SLAC-PUB-1284
(T)
August 1973

ON THE RADIATIVE CORRECTIONS TO THE LOW ENERGY THEOREM FOR $\pi_0 \to \gamma\gamma$ IN A NONABELIAN GAUGE THEORY*

Michael S. Chanowitz
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We investigate a class of possible radiative corrections to the low energy theorem for $\pi_0 \to \gamma\gamma$ in theories with massive charged vector mesons. The absence of these corrections is verified to fourth order in a spontaneously broken, non-Abelian gauge model.

(Submitted to Phys. Rev. D Comments and Addenda.)

*Work supported by the U. S. Atomic Energy Commission.

The decay rate $\Gamma(\pi_0 \to \gamma\gamma)$ occupies a privileged position among low energy phenomena as a probe of the quantum numbers of the constituents of hadronic currents.[1] Using PCAC and current algebra assumptions and, in addition, a remarkable theorem[2] due to Adler and Bardeen, the experimental rate is interpreted to support models with three triplets of quarks. The theorem, verified in electrodynamics and the sigma model, tells us that the amplitude $\mathcal{M}(\pi_0 \to \gamma\gamma)$ at $p_\pi^2 = 0$ is determined to any finite order in perturbation theory by the lowest order fermion loop diagram (Fig. 1). In this note, I will discuss the verification of the theorem in models containing fermions and charged vector mesons. Since it is necessary to consider a renormalizable model, I will consider a spontaneously broken gauge model, from the class of such models in which the hadron anomaly is cancelled by the lepton anomaly.[3]

The proof of the theorem is in two steps. In the first step one establishes that the anomaly in chiral Ward-Takahashi identities is unaltered by radiative corrections. For instance, in lepton electrodynamics the anomaly is to any finite order

$$\partial_\mu j_5^\mu = 2mi j_5 + \frac{\alpha}{4\pi} F^{\mu\nu} F^{\alpha\beta} \epsilon_{\mu\nu\alpha\beta} \tag{1}$$

where $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$, $j_5 \equiv \bar{\psi}\gamma_5\psi$, and $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. Similarly in a gauge theory, the existence of an appropriate regularization method presumably assures us that to any finite order in the gauge coupling constant the hadron anomaly is given by just the lowest order contribution.[4,5]

In the second step of the proof of the low energy theorem, in the case of Q.E.D. one uses Eq. (1) to calculate $\mathcal{F}(0)$ defined by

$$\langle \Omega | 2mi\, j_5 | \gamma_1\gamma_2 \rangle \equiv \epsilon^{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \epsilon_1^\mu \epsilon_2^\nu \mathcal{F}(k_1 \cdot k_2) \tag{2}$$

where k_i are photon momenta, $k_i^2 = 0$, and ϵ_i are photon polarizations. Since $<\Omega|\partial_\mu j_5^\mu|\gamma_1\gamma_2>$ is of third order in the k_i, it suffices to calculate

$$<\Omega|F^{\mu\nu}F^{\alpha\beta}|\gamma_1\gamma_2> = -2k_1^\mu k_2^\alpha \epsilon_1^\nu \epsilon_2^\beta + 0(\alpha) \tag{3}$$

so that

$$\mathscr{F}(0) = \frac{\alpha}{2\pi} + 0(\alpha^2) \quad . \tag{4}$$

The proof is complete if it is shown that there is no term $0(\alpha)$ in (3) of the form $k_1^\mu k_2^\alpha \epsilon_1^\nu \epsilon_2^\beta$, so that (4) becomes

$$\mathscr{F}(0) = \frac{\alpha}{2\pi} \tag{5}$$

to any order in α. A contribution to (3) of order $0(\alpha)$ would arise from "final state scattering" of the two photons as in Fig. 2. But Fig. 2 is of third order in the k_i: one power of k_i comes from the effective anomaly vertex and one power of k_i each from the coupling of the external photon legs to the photon-photon scattering amplitude through the field strength tensors $F^{\mu\nu}$.

To complete the second step of the proof in a model containing charged vector mesons which couple to the fermions, we must consider in addition to Fig. 2 diagrams of the type shown in Fig. 3. Figure 3 represents possible fourth order contributions to the low energy theorem, in which charged vector mesons emerge from the fermion loop and then interact to produce a two photon final state. Unlike the electrodynamics contribution of Fig. 2, there is no simple power-counting argument which shows that Fig. 3 cannot contribute to the low energy theorem. For instance, the "final state scattering" of Fig. 3a, shown in Fig. 4, is one of the terms which contribute to the low energy theorem for γ-W scattering,[6] in which the coupling is of the A^μ rather than the $F^{\mu\nu}$ type. It might be conjectured that summing all possible insertions of the photons, which here means adding Figs. 3a and 3b, might construct $F^{\mu\nu}$ type couplings for the two photons. But

since in Fig. 3 the closed charge loop to which the photons couple includes the fermion loop which gives rise to the anomaly, it is not obvious that a third, separate factor of photon momentum must arise from the fermion loop, and without an explicit calculation we cannot rule out the possibility that Fig. 3 does contribute radiative corrections to the low energy theorem. The essential element of simplicity in Fig. 2 — that the charge loop to which the photons couple may be factorized from the fermion loop which causes the anomaly — is absent from Fig. 3.

We therefore consider an explicit calculation of the diagrams of Fig. 3. We shall find that Fig. 3 does not contribute to the low energy theorem, thus verifying the absence of these radiative corrections to fourth order in the gauge coupling constant. The calculation is simple and straightforward. The one remarkable feature has to do with the regularization procedure. In the model defined below, Fig. 3a makes a quadratically divergent contribution and Fig. 3b makes a logarithmically divergent contribution to the low energy theorem. But when Figs. 3a and 3b are defined by the dimensional regularization procedure, they are found to be equal and opposite.

The model chosen for the calculation contains an SU(2) triplet of gauge vector fields A_i^μ, an SU(2) triplet of real scalar fields ϕ_i, and an SU(2) doublet of (hadronic) fermions ψ. The unbroken Lagrangian, invariant under SU(2) gauge transformations, is (with lepton terms suppressed)

$$= \bar{\psi}\left(i\not{\partial} + e\,\frac{\tau_a}{2}\,\not{A}_a + g\,\frac{\tau_a}{2}\,\phi_a - m\right)\psi + \frac{1}{4}\,F_a^{\mu\nu}\,F_{a\mu\nu}$$

$$+ \frac{1}{2}\left(\partial^\mu\phi_a - e\,\epsilon_{abc}\,\phi_b A_c^\mu\right)^2 + f(\phi) \quad , \tag{6}$$

where $F_a^{\mu\nu} \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\,\epsilon_{abc}\,A_b^\mu A_c^\nu$ and f is an SU(2) invariant, renormalizable polynomial in the scalar fields with a minimum at $\vec{\phi} = (0,0,v)$. Redefining the

scalar fields to have vanishing vacuum expectation values, $A^{\mu}_{\pm}$ acquire a mass $\lambda = ev$ and the remaining U(1) gauge symmetry is the "electromagnetic" gauge invariance of the model. Notice that the "photon" A^{μ}_{3} is pure isovector, so that the "proton" and "neutron" have charges $\pm 1/2$.

For convenience we consider the low energy theorem for the isoscalar axial current

$$j^{\mu}_{5} \equiv \frac{1}{2}\ \bar{\psi}\gamma^{\mu}\gamma_{5}\psi$$

and we investigate the diagrams of Fig. 3 to fourth order in the gauge coupling constant (and to all orders in the Yukawa coupling constant g). It is most convenient to calculate in the unitary gauge, because it is only in this gauge that the vacuum polarization diagrams, Fig. 5, correspond precisely to the charge renormalization of the lowest order contribution, Fig. 1.[7] In other gauges, the charge renormalization is accomplished by contributions from Figs. 3 and 5, so that the calculation is more complicated.

The regularization prescription used is the one advocated by Bardeen[4]: the integral is continued to n dimensions, $g_{\mu\nu}g^{\mu\nu} = n$, but Dirac matrices are kept four-dimensional. The inner product of a Dirac matrix with an n-dimensional momentum vector is defined by keeping only the first four components of the n-vector. Consequently, fermion lines are unmodified, and the anomaly, which involves only fermion lines, is given by the usual expression,[8] with all tensors and momenta defined in four dimensions:

$$\partial_{\mu} j^{\mu}_{5} = 2mij_{5} + \frac{e^{2}}{32\pi^{2}}\ \epsilon^{\mu\nu\sigma\tau}\ \mathrm{Tr}\ \mathscr{F}^{\mu\nu}\ \mathscr{F}^{\sigma\tau} \tag{7}$$

where

$$\mathscr{F}^{\mu\nu} \equiv \left(\partial^{\mu}A^{\nu}_{a} - \partial^{\nu}A^{\mu}_{a}\right)\frac{\tau_{a}}{2} + \epsilon_{abc}A^{\mu}_{a}A^{\nu}_{b}\frac{\tau_{c}}{2}$$

and the τ_i are normalized to $\left[\tau_a, \tau_b\right] = 2i\epsilon_{abc}\tau_c$. The regularization scheme is presumed sufficient to guarantee that Eq. (7) is correct to any finite order in e.[4]

We now wish to calculate $<\Omega|\mathscr{F}^{\mu\nu}\mathscr{F}^{\sigma\tau}|\gamma_1\gamma_2>$ to order e^2. Following the regularization prescription, the boson loop integrals encountered here are defined by continuation in the space-time dimension.[9] As in Bardeen's definition of $p_\mu\gamma^\mu$ with p an n-vector, the inner product of an n-tensor with a 4-tensor is defined by keeping just the first four components of each index of the n-tensor. Thus in evaluating the contribution of $<\Omega|\mathscr{F}^{\mu\nu}\mathscr{F}^{\sigma\tau}|\gamma_1\gamma_2>$ to Eq. (7), we can discard terms which are symmetric in any pair of Lorentz indices. For this reason (among others), diagrams such as Fig. 6 cannot contribute, since the vector meson propagator is symmetric in its Lorentz indices.

Thus the only possible fourth order diagrams in addition to those represented by Figs. 2 and 5 are the diagrams of Fig. 3. The calculation is trivially exact to all orders in g, because it is impossible to draw diagrams with Yukawa couplings if only fourth order in the gauge coupling constant e is allowed (except for contributions to the charge renormalization, Fig. 5).

The calculation is straightforward and the results are as follows: the contribution from Fig. 3a to the low energy theorem is

$$\mathscr{F}_a(0) = -i\,\frac{e^4}{4\pi^2}\int\frac{d^n w}{(2\pi)^4}\,\frac{1}{(w^2-\lambda^2)^3}\left\{\frac{20}{n}\,w^2 - \frac{w^2}{\lambda^2}\,(w^2-\lambda^2)\right\} \tag{8}$$

and from Fig. 3b it is

$$\mathscr{F}_b(0) = i\,\frac{e^4}{4\pi^2}\int\frac{d^n w}{(2\pi)^4}\,\frac{1}{(w^2-\lambda^2)^2}\left\{3 + \frac{w^2}{\lambda^2}\left(1-\frac{4}{n}\right)\right\}. \tag{9}$$

Notice that for n=4, $\mathscr{F}_a(0)$ diverges quadratically while $\mathscr{F}_b(0)$ diverges logarithmically. Thus if we had performed our calculation in four dimensions and defined

the integrals with a cutoff, we would have concluded that $\mathscr{F}_a(0) + \mathscr{F}_b(0)$ is a cutoff dependent, nonvanishing quantity. However, evaluating (8) and (9) by the rules for performing integrations in n dimensions,[9] we find that

$$\mathscr{F}_a(0) = - \mathscr{F}_b(0)$$

$$= \frac{e^4}{64\pi^4} \left(\frac{1}{\lambda^2}\right)^{2-\frac{n}{2}} \left\{4\Gamma\left(2-\frac{n}{2}\right) + \Gamma\left(1-\frac{n}{2}\right)\right\} \tag{10}$$

This miracle — the cancellation of a quadratic with a logarithmic divergence — occurs because of the following unlikely expression for zero:

$$\int \frac{d^n w}{(w^2-\lambda^2)^2} \left\{\frac{4}{n} \frac{w^2}{(w^2-\lambda^2)} - \left(1-\frac{2}{n}\right) \frac{w^2}{\lambda^2}\right\}$$

$$= \frac{i\pi^2}{(\lambda^2)^{2-\frac{n}{2}}} \left\{\frac{4}{n} \cdot \frac{n}{4} \Gamma\left(2-\frac{n}{2}\right) + \left(1-\frac{2}{n}\right) \frac{n}{2} \Gamma\left(1-\frac{n}{2}\right)\right\}$$

$$= 0 \quad . \tag{11}$$

It is not surprising that regulating with a cutoff, we would obtain an ill-defined, cutoff dependent result. The question of renormalizability aside, if we had regulated the theory with a cutoff then Eq. (7) would not be the complete expression for the anomaly, since we would also expect singular subintegrations involving massive vector meson propagators to contribute to the anomaly. (If we regulate with a cutoff, the anomaly arises from surface terms which are generated by shifting the origin of integration, and we would expect such surface terms not only from fermion loops but also from loops containing massive vector meson propagators.) In this case, the calculation which has been presented here would be a <u>non-sequitur,</u> since it is based on the validity of Eq. (7).

To summarize, we have shown to fourth order in the gauge coupling constant that there are no radiative corrections to the low energy theorem for $<\Omega|j_5|\gamma_1\gamma_2>$ due to "final state scattering" of the gauge mesons. However before we can conclude that there are no radiative corrections which modify the low energy theorem, additional work is required on two questions:

(1) It must be shown that "final state" radiative corrections of the type considered in this paper vanish in any finite order.

(2) Considering Ref. 5, it seems desirable to verify the argument[4] that the existence of an adequate regularization scheme is sufficient to guarantee that the Ward identity anomaly in the hadron sector is given by just the lowest order hadron anomaly.

In view of the significance attributed to the low energy theorem for $\Gamma(\pi_0 \to \gamma\gamma)$, it would be very interesting to resolve these two questions.

Acknowledgements

I have had helpful discussions about gauge theories and anomalies with many kind people, among them T. Appelquist, W. Bardeen, J. Bjorken, L.-F. Li, and H. Quinn.

REFERENCES

1. For reviews see S. Adler, 1970 Brandeis Summer Inst. Lectures (MIT Press, Cambridge, 1971), or R. Jackiw in Lectures on Current Algebra and its Applications (Princeton U. Press, Princeton, N. J., 1972).
2. S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969).
3. For a review and references, see B. W. Lee in Proceedings of the XVI International Conference on High Energy Physics (NAL, Batavia, Ill., 1972), Vol. 4, p. 249.
4. W. A. Bardeen, Proceedings of the XVI International Conference on High Energy Physics (NAL, Batavia, Ill., 1972), Vol. 2, p. 295.
5. This presumption is supported by one calculation — S. S. Shei and A. Zee, Cornell preprint CLNS 213 (January 1973) — and left in question by another — D. Corrigan, U. of Utah preprint (May 1973) — so it merits further study. Both calculations consider Abelian models, so they do not encounter the diagrams considered here. Furthermore it is possible that the radiative corrections encountered by Corrigan are not to the Ward identity anomaly (referred to as step 1 in the text) but are due to "final state interactions" (step 2 in the text) which may be peculiar to the Abelian model. In this case there would be no contradiction with Ref. 4. I thank D. Corrigan for discussions.
6. W. Thirring, Phil. Mag. 41, 1193 (1950).
7. The Ward identity relating the fermion self-energy to the fermion electromagnetic vertex depends on the gauge chosen for the massive vector mesons, and only in the unitary gauge does one find the usual Ward identity, $Z_1=Z_2$. This is consistent with the well known dependence of the vacuum polarization on the gauge chosen for the massive vector mesons — see for instance B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3155 (1972).

8. W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

R. W. Brown, C.-C. Shih and B. L. Young, Phys. Rev. 186, 1491 (1969).

9. C. G. Bollini, J. J. Giambiagi, and A. Gonzales Dominguez, Nuovo Cimento 31, 550 (1964); G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); G. M. Cicuta and E. Montaldi, Lettere al Nuovo Cimento 4, 329 (1972).

FIGURE CAPTIONS

1. Lowest order contribution to the low energy theorem.
2. A possible source of radiative corrections due to $\gamma\gamma$ "final state" scattering. The crossed vertex represents the effective vertex due to the anomaly.
3. Possible sources of radiative corrections due to "final state" interactions among the gauge mesons. Charged gauge mesons are denoted by "W".
4. A contribution to the lowest order γW scattering amplitude.
5. The contribution of vacuum polarization to the low energy theorem.
6. The contribution of the pentagon anomaly, which vanishes identically.

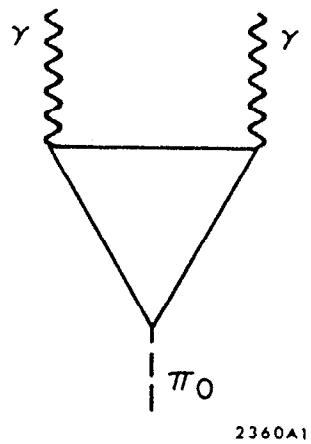

FIG. 1

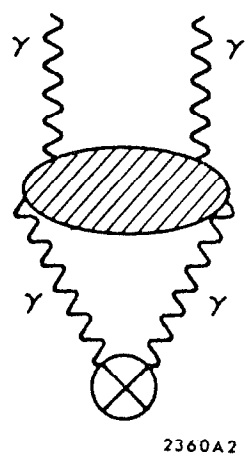

FIG. 2

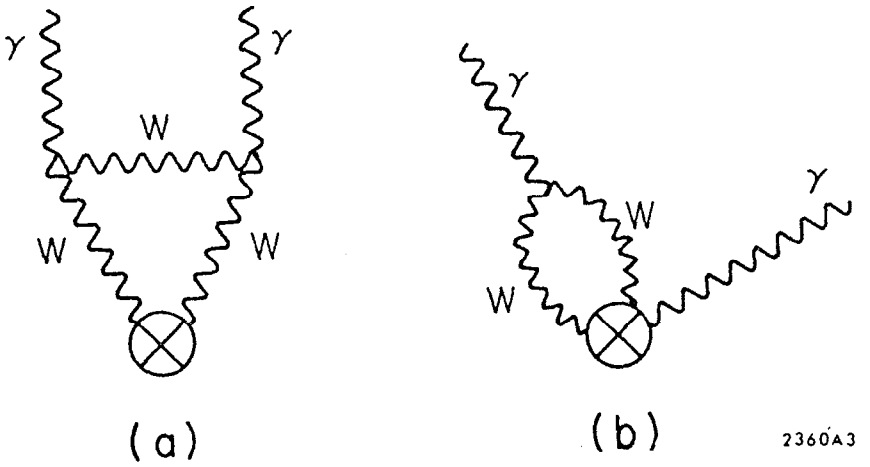

FIG. 3

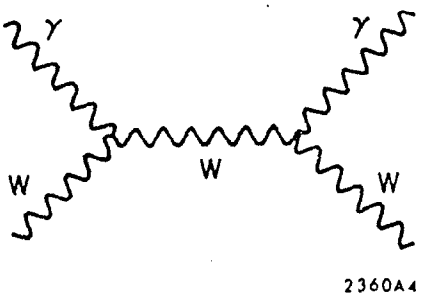

FIG. 4

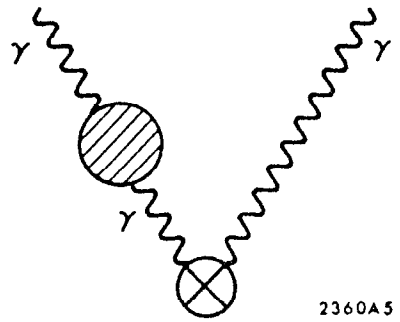

FIG. 5

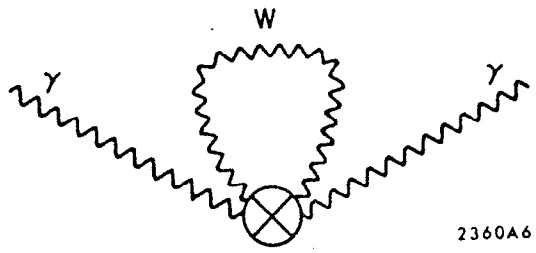

FIG. 6