on the right of (4.18). Analogous to (4.8) we define the matrix element of $\mathscr{B}_{0\beta}$ with respect to $\tilde{\mathscr{D}}^{J*}_{M\lambda_\beta}(R_\beta)$ and $Y_{JM}(\hat{p}_\beta)$, viz.

$$\begin{aligned}
\mathscr{B}^J_{0\beta;\lambda_\beta}(p_\alpha p_\beta p_\gamma; p'_\beta; z) &= \int \tilde{\mathscr{D}}^{J*}_{M\lambda_\beta}(R_\beta) \mathscr{B}_{0\beta}(\vec{p}, \vec{q}; \vec{p}'_\beta; z)\; Y_{JM}(\hat{p}'_\beta)\, dR_\beta d\hat{p}'_\beta \\
&= -\sum_{\alpha\lambda_\alpha} d^J_{\lambda_\beta\lambda_\alpha}(\theta_{\alpha\beta})\, \mathscr{K}^J_{\alpha\beta;\lambda_\alpha}(P_\alpha; p'_\beta; z)
\end{aligned} \tag{4.21}$$

And given $\mathscr{B}^{J}_{0\beta;\lambda_\beta}$ we have the representation for $\mathscr{B}_{0\beta}$,

$$\mathscr{B}_{0\beta}(\vec{p}, \vec{q}; \vec{p}'_\beta; z) = \sum_{JM\lambda_\beta} \widetilde{\mathscr{D}}^{J}_{M\lambda_\beta}(R_\beta)\, \mathscr{B}^{J}_{0\beta;\lambda_\beta}(p_\alpha p_\beta p_\gamma; p'_\beta; z) \quad Y^*_{JM}(\hat{p}'_\beta) \tag{4.22}$$

Putting (4.22) in Eq. (4.18) and integrating with respect to the Y_{JM}'s and $\widetilde{\mathscr{D}}^{J*}_{M\lambda_\beta}$ yields,

$$-2\pi i \int \sum_{\lambda_\beta} \mathscr{B}^{J*}_{0\alpha;\lambda_\beta}(p''_\alpha p''_\beta p''_\gamma; p_\alpha; E'_\beta + i0)\ \delta\left(\frac{p''^2_\alpha}{2m_\alpha} + \frac{p''^2_\beta}{2m_\beta} + \frac{p''^2_\gamma}{2m_\gamma} - E'_\beta\right)$$

$$\mathscr{B}^{J}_{0\beta;\lambda_\beta}(p''_\alpha, p''_\beta, p''_\gamma; p'_\beta; E'_\beta + i0)\ p''_\alpha\, dp''_\alpha\, p''_\beta\, dp''_\beta\, p''_\gamma\, dp''_\gamma \tag{4.23}$$

Equating the term (4.19) to the sum of (4.20) and (4.23) gives us the off-shell partial-wave unitarity relations for $\mathscr{H}^{J}_{\alpha\beta}$. In fact, because of the time-reversal property of $\mathscr{H}_{\alpha\beta}$ demonstrated in Section III and $\mathscr{H}_{\alpha\beta}(z) = \mathscr{H}^{*}_{\beta\alpha}(\bar{z})$ the term (4.19) can be written as $2i\ \mathrm{Im}\ \mathscr{H}^{J}_{\alpha\beta}(p_\alpha; p'_\beta; E'_\beta + i0)$.

Specializing to the case where $\alpha = \beta$, putting the relation on-shell by setting $p_\beta = p'_\beta$, we have

$$2i\ \mathrm{Im}\ \mathscr{H}^{J}_{\beta\beta}(p'_\beta, p'_\beta; E'_\beta + i0)$$

$$= -2\pi i \int \sum_{\lambda_\beta} \left| \mathscr{B}^{J}_{0\beta;\lambda_\beta}(p''_\alpha, p''_\beta, p''_\gamma; p'_\beta; E'_\beta + i0) \right|^2$$

$$\delta\left(\frac{p''^2_\alpha}{2m_\alpha} + \frac{p''^2_\beta}{2m_\beta} + \frac{p''^2_\gamma}{2m_\gamma}\right) \quad p''_\alpha\, dp''_\alpha\, p''_\beta\, dp''_\beta\, p''_\gamma\, dp''_\gamma$$

$$-2\pi i \sum_{\gamma} \int \left| \mathscr{H}^{J}_{\gamma\beta}(p''_\gamma; p'_\beta; E'_\beta + i0) \right|^2 \delta(\widetilde{p''^2_\gamma} - \chi^2_\gamma - E'_\beta)\ p''^2_\gamma\, dp''_\gamma \tag{4.24}$$

To obtain the optical theorem we substitute the partial-wave cross sections (4.12) and (4.17) into (4.24) and perform the indicated integrations.

We obtain,

$$\frac{2J+1}{4\pi} \text{ Im } \mathscr{H}^J_{\beta\beta}(p'_\beta;\, p'_\beta;\, E'_\beta) = \frac{-1}{(2\pi)^3} \frac{p'_\beta}{2n_\beta} \sum_{\gamma=0}^{3} \sigma^J_{\gamma\beta} \tag{4.25}$$

If one replaces the transition amplitude, $\mathscr{H}^J_{\beta\beta}$, by the traditional scattering amplitude, f^J_β, where

$$f^J_\beta = -\, 2n_\beta\, 2\pi^2 \mathscr{H}^J_{\beta\beta} \tag{4.26}$$

one obtains that

$$\sigma^J_{TOT} = \sum_{\gamma=0}^{3} \sigma^J_{\gamma_\beta} = \frac{2J+1}{p'_\beta} \text{ Im } f^J_\beta \tag{4.27}$$

We conclude this paper by giving a phase-shift parametrization of our reduced amplitudes. The most direct approach is to parametrize the S-matrices. Thus we consider the multichannel unitarity relation[1,4] in its operator form

$$\delta_{\alpha\beta} = \sum_{\gamma=0}^{3} S^\dagger_{\gamma\alpha} S_{\gamma\beta} \tag{4.28}$$

The operators in this relation have the well known kernel definitions[1,4,11]

$$\begin{aligned}
S_{\alpha\beta}(\vec{p}_\alpha;\, \vec{p}'_\beta) &= \delta_{\alpha\beta}\delta(\vec{p}_\alpha - \vec{p}'_\beta) - 2\pi i\, \delta(E_\alpha - E'_\beta)\, \mathscr{H}_{\alpha\beta}(\vec{p}_\alpha;\, \vec{p}'_\beta;\, E'_\beta + i0) \\
S_{0\beta}(\vec{p},\, \vec{q};\, \vec{p}'_\beta) &= -\, 2\pi i\, \delta(\tilde{p}^2 + \tilde{q}^2 - E'_\beta)\, \mathscr{B}_{0\beta}(\vec{p},\, \vec{q};\, \vec{p}'_\beta;\, E'_\beta + i0) \\
S_{\beta 0}(\vec{p}'_\beta;\, \vec{p},\, \vec{q}) &= -\, 2\pi i\, \delta(\tilde{p}^2 + \tilde{q}^2 - E'_\beta)\, \mathscr{B}_{\beta 0}(\vec{p}'_\beta;\, \vec{p},\, \vec{q};\, E'_\beta + i0)
\end{aligned} \tag{4.29}$$

where $E_\alpha = \tilde{p}^2_\alpha - \chi^2_\alpha$ and $E'_\beta = \tilde{p}'^2_\beta - \chi^2_\beta$.

In terms of these kernels the relation (4.28) becomes,

$$\begin{aligned}
\delta_{\alpha\beta}\delta(\vec{p}_\alpha - \vec{p}'_\beta) = &\sum_{\gamma=1}^{3} \int d\vec{p}''_\gamma\, S^\dagger_{\gamma\alpha}(\vec{p}_\alpha;\, \vec{p}''_\gamma)\, S_{\gamma\beta}(\vec{p}''_\gamma;\, \vec{p}'_\beta) \\
&+ \int d\vec{p}''\, d\vec{q}''\, S^\dagger_{0\alpha}(\vec{p}_\alpha;\, \vec{p}'',\vec{q}'')\, S_{0\beta}(\vec{p}'',\, \vec{q}'';\, \vec{p}'_\beta)
\end{aligned} \tag{4.30}$$

The partial wave two-body like S-matrices are defined by

$$S^J_{\alpha\beta}(p_\alpha;\, p'_\beta) \equiv \int Y^*_{JM}(\hat{p}_\alpha)\, S_{\alpha\beta}(\vec{p}_\alpha;\, \vec{p}'_\beta)\, Y_{JM}(\hat{p}'_\beta)\, d\hat{p}_\alpha d\hat{p}'_\beta \tag{4.31}$$

The associated breakup partial-wave S-matrix is

$$S^J_{0\beta;\lambda_\beta}(P;\, p'_\beta) = \int \tilde{\mathscr{D}}^{J*}_{M\lambda_\beta}(R_\beta)\, S_{0\beta}(\vec{p},\, \vec{q};\, \vec{p}'_\beta)\, Y_{JM}(\hat{p}'_\beta)\, dR_\beta\, d\hat{p}'_\beta \tag{4.32}$$

The definitions (4.31) and (4.32) imply

$$S^J_{\alpha\beta}(p_\alpha;\, p'_\beta) = \delta_{\alpha\beta}\, \frac{\delta(p_\alpha - p'_\beta)}{p'^2_\beta} - 2\pi i\, \delta(E_\alpha - E'_\beta)\, \mathscr{H}^J_{\alpha\beta}(p_\alpha;\, p'_\beta;\, E'_\beta + i0) \tag{4.33}$$

and

$$S^J_{0\beta;\lambda_\beta}(P;\, p'_\beta) = -2\pi i\, \delta(\tilde{p}^2 + \tilde{q}^2 - E'_\beta)\, \mathscr{B}^J_{0\beta;\lambda_\beta}(P;\, p'_\beta;\, E'_\beta + i0) \tag{4.34}$$

where $\mathscr{B}^J_{0\beta;\lambda_\beta}$ is given by (4.21).

Now if we integrate Eq. (4.30) with respect to $Y^*_{JM}(\hat{p}_\alpha)\, Y_{JM}(\hat{p}'_\beta)\, d\hat{p}_\alpha d\hat{p}'_\beta$ and use the representations that follow from (4.31) and (4.32) we obtain

$$\delta_{\alpha\beta}\, \frac{\delta(p_\alpha - p'_\beta)}{p'^2_\beta} = \sum_\gamma \int S^{*J}_{\gamma\alpha}(p''_\gamma;\, p_\alpha)\, S^J_{\gamma\beta}(p''_\gamma;\, p'_\beta)\, p''^2_\gamma\, dp_\gamma$$

$$+ \sum_{\lambda_\beta} \int S^{*J}_{0\alpha,\lambda_\beta}(P'';\, p_\alpha)\, S^J_{0\beta,\lambda_\beta}(P'';\, p'_\beta)\, dP'' \tag{4.35}$$

Because of the presence of the energy conserving δ-functions the partial-wave S-matrix forms (4.33) and (4.34) are not suited for a phase-shift representation. We therefore factor out these δ-functions and define an appropriate reduced

S- matrix, $s^J_{\alpha\beta}$ as follows,

$$S^J_{\alpha\beta}(p_\alpha;\, p'_\beta) \equiv \frac{\delta(E_\alpha - E'_\beta)}{(n_\alpha p_\alpha n_\beta p'_\beta)^{1/2}}\; s^J_{\alpha\beta}(p_\alpha;\, p'_\beta) \tag{4.36}$$

where it follows from (4.33) that $s^J_{\alpha\beta}$ is

$$s^J_{\alpha\beta}(p_\alpha;\, p'_\beta) = \delta_{\alpha\beta} - 2\pi i(n_\alpha p_\alpha\, n_\beta p'_\beta)^{1/2}\; \mathscr{H}^J_{\alpha\beta}(p_\alpha;\, p'_\beta;\, E'_\beta + i0) \tag{4.37}$$

The choice of factors in the square root here have been determined on the basis that $s^J_{\alpha\beta}$ should share the time reversal property that $S^J_{\alpha\beta}$ possesses. If we insert (4.37) into (4.35) we get for the case where $\alpha = \beta$,

$$\frac{\delta(E_\beta - E'_\beta)}{n_\beta p'_\beta} = \frac{\delta(E_\beta - E'_\beta)}{(n_\beta p_\beta n_\beta p'_\beta)^{1/2}} \Bigg\{ \sum_\gamma s^{*J}_{\gamma\beta}(p^f_\gamma;\, p'_\beta)\, s^J_{\gamma\beta}(p^f_\gamma;\, p'_\beta)$$

$$+ \int 4\pi^2 n_\beta (n_\beta p_\beta n_\beta p'_\beta)^{1/2} \sum_{\lambda_\beta} \mathscr{B}^{*J}_{0\beta;\lambda_\beta}(p''_\alpha, p^f_\beta, p''_\gamma; p'_\beta)\, \mathscr{B}^J_{0\beta;\lambda_\beta}(p''_\alpha, p^f_\beta;\, p''_\gamma;\, p'_\beta)\, p''_\alpha p''_\gamma\, dp''_\alpha\, dp''_\gamma \Bigg\} \tag{4.38}$$

Equating coefficients of the δ-function gives

$$1 = \sum_{\gamma=1}^{3} \left| s^J_{\gamma\beta}(p'_\beta) \right|^2 + 4\pi^2 n_\beta p'_\beta \int n_\beta \sum_{\lambda_\beta} \left| \mathscr{B}^J_{0\beta}(p''_\alpha, p^f_\beta, p''_\gamma;\, p'_\beta) \right|^2 p''_\alpha\, p''_\gamma\, dp''_\alpha\, dp''_\gamma \tag{4.39}$$

In the first term on the right we indicate just one momentum argument for $s^J_{\gamma\beta}$. We have used the on-shell energy condition $\tilde{p}^f_\gamma = (\tilde{p}'^2_\beta - \chi^2_\beta + \chi^2_\gamma)^{1/2}$ to eliminate the p^f_γ dependence. The second term on the right may be expressed in terms of the breakup cross section. Using (4.17) one has for this last term

$$\frac{1}{2J+1}\; \frac{1}{\pi}\; p'^2_\beta\; \sigma^J_{0\beta}$$

The final form (4.39) takes is then

$$1 = \sum_{\gamma=1}^{3} \left| s^J_{\gamma\beta}(p'_\beta) \right|^2 + \frac{1}{2J+1} \; \frac{1}{\pi} \; p'^2_\beta \; \sigma^J_{0\beta} \tag{4.40}$$

This result justifies a phase shift representation for $s^J_{\alpha\beta}$. Specifically since

$$\left| s^J_{\alpha\beta}(p'_\beta) \right|^2 \leq 1 \qquad \alpha = 1,2,3 \text{ all } p'_\beta \tag{4.41}$$

we may introduce

$$s^J_{\alpha\beta}(p'_\beta) = \eta^J_{\alpha\beta}(p'_\beta) \; e^{2i\,\delta^J_{\alpha\beta}(p'_\beta)} \qquad \alpha = 1,2,3. \tag{4.42}$$

where $\delta^J_{\alpha\beta}$ is real and the absorbtion parameter $\eta^J_{\alpha\beta}$ is

$$\left| \eta^J_{\alpha\beta} \right| \leq 1 \tag{4.43}$$

Using (4.37) one finds that the transition amplitudes are parametrized by

$$\mathscr{H}^J_{\beta\beta}(p'_\beta;\, p'_\beta;\, E'_\beta + i0) = \frac{-1}{2\pi i n_\beta p'_\beta} \left(\eta^J_{\beta\beta} \, e^{2i\,\delta^J_{\beta\beta}} - 1 \right), \tag{4.44}$$

$$\mathscr{H}^J_{\alpha\beta}(p^f_\alpha;\, p'_\beta;\, E'_\beta + i0) = - \frac{\eta^J_{\alpha\beta} \, e^{2i\,\delta^J_{\alpha\beta}}}{2\pi i (n_\beta p'_\beta n_\alpha p^f_\alpha)^{1/2}} \, . \tag{4.45}$$

The parametrization given in Eq. (4.44) is the usual elastic channel phase-shift representation. One could also obtain this phase-shift representation starting from the asymptotic wave function form (4.2). In this alternate approach $j_J(p'_\beta x_\beta)$ is expanded for large x_β. The resultant coefficient for the term with the behavior $\dfrac{e^{ip'_\beta x_\beta}}{x_\beta}$ is proportional to the elastic channel S-matrix. The result is identical with (4.37) for $\alpha = \beta$. However, this wave-function approach does not give one any guidance about how to construct a reduced S-matrix appropriate for rearrangement scattering.

One final observation about the unitarity representation (4.40) is that it gives a unitary bound on the partial-wave breakup cross section plus rearrangement cross-sections, namely

$$\sigma^J_{\alpha\beta} + \sigma^J_{\gamma\beta} + \sigma^J_{0\beta} = \frac{(2J+1)\pi}{p'^2_\beta}\left(1 - \left|\eta^J_{\beta\beta}\right|^2\right) \leq \frac{(2J+1)\pi}{p'^2_\beta} \tag{4.46}$$

In obtaining this result from Eq. (4.40) it is necessary to use Eq. (4.44) and (4.45) together with the partial-wave cross-section definitions (4.12).

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APPENDIX A

We give here some of the Jacobians that are useful in our angular momentum reductions. Consider first the integral

$$\int d\vec{p}_\alpha \int d\vec{q}_\alpha = \int p_\alpha^2 \, dp_\alpha \, d\Omega_{\hat{p}_\alpha} \int q_\alpha^2 \, dq_\alpha \, d\Omega_{\hat{q}_\alpha} \tag{A.1}$$

with our convention for vectors

$$\hat{p}_\alpha = R(\psi_\alpha, \theta_\alpha, \phi_\alpha)\,\hat{z}, \qquad \hat{q}_\alpha = R(\psi_\alpha, \theta_\alpha, \phi_\alpha) R_y(\Gamma_\alpha)\,\hat{z}$$

so

$$\begin{aligned} d\Omega_{\hat{p}_\alpha} &= d\cos\theta_\alpha \, d\psi_\alpha \\ d\Omega_{\hat{q}_\alpha}\Big|_{\hat{p}_\alpha} &= d\cos\Gamma_\alpha \, d\phi_\alpha \end{aligned} \tag{A.2}$$

Thus the integral (A.1) becomes,

$$\int d\vec{p}_\alpha \int d\vec{q}_\alpha = \int p_\alpha^2 dp_\alpha \, q_\alpha^2 dq_\alpha \, d\cos\Gamma_\alpha \, dR_\alpha, \tag{A.3}$$

where $dR_\alpha = d\psi_\alpha \, d\cos\theta_\alpha \, d\phi_\alpha$. One may obtain another form for this integral by using (B.4) to show

$$\frac{\partial \cos\Gamma_\alpha}{\partial p_\beta}\Bigg|_{p_\alpha, q_\alpha} = \frac{m_\gamma}{\mu_\alpha} \frac{p_\beta}{p_\alpha q_\alpha}$$

So,

$$\int d\vec{p}_\alpha \int d\vec{q}_\alpha = \int p_\alpha q_\alpha \, dp_\alpha \, dq_\alpha \, p_\beta \, dp_\beta \, \frac{m_\gamma}{\mu_\alpha} \, dR_\alpha \tag{A.4}$$

Now if one takes the partial derivative of the energy relation

$$\frac{q_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha} = \frac{p_\alpha^2}{2m_\alpha} + \frac{p_\beta^2}{2m_\beta} + \frac{p_\gamma^2}{2m_\gamma}$$

for fixed p_α, p_β one has

$$q_\alpha dq_\alpha \Big|_{p_\alpha, p_\beta} = \frac{\mu_\alpha}{m_\gamma} p_\gamma dp_\gamma$$

Thus the integral can be written

$$\int d\vec{p}_\alpha \int d\vec{q}_\alpha = \int p_\alpha p_\beta p_\gamma \, dp_\alpha \, dp_\beta \, dp_\gamma \, dR_\alpha \tag{A.5}$$

Further by comparing (A.5) with (A.3) it is clear that our three coordinate systems p_α, q_α, $\cos \Gamma_\alpha$ where $\alpha = 1, 2, 3$ have an invariant measure, i.e.,

$$p_1 p_2 p_3 \, dp_1 \, dp_2 \, dp_3 = p_\alpha^2 \, dp_\alpha \, q_\alpha^2 \, dq_\alpha \, d\cos \Gamma_\alpha, \qquad \alpha = 1, 2, 3 \, . \tag{A.6}$$

APPENDIX B

In this Appendix we give an explicit algebraic expression for the kinematic kernel V^J which appears in Section III. The kernel in question is

$$V^J(p_\alpha; p''_\gamma, q''_\gamma)_{\ell_\alpha \lambda_\alpha; \ell''_\gamma \lambda''_\gamma} = \frac{(2\pi) p''_\gamma q''_\gamma}{p_\alpha^2} \int_{-1}^{+1} \delta(p_\alpha - p''_\alpha) Y^*_{\ell_\alpha \lambda_\alpha}(\Gamma''_\alpha, 0)\, d^J_{\lambda_\alpha \lambda''_\gamma}(\theta''_{\gamma\alpha})\, Y_{\ell''_\gamma \lambda''_\gamma}(\Gamma''_\gamma, 0)\, d\cos\Gamma''_\gamma,$$

$$\gamma \neq \alpha. \qquad \text{(B.1)}$$

In the integral of (B.1) we must express all the angle dependencies of Γ''_α, Γ''_γ and $\theta''_{\gamma\alpha}$ as a function of the variables $(p_\alpha, p''_\gamma, q''_\gamma)$. Further in order to exploit the δ function in (B.1) we need the Jacobian relating p''_α and $\cos\Gamma''_\gamma$ for fixed p''_γ, q''_γ.

We examine first some of the kinematics needed for this calculation. The internal momentum of cluster α is $\vec{q}_\alpha$ and is defined by

$$\vec{q}_\alpha = \frac{m_\beta \vec{p}_\gamma - m_\gamma \vec{p}_\beta}{m_\beta + m_\gamma}, \qquad \alpha = 1,2,3;\ \beta \neq \alpha \qquad \text{(B.2)}$$

If we use $\vec{p}_\alpha + \vec{p}_\beta + \vec{p}_\gamma = 0$ to re-express the momentum $\vec{p}_\beta$ in (B.2) we find the following representation for $\vec{q}_\alpha$.

$$\vec{q}_\alpha = \pm \vec{p}_{\alpha\pm1} \pm \frac{m_{\alpha\pm1}}{m_{\alpha+1} + m_{\alpha\mp1}} \vec{p}_\alpha, \qquad \alpha = 1,2,3\,. \qquad \text{(B.3)}$$

Equation (B.3) represents 6 relations between the $\vec{p}_\alpha$ and $\vec{q}_\alpha$. The convention for the right-hand side is that either all the upper or all the lower signs must be used together. Furthermore the indices on the masses and momenta are understood to be cyclical over (1,2,3). If we take the second term on the right of (B.3) to the left and square that equation we have,

$$p^2_{\alpha\pm1} = q^2_\alpha + \left(\frac{\mu_\alpha p_\alpha}{m_{\alpha\mp1}}\right)^2 \mp 2\,\frac{\mu_\alpha}{m_{\alpha\mp1}}\, p_\alpha q_\alpha \cos\Gamma_\alpha\,. \qquad \text{(B.4)}$$

From (B.4) it is easy to show that,

$$\cos\Gamma_{\alpha+1} = \pm \frac{p_\alpha^2 - q_{\alpha\pm1}^2 - \left(\frac{m_\alpha p_{\alpha\pm1}}{m_\alpha + m_{\alpha\mp1}}\right)^2}{2\left(\frac{m_\alpha}{m_\alpha + m_{\alpha\mp1}}\right) p_{\alpha\pm1} q_{\alpha\pm1}} \quad . \tag{B.5}$$

If in the integral we specify p''_γ, q''_γ and $p''_\alpha = p_\alpha$ then these three variables fix the momenta variables p''_1, p''_2 and p''_3 and all angles associated with these vectors, i.e., $\theta''_{\gamma\alpha}$, Γ''_γ. Let us denote the so determined angles by $\theta^{o''}_{\alpha\gamma}$, $\Gamma^{o''}_\gamma$. We evaluate (B.1) by using,

$$\int_a^b \delta(f(x))\, g(x)\, dx = \sum_i \frac{g(x_i)}{\left|\frac{\partial f}{\partial x}\right|_{x=x_i}} \quad , \tag{B.6}$$

where $x_i \in [a,b]$ and are such that $f(x_i) = 0$. Here we have,

$$\left|\frac{\partial p''_\alpha}{\partial \cos\Gamma''_{\alpha\pm1}}\right|_{\cos\Gamma''_{\alpha\pm1} = \cos\Gamma^{o''}_{\alpha\pm1}} = \frac{m_\alpha q''_{\alpha\pm1} p''_{\alpha\pm1}}{(m_\alpha + m_{\alpha\mp1}) p_\alpha} \quad , \tag{B.7}$$

where,

$$\cos\Gamma^{o''}_{\alpha\pm1} \in [-1, 1] \tag{B.8}$$

This last condition implies the δ function integral is zero unless the constraint

$$p_\alpha - \frac{m_\alpha}{m_\alpha + m_{\alpha\mp1}}\, p''_{\alpha\pm1} \le q''_{\alpha\pm1} \le p_\alpha + \frac{m_\alpha}{m_\alpha + m_{\alpha\mp1}}\, p''_{\alpha\pm1} \tag{B.9}$$

is satisfied. So our expression for V^J is, when (B.8) is satisfied,

$$V^J(p_\alpha;\, p''_{\alpha\pm1}, q''_{\alpha\pm1})_{\ell_\alpha \lambda_\alpha;\, \ell''_{\alpha\pm1}, \lambda''_{\alpha\pm1}}$$

$$= \frac{(2\pi)}{p_\alpha} \left(\frac{m_\alpha + m_{\alpha\mp1}}{m_\alpha}\right) Y^*_{\ell_\alpha \lambda_\alpha}(\Gamma^{o''}_\alpha, 0)\, d^J_{\lambda_\alpha \lambda''_{\alpha\pm1}}(\theta^{o''}_{\alpha\pm1,\alpha})\, Y_{\ell''_{\alpha\pm1}, \lambda''_{\alpha\pm1}}(\Gamma^{o''}_{\alpha\pm1}, 0), \tag{B.10}$$

otherwise the value of V^J is zero.

The only remaining problem is to determine the three angles $\Gamma^{o''}_{\alpha\pm1}$, $\Gamma^{o''}_{\alpha}$, $\theta^{o''}_{\alpha\pm1,\alpha}$. The expression (B.4) immediately gives the first angle, viz.

$$\cos\Gamma^{o''}_{\alpha\pm1} = \pm \frac{p^2_\alpha - q''^2_{\alpha\pm1} - [m_\alpha p''_{\alpha\pm1}/(m_\alpha + m_{\alpha\mp1})]^2}{2q''_{\alpha\pm1}p''_{\alpha\pm1}(m_\alpha+m_{\alpha\mp1})^{-1}m_\alpha} . \qquad (B.11)$$

Next, let us consider the angle $\theta^{o''}_{\alpha\pm1,\alpha}$. We recall the formula relating $\sin^2\theta_{\gamma\alpha}$ and $\sin^2\Gamma_\gamma$ derived by Omnes,[5]

$$\sin^2\theta''_{\alpha\pm1,\alpha} = \sin^2\Gamma^{o''}_{\alpha\pm1} \frac{q''^2_{\alpha\pm1}}{p''^2_\alpha} \qquad (B.12)$$

This relation gives us $\sin\theta''_{\alpha\pm1,\alpha}$ in terms of $p''_{\alpha\pm1}$, $q''_{\alpha\pm1}$ and p_α.

Finally we need to find $\cos\Gamma^{o''}_{\alpha}$. From (B.4) we have

$$\cos\Gamma^{o''}_{\alpha} = \mp \frac{p''^2_{\alpha\pm1} - q''^2_\alpha - \left(\frac{m_{\alpha\pm1}}{m_{\alpha+1}+m_{\alpha-1}}\right)^2 p^2_\alpha}{2\left(\frac{2m_{\alpha\pm1}}{m_{\alpha+1}+m_{\alpha-1}}\right)q''_\alpha p_\alpha} \qquad (B.13)$$

We note that q''_α occurs on the right hand side of Eq. (B.13). This must be expressed in terms of p_α, $p''_{\alpha\pm1}$, $q''_{\alpha\pm1}$. This can be done by using the condition associated with energy conservation, viz.

$$\frac{q''^2_\alpha}{2\mu_\alpha} = \frac{q''^2_{\alpha\pm1}}{2\mu_{\alpha\pm1}} + \frac{p''^2_{\alpha\pm1}}{2n_{\alpha\pm1}} - \frac{p^2_\alpha}{2n_\alpha} \qquad (B.14)$$

Substituting (B.14) into (B.13) give us a closed expression for $\cos\Gamma^{o''}_{\alpha}$ in terms of p_α, $p''_{\alpha\pm1}$, $q''_{\alpha\pm1}$.

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11. Reference 1, p. 81. The relation between the breakup S-matrix and the breakup transition amplitude, $-\sum_{\alpha=1}^{3} \mathcal{K}_{\alpha\beta}$, given by Faddeev has an incorrect sign. Eq. (4.29) in the text gives the correct sign.

FIGURE CAPTION

1. Angular Conventions for Three Body Momentum Variables.

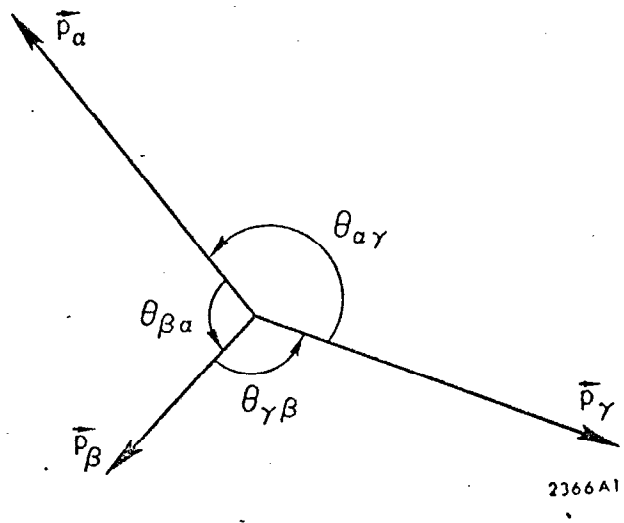

Fig. 1