# PI-MUONIUM: A DETERMINATION OF THE PION CHARGE RADIUS 

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## Abstract

The $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ energy level shift of the $\pi-\mu$ atom is studied. It is shown that an experimental determination of the $2 S_{1 / 2}-2 P_{1 / 2}$ energy level shift accurate to one part per thousand can provide an independent determination of the pion charge radius.

Recently, an experiment was proposed and planned ${ }^{(1)}$ to measure the $2 S_{1 / 2}-2 P_{1 / 2}$ energy level shift in the $\pi-\mu$ atom (pi-muonium) produced in the decay of $\mathrm{K}_{\mathrm{L}}^{\mathrm{O}} \longrightarrow \pi \mu \nu . \dagger$ One of the contributions to the $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ energy splitting comes from the finite size of the pion which can be written as ${ }^{(3)}$
$\Delta E_{2 S}($ size $)-\Delta E_{2 P}($ size $)=\frac{\alpha^{4} \mu^{3}}{12} r_{\pi}^{2} \quad$.
where the reduced mass $\mu=m_{\pi} m_{\mu} /\left(m_{\pi}+m_{\mu}\right)$, and $r_{\pi}$ is defined to be the pion charge radius. Up to now, $r_{\pi}$ is not a very well measured quantity. Electroproduction of a charged pion ${ }^{(4)}$ and $\mathrm{e}^{+} \mathrm{e}^{-} \longrightarrow \pi^{+} \pi^{-}$data ${ }^{(5)}$ favors a small $r_{\pi}$ of about .6F. However, recent UCLA - Serpukhov $\pi^{+} \mathrm{e}^{-}$data ${ }^{(6)}$ seems to favor a large $\mathrm{r}_{\pi}$ of about . $9 \mathrm{~F} . \dagger \dagger$ A measurement of the $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ shift in pi-muonium can provide an independent measurement of the pion charge radius. From Eq. (1), one finds that $\Delta E$ (size) is of the order .5 to $1.0 \times 10^{-3} \mathrm{eV}$. Therefore, in order to obtain the pion charge radius from the $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ shift, one must calculate all QED contributions to the energy splitting accurate to the order $10^{-4} \mathrm{eV}$. In this note, we calculate all the QED contributions up to $10^{-4} \mathrm{eV}$. This includes all $\alpha^{2}$ Ry and $\alpha^{3} \mathrm{Ry} \dagger \dagger \dagger$ contributions and the dominant $\alpha^{4}$ Ry contribution comes from the fourth order $\mathrm{e}^{+} \mathrm{e}^{-}$vacuum polarization.

A few words about our general philosophy are in order here. We adopt the effective potential method of reference ${ }^{(7)}$ where an effective bound state potential for a Schrodinger-like eq. is inferred from an on-shell scattering theory. As a first approximation, off-shell behavior of the lowest order potential is chosen so as to account for as much as possible of the second order Feynman scattering diagrams. For our particular recoil calculations, this method reproduces the results of the covariant perturbation theory of Saltpeter ${ }^{(8)}$ and uses an effective potential in the Schrodinger-like equation which is well-known as the Breit potential.

To order $\alpha^{2} \mathrm{Ry}$, the Breit equation solution gives rise to a $2 \mathrm{~S}_{\frac{1}{2}}-2 \mathrm{P}_{\frac{1}{2}}$ energy splitting displayed in the last row of Table A. ${ }^{(10)}$ Note that this formula gives the correct infinite mass limits for an atom with two spin $\frac{1}{2}$ particles where this term vanishes and for an atom with two spin 0 particles where it is known to exist. The solution of the Breit equation has no $\alpha^{3}$ Ry term in its expansion. However, vacuum polarization and other QED effects modify the Breit equation result and give rise to the energy splitting to order $\alpha^{3}$ Ry. The only significant (to order $10^{-4} \mathrm{eV}$ of the energy splitting) $\alpha^{4} \mathrm{Ry}$ contribution is the fourth order vacuum polarization.

We display the formula and results for the various contributions aside from the pion finite size effect, to the pi-muonium Lamb shift up to order $10^{-4} \mathrm{eV}$ in Table A. We briefly describe these terms below.
(A) Second and fourth order $\mathrm{e}^{+} \mathrm{e}^{-}$vacuum polarization. The $\mathrm{e}^{+} \mathrm{e}^{-}$vacuum polarization gives the dominant contribution to the pimuonium Lamb shift. This is because the $\mathrm{e}^{+} \mathrm{e}^{-}$vacuum polarization modifies the Coulomb potential at a distance of the order of an electron Compton wave length $\lambda_{e}$. The Bohr radius of the pi-muonium is equal $4.5 \times 10^{-11} \mathrm{~cm}$. This is comparable to $\lambda_{\mathrm{e}}$ which is equal to $3.9 \times 10^{-11}$ cm . So the pion and the muon spend a great fraction of time in the region in which the Coulomb potential is modified. The second and fourth order $\mathrm{e}^{+} \mathrm{e}^{-}$vacuum polarization were calculated by Di Giacomo in the muonic hydrogen problem. ${ }^{(11)}$ Note that this energy level shift is of opposite sign to that of the hydrogen atom energy $2 \mathrm{~S}_{\frac{1}{2}}-2 \mathrm{P}_{\frac{1}{2}}$ level shift.
(B) Second order vertex correction and vacuum polarization.

The second order vertex correction for the muon including the correct
reduced mass effect and second order $\mu^{+} \mu^{-}$vacuum polarization are given by Erickson and Yennie ${ }^{(3)} . \ddagger$ The second order vertex correction for pion and $\pi^{+} \pi^{-}$vacuum polarization are calculated using scalar electrodynamics for a point scalar particle. Ignoring the pion structure in the calculation of these corrections introduces small errors because the vertex correction is basically a non-relativistic low-frequency effect, and the $\pi^{+} \pi^{-}$vacuum polarization effect is negligible even for a point pion. In all these radiative corrections, we consider only Coulomb interaction between the pion and the muon. Radiative corrections with transverse photon exchange is of order $\alpha^{2}$ smaller. We also ignore relativistic corrections to the wave function which introduces corrections of order $\alpha^{2} \ln ^{2} \alpha \quad$ smaller to the vacuum polarization term ${ }^{(12)}$. Errors in the measured masses also propagate negligibly in the formulas.
(C) Correction to the Breit Interaction.

The usual Breit interaction term ignores the energy difference between the intermediate and the initial unperturbed atomic states. The correction to this term is the same for spinor-spinor system and scalar-spinor system to order $\alpha^{3} \mathrm{Ry}$, and is given in Eq. (4.28) - (4.29) of Fulton and Martin ${ }^{(13)}$. Note that the pion and muon have almost equal mass, and one cannot make the approximation used in the hydrogen atom problem where one particle is much more massive than the other.
(D) Correction to the iteration of the Breit equation for two photon exchange. The Breit equation is a good approximation in first order perturbation theory. However, one can get quite wrong results when the Breit equation is solved to order higher than $\alpha^{2}$ Ry. This is because the Breit equation when interated to higher order allows scattered waves of negative frequency to propagate forward in time while the rules of QED,
do not allow this. In order to obtain the correct QED contribution to the $2 S_{\frac{1}{2}}-2 P_{\frac{1}{2}}$ shift, we calculate the two photon exchange terms using the rule of QED, and subtract away from it the Breit equation result iterated to that order. $\ddagger \ddagger$ Since the Breit equation does not contribute to the $2 \mathrm{~S}_{\frac{1}{2}}-2 \mathrm{P}_{\frac{1}{2}}$ energy splitting in order $\alpha^{3}$ Ry, the difference between the QED solution and the Breit solution can be used to obtain the $2 \mathrm{~S}_{\frac{1}{2}}$ $2 \mathbf{P}_{\frac{1}{2}}$ splitting from two photon exchange. We calculate the double Coulomb exchange contribution $\Delta \mathrm{E}_{\mathrm{cc}}$ and double transverse photon exchange contribution $\Delta \mathrm{E}_{\mathrm{TT}}$ separately. The results aside from logarithm terms are similar to those given by Fulton and Martin ${ }^{(13)}$ in their calculation of the positronium problem. Note that as $\mathrm{m}_{\pi} \rightarrow \infty$, one obtains the same results as that of the spinor-spinor case with one particle very massive ${ }^{(7),(8)}$.
Adding up all the QED contributions to the $2 \mathrm{~S}_{\frac{1}{2}}-2 \mathrm{P}_{\frac{1}{2}}$ shift in Table $A$, one finds $\Delta \mathrm{E}_{2 \mathrm{~S}_{\frac{1}{2}}}(\mathrm{QED})-\Delta \mathrm{E}_{2 \mathrm{P} \frac{1}{2}}(\mathrm{QED})=-.07945 \mathrm{eV}$. From our earlier estimate using Eq. (1), we see that the pion size effect is about a percent of the total energy shift. The fourth order $e^{+} e^{-}$vacuum polarization is of the same order. So if the $2 S_{\frac{1}{2}}-2 P_{\frac{1}{2}}$ energy shift can be measured to one part in a thousand, and assuming QED to be valid to $\alpha^{4} \mathrm{Ry}$, one can have an independent measurement of the pion charge radius. $\ddagger \ddagger \$$ An experiment of this type could also check for the existence of any anomalous $\mu-\pi$ interaction.

The remaining question is: Can one measure the $2 \mathrm{~S}_{\frac{1}{2}}-2 \mathrm{P}_{\frac{1}{2}}$ energy shift in pi-muonium accurate to one part in a thousand? We leave that as a challenge to the experimentalists!

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$\dagger$ The branching ration for $\mathrm{K}_{\mathrm{L}}^{\mathrm{o}}-(\pi \mu)_{\text {atom }} \nu$ to $\mathrm{K}_{\mathrm{L}}^{\mathrm{o}} \rightarrow \pi \mu \nu$ is of the order $10^{-7}$ (Ref. 2) and the $\pi-\mu$ atom can be quite readily detected. (Ref. 1)
$\dagger \dagger$ The problem with this experiment is that the square of the 4 -momentum transfer is limited to about. $04 \mathrm{GeV}^{2}$.
$\dagger \dagger \dagger$ Ry is defined as usual to be the Rydberg of the system and is equal to $\alpha^{2}{ }^{2} \mu / 2$. We set $\mathrm{h}=\mathrm{c}=1$ in all our formulae.
$\ddagger$ See Eq. (4.1a) - (4.2) of Ref. 3. Note that the reduced mass effect for this contribution is incorrectly given in many places in the literature. $\ddagger \ddagger$ One can use the usual perturbation formula $\sum_{n} H_{f n}^{(2)} H_{n i}^{(2)} /\left(E_{i}-E_{n}\right)$, where $i, n$, and $f$ denotes initial, intermediate, and final states, and $H^{(2)}$ are the second order muon-pion scattering matrix element. Note the $n$ can involve both positive and negative energy states. $\ddagger \ddagger \ddagger$ On the other hand, if reliable measurement of the pion charge radius can be obtained from other experiments, the measurement of the $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ splitting may be used as a good test of quantum electrodynamics.


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Fig. 2, $(\alpha)^{2}{ }^{2}{ }^{2} \mathrm{Z} \alpha$ correction and section VI.
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TABLE A
VARIOUS CONTRIBUTIONS TO THE $2 \mathrm{~S}_{1 / 2}-2 \mathrm{P}_{1 / 2}$ SPLITTING IN PI-MUONIUM ${ }^{\text {(a) }}$

(a) Constants $\alpha^{-1}=137.03602, \mu=60.13624 \mathrm{MeV}, \mathrm{m}_{\pi}=139.576 \mathrm{MeV}, \mathrm{m}_{\mu}-105.6594 \mathrm{MeV} . \quad \mathrm{Ry}-1.601168 \mathrm{keV}$.
(b) $\rho\left(\frac{a^{2}}{x}\right)=\frac{\alpha^{2}}{3 \pi^{2}} F(\sqrt{1-x})$.

$$
\begin{aligned}
F(Z)= & Z\left[-\frac{19}{24}+\frac{55}{72} Z^{2}-\frac{1}{3} Z^{2}-\frac{1}{2}\left(3-Z^{2}\right) \ln \frac{64 Z^{4}}{\left(1-Z^{2}\right)^{3}}\right]+\ln \left(\frac{1+Z}{1-Z}\right)\left[\frac{33}{16}+\frac{23}{8} Z^{2}-\frac{23}{16} Z^{4}+\frac{1}{6} Z^{6}+\left(\frac{3}{2}+Z^{2}-\frac{1}{2} Z^{4}\right) \ln \frac{(1+Z)^{3}}{8 Z^{2}}\right] \\
& +\left(\frac{3}{2}+Z^{2}-\frac{1}{2} Z^{4}\right)\left[4 \operatorname{Li}\left(\frac{1-Z}{1+Z}\right)+2 \operatorname{Li}\left(\frac{Z-1}{Z+1}\right)\right]
\end{aligned}
$$

(c) The last term in $\Delta E_{2 S}$ comes from $\mu^{+} \mu^{-}$vacuum polarization.
(d) The last term in $\Delta E_{2 S}$ comes from $\pi^{+} \pi^{--}$vacuum polarization.

