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WALSH SUMMING AND DIFFERENCING TRANSFORMS*

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ABSTRACT

Analogous to Fourier frequency transforms of the integration and differentiation of a continuous-time function, Walsh sequency transforms of the summing and differencing of an arbitrary discrete-time function have been derived. These transforms can be represented numerically in the form of matrices of simple recursive structure. The matrices are not orthogonal, but they are the inverse of each other, and the value of their determinants is one.

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INTRODUCTION

The existence of Fourier frequency transforms of the integration and differentiation of continuous-time functions suggests seeking Walsh sequency transforms of the summing and differencing of discrete-time (or other discrete-data) functions.

In brief, if s and d represent the summing Σ and differencing Δ of an arbitrary discrete-time function f, and if S, D, and F represent their respective Walsh transforms, we seek transforms $\bar{\Sigma}$ and $\bar{\Delta}$ that carry F into S and D respectively, as depicted in Fig. 1.

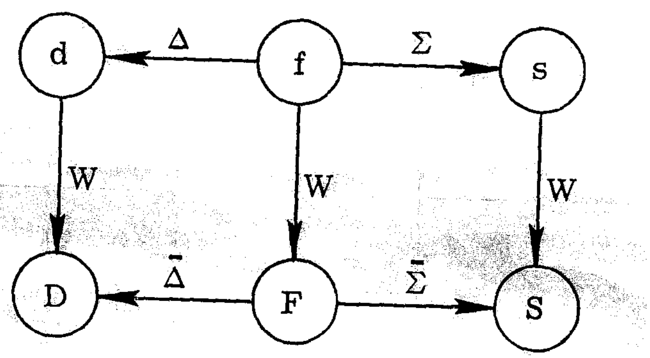

Fig. 1. Relations between existing and desired sequency transforms.

The desired transforms are easily derived, and can be represented numerically in matrix form. If the Walsh transform is defined as a matrix W of sequency-ordered Walsh functions, then the desired matrices $\bar{\Sigma}$ and $\bar{\Delta}$ are of simple recursive structure. Although not orthogonal, they are the inverse of each other, and the value of their determinants is 1.

DEFINITIONS

Let f_i denote the value of an arbitrary discrete-time function f in the ith subinterval ($i = 0, 1, \ldots\ 2^n - 1$) of the finite discrete-time interval (0, T).

Let

$$s_j = \sum_{i=0}^{j} f_i \qquad (j = 0,\ 1,\ \ldots\ 2^n - 1) \tag{1}$$

denote the forward sum of f, so that s_j is the value of the sum at the end of the jth subinterval.

Let

$$d_i = f_i - f_{i-1} \qquad (f_{-1} = 0) \tag{2}$$

denote the backward difference of f, so that d_i is the value of the difference at the beginning of the ith subinterval.

In matrix form (1) and (2) are simply

$$s = \Sigma \cdot f \tag{3}$$

$$d = \Delta \cdot f \tag{4}$$

where s, d, and f are all column matrices of 2^n elements, and Σ and Δ are square matrices of order 2^n.

The summing matrix Σ is a lower triangular matrix:

$$\Sigma = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1 \end{vmatrix} \tag{5}$$

The differencing matrix Δ is one whose elements are 1 on the principal diagonal, -1 on the first subdiagonal, and 0 elsewhere:

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \tag{6}$$

The definitions of Σ and Δ in terms of forward sum and backward difference yield the convenient result that

$$\Sigma \cdot \Delta = \Delta \cdot \Sigma = I \tag{7}$$

where I is the identity matrix, and consequently that Σ and Δ are the inverse of each other.

It is also useful to observe that since both are triangular matrices whose elements on the principal diagonal are all 1,

$$\det \Sigma = \det \Delta = 1 \tag{8}$$

They are not orthogonal, however, since the transpose of either matrix is not its inverse.

In matrix form the Walsh transform of f is

$$F = W \cdot f \tag{9}$$

where W is (herein) the sequency-ordered Walsh matrix [1–3].

If n = 3, for example,

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \tag{10}$$

Inasmuch as f is an arbitrary discrete-time function, f in (9) can be replaced by s from (3) or d from (4), and the resulting transforms defined as

$$S = W \cdot s = W \cdot \Sigma \cdot f \tag{11}$$

$$D = W \cdot d = W \cdot \Delta \cdot f \tag{12}$$

We now wish to derive and determine the nature of matrices $\bar{\Sigma}$ and $\bar{\Delta}$ defining operations in the sequency domain corresponding to those defined by Σ and Δ in the discrete-time domain, so that

$$S = \bar{\Sigma} \cdot F \tag{13}$$

$$D = \bar{\Delta} \cdot F \tag{14}$$

Thus, if we know the numerical nature of $\bar{\Sigma}$ and $\bar{\Delta}$, (13) and (14) provide alternative means of computing the Walsh transforms of the forward sum and backward difference of an arbitrary discrete-time function.

DERIVATIONS

Substitution of (9) in (13) and (14) gives

$$S = \bar{\Sigma} \cdot W \cdot f \tag{15}$$

$$D = \bar{\Delta} \cdot W \cdot f \tag{16}$$

Since f is generally non-null, it follows from equating the right sides of (11) and (15), and of (12) and (16), that

$$\bar{\Sigma} \cdot W = W \cdot \Sigma \tag{17}$$

$$\bar{\Delta} \cdot W = W \cdot \Delta \tag{18}$$

As is well known [1],

$$W^{-1} = 2^{-n} W \tag{19}$$

Postmultiplication of both sides of (17) and (18) by W^{-1} and substitution of (19) thus gives the desired matrices:

$$\bar{\Sigma} = 2^{-n} W \cdot \Sigma \cdot W \tag{20}$$

$$\bar{\Delta} = 2^{-n} W \cdot \Delta \cdot W \tag{21}$$

NUMERICAL NATURE OF THE TRANSFORM MATRICES

We now have only to perform the operations specified by (20) and (21) to determine the numerical nature of $\bar{\Sigma}$ and $\bar{\Delta}$.

The following results are stated without proof, having been obtained empirically.

The matrix $\bar{\Sigma}$ is the simpler of the two, and can easily be written from memory.

If n = 3, for example,

$$\bar{\Sigma} = 2^{-1} \begin{array}{|cc|cc|cccc|}
\hline
9 & 4 & & 2 & & & & 1 \\
-4 & 1 & 2 & & & & 1 & \\
\hline
 & -2 & 1 & & & 1 & & \\
-2 & & & 1 & 1 & & & \\
\hline
 & & & -1 & 1 & & & \\
 & & -1 & & & 1 & & \\
 & -1 & & & & & 1 & \\
-1 & & & & & & & 1 \\
\hline
\end{array}$$

$$= 2^{-1} \left(\begin{array}{|cc|cc|cccc|}
\hline
8 & 4 & & 2 & & & & 1 \\
-4 & & 2 & & & & 1 & \\
\hline
 & -2 & & & & 1 & & \\
-2 & & & & 1 & & & \\
\hline
 & & & -1 & & & & \\
 & & -1 & & & & & \\
 & -1 & & & & & & \\
-1 & & & & & & & \\
\hline
\end{array} + \begin{array}{|cccccccc|}
\hline
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1 \\
\hline
\end{array} \right)$$

$$= K_{\Sigma}\left(\bar{\Sigma}_1 + \bar{\Sigma}_2\right) \tag{22}$$

where $\bar{\Sigma}_1$ has a more or less obvious recursive structure, and

$$\bar{\Sigma}_2 = I \tag{23}$$

The matrix $\bar{\Sigma}_1$ can be constructed as a series of upper left square sub-matrices of order 2^p ($p = 0, 1, \ldots n$), as shown in (22), by some very simple recursion formulas.

Let $A_{11}^{(2^p)}$ denote the upper left square submatrix of order 2^p of $\bar{\Sigma}_1$.

For $p = 0$,

$$A_{11}^{(2^0)} = 2^n \tag{24}$$

For $1 \leqq p \leqq n$,

$$A_{11}^{(2^p)} = \begin{array}{|c|c|} \hline A_{11}^{(2^{p-1})} & A_{12}^{(2^{p-1})} \\ \hline A_{21}^{(2^{p-1})} & A_{22}^{(2^{p-1})} \\ \hline \end{array} \tag{25}$$

in which $A_{11}^{(2^{p-1})}$ has already been determined from the preceding recursion step, and

$$A_{12}^{(2^{p-1})} = 2^{n-p} \cdot P \tag{26}$$

$$A_{21}^{(2^{p-1})} = -2^{n-p} \cdot P \tag{27}$$

$$A_{22}^{(2^{p-1})} = N \tag{28}$$

where P is a permutation matrix whose elements are 1 on the secondary diagonal and 0 elsewhere, and N is a null matrix.

Finally, for any value of n,

$$K_\Sigma \equiv 2^{-1} \tag{29}$$

The matrix $\bar{\Delta}$ is only slightly more complicated, although it may appear more formidable at first glance because it is not sparse.

If n = 3, for example,

$$\bar{\Delta} = 2^{-3}\left[\begin{array}{cccc|cccc}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 3 & -3 & -1 & 1 & 3 & -3 & -1 \\
1 & 3 & 5 & -1 & 1 & -5 & -3 & -1 \\
1 & -1 & 1 & 7 & -7 & -1 & 1 & -1 \\
\hline
1 & -1 & 1 & 7 & 9 & -1 & 1 & -1 \\
1 & 3 & 5 & -1 & 1 & 11 & -3 & -1 \\
1 & 3 & -3 & -1 & 1 & 3 & 13 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 15
\end{array}\right]$$

$$= 2^{-3}\left(\left[\begin{array}{cccc|cccc}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 3 & -3 & -1 & 1 & 3 & -3 & -1 \\
1 & 3 & 5 & -1 & 1 & -5 & -3 & -1 \\
1 & -1 & 1 & 7 & -7 & -1 & 1 & -1 \\
\hline
1 & -1 & 1 & 7 & -7 & -1 & 1 & -1 \\
1 & 3 & 5 & -1 & 1 & -5 & -3 & -1 \\
1 & 3 & -3 & -1 & 1 & 3 & -3 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{array}\right] + \left[\begin{array}{cccc|cccc}
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
\hline
 & & & & 16 & & & \\
 & & & & & 16 & & \\
 & & & & & & 16 & \\
 & & & & & & & 16
\end{array}\right]\right)$$

$$= K_{\Delta}\left(\bar{\Delta}_1 + \bar{\Delta}_2\right) \tag{30}$$

The matrix $\bar{\Delta}_1$ has a very simple semirecursive structure. Consider its partitioning into four square submatrices of order 2^{n-1}, and observe that the two right submatrices are simply negative horizontal reflections of the respective left submatrices (i.e., with all element signs changed, and the order of the columns reversed), and that the two lower submatrices are simply positive vertical reflections of the respective upper submatrices (i.e., with no element signs changed, but the order of the rows reversed).

If the upper left submatrix is partitioned into four square submatrices of order 2^{n-2}, the same statement is true, except on the principal diagonal of the new (order 2^{n-2}) lower right submatrix. The correct elements there can be obtained by adding to the lower right submatrix a scalar matrix of order 2^{n-2} whose elements on the principal diagonal are all 2^n (=8 in this example).

Thus, the matrix $\bar{\Delta}$ can be constructed as a series of upper left square submatrices of order 2^q ($q = 0, 1, \ldots n$), as shown in (30), by some very simple recursion formulas. In this case, however, a matrix $\bar{\Delta}_2$ of order 2^q must be added to the matrix $\bar{\Delta}_1$ of order 2^q in each recursion step.

Let $B_{11}^{(2^q)}$ denote the upper left square submatrix of order 2^q of $\bar{\Delta}_1$, and $C_{11}^{(2^q)}$ that of $\bar{\Delta}_2$.

For $q = 0$,

$$B_{11}^{(2^0)} = 1 \tag{31}$$

$$C_{11}^{(2^0)} = 0 \tag{32}$$

$$\bar{\Delta}_{11}^{(2^0)} = B_{11}^{(2^0)} + C_{11}^{(2^0)} = 1 \tag{33}$$

For $1 \leqq q \leqq n$,

$$B_{11}^{(2^q)} = \begin{array}{|c|c|} \hline B_{11}^{(2^{q-1})} & B_{12}^{(2^{q-1})} \\ \hline B_{21}^{(q-1)} & B_{22}^{(q-1)} \\ \hline \end{array} \tag{34}$$

$$C_{11}^{(2^q)} = \begin{array}{|c|c|} \hline C_{11}^{(2^{q-1})} & C_{12}^{(2^{q-1})} \\ \hline C_{21}^{(2^{q-1})} & C_{22}^{(2^{q-1})} \\ \hline \end{array} \tag{35}$$

$$\bar{\Delta}_{11}^{(2^q)} = B_{11}^{(2^q)} + C_{11}^{(2^q)} \tag{36}$$

in which $B_{11}^{(q-1)}$ has already been determined from the preceding recursion step, and

$$B_{12}^{(2^{q-1})} = - B_{11}^{(2^{q-1})} \cdot P \tag{37}$$

$$B_{21}^{(2^{q-1})} = P \cdot B_{11}^{(2^{q-1})} \tag{38}$$

$$B_{22}^{(2^{q-1})} = - P \cdot B_{11}^{(2^{q-1})} \cdot P \tag{39}$$

$$C_{11}^{(2^{q-1})} = C_{12}^{(2^{q-1})} = C_{21}^{(2^{q-1})} = N \tag{40}$$

$$C_{22}^{(2^{q-1})} = 2^{q+1}\, I \tag{41}$$

where P is a permutation matrix whose elements are 1 in the secondary diagonal and 0 elsewhere, and N is a null matrix.

Finally,

$$K_{\Delta} = 2^{-n} \tag{42}$$

OTHER PROPERTIES OF THE TRANSFORM MATRICES

As with Σ and Δ, it can be shown easily that

$$\bar{\Sigma} \cdot \bar{\Delta} = I \tag{43}$$

and consequently that $\bar{\Sigma}$ and $\bar{\Delta}$ too are the inverse of each other. Substitution of (20) and (21) in (43) gives

$$\left(2^{-n}\, W \cdot \Sigma \cdot W\right) \cdot \left(2^{-n}\, W \cdot \Delta \cdot W\right) = I \tag{44}$$

With the use of (19) and (7), (44) can be reduced readily to an identity, thus verifying (43).

As with Σ and Δ, it can also be shown easily that

$$\det \bar{\Sigma} = \det \bar{\Delta} = 1 \tag{45}$$

Substitution of (19) in (20) and (21) gives

$$\bar{\Sigma} = W^{-1} \cdot \Sigma \circ W \tag{46}$$

$$\bar{\Delta} = W^{-1} \circ \Delta \cdot W \tag{47}$$

As is well known, the determinant of a product of square matrices is equal to the product of the determinants of the factor matrices [4], so

$$\det \bar{\Sigma} = (\det W^{-1})\ (\det \Sigma)\ (\det W) \tag{48}$$

$$\det \bar{\Delta} = (\det W^{-1})\ (\det \Delta)\ (\det W) \tag{49}$$

As is also well known, the determinant of the inverse of a nonsingular matrix is equal to the reciprocal of the determinant of the matrix [5], so (48) and (49) can be rewritten as simply

$$\det \bar{\Sigma} = \det \Sigma \tag{50}$$

$$\det \bar{\Delta} = \det \Delta \tag{51}$$

Reference to (8) verifies (45).

Finally, like Σ and Δ, $\bar{\Sigma}$ and $\bar{\Delta}$ are not orthogonal, since the transspace of either matrix is not its inverse.

POSSIBLE APPLICATIONS

Specific applications of these new transform matrices is beyond the scope of this paper, but one possibility is a new approach to the analysis of interference in pulse systems. Capacitive interference is a result of pulse sums (sums of products of pulse amplitudes and durations), while inductive interference is a result of pulse differences (changes in amplitudes at pulse leading and trailing edges). Thus in the case of an interference-causing pulse pattern there is a potential interest not only in its own sequency spectrum, but also in that of its differences or sums during the pattern period. It is hoped that use of these matrices will render a clearer understanding of the nature of such problems and facilitate better solutions than is possible through the use of nonsinusoidal Fourier methods.

Another possible application is further development of the concepts of nonsinusoidal (pulsed) electromagnetic waves [6].

Finally, it is hoped that they will contribute to a further understanding of the Walsh functions themselves.

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