

Is $\mathcal{N} = 8$ Supergravity Ultraviolet Finite?

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Conventional wisdom holds that no four-dimensional gravity field theory can be ultraviolet finite. This understanding is based mainly on power counting. Recent studies confirm that one-loop $\mathcal{N} = 8$ supergravity amplitudes satisfy the so-called “no-triangle hypothesis”, which states that triangle and bubble integrals cancel from these amplitudes. A consequence of this hypothesis is that for any number of external legs, at one loop $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills have identical superficial degrees of ultraviolet behavior in D dimensions. We describe how the unitarity method allows us to promote these one-loop cancellations to higher loops, suggesting that previous power counts were too conservative. We discuss higher-loop evidence suggesting that $\mathcal{N} = 8$ supergravity has the same degree of divergence as $\mathcal{N} = 4$ super-Yang-Mills theory and is ultraviolet finite in four dimensions. We comment on calculations needed to reinforce this proposal, which are feasible using the unitarity method.

INTRODUCTION

Conventional wisdom holds that it is impossible to construct a finite field theory of quantum gravity. Indeed, all widely accepted studies to date have concluded that all known gravity field theories are ultraviolet divergent and non-renormalizable [1–4]. If one were able to find a finite four-dimensional quantum field theory of gravity, it would have profound implications. In particular, finiteness would seem to imply that there should be an additional symmetry hidden in the theory.

Although power counting arguments indicate that all known gravity field theories are non-renormalizable, there are very few explicit calculations establishing their divergence properties. For pure gravity, a field redefinition removes the potential on-shell one-loop divergence [1, 2], but the calculation of Goroff and Sagnotti [5], confirmed by van de Ven [6], explicitly shows that pure Einstein gravity has an ultraviolet divergence at two loops. If generic matter fields are added [1, 2] a divergence appears already at one loop. If the matter is added so as to make the theory supersymmetric, the divergences are in general delayed until at least three loops (see e.g. refs. [3, 4]). However, no complete calculations have been performed to confirm that the coefficients of the potential divergences in supersymmetric theories are actually non-vanishing.

One approach to dealing with the calculational difficulties [7–11] makes use of the unitarity method [12–16], as well as the Kawai, Lewellen and Tye (KLT) relations between open- and closed-string tree-level amplitudes [17]. In the low-energy limit the KLT relations express grav-

ity tree amplitudes in terms of gauge theory tree amplitudes [18]. Combining the KLT representation with the unitarity method, which builds loop amplitudes from tree amplitudes, massless gravity scattering amplitudes – including their ultraviolet divergences – are fully determined to any loop order starting from gauge theory tree amplitudes. In particular, for the case of $\mathcal{N} = 8$ supergravity, the entire perturbative expansion can be built from $\mathcal{N} = 4$ super-Yang-Mills tree amplitudes [7]. It is rather striking that one can obtain all the amplitudes of $\mathcal{N} = 8$ supergravity from the tree amplitudes of an ultraviolet-finite conformal field theory.

The KLT relations between gauge and gravity amplitudes are especially useful for addressing the question of the ultraviolet divergences of gravity theories because, from a technical viewpoint, perturbative computations in gauge theories are much simpler than in gravity theories. With the unitarity method, these relations are promoted to relations on the unitarity cuts. This strategy has already been used [7] to argue that the first potential divergence in $\mathcal{N} = 8$ supergravity would occur at five loops, instead of the three loops previously predicted using superspace power counting arguments [4]. Using harmonic superspace, Howe and Stelle have confirmed this result [19]. Very interestingly, they also speculate that the potential divergences may be delayed an additional loop order.

In this note we reexamine the power counting of ref. [7] for $\mathcal{N} = 8$ supergravity. We demonstrate that there are additional unexpected cancellations beyond those identified in that paper. Our analysis of the amplitudes is based on unitarity cuts which slice through three or more lines representing particles, instead of the iterated two-

particle cuts focused on in ref. [7]. It suggests that $\mathcal{N} = 8$ supergravity may have the same ultraviolet behavior as $\mathcal{N} = 4$ super-Yang-Mills theory, *i.e.* that it is finite in four dimensions. We will also outline calculations, feasible using the unitarity method, that should be done to shed further light on this issue.

Our motivation for carrying out this reexamination stems from the recent realization that in the $\mathcal{N} = 8$ theory unexpected one-loop cancellations first observed for the special class of maximally helicity violating (MHV) amplitudes [8], in fact hold more generally [10, 11, 20]. The potential consequences of these cancellations for the ultraviolet divergences at higher loops were noted in the latter references. See also ref. [21].

It has been known for a while that the one-loop MHV amplitudes with four, five and six external gravitons are composed entirely of scalar box integrals [8, 22], lacking all triangle and bubble integrals. Although there are no complete calculations for more than six external legs, factorization arguments make a strong case that the same property holds for MHV amplitudes with an arbitrary number of external legs. Factorization puts rather strong constraints on amplitudes and has even been used to determine their explicit form in some cases (see, for example, refs. [8, 23]). Because the scalar box integral functions are the same ones that appear in the corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitudes, the D -dimensional ultraviolet behavior is identical: The amplitudes all begin to diverge at $D = 8$.

As the number of external legs increases, one might have expected the ultraviolet properties of supergravity to become relatively worse compared to super-Yang-Mills theory, as a consequence of the two-derivative couplings of gravity. From recent explicit calculations of six and seven graviton amplitudes [10, 11, 20], it is now clear that the cancellations which prevent triangle or bubble integrals from appearing extend beyond the MHV case. This property has been referred to as the “no-triangle hypothesis”. This hypothesis puts an upper bound on the number of loop momenta that can appear in the numerator of any one-loop integral. Under integral reductions [24] any power of loop momentum l appearing in the numerator can be used to reduce an m -gon integral to an $(m - 1)$ -gon integral. The $(m - 1)$ -gon integral has one less power of l in its numerator. Inverse propagators (such as l^2) present in the numerator are special: they can also be used to reduce m -gon integrals to $(m - 1)$ -gon integrals with *two* less powers of l .

For example, consider a pentagon integral with a factor of $2l \cdot k_1$ in the numerator, denoted by $I_5[2l \cdot k_1]$. If l^2 and $(l - k_1)^2$ are two propagators in the pentagon integral and k_1 is the momentum of an external line, we can rewrite the numerator factor as a difference of two inverse propagators,

$$2l \cdot k_1 = l^2 - (l - k_1)^2. \quad (1)$$

This equation immediately reduces the linear pentagon integral to a difference of two scalar box integrals,

$$I_5[2l \cdot k_1] = I_4^{(1)} - I_4^{(2)}, \quad (2)$$

where $I_4^{(1)}$ and $I_4^{(2)}$ are the box integrals obtained from the pentagon by removing the l^2 and $(l - k_1)^2$ propagators, respectively. More generally, integral reductions bound the maximum power of loop momenta in the numerator, in order that triangle integrals not appear in the final result. For a pentagon integral the maximum is one inverse propagator, or one generic power of l ; for a hexagon it is two inverse propagators, or two generic powers of l ; and so forth.

For six-graviton non-MHV amplitudes, the computations of ref. [11] constitute a proof of the no-triangle hypothesis. For seven gravitons they demonstrate that the box integrals correctly account for infrared divergences, making it unlikely for infrared-divergent triangle integrals to appear. For larger numbers of external gravitons, the same factorization arguments used for MHV amplitudes [8, 10] make a strong case that the required cancellations continue to hold for all graviton helicity configurations. Supersymmetry relations suggest that the no-triangle hypothesis should be extendable from graviton amplitudes to amplitudes for all possible external states, because all $\mathcal{N} = 8$ supergravity states belong to the same supermultiplet. The additional cancellations implied by the no-triangle hypothesis are rather non-trivial. For example, one-loop integrands constructed using the KLT representation of tree amplitudes naively violate the no-triangle bound [8, 10]. In the rest of this letter we will assume that the no-triangle hypothesis holds for all one-loop $\mathcal{N} = 8$ amplitudes and examine the consequences for higher loops.

Will these unexpected one-loop cancellations continue to higher orders? Using conventional Feynman diagram techniques, it is not at all clear how to extract from these one-loop on-shell cancellations useful higher-loop statements, given that the formalism is inherently off shell. The unitarity method [12–16], however, provides a means for doing so. Because of the direct way that lower-loop on-shell amplitudes are used to construct the higher-loop ones, it is clear that the one-loop cancellations will continue to be found in higher-loop amplitudes. The main question then is whether the cancellations are sufficient to imply finiteness of the theory to all loop orders.

Besides the implications of the no-triangle hypothesis for higher loops via unitarity, there are a number of other clues pointing to a better than expected ultraviolet behavior. One interesting clue comes from the fact that the only complete $\mathcal{N} = 8$ calculation at two loops – the four-graviton amplitude – has exactly the same power counting as the corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitude [7]. This example clearly shows that a common degree of finiteness between the two theories in D dimensions is not limited to one loop.

An indirect hint of additional cancellations at higher loops comes from M theory dualities. In refs. [25], by using duality (defined in an appropriate way in the low-energy limit using eleven-dimensional supergravity with counterterms and a particular string-inspired regulator), it was argued that in type II string theory the $\partial^4 R^4$ term in the effective action does not suffer from renormalization beyond two loops. Because $D = 4$, $\mathcal{N} = 8$ supergravity may be obtained from type II supergravity through compactification of six of the dimensions, it hints that the former theory may also have additional cancellations. A very recent paper from Green, Russo and Vanhove uses restrictions from M and string theory dualities to argue that $\mathcal{N} = 8$ supergravity is no worse than logarithmically divergent in the ultraviolet [26]. (A related analysis of the four-graviton ten-, twelve- and fourteen-derivative terms has also just appeared [27].) Statements linking string dualities to improved ultraviolet behavior, and perhaps ultraviolet finiteness, in $\mathcal{N} = 8$ supergravity may also be found in an earlier paper by Chalmers [28].

An important new clue for the existence of improved ultraviolet behavior in $\mathcal{N} = 8$ supergravity comes from the recent work of Berkovits on the multi-loop effective action of type II string theories [29]. By analyzing string theory amplitudes, Berkovits found that $\partial^n R^4$ terms in the effective action do not receive perturbative contributions from above $n/2$ loops for $0 < n < 12$, because more than $n + 8$ powers of external momenta come out of the string integrand. Here R^4 denotes an $\mathcal{N} = 8$ supersymmetric contraction of Riemann tensors [3], and ∂ denotes a generic spacetime derivative. Assuming that there are no cancellations between the massless and higher-mass states of the string in the loops for small external momenta, the string amplitude properties can be applied to supergravity amplitudes, providing an indication of additional cancellations in ten-dimensional type II supergravity. If true, then the fact that type II supergravity corresponds to $\mathcal{N} = 8$ supergravity oxidized to ten dimensions would indicate the existence of additional cancellations in four dimensions, beyond those of refs. [7, 19], supporting the speculation of Howe and Stelle.

Finally, the new twistor structure uncovered for gravity theories [10, 30] implies a rich set of constraints on the form of gravity amplitudes. If $\mathcal{N} = 8$ supergravity loop amplitudes could be obtained from a topological string theory, it might lead to a natural explanation for ultraviolet finiteness. Recent developments in constructing a topological twistor string for gravity theories may be found in ref. [31].

All these indirect results point to the need to reinvestigate the ultraviolet properties of the $\mathcal{N} = 8$ theory directly. A first complete test would be to compute the full three-loop four-graviton amplitude, in order to confirm that 14 powers of external momentum do in fact come out of the $\mathcal{N} = 8$ loop momentum integrals. Here we shall perform a preliminary examination, studying types

of cancellations occurring in three- and higher-loop amplitudes.

UNITARITY CUTS

The unitarity method offers a powerful way to determine ultraviolet properties of gravity theories. The higher-loop study in ref. [7] relied on using two-particle cuts. Because of a remarkable recycling property, it is possible to iterate the two-particle cuts to *all* loop orders. Although two-particle iteration provides a wealth of information on the structure of the amplitudes, it is only for a limited set of contributions. Based on the contributions to the iterated two-particle cuts, as well as an observed “squaring” structure compared to the super-Yang-Mills case, the conclusion of ref. [7] is that the $\mathcal{N} = 8$ supergravity amplitudes should be ultraviolet finite for

$$D < \frac{10}{L} + 2 \quad (L > 1), \quad (3)$$

where D is the dimension and L the loop order. (The case of one-loop, $L = 1$, is special and the amplitudes are ultraviolet finite for $D < 8$, not $D < 12$.) This formula implies that in $D = 4$ the first potential divergence is at five loops. This result was confirmed by studying all cuts, but restricted to MHV helicity configurations crossing the cuts. It is also in agreement with the more recent harmonic superspace analysis of Howe and Stelle [19]. In contrast, the finiteness condition for $\mathcal{N} = 4$ super-Yang-Mills theory, found in refs. [7, 19], is

$$D < \frac{6}{L} + 4 \quad (L > 1). \quad (4)$$

(For $L = 1$, again the amplitudes are finite for $D < 8$.) The bound in eq. (4) differs somewhat from the earlier superspace power counting bound [32], though all bounds confirm that $\mathcal{N} = 4$ super-Yang-Mills theory is ultraviolet finite in $D = 4$. In the planar limit the complete expressions for the $\mathcal{N} = 4$ super-Yang-Mills amplitudes are known through four loops [7, 33, 34]. We have evaluated all logarithmic singularities of the planar contributions in the critical dimensions $D_c = 7, 6, 11/2$ corresponding to two, three and four loops, directly confirming eq. (4), at least in the limit of a large number of colors. For the $\mathcal{N} = 8$ supergravity case there are no complete calculations beyond two loops, so the finiteness condition (3) is much less certain.

A key assumption behind the finiteness condition (3) is that there are no cancellations with terms not present in iterated two-particle cuts. As discussed above, there are good reasons to reexamine this assumption.

To do so, consider an $\mathcal{N} = 8$ supergravity contribution from the iterated two-particle cuts used in the power

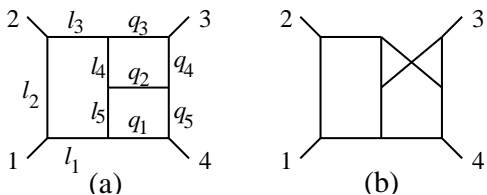


FIG. 1: Diagram (a) corresponds to a contribution appearing in the iterated two-particle cut of fig. 2(a). In $\mathcal{N} = 4$ super-Yang-Mills the iterated two-particle cuts give a numerator factor of $(l_1 + k_4)^2$. In $\mathcal{N} = 8$ supergravity it is $[(l_1 + k_4)^2]^2$. Diagram (b) contains a non-planar contribution which is not detectable in the iterated two-particle cut of fig. 2(a), but is detectable in the cut of fig. 2(b).

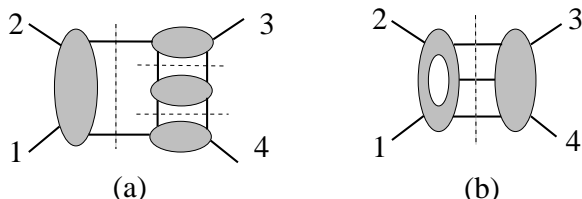


FIG. 2: An iterated two-particle cut (a) and a three-particle cut (b).

count of ref. [7]. From this reference, the planar contribution, displayed in fig. 1(a), is

$$[s^2 t A_4^{\text{tree}}]^j \int \frac{d^D l_1}{(2\pi)^D} \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{[(l_1 + k_4)^2]^j}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 q_1^2 q_2^2 q_3^2 q_4^2 q_5^2}, \quad (5)$$

up to overall normalization factors not depending on momenta. Here $A_4^{\text{tree}} \equiv A_4^{\text{tree}}(1, 2, 3, 4)$ is a color-ordered four-point super-Yang-Mills tree amplitude, $j = 1$ for $\mathcal{N} = 4$ super-Yang-Mills and $j = 2$ for $\mathcal{N} = 8$ supergravity. The k_i are external momenta, labeled by i in fig. 1(a). The Mandelstam invariants are $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ and the l_m are the momenta of the left pentagon subintegral in fig. 1(a), while the q_m are the momenta appearing in the other loops. The other contributions to the iterated two-particle cuts are similar. For the $\mathcal{N} = 4$ case, an examination of the three-particle cuts confirms [7, 33] that the correct numerator factor is $(l_1 + k_4)^2$, instead of, for example $2l_1 \cdot k_4$ which is equivalent using the on-shell conditions of the iterated two-particle cut in fig. 2(a). No analogous check has been performed on the $\mathcal{N} = 8$ amplitudes. For the supergravity case, we can rewrite the prefactor in front of the integral directly in terms of the supergravity tree amplitude, using the KLT-like relation

$$i[st A_4^{\text{tree}}(1, 2, 3, 4)]^2 = st(s + t)M_4^{\text{tree}}(1, 2, 3, 4), \quad (6)$$

where M_4^{tree} is the $\mathcal{N} = 8$ four-point tree amplitude.

The scaling of the integral in eq. (5) is that it is finite for

$$3D < 20 - 2j, \quad (7)$$

so that it corresponds to eq. (3) and eq. (4) with $L = 3$, for $j = 2$ and $j = 1$ respectively.

Is this power counting consistent with the three-particle cut in fig. 2(b)? On the left-hand side of the cut we have a one-loop pentagon integral contribution proportional to

$$\int \frac{d^D l_1}{(2\pi)^D} \frac{[(l_1 + k_4)^2]^j}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2}. \quad (8)$$

If we perform an integral reduction [24], in the Yang-Mills case, $j = 1$, eq. (8) reduces to a sum over box integrals. For the $\mathcal{N} = 8$ supergravity case, $j = 2$, a similar reduction leads also to triangle integrals because of the higher power of loop momentum in the numerator. It is important to note that replacing $[(l_1 + k_4)^2]^2$ with $(2l_1 \cdot k_4)^2$ also leads, under integral reduction, to a violation of the no-triangle hypothesis, although it has a better ultraviolet behavior.

We may compare this power counting to the results from an evaluation of the three-particle cuts. On the left-hand side of the cut in fig. 2(b), the one-loop five-point supergravity amplitude is given by [8]

$$\begin{aligned} M_5^{1\text{-loop}}(1, 2, q_1, q_2, q_3) &= -\frac{1}{2} \sum_{\text{perms}} s_{q_2 q_1} s_{12}^2 s_{23}^2 A_5^{\text{tree}}(1, 2, q_3, q_2, q_1) \\ &\times A_5^{\text{tree}}(1, 2, q_3, q_1, q_2) \int \frac{d^D l_1}{(2\pi)^D} \frac{1}{l_1^2 l_2^2 l_3^2 l_4^2}, \end{aligned} \quad (9)$$

where A_5^{tree} are color-ordered five-point super-Yang-Mills tree amplitudes and $s_{2q_3} = (k_2 + q_3)^2$, etc. In eq. (9) we have dropped the contributions that vanish away from four-dimensions, since they are suppressed by a power of $\epsilon = (4 - D)/2$ and should be irrelevant as far as the leading ultraviolet behavior near $D = 4$ is concerned. At five points all amplitudes are MHV, and there are simple supersymmetric Ward identities [35] relating the various five-point amplitudes involving the superpartners; these are described in appendix E of ref. [7]. The appearance of exactly four propagators in each term in eq. (9) implies that pentagon integrals have canceled down to boxes, but with no triangle integrals present. Alternatively, instead of carrying out the integration one can use the merging procedure on the cut integrands discussed in ref. [15] to algebraically arrive at the same conclusion. The coefficients of these box integrals may also be readily obtained using an observation due to Britto, Cachazo and Feng that the quadruple cuts freeze box integral loop momenta, allowing for their simple algebraic determination [16]. The lack of triangle integrals involves a rather non-trivial set of cancellations: The permutation sum is over 30 contributions corresponding to the distinct scalar box integrals with one external massive leg.

Comparing eq. (9) to eq. (8) we see that the one-loop amplitude entering the cut is much better behaved in

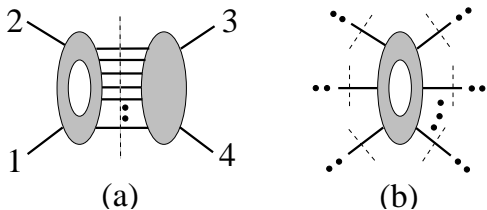


FIG. 3: From the “no-triangle hypothesis” all one-loop subamplitudes appearing in unitarity cuts in $\mathcal{N} = 8$ supergravity have the same degree of divergence as in $\mathcal{N} = 4$ super-Yang-Mills theory. The cut (a) is an L -particle cut of an L -loop amplitude. Cut (b) makes use of generalized unitarity; if a leg is external to the entire amplitude, it should not be cut.

the ultraviolet than is implied by the result (5). Because eq. (8) violates the no-triangle hypothesis, which we know is correct at five points [8], some of the powers of loop momenta must cancel. However, this cancellation is not visible in the contribution (5). A crucial difference between the iterated two-particle cut depicted in fig. 2(a) and the three-particle cut depicted in fig. 2(b) is that the latter includes also a variety of other diagrammatic topologies. For example, the non-planar diagram (b) of fig. 1 is not detected in the iterated two-particle cut of fig. 2(a), but it is included in the three-particle cut of fig. 2(b). These cancellations might involve integrals that are detectable in other iterated two-particle cuts, or integrals that are visible only in higher-particle cuts. In $\mathcal{N} = 4$ super-Yang-Mills theory, cancellations between planar and non-planar topologies cannot happen, because planar and non-planar contributions carry different color factors and can therefore be treated independently.

Cancellations at higher loops are also dictated by the no-triangle hypothesis. Consider the L -particle cut of the L -loop amplitudes shown in fig. 3(a). The iterated two-particle cut analysis and squaring assumption of ref. [7] give the numerator factor appearing in the diagrammatic contribution shown in fig. 4 as $[(l + k_1)^2]^{2(L-2)}$ for the $\mathcal{N} = 8$ case. This factor is the square of the $\mathcal{N} = 4$ super-Yang-Mills factor $[(l + k_1)^2]^{(L-2)}$. This proposed $\mathcal{N} = 8$ numerator factor is at the origin of the power count (3). However, this factor leads to a violation of the no-triangle hypothesis for the $(L + 2)$ -leg amplitude on the left-hand side of the L -particle cut in fig. 3(a). That is, we find that the cancellations which reduce the one-loop amplitude to a sum over box integrals have not been taken into account. Since there is strong evidence in favor of the one-loop no-triangle hypothesis [11], we conclude that the finiteness condition (3) is probably too conservative.

One can also extend this analysis using generalized unitarity, which provides a powerful way to construct amplitudes [14–16]. For all possible one-loop subamplitudes isolated by cutting internal lines in a higher-loop ampli-

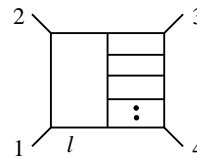


FIG. 4: An L -loop contribution as proposed in ref. [7]. In $\mathcal{N} = 4$ super-Yang-Mills theory the numerator factor is $[(l + k_4)^2]^{(L-2)}$, while in $\mathcal{N} = 8$ supergravity the factor is $[(l + k_4)^2]^{2(L-2)}$.

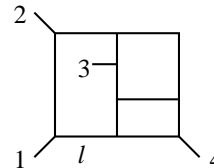


FIG. 5: An example of a potential $\mathcal{N} = 8$ supergravity contribution where the no-triangle hypothesis does not provide sufficient information to rule out a violation of the bound (4). The numerator factor proposed in ref. [7] is $[(l + k_4)^2]^2$, although when evaluated on the iterated two-particle cuts used in its construction, it is indistinguishable from $(2l \cdot k_4)^2$, which is consistent with the no-triangle bound.

tude, as depicted in fig. 3(b), the no-triangle hypothesis implies that they have the same degree of divergence as the $\mathcal{N} = 4$ super-Yang-Mills theory. Because this result holds for all possible generalized cuts which isolate a one-loop amplitude, we obtain a rather non-trivial set of consistency conditions limiting the ultraviolet behavior of the higher-loop amplitudes. If a specific set of cuts points to bad ultraviolet behavior in a given loop momentum, we can isolate that loop via generalized unitarity. Every such one-loop subamplitude has a power count no worse than that of $\mathcal{N} = 4$ super-Yang-Mills theory (assuming the no-triangle hypothesis), suggesting that the entire amplitude may have this property.

In order to construct a proof that the overall degree of divergence matches that of $\mathcal{N} = 4$ super-Yang-Mills, it is crucial to track the critical dimension where logarithmic divergences first arise. In such a proof one would need to rule out contributions where the no-triangle hypothesis is not violated, yet the overall finiteness bound (4) is violated. An example of such a potential contribution is given in fig. 5. The numerator factor proposed in ref. [7], $[(l + k_4)^2]^2$, would not violate the no-triangle hypothesis for the one-loop hexagon subdiagram, yet would violate the overall bound. On the other hand, the iterated two-particle cut analysis used in its construction [7] does not distinguish $[(l + k_4)^2]^2$ from $(2l \cdot k_4)^2$ — the latter form *is* consistent with the overall bound (4) — nor does it take into account any cancellations of the type found for the planar contribution in fig. 1. A complete construction of the three-loop amplitude would, of course, resolve this situation.

By power counting Feynman diagrams, one can see that if the finiteness condition of $\mathcal{N} = 8$ supergravity is identical to that of $\mathcal{N} = 4$ super-Yang-Mills theory, then the L -loop contribution to the one-particle irreducible effective action would start with the form $\partial^{2L} R^4 / \partial^6$, with an ultraviolet-finite (though infrared-singular) coefficient in $D = 4$. The nonlocal factor of $1/\partial^6$ arises from the loop integrals, by dimensional analysis. A discussion of power counting in effective actions, and implications for the degree of divergence of the theory in $D = 4$ may be found in ref. [26]. The precise derivative factors that actually appear would need to be calculated. At two loops, the explicit integrand for the amplitude [7], as well as the values of the dimensionally regularized infrared singular integrals in $D = 4$ [36], are known.

DISCUSSION

In this note we discussed evidence that four-dimensional $\mathcal{N} = 8$ supergravity may be ultraviolet finite. Given the additional cancellations we observe at higher loops, as well as the other clues described in the introduction [10, 11, 19, 20, 25, 26, 28, 29], there is good reason to believe that the finiteness bound of refs. [7, 19] is too conservative. Clearly a closer direct reexamination of the ultraviolet properties of $\mathcal{N} = 8$ supergravity is needed. A number of calculations should be carried out to this end. The most important task is to construct complete amplitudes beyond two loops. One could then check directly whether they satisfy the same D -dimensional finiteness condition (4) obeyed by $\mathcal{N} = 4$ super-Yang-Mills amplitudes.

Using the unitarity method it is feasible to construct complete four-point integrands at three and perhaps higher loops. As explained in ref. [7], higher-loop calculations of the $\mathcal{N} = 8$ supergravity unitarity cuts are essentially double copies of $\mathcal{N} = 4$ super-Yang-Mills cuts, due to the KLT relations. In the super-Yang-Mills case, the complete planar four-point three- and four-loop integrands are known [33, 34]. For supergravity, one also needs non-planar contributions, which are more complicated than planar ones.

In order to confirm that the cancellations are not limited to divergences of the form $\partial^n R^4$, but extend to operators with more powers of R , it is important to construct integrands for higher-point amplitudes. Given that the five-point two-loop planar $\mathcal{N} = 4$ super-Yang-Mills integrand has already been determined [37], it should also be feasible to obtain the five-point two-loop $\mathcal{N} = 8$ supergravity amplitude.

It should also be possible to carry out all-order studies using the unitarity method, given the recursive nature of the formalism. Tracking potential logarithmic divergences that arise in the critical dimension D_c is crucial. Such divergences are unambiguous, whereas power

divergences can depend on details of the regularization scheme.

Although there is already rather strong evidence that in D dimensions one-loop $\mathcal{N} = 8$ supergravity amplitudes have the same degree of divergence as their $\mathcal{N} = 4$ super-Yang-Mills counterparts [8, 10, 11, 20], it is important to construct a complete proof, because the result is a key ingredient for using the unitarity method in higher-loop analyses.

At the multi-loop level, besides carrying out explicit constructions of complete amplitudes, it would also be important to identify an underlying dynamical principle or symmetry explaining the additional cancellations observed.

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