

Extraction of $|V_{ub}|$ with Reduced Dependence on Shape Functions.

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Abstract

Using *BABAR* measurements of the inclusive electron spectrum in $B \rightarrow X_u e \nu$ decays and the inclusive photon spectrum in $B \rightarrow X_s \gamma$ decays, we extract the magnitude of the CKM matrix element V_{ub} . The extraction is based on several theoretical calculations designed to reduce the theoretical uncertainties by exploiting the assumption that the leading shape functions are the same for all $b \rightarrow q$ transitions (q is a light quark). The current results agree well with the previous analysis, have indeed smaller theoretical errors, but are presently limited by the knowledge of the photon spectrum and the experimental errors on the lepton spectrum.

1 Introduction

The determination of the magnitude of the Cabibbo-Kobayashi-Maskawa matrix element V_{ub} from charmless semileptonic B decays is complicated by the fact that over most of the phase space $B \rightarrow X_c \ell \nu$ decays dominate and are very difficult to distinguish from the signal $B \rightarrow X_u \ell \nu$ decays. To reduce the dominant $B \rightarrow X_c \ell \nu$ background, signal decays are selected in restricted regions of phase space where these background decays are forbidden or highly suppressed, for instance, near the endpoint of the lepton-energy spectrum, $E_\ell > E_0$. The partial decay rate is extrapolated to the total by comparing the experimentally measured rate with a theoretical prediction.

Theoretically, the most precise predictions can be made for the total $B \rightarrow X_u \ell \nu$ decay rate. Accounting for restriction in phase space is difficult because decay spectra close to the kinematic limit are susceptible to non-perturbative strong-interaction effects. Theoretical tools for the calculations of the partial inclusive decay rates are QCD factorization and local operator product expansions (OPE). These calculations separate non-perturbative from perturbative quantities, and use expansions in inverse powers of the b -quark mass, m_b , and in powers of α_s [1]. At leading order in Λ_{QCD}/m_b , the non-perturbative bound-state effects are accounted for by a shape function describing the “Fermi-motion” of the b quark inside the B meson. These shape functions cannot be calculated. Different shapes have been proposed, and parameters defining these shapes have to be extracted from data. This introduces significant additional hadronic uncertainties. At leading order, these shape functions are assumed to be universal functions for $b \rightarrow q$ transitions, where q is a light quark, either s or u .

In the past, we have extracted the non-perturbative parameters of these shape functions, the b -quark mass m_b and its mean kinetic energy squared, $\mu_\pi^2(\mu)$, from the inclusive photon spectrum in $B \rightarrow X_s \gamma$ decays. These parameters depend on the choice of the renormalization scale, μ .

It was suggested many years ago [2] that $|V_{ub}|$ can be extracted with smaller theoretical uncertainties by combining integrals over the lepton-endpoint spectrum in $B \rightarrow X_u \ell \nu$ decays with weighted integrals over the photon-endpoint spectrum from $B \rightarrow X_s \gamma$ decays. The underlying assumption is that the QCD interactions affecting these two processes are the same and thus will cancel to first order in the appropriate ratio of weighted rates. The advantage of this approach is that it reduces the sensitivity to the choice of the shape-function parameterization and thus avoids uncertainties that are difficult to quantify.

In this note, we present the extraction of $|V_{ub}|$ based on different prescriptions proposed by Leibovich, Low, and Rothstein [3, 4, 5], by Neubert [6], and more recent calculations by Lange, Neubert, and Paz [7, 8]. The three prescriptions use different calculations of the weight functions, and thus result in different estimates of the theoretical uncertainties.

The *BABAR* Collaboration was the first to apply the prescription of refs. [3, 4, 5] to extract $|V_{ub}|$, based on measurements of the hadron mass spectrum in charmless semileptonic B decays and the inclusive photon spectrum [9].

2 Experimental Input

The experimental inputs for this analysis are the published *BABAR* measurements of the inclusive electron energy spectrum in $B \rightarrow X_u e \nu$ decays [10], and of the inclusive photon-energy spectrum in $B \rightarrow X_s \gamma$ decays [11]. These measurements are based on a data sample corresponding to a

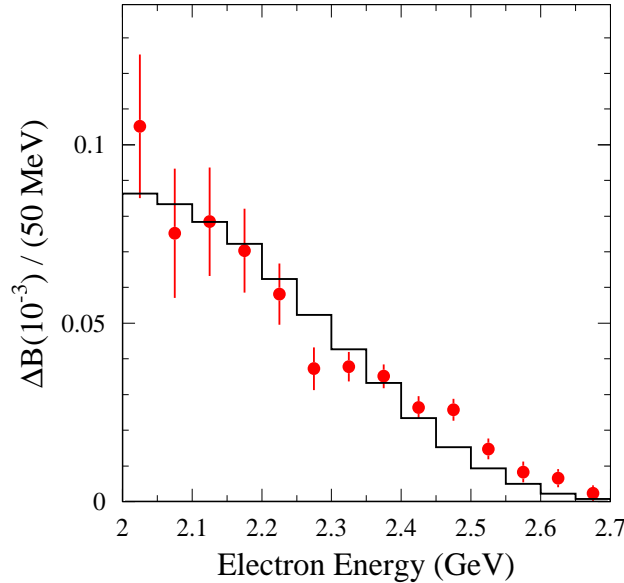


Figure 1: The normalized inclusive electron spectrum in $B \rightarrow X_u e \nu$ decays [10] measured in the $\Upsilon(4S)$ rest frame. The error bars indicate the statistical errors. The histogram represents the theoretical prediction [12].

total integrated luminosity of about 80 fb^{-1} . The measured spectra will be integrated above an energy E_0 , measured in the B -meson rest frame.

2.1 Inclusive Lepton Spectrum in $B \rightarrow X_u e \nu$ Decays

The inclusive electron-energy spectrum above 2.0 GeV , measured in the $\Upsilon(4S)$ rest frame, is shown in Fig.1. The data are fully corrected for efficiencies and radiative effects. They are normalized to the total number of charged and neutral B -meson decays and presented as partial branching fractions.

For the extraction of $|V_{ub}|$ we need to transform the measured branching fractions from the $\Upsilon(4S)$ rest frame to the branching fraction in the B -meson rest frame with a lower electron energy cut at E_0 . This is done using correction factors derived from the predicted electron spectrum [12], calculated with the shape-function parameters based on a global fit [13] to moments of inclusive distributions in semileptonic and radiative B -meson decays. The systematic uncertainties for this transformation are estimated by varying the shape-function parameters within their experimental uncertainties. The resulting partial branching fractions for different values of the in the electron-energy cut-offs, E_0 , measured in the $\Upsilon(4S)$ and the B -meson rest frame, and correction factors relating the two, are listed in Table 1.

2.2 Inclusive Photon Spectrum from $B \rightarrow X_s \gamma$ Decays

We use the *BABAR* measurements of the semi-inclusive $B \rightarrow X_s \gamma$ decay rate [11]. This photon spectrum is derived from the sum of the photon spectra of 38 exclusive decay modes and is measured in the B -meson rest frame. The data are fully corrected for efficiencies and radiative

Table 1: Partial branching fractions for $B \rightarrow X_u e \nu$ decays in units 10^{-3} , integrated over the energy range from E_0 to $2.6 \text{ GeV}/c$, both in the $\Upsilon(4S)$ rest frame and the B -meson rest frame, as well as the correction factor relating the two. The uncertainty of the correction factor reflects the uncertainty in the assumed shape of the spectrum.

E_0 [GeV]	$\Delta\mathcal{B}(E_0) \cdot 10^3$ $\Upsilon(4S)$ rest frame	Correction factor	$\Delta\mathcal{B}(E_0) \cdot 10^3$ B rest frame
2.0	$0.572 \pm 0.041 \pm 0.065$	1.002 ± 0.005	0.573 ± 0.077
2.1	$0.392 \pm 0.023 \pm 0.038$	0.994 ± 0.008	0.390 ± 0.044
2.2	$0.243 \pm 0.011 \pm 0.020$	0.973 ± 0.016	0.236 ± 0.023
2.3	$0.148 \pm 0.006 \pm 0.010$	$0.915^{+0.041}_{-0.034}$	0.135 ± 0.012
2.4	$0.075 \pm 0.004 \pm 0.006$	$0.772^{+0.075}_{-0.084}$	0.058 ± 0.008

effects. They are normalized to the total number of charged and neutral B -meson decays and presented as differential branching fractions. The measured $B \rightarrow X_s \gamma$ differential branching fraction is shown in Fig. 2.

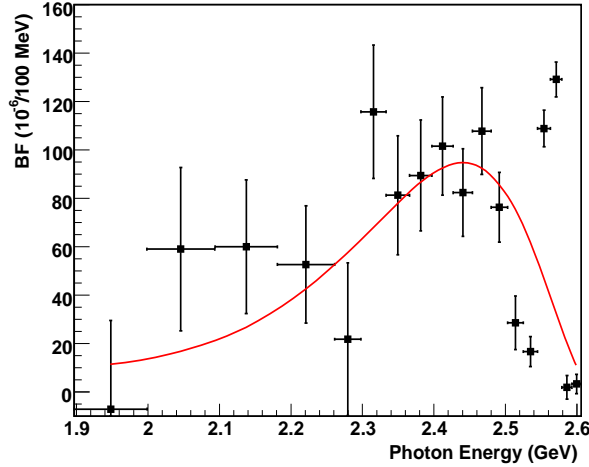


Figure 2: The differential photon energy spectrum in the B rest frame, derived from the sum the photon spectra of 38 exclusive $B \rightarrow X_s \gamma$ decay modes. The error bars indicate the statistical errors. The data are compared to the theoretical prediction [12] with the shape function parameters $m_b = 4.67 \text{ GeV}$, $\mu_\pi^2 = 0.16 \text{ GeV}^2$ obtained from the fit to E_γ spectrum [11].

3 Theoretical Framework for the Extraction of $|V_{ub}|$

3.1 Method I proposed by Leibovich, Low, Rothstein

A. K. Leibovich, I. Low, and I. Z. Rothstein [3, 4] proposed a method for the extraction of the ratio $|V_{ub}|^2/|V_{tb}V_{ts}^*|^2$ without invoking knowledge of the shape function. Their calculation relates

$|V_{ub}|^2/|V_{tb}V_{ts}^*|^2$ to the experimentally measured differential branching fractions for $B \rightarrow X_u l \nu$ and $B \rightarrow X_s \gamma$ decays,

$$\frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} = \frac{3\alpha|C_7^{(0)}(m_b)|^2}{\pi}(1 + H_{\text{mix}}^\gamma) \left(\int_{x_B^c}^1 dx_B \frac{d\Gamma}{dx_B} \right) / \left(\int_{x_B^c/\rho}^1 du_B \frac{d\Gamma^\gamma}{du_B} w(u_B, x_B^c, \rho) \right), \quad (1)$$

the integration variables are

$$x_B = 2E_l/M_B, \quad u_B = 2E_\gamma/M_B,$$

with limits x_B^c and x_B^c/ρ ; M_B refers to the B -meson mass. The weight function is approximately linear as a function of u_B ,

$$w(u_B, x_B^c, \rho) = u_B^2 \int_{x_B^c}^{\rho u_B} dx_B K\left(x_B; \frac{4}{3\pi\beta_0} \ln(1 - \alpha_s \beta_0 l_{x_B/u_B})\right), \quad (2)$$

and

$$\begin{aligned} H_{\text{mix}}^\gamma &= \frac{\alpha_s}{2\pi C_7^{(0)}} \left[C_7^{(1)} + C_2^{(0)} \Re(r_2) + C_8^{(0)} \left(\frac{44}{9} - \frac{8\pi^2}{27} \right) \right], \\ \Re(r_2) &\approx -4.092 + 12.78(m_c/m_b - 0.29), \\ K(x; y) &= 6 \left\{ \left[1 + \frac{4\alpha_s}{3\pi} (1 - \psi'(4+y)) \right] \frac{1}{(y+2)(y+3)} - \right. \\ &\quad \left. - \frac{\alpha_s}{3\pi} \left[\frac{1}{(y+2)^2} - \frac{7}{(y+3)^2} \right] - \frac{4\alpha_s}{3\pi} \left[\frac{1}{(y+2)^3} - \frac{1}{(y+3)^3} \right] \right\} - 3(1-x)^2, \\ \psi(z) &= \frac{1}{\Gamma(z)} \frac{d}{dz} \Gamma(z), \quad l_{x/u} = -\ln[-\ln(x/u)]. \end{aligned}$$

The Wilson coefficients $C_7^{(0)}(m_b)$, $C_8^{(0)}(m_b)$, *etc.*, computed in NLO, can be found in [14, 15]. The argument of K , $y = \frac{4}{3\pi\beta_0} \ln(1 - \alpha_s \beta_0 l_{x_B/u_B})$ diverges for $\alpha_s \beta_0 l_{x_B/u_B} = 1$. To avoid this pole, the integration limits are set to $x_B \leq \rho u_B$ with $\rho < 0.999$. For smaller values of ρ , the weight function deviates and thus the extracted value of $|V_{ub}|$ changes, and the uncertainty of this scheme increases.

To check the impact of the resummation of the Sudakov logarithms we also tried a non-resummed version of the LLR weight function [5]:

$$w(u_B, x_B^c) = u_B^2 \int_{x_B^c}^{u_B} dx_B \left(1 - 3(1-x_B)^2 + \frac{\alpha_s}{\pi} \left(\frac{7}{2} - \frac{2\pi^2}{9} - \frac{10}{9} \ln(1 - \frac{x_B}{u_B}) \right) \right). \quad (3)$$

This calculation (Eq. 1 – 3) includes NLO perturbative corrections, but does not take into account power corrections. Estimates of the theoretical uncertainties are discussed in Section 4. The weight function is shown in Fig. 3, for $E_0 = 2.0$ GeV and $E_0 = 2.3$ GeV and different values of the parameter ρ .

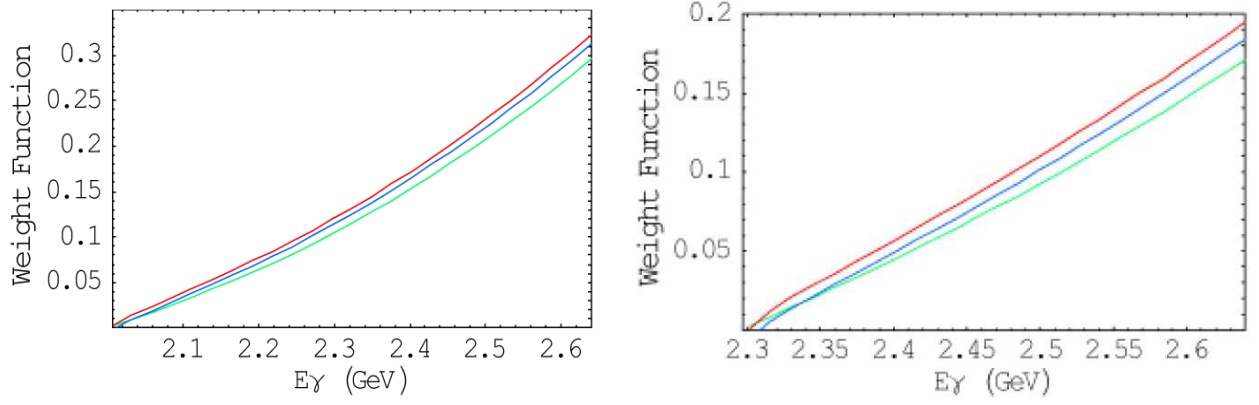


Figure 3: The weight function for Method I (LLR) for two different values $\rho = 0.9983$ (red), and $\rho = 0.995$ (blue), and non-resummed weight function (green); separately for $E_0 = 2.0$ GeV (left) and $E_0 = 2.3$ GeV (right).

3.2 Method II proposed by Neubert

This calculation by M. Neubert [6] is based on an approach that is quite similar to the one proposed by A. K. Leibovich, I. Low, and I. Z. Rothstein. The relation between $|V_{ub}|^2/|V_{tb}V_{ts}^*|^2$ and the differential branching fractions for $B \rightarrow X_u l \nu$ and $B \rightarrow X_s \gamma$ decays is

$$\frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} = \frac{3\alpha}{\pi} K_{\text{pert}} \frac{B_u(E_0)}{B_s(E_0)} + \mathcal{O}(\Lambda_{\text{QCD}}/m_b), \quad (4)$$

where the integrals over the spectra

$$B_u(E_0) = \int_{E_0}^{M_B/2} dE_l \frac{d\mathcal{B}(B \rightarrow X_u l \nu)}{dE_l} \quad (5)$$

and

$$B_s(E_0) = \frac{2}{M_B} \int_{E_0}^{M_B/2} dE_\gamma w(E_\gamma, E_0) \frac{d\mathcal{B}(B \rightarrow X_s \gamma)}{dE_\gamma} \quad (6)$$

extend from E_0 to the maximum $M_B/2$. They are to be determined from experiment. At order α_s , the weight function applied to the photon spectrum is given by

$$w(E_\gamma, E_0) = (E_\gamma - E_0) \left[1 - \frac{10}{9} \frac{\alpha_s}{\pi} \ln \left(1 - \frac{E_0}{E_\gamma} \right) \right]. \quad (7)$$

The more precise expression for the weight function, which includes resummation of the next-to-leading order Sudakov logarithms, which become large as the energy cut-off approaches $M_B/2$, is the following:

$$w_{\text{res}} = \left(1 - \frac{\alpha_s(m_b)}{2\pi} \left(-\frac{83}{9} + \frac{4\pi^2}{9} \right) \right)^{-1} \int_{E_0}^{E_\gamma} dE K \left[\frac{2E}{M_B}; \frac{16}{3\beta_0} \ln \left(1 + \frac{\beta_0 \alpha_s(m_b)}{4\pi} \ln \left(\ln \frac{E_\gamma}{E} \right) \right) \right]. \quad (8)$$

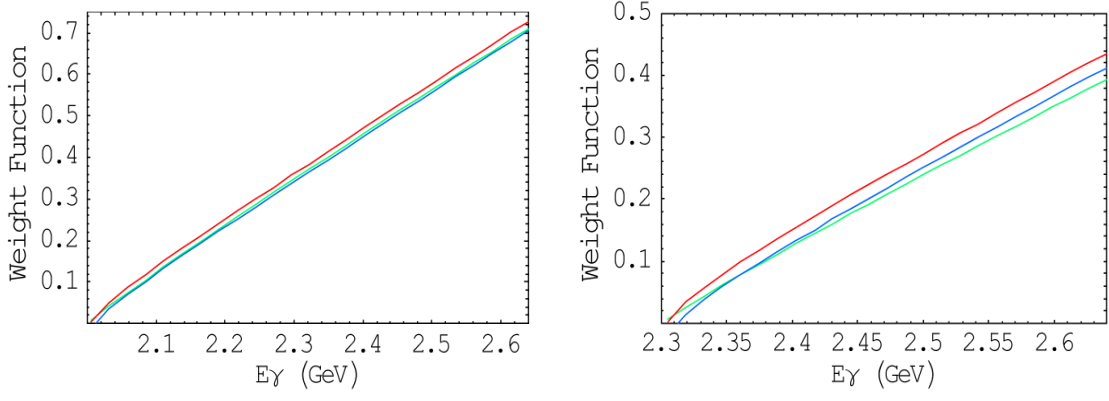


Figure 4: The weight functions for Method II: non-resummed (green) and resummed weight functions for $\rho = 0.995$ (blue), $\rho = 0.9983$ (red); separately for $E_0 = 2.0$ GeV (left) and $E_0 = 2.3$ GeV (right).

The matching corrections are given by

$$\begin{aligned}
K_{\text{pert}} = & (C_7^{(0)})^2 \left\{ 1 + \frac{\alpha_s}{2\pi} \left[-\frac{83}{9} + \frac{4\pi^2}{9} + \frac{32}{3} \ln \frac{m_b}{\mu} + \frac{C_7^{(1)}}{C_7^{(0)}} \right] \right\} + \\
& + C_2^{(0)} C_7^{(0)} \left[\frac{\alpha_s}{2\pi} \left(\Re(r_2) + \frac{416}{81} \ln \frac{m_b}{\mu} \right) - \frac{\lambda_2}{9m_c^2} \right] + C_8^{(0)} C_7^{(0)} \frac{\alpha_s}{2\pi} \left(\frac{44}{9} - \frac{8\pi^2}{27} - \frac{32}{9} \ln \frac{m_b}{\mu} \right), \\
\Re(r_2) \approx & -4.092 - 12.78(0.29 - m_c/m_b).
\end{aligned}$$

At tree level, the weight function is a linear function of the photon energy, $w(E_\gamma, E_0) = (E_\gamma - E_0)$. The Wilson coefficients, C_i , can be found in Ref. [14, 15]. This calculation contains NLO perturbative corrections and does not take into account power corrections. The expressions are scale and scheme dependent. The weight functions calculated for $E_0 = 2.0$ GeV and $E_0 = 2.3$ GeV and different values of ρ are shown in Fig. 4, using either the “non-resummed” weight function of Eq. 7 or the “resummed” weight function of Eq. 8.

3.3 Method III proposed by Bosch, Lange, Neubert, and Paz

The third method for the extraction of $|V_{ub}|$ is an application of the more recent two-loop calculations by B. Lange, M. Neubert, and G. Paz [7], and by B. Lange [8], performed for the high end of the lepton and photon energy spectra in $B \rightarrow X_u \ell \nu$ and $B \rightarrow X_s \gamma$ decays.

Unlike the previously described methods, this one relates $|V_{ub}|$ to the measured partial $B \rightarrow X_u \ell \nu$ branching fraction and normalized photon spectrum in the $B \rightarrow X_s \gamma$ decay,

$$|V_{ub}|^2 = \frac{\frac{1}{\tau_B} \int_{E_0}^{M_B/2} dE_l (dB(B \rightarrow X_u \ell \nu)/dE_l)}{\int_{E_0}^{M_B/2} dE_\gamma w(E_\gamma, E_0) S(E_\gamma) + \Gamma_{\text{rh}}(E_0)}, \quad (9)$$

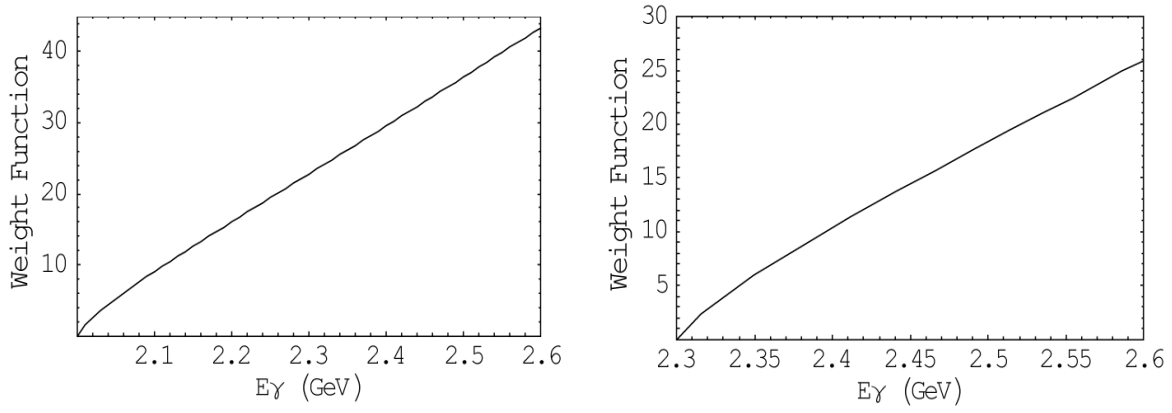


Figure 5: The weight function (in units of ps^{-1}) for Method III, separately for $E_0 = 2.0 \text{ GeV}$ (left) and $E_0 = 2.3 \text{ GeV}$ (right).

$$S(E_\gamma) = \frac{1}{\Gamma(B \rightarrow X_s \gamma)_{E_\gamma > E_{\min}}} \frac{d\Gamma(B \rightarrow X_s \gamma)}{dE_\gamma}, \quad (10)$$

where $\Gamma_{\text{rhc}}(E_0)$ represents residual hadronic power corrections, which in [7] were absorbed into the weight function. $E_{\min} = 1.90 \text{ GeV}$ is chosen as the lower limit for the normalization of the photon energy spectrum; the corresponding branching fraction is 3.95×10^{-4} .

This calculation contains perturbative corrections at NNLO at the so-called “jet scale”, $\mu_i \sim \sqrt{m_b \Lambda_{\text{QCD}}}$, and at NLO at the so-called “hard scale”, $\mu_h \sim m_b$. Also included are the first-order power corrections, which are separated into two parts: kinematic corrections and residual hadronic corrections. The kinematic corrections do not introduce hadronic uncertainties and are applied directly to the weight function. The residual hadronic corrections include subleading shape functions for which the functional form is unknown. This introduces significant theoretical uncertainties. The weight functions, calculated for $E_0 = 2.0 \text{ GeV}$ and $E_0 = 2.3 \text{ GeV}$, are shown in Fig 5.

4 Determination of $|V_{ub}|$

4.1 Calculation of weighted integrals of $B \rightarrow X_s \gamma$ photon spectrum

For each of the three methods we extract $|V_{ub}|$ from a ratio of the $B \rightarrow X_u l \nu$ partial branching fraction and a weighted integral over the photon spectrum for $B \rightarrow X_s \gamma$. Each method introduces a specific weight function $w(E_\gamma, E_0)$. The value of the weighted integral of the photon spectrum above a minimum energy E_0 ,

$$I(E_0) = \int_{E_0}^{M_B/2} dE_\gamma w(E_\gamma, E_0) d\Gamma(E_\gamma)/dE_\gamma, \quad (11)$$

is taken as a sum over bins in the photon energy in the B -rest frame,

$$\tilde{I}(E_0) = \sum_i w(E_{\gamma i}, E_0) \left(\frac{d\Gamma(E_\gamma)}{dE_\gamma} \right)_i \Delta E_{\gamma i}. \quad (12)$$

The uncertainty for this sum is estimated using the standard error propagation for a linear combination of random variables:

$$\sigma^2 = \sum_{ij} w(E_{\gamma i}, E_0) w(E_{\gamma j}, E_0) \Delta E_{\gamma i} \Delta E_{\gamma j} V^{ij}, \quad (13)$$

where V^{ij} is a covariance matrix of the differential rate $d\Gamma(E_\gamma)/dE_\gamma$ for different photon energy bins. Assuming uncorrelated statistical errors and correlated systematic errors with known correlation matrix R^{ij} , the covariance matrix is of the form

$$V^{ij} = \sigma_{\text{stat}}^i \sigma_{\text{stat}}^j \delta^{ij} + \sigma_{\text{syst}}^i \sigma_{\text{syst}}^j R^{ij}. \quad (14)$$

The correlation matrix R^{ij} was provided by the *BABAR* Collaboration [11].

4.2 Results on $|V_{ub}|$ and Error Estimation

4.2.1 Method I

The results of the extraction of $|V_{ub}|/|V_{tb}V_{ts}^*|$ based on Method I are presented in Table 2. The theoretical uncertainties of the NLO calculations have been estimated by Leibovich et al. [3, 4] to be $\mathcal{O}(\Lambda_{\text{QCD}}/M_B)$, resulting in a relative error of about 6% for $|V_{ub}|/|V_{tb}V_{ts}^*|$. The extraction of $|V_{ub}|/|V_{tb}V_{ts}^*|$ was done for two forms of the weight function, one with resummed Sudakov logarithms and another without resummation. The numerical effect of the resummation is $\sim 6\%$ at $E_0 = 2.0 \text{ GeV}$ and increases up to $\sim 12\%$ at 2.4 GeV .

Table 2: The results for $|V_{ub}|/|V_{tb}V_{ts}^*|$ for Method I. The extraction is performed for two different weight functions: one with resummed Sudakov logarithms and $\rho = 0.9983$, and the other without resummation. The first error represents the error from the measured $B \rightarrow X_u e \nu$ partial branching fraction, the second error is from the measured $B \rightarrow X_s \gamma$ spectrum, and the third is the estimated theoretical error.

$E_0 \text{ [GeV]}$	$ V_{ub} / V_{tb}V_{ts}^* $ Resummed WF ($\rho = 0.9983$)	$ V_{ub} / V_{tb}V_{ts}^* $ Non-resummed WF
2.0	$0.106 \pm 0.007 \pm 0.007 \pm 0.006$	$0.112 \pm 0.008 \pm 0.007 \pm 0.007$
2.1	$0.100 \pm 0.006 \pm 0.006 \pm 0.006$	$0.107 \pm 0.006 \pm 0.006 \pm 0.006$
2.2	$0.093 \pm 0.005 \pm 0.005 \pm 0.006$	$0.101 \pm 0.005 \pm 0.006 \pm 0.006$
2.3	$0.091 \pm 0.004 \pm 0.005 \pm 0.006$	$0.100 \pm 0.004 \pm 0.005 \pm 0.006$
2.4	$0.090 \pm 0.006 \pm 0.004 \pm 0.006$	$0.101 \pm 0.007 \pm 0.005 \pm 0.006$

To translate these results into $|V_{ub}|$ we exploit the constraints of the unitarity of the CKM matrix resulting in $|V_{tb}| \cong 1 + \mathcal{O}(\lambda^4)$, where $\lambda = 0.226$ is the sine of the Cabibbo angle. Using $|V_{ts}| = (40.6 \pm 2.7) \cdot 10^{-3}$ [16], we calculate $|V_{ub}|$ (see Table 3).

The results show high stability with respect to variations of the lepton energy cut in the range from 2.0 to 2.4 GeV. The partial charmless semileptonic branching fraction decreases in this range from 25% to 2.3% of the total $B \rightarrow X_u l \nu$ branching fraction [10]. The observed stability is quite surprising because near the endpoint of the lepton energy spectrum the theoretical calculations are expected to break down, and this should lead to increasing theoretical uncertainties.

Table 3: The results for $|V_{ub}|$ based on Method I. The first error is from the measured $B \rightarrow X_u e \nu$ partial branching fraction, the second is from the measured $B \rightarrow X_s \gamma$ spectrum, the third is a theoretical uncertainty, and the last one from the error on $|V_{ts}|$.

E_0 [GeV]	$ V_{ub} \cdot 10^3$
2.0	$4.28 \pm 0.29 \pm 0.29 \pm 0.26 \pm 0.28$
2.1	$4.06 \pm 0.23 \pm 0.25 \pm 0.24 \pm 0.27$
2.2	$3.78 \pm 0.18 \pm 0.21 \pm 0.23 \pm 0.25$
2.3	$3.69 \pm 0.16 \pm 0.19 \pm 0.22 \pm 0.25$
2.4	$3.64 \pm 0.25 \pm 0.17 \pm 0.22 \pm 0.24$

4.2.2 Method II

The results of the extraction of $|V_{ub}|/|V_{tb}V_{ts}^*|$ based on Method II [6] are presented in Table 4 for two different weight functions, the simpler “unresummed” according to Eq. 6, and the “resummed” with $\rho = 0.9983$ based on Eq. 7. The simpler ansatz results in a more stable value of $|V_{ub}|$, whereas the improved ansatz for the weight function results in a value of $|V_{ub}|$ that decreases with E_0 , up to 10% at the highest values of E_0 .

Table 4: The results for $|V_{ub}|/|V_{tb}V_{ts}^*|$ for Method II with the weight functions defined in Eqs. 7 and 8. The first error represents the error from the measured $B \rightarrow X_u e \nu$ partial branching fraction, the second error is from the measured $B \rightarrow X_s \gamma$ spectrum, the third is the estimated theoretical error.

E_0 [GeV]	$ V_{ub} / V_{tb}V_{ts}^* $ ($\rho = 0.9983$), Resummed WF	$ V_{ub} / V_{tb}V_{ts}^* $, Non-resummed WF
2.0	$0.099 \pm 0.007 \pm 0.007 \pm 0.008$	$0.101 \pm 0.007 \pm 0.007 \pm 0.008$
2.1	$0.095 \pm 0.005 \pm 0.006 \pm 0.008$	$0.098 \pm 0.006 \pm 0.006 \pm 0.008$
2.2	$0.089 \pm 0.004 \pm 0.005 \pm 0.007$	$0.094 \pm 0.005 \pm 0.005 \pm 0.008$
2.3	$0.088 \pm 0.004 \pm 0.005 \pm 0.007$	$0.095 \pm 0.004 \pm 0.005 \pm 0.008$
2.4	$0.088 \pm 0.006 \pm 0.004 \pm 0.007$	$0.097 \pm 0.007 \pm 0.005 \pm 0.008$

The theoretical error on $|V_{ub}|^2/|V_{tb}V_{ts}^*|^2$, as quoted by Neubert, has two contributions, one originating from the perturbative correction, $\sim 6\%$, and one from the higher order hadronic corrections, $\mathcal{O}(\Lambda_{\text{QCD}}/M_B)$, also estimated to be $\sim 6\%$, resulting in a total uncertainty of $\sim 8\%$. As described above, we translate the results for $|V_{ub}|/|V_{tb}V_{ts}^*|$ into $|V_{ub}|$ and present the results in Table 5.

4.2.3 Method III

The results on the extraction of $|V_{ub}|$ based on Method III are presented in Table 6. The total theoretical uncertainty is estimated as sum of the squares of the individual contributions [8]. The first order non-perturbative hadronic power corrections, not considered in Methods I and II, are split into two contributions: kinematic corrections, which are included in the weight function and depend on the scale for the calculation of kinematic power corrections, $\bar{\mu}$, but are

Table 5: The results for $|V_{ub}|$ from Method II. The first error is due to the experimental error of the $B \rightarrow X_u e \nu$ partial branching fraction, the second is from the error on $B \rightarrow X_s \gamma$, the third is a theoretical uncertainty, and the last one from the error on $|V_{ts}|$.

E_0 [GeV]	$ V_{ub} \cdot 10^3$
2.0	$4.01 \pm 0.27 \pm 0.29 \pm 0.32 \pm 0.27$
2.1	$3.84 \pm 0.22 \pm 0.24 \pm 0.31 \pm 0.26$
2.2	$3.62 \pm 0.18 \pm 0.21 \pm 0.29 \pm 0.24$
2.3	$3.57 \pm 0.16 \pm 0.19 \pm 0.29 \pm 0.24$
2.4	$3.58 \pm 0.25 \pm 0.17 \pm 0.29 \pm 0.24$

independent of intermediate scale, μ_i , the hard scale, μ_h , and residual hadronic corrections. These corrections depend on unknown subleading shape functions, which affect the $B \rightarrow X_u l \nu$ and $B \rightarrow X_s \gamma$ spectra in different ways and thus do not cancel in the weighted ratio in the integrals. This uncertainty, σ_{hadr} , is estimated following a suggestion by Lange [8], namely by varying the shape and the parameters of the subleading shape functions.

The error σ_{pert} is the uncertainty of the NNLO approximation of the shape function and the LO approximation of the power corrections. It was estimated from the dependence of the sum of the weighted integral over the photon spectrum and residual hadronic correction on the hard scale, $\mu_h = m_b/\sqrt{2}$, on the intermediate scale, $\mu_i = 1.5$ GeV, and on the scale for the power corrections, $\bar{\mu} = 1.5$ GeV. All three scales are varied by factors of $\sqrt{2}$ and $1/\sqrt{2}$ relative to their default values, and the largest variation of the weighted integral is taken as the estimate of the perturbative uncertainty.

The errors σ_{m_b} and σ_{param} are due to uncertainties of the parameters that are inputs to the calculation and were varied within their stated errors: $m_b = 4.61 \pm 0.06$ GeV and $m_c/m_b = 0.222 \pm 0.027$, $m_s = 90 \pm 25$ MeV, $\lambda_2 = 0.12$ GeV², $\mu_\pi^2 = 0.25 \pm 0.10$ GeV². Here m_b and μ_π^2 are defined in the shape-function scheme at a scale $\mu_* = 1.5$ GeV, m_s and the ratio m_c/m_b are evaluated in the \overline{MS} scheme at 1.5 GeV, where the ratio is scale-invariant.

The error σ_{norm} represents the uncertainty of the normalization of the photon spectrum, which is estimated to be about 6% [8].

Table 6: The results for $|V_{ub}|$ based on Method III. The first error represents the error from the measured $B \rightarrow X_u e \nu$ partial branching fraction, and the second from the measured $B \rightarrow X_s \gamma$ spectrum. The contributions to the theoretical error, as described in the text, are given together with the total theoretical error. All values in the table are multiplied by 10^3 .

E_0 [GeV]	$ V_{ub} $	σ_{hadr}	σ_{pert}	σ_{m_b}	σ_{param}	σ_{norm}	σ_{theory}
2.0	$4.40 \pm 0.30 \pm 0.41$	+0.08 -0.07	+0.03 -0.01	+0.13 -0.12	+0.01 -0.00	± 0.17	+0.23 -0.22
2.1	$4.55 \pm 0.26 \pm 0.45$	+0.15 -0.13	+0.11 -0.05	+0.16 -0.15	± 0.01	± 0.21	+0.32 -0.29
2.2	$5.01 \pm 0.24 \pm 0.60$	+0.42 -0.33	+0.40 -0.21	+0.25 -0.22	± 0.01	± 0.32	+0.71 -0.55
2.3	$6.99 \pm 0.31 \pm 1.60$	+3.90 -1.44	+3.02 -1.00	+0.75 -0.58	+0.04 -0.03	± 0.92	+5.07 -2.05

We observe in Table 6 a significant increase in the extracted value of $|V_{ub}|$ for $E_0 \geq 2.2$ GeV, and the theoretical uncertainties also show a rapid growth for $E_0 > 2.1$ GeV, with errors

reaching 100% above $E_0 \simeq 2.3$ GeV. This is due to the denominator in Eq. 8, the sum of the weighted integral and the residual hadronic correction $\Gamma_{\text{rhc}}(E_0)$. This correction is negative and almost independent of E_0 . As E_0 increases, the integral over the photon spectrum decreases and its contribution to the total uncertainty increases. The same is the case for the uncertainty of the Γ_{rhc} . This is not unexpected, because the effect of non-perturbative hadronic corrections must increase in the region close to kinematic endpoints for both decays.

5 Conclusion

We have extracted the CKM matrix element $|V_{ub}|$ using published *BABAR* measurements of the inclusive lepton spectrum in $B \rightarrow X_u e \nu$ decays and inclusive photon spectrum in $B \rightarrow X_s \gamma$ decays. By using the ratios of the weighted spectra for these two decays, the results are expected to be less model dependent than previous measurements relying on the extraction of the shape functions from data and specific parameterizations of these functions.

Table 7: Comparison of the $|V_{ub}|$ extraction for $E_0 = 2.0$ GeV for all methods considered. The first error reflects the uncertainty in the measurements of the $B \rightarrow X_u l \nu$ lepton spectrum, the second error is due to the measurement of the $B \rightarrow X_s \gamma$ photon spectrum. For the shape-function based analysis, the second error originates from the extraction of the shape-function parameters, in this case based on both the inclusive photon spectrum as well as hadron-mass and lepton-energy moments from $B \rightarrow X_c l \nu$ decays. The third is the theoretical uncertainty. The fourth error for Methods I and II is due to the uncertainty of $|V_{ts}|$.

Method	$ V_{ub} \cdot 10^3$
LLR [3, 4]	$4.28 \pm 0.29 \pm 0.29 \pm 0.26 \pm 0.28$
Neubert [6]	$4.01 \pm 0.27 \pm 0.29 \pm 0.32 \pm 0.27$
BLNP [7, 8]	$4.40 \pm 0.30 \pm 0.41 \pm 0.23$
SF-based analysis [10]	$4.44 \pm 0.25^{+0.42}_{-0.38} \pm 0.22$

For comparison, the results for $E_0 = 2.0$ GeV for all three methods are presented in Table 7, together with the *BABAR* shape-function based measurement [10]. Figure 6 shows the dependence of the results on the lepton energy cut-off, E_0 .

All measurements agree well within their stated uncertainties. Methods I and II result in values of $|V_{ub}|$ and errors that appear to be unaffected by the restriction of the phase space near the kinematic limit. The theoretical errors are somewhat smaller than for Method III, and also smaller than for the previous extraction method, but there is an additional error due to the uncertainty of $|V_{ts}|$.

The values of $|V_{ub}|$ extracted by using Method III for the lepton energy cut-off at $E_0 = 2.0$ GeV and 2.1 GeV agree well with those of the other methods. The lowest total theoretical error is obtained for $E_0 = 2.0$ GeV; it is about 5%, and according to the authors, this error is more reliably estimated. However, at larger E_0 , there are several increasing contributions to the theoretical uncertainties. Furthermore, the error contributions from the integral over the photon spectrum increase very rapidly as the phase space is more restricted.

At present, for $E_0 = 2.0$ GeV, the experimental errors on both the $B \rightarrow X_u l \nu$ branching fraction and the integral over $B \rightarrow X_s \gamma$ are about 12%; this means that there are opportunities

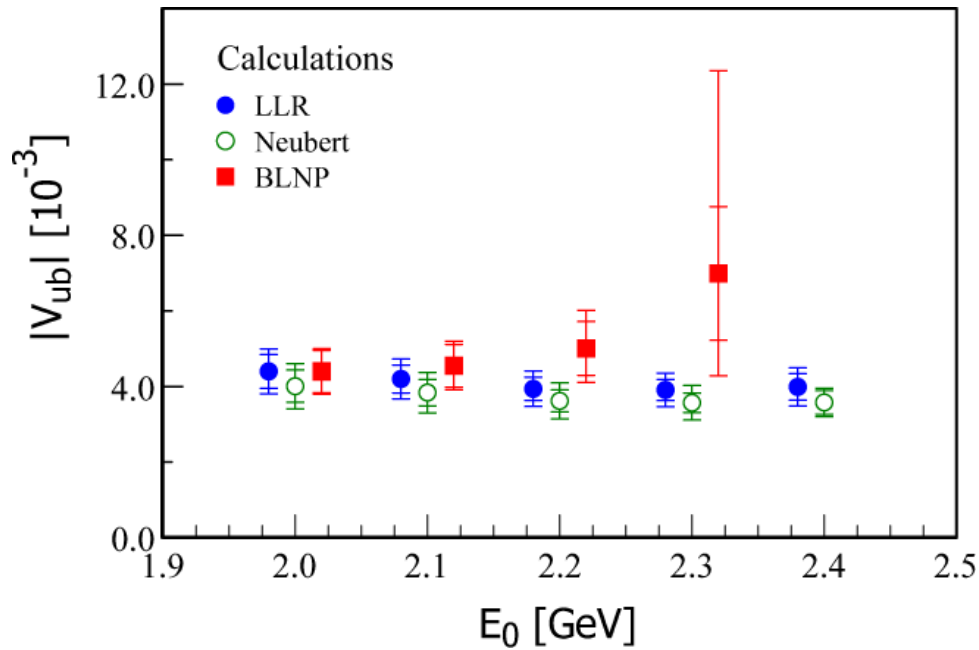


Figure 6: Comparison of $|V_{ub}|$ values extracted from the three different calculations as a function of the lepton energy cut-off, E_0 .

to improve the accuracy of $|V_{ub}|$ significantly by reducing the experimental errors of these two spectra with more data available now and in the future. Also, extending E_0 to 1.9 GeV or lower, would allow us to establish more clearly the stability of the results in this region.

It would be of interest to perform a similar analysis for the hadron-mass distribution in $B \rightarrow X_u \ell \nu$ decays, with increasingly tighter restrictions on the mass of X_u , and assess whether this measurement is indeed more robust than the measurement of the lepton spectrum near the kinematic endpoint.

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