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Study of collective beam instabilities using a correlation-moment analysis

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Abstract

A general formalism for treating simultaneously the transverse coupled bunch and transverse coupled mode instabilities is presented. In this approach, the equations of motion of a coupled multi-bunch beam are expanded to yield a system of equations involving correlation-moments of the transverse and longitudinal motions. After a proper truncation, the system of equations is closed and can be solved. This approach allows us to formulate within one framework several known instability mechanisms including the single bunch mode coupling instability, the coupled bunch instability, the mode coupling instability, and the coupled mode coupled bunch instability as particular cases.

1 Introduction

In a storage ring, a train of bunches is subject to various collective instabilities if the beam current is sufficiently high. For a single-bunch beam, instabilities are traditionally analyzed by decomposing the collective motion of the particles in the bunch into collective modes; the instability then results from the coupling among these modes. This is referred to as the mode coupling instability in the literature. Another mechanism deals with a train of multiple bunches; the corresponding instabilities can be analyzed in terms of the coupled motion among the interacting bunches while basically ignoring the internal degrees of freedom within each individual bunch. This is referred to as coupled bunch instability in the literature. Traditional treatment of collective instabilities in storage rings considers the mode coupling instability and the coupled bunch instability separately [1].

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An exception is the recently considered coupled mode coupled bunch (CMCB) instability [2] where the effect of bunch coupling is taken into account in description of the mode coupling instability. In the present paper, we develop an alternative formalism of collective instabilities where these two types of instabilities are treated systematically within one framework.

One traditional way of analyzing the beam stability is based on the linearized Vlasov equation [1, 2]. The formalism we present here adopts an alternative approach, based on expanding the equations of motion into a system of equations for correlation-moments, to be defined later, of the transverse and longitudinal degrees of freedom. After a proper truncation, the system of equations is closed and can be solved. Due to the process of truncation, our approach is limited to the lowest few modes, while the modes higher than the quadrupole modes are ignored. However, the latter have higher thresholds and, usually, are less important.

We will limit ourselves to the transverse collective instabilities, even though these instabilities necessarily involve the longitudinal beam dynamics as well.

2 Equation of Motion

Let us consider a train of n_b bunches in a storage ring with circumference $C = 2\pi R$ and the revolution period $T_0 = C/c$. For simplicity, we assume that the equidistant bunches are separated by $s_B = c\tau_b$. In the absence of collective motion, the center of the leading bunch in the train is at the location $s = ct$ around the ring at time t. We consider the ultra-relativistic case when particles move with the speed of light c. The center of the N-th bunch is at the distance $s_N = (N-1)s_B > 0$ behind the leading bunch for evenly spaced bunches. The variable s $(-\infty < s < \infty)$, rather than the time t, is chosen as the independent variable because the properties of the focusing system are defined by their location s. Position $s_{i,N}(t)$ of the i-th particle in the N-th bunch, $N = 1, 2, ..., n_b$, is

$$s_{i,N}(t) = ct - s_N + z_{i,N}(s), \tag{1}$$

where $z_{i,N}$ is longitudinal displacement due to synchrotron motion with a sign convention that $z_{i,N} > 0$ indicates the displacement is toward the head of the bunch.

We want to study effects of coupling of the longitudinal and transverse oscillations assuming that the bunch current is below the threshold of the longitudinal microwave instability [3]. The longitudinal distribution of particles in this case is a steady-state Haissinski distribution [4], and the single particle trajectory gives

$$z_{i,N}(s) = \zeta_N(s) + a_{i,N} \sin(\omega_s s/c + \phi_{i,N}), \tag{2}$$

where ζ_N describes the motion of the bunch centroid. We assume below that $\zeta_N = 0$ neglecting the longitudinal coupled-bunch effect (such as that due to beam loading) although the consideration can be easily generalized to take it into account. Generally, the trajectory $z_{i,N}(s)$ in the nonlinear steady-state Haissinski potential has harmonics multiple of the synchrotron frequency ω_s. However, the head-tail instability due to the coupling is, usually, caused by the crossings of the lowest two monopolar and dipolar modes. Therefore, it is sufficient to consider only the lowest harmonics of the synchrotron frequency ω_s but taking into account correlation of the longitudinal and transverse motion in the same bunch.

Collective instabilities are caused by the wake fields generated by the beam at irregularities of the vacuum chamber. If t is the time when the i-th particle in the N-th bunch is at the position $s_{i,N}(t)$ of the vacuum system, the time $t_{j,M,k}$ when a particle j of the M-th bunch is at the same location on the turn number k $(-\infty < k < \infty)$ is

$$t_{j,M,k} = t^0_{M,k} + \frac{1}{c}\left[z_{i,N}(s) - z_{j,M}(s - kC)\right], \tag{3}$$

where

$$t^0_{M,k} = t - kT_0 + (M - N)\tau_b. \tag{4}$$

The transverse offset of the i-th particle in the N-th bunch around the ring is

$$y_{i,N}(s) = A_{i,N}(s)e^{-i\psi_{i,N}(s)} + c.c.\,, \tag{5}$$

where the transverse phase advance $\psi_{i,N}$ and the betatron frequency $\omega^y_{i,N}$ are

$$\psi_{i,N}(s) = \frac{\omega_y s}{c} - \omega_\xi \frac{z_{i,N}(s)}{c}, \qquad \frac{\omega^y_{i,N}}{c} = \frac{d\psi_{i,N}}{ds}. \tag{6}$$

Here, ω_y is the nominal betatron frequency, and [5]

$$\omega_\xi = \frac{\xi}{\alpha}\omega_y, \qquad \xi = \frac{1}{\omega_y}\frac{d\omega_y}{d\delta} \tag{7}$$

are the chromatic head-tail frequency and the relative chromaticity, respectively; α is the momentum compaction factor, and $\delta = \Delta E/E$ is the relative energy shift of a particle. In general, the amplitude $A_{i,N}$ is complex and is slowly varying in time.

The equation of motion in the smooth focusing approximation is

$$\frac{d^2 y_{i,N}(s)}{ds^2} + \left(\frac{\omega^y_{i,N}}{c}\right)^2 y_{i,N}(s) = \frac{r_e}{\gamma C}\sum_{j,M,k} W_y(t - t_{j,M,k})\, y_{j,M}(s - kC), \tag{8}$$

where r_e is the classical radius of the particle, and γ is the relativistic factor. The right-hand-side (RHS) is due to the wake fields and is responsible for the collective instabilities; $W_y(t)$ is the transverse wake per turn (dimension V/pC/m or 1/cm^2). For an ultra-relativistic beam, causality requires that $W_y(t) = 0$ for $t < 0$. Summation over k takes into account long-range wake fields that last multiple number of revolutions.

Usually, the coherent frequency shift and the instability growth rate are small compared to ω_y. In this case, Eq. (8) can be averaged over fast oscillations, giving equation for the amplitudes,

$$\frac{dA_{i,N}(s)}{ds} = i\lambda_y \sum_{j,M,k} W_y(t - t_{j,M,k})\, A_{j,M}(s - kC)\, e^{ik\omega_y T_0 - i\frac{\omega_\xi}{c}\,[z_{i,N}(s) - z_{j,M}(s-kC)]}, \qquad (9)$$

where we have dropped terms $d^2A_{i,N}/ds^2$ and $d\omega^y_{i,N}/ds$, and introduced

$$\lambda_y = \frac{r_e c}{2\gamma C\omega_y}. \qquad (10)$$

3 Correlation-Moment Expansion

Measurements of the beam dynamics, usually, detect only the bunch centroid motion. We want to reduce Eq. (9) to the system of equations for the quantities averaged over N_b particles in a bunch

$$\langle A_M(s)\rangle = \frac{1}{N_b}\sum_{i=1}^{N_b} A_{i,M}(s). \qquad (11)$$

Equation (9) shows that the transverse and longitudinal motions are correlated. Therefore, the averaging denoted by the angular brackets means averaging over the full distribution function depending on the transverse and longitudinal coordinates. We assume, however, that the correlation is week and only the lowest order correlation-moments need to be taken into account. For example, the average

$$\langle A_M z_M^l\rangle = \frac{1}{N_b}\sum_j A_{j,M} z_{j,M}^l \qquad (12)$$

can be expanded over the correlation-moments $\langle A_M\rangle$, $\langle A_M z_M\rangle$, etc.,

$$\langle A_M z_M^l\rangle = \langle A_M\rangle\langle\langle z_M^l\rangle\rangle + \langle A_M z_M\rangle\langle\langle z_M^{l-1}\rangle\rangle\frac{l!}{1!(l-1)!} + \cdots \qquad (13)$$

where double angular brackets mean averaging over the longitudinal uncorrelated motion. Such averaging can be carried out by convolution with the longitudinal distribution function $\rho_M(z, p_z, s)$ of the M-th bunch, for example,

$$\langle\langle z_M^{l-1}\rangle\rangle = \int dz dp_z \rho_M(z, p_z, s) z^{l-1}. \qquad (14)$$

Expanding over correlation-moments, we assume that there is no correlation of the longitudinal and transverse motion of particles belonging to different bunches, i.e. $\langle z_{iM} z_{jN}\rangle = 0$, $\langle z_{iM} A_{jN}\rangle = 0$ if $N \neq M$, and neglect higher order correlation-moments such as

$\langle z_{iM} z_{jN}^2 \rangle$. These assumptions do not preclude bunch-to-bunch coupling of the amplitudes $\langle A_M(s) \rangle$ of different bunches. Using these assumptions, we can reduce Eq. (9) to a system of equations for the lowest order correlation-moments.

It is convenient to write Eq. (9) in terms of the transverse impedance per turn Z_y,

$$\frac{dA_{i,N}(s)}{ds} = \frac{1}{N_b} \sum_{k,M,j} \int \frac{d\omega}{2\pi} F_k(\omega, N-M)\, A_{j,M}(s-kC)\, e^{i\kappa\,[z_{i,N}(s) - z_{j,M}(s-kC)]}, \tag{15}$$

where $\kappa = (\omega - \omega_\xi)/c$, and

$$F_k(\omega, N-M) = -\lambda_y N_b Z_y(\omega)\, e^{-i(\omega-\omega_y)kT_0 - i\omega\tau_b(N-M)}. \tag{16}$$

Equation for $d\langle A_M(s) \rangle/ds$ can be obtained from Eq. (15) using definition Eq. (11).

Expanding exponents in the RHS in series over $z_{i,N}$ and $z_{j,M}$,

$$e^{i\frac{\omega-\omega_\xi}{c}[z_{i,N}(s)-z_{j,M}(s-kC)]} = \sum_{l=0}^{\infty} \frac{i^l}{l!} \left(\frac{\omega-\omega_\xi}{c}\right)^l [z_{i,N}(s)]^l \sum_{l'=0}^{\infty} \frac{(-i)^{l'}}{l'!} \left(\frac{\omega-\omega_\xi}{c}\right)^{l'} [z_{j,M}(s-kC)]^{l'} \tag{17}$$

and calculating the average using Eq. (13), we get

$$\begin{aligned} & \langle A_{j,M}(s-kC)\, e^{i\kappa\,[z_{i,N}(s)-z_{j,M}(s-kC)]} \rangle \\ = \; & |G(\kappa)|^2 \left\{ A^0_M(s-kC) - \frac{\kappa}{2} \left[Q^+_M(s-kC)\, e^{i\frac{\omega_s}{c}(s-kC)} - Q^-_M(s-kC)\, e^{-i\frac{\omega_s}{c}(s-kC)} \right] \right\}. \end{aligned} \tag{18}$$

Here we have used Eq. (2) for $z_N(s)$ and notations

$$\begin{aligned} A^0_M(s) &= \langle A_M(s) \rangle, \\ Q^{\pm}_M(s) &= \langle a_M(s) e^{\pm i\phi_M} A_M(s) \rangle, \\ G(\kappa) &= \langle\langle e^{-i\kappa z_N} \rangle\rangle. \end{aligned} \tag{19}$$

Because the time dependence of a trajectory $z_M(s)$ can be considered as canonical transform preserving the phase volume, the average $\langle\langle z_M^{l-1} \rangle\rangle$ and, therefore, $G(\omega-\omega_\xi)$ for the steady-state distribution $\rho(z, p_z)$ is time independent. If all bunches have the same longitudinal profile, then G is also independent of the bunch number. For the Gaussian distribution with the rms bunch length σ, $G(\Delta\omega)$ is easy to calculate,

$$G(\frac{\Delta\Omega}{c}) = \int \frac{dz dp}{2\pi\sigma\delta}\, e^{-\frac{p^2}{2\delta^2} - \frac{z^2}{2\sigma^2}}\, e^{-i\frac{\Delta\Omega}{c} a \sin(\omega_s s/c + \phi)} = e^{-\frac{1}{2}(\frac{\Delta\Omega\sigma}{c})^2}. \tag{20}$$

Equation (15) after averaging takes the form

$$\begin{aligned} \frac{dA^0_N(s)}{ds} &= \sum_{k,M} \int \frac{d\omega}{2\pi} F_k(\omega, N-M)\, |G(\kappa)|^2 \\ &\times \left\{ A^0_M(s-kC) - \frac{\kappa}{2} \left[Q^+_M(s-kC)\, e^{i\frac{\omega_s}{c}(s-kC)} - Q^-_M(s-kC)\, e^{-i\frac{\omega_s}{c}(s-kC)} \right] \right\}. \end{aligned} \tag{21}$$

Similarly, multiplying Eq. (15) by $z_{i,N}(s')$ and calculating the sum over i, we get in the LHS,

$$\frac{d}{ds}\frac{1}{N_b}\sum_i A_{i,N}(s) z_{i,N}(s') = \frac{1}{2i}\frac{d}{ds}\left\{Q_N^+(s)\, e^{i\omega_s s'/c} - Q_N^-(s)\, e^{-i\omega_s s'/c}\right\}. \tag{22}$$

The average of the RHS differs from calculations described above by the factor

$$\langle z_N(s') e^{i\kappa z_N(s)}\rangle = -i\frac{\partial}{\partial\kappa} G^*(\kappa)\cos[\frac{\omega_s}{c}(s-s')]. \tag{23}$$

Separating terms proportional to $e^{\pm i\omega_s s'/c}$, we get two equations

$$\begin{aligned}\pm\frac{1}{2i}\frac{dQ_N^\pm(s)}{ds} &= -\frac{i}{2}\sum_{k,M}\int\frac{d\omega}{2\pi}\, F_k(\omega, N-M)\, G(\kappa)\,\frac{dG^*(\kappa)}{d\kappa}\, e^{\mp i\omega_s s/c} \\ &\times \left\{A_M^0(s-kC) - \frac{\kappa}{2}\left[Q_M^+(s-kC)\, e^{i\frac{\omega_s}{c}(s-kC)} - Q_M^-(s-kC)\, e^{-i\frac{\omega_s}{c}(s-kC)}\right]\right\}.\end{aligned} \tag{24}$$

It is convenient to write Eqs. (21) and (24) in the frequency domain for the Fourier components $\tilde{A}_M^0(\Omega)$ and $\tilde{Q}^\pm(\Omega)$,

$$\begin{aligned}A_M^0(s) &= \int\frac{d\Omega}{2\pi}\,\tilde{A}_M^0(\Omega)\, e^{-i\frac{\Omega s}{c}}, \\ Q_M^\pm(s) &= \int\frac{d\Omega}{2\pi}\,\tilde{Q}_M^\pm(\Omega)\, e^{-i\frac{\Omega s}{c}}.\end{aligned} \tag{25}$$

Fourier harmonics $\tilde{A}_N^0(\Omega)$ satisfy the following equation,

$$\begin{aligned}\Omega\tilde{A}_N^0(\Omega) &= ic\sum_{k,M}\int\frac{d\omega}{2\pi}\, F_k(\omega, N-M)\,|G(\kappa)|^2\, e^{i\Omega k T_0} \\ &\times \left\{\tilde{A}_M^0(\Omega) - \frac{\kappa}{2}\left[\tilde{Q}_M^+(\Omega+\omega_s) - \tilde{Q}_M^-(\Omega-\omega_s)\right]\right\}.\end{aligned} \tag{26}$$

The sum over turns k can be calculated using

$$\sum_k e^{i\nu T_0 k} = \omega_0\sum_k \delta[\nu + k\omega_0], \tag{27}$$

where $\delta(x)$ is the δ-function. Hence,

$$\begin{aligned}\Omega\tilde{A}_N^0(\Omega) &= -i\frac{\lambda_y\omega_0 c N_b}{2\pi}\sum_{M,k} Z_y(\omega)|G(\kappa)|^2\, e^{-i\omega\tau_b(N-M)} \\ &\times \left\{\tilde{A}_M^0(\Omega) - \frac{\kappa}{2}\left[Q_M^+(\Omega+\omega_s) - \tilde{Q}_M^-(\Omega-\omega_s)\right]\right\},\end{aligned} \tag{28}$$

where $\omega = \omega_y + \Omega + k\omega_0$.

Similarly, for harmonics $\tilde{Q}^{\pm}(\Omega)$ we get

$$
\begin{aligned}
(\Omega+\omega_s)\tilde{Q}^{+}_{N}(\Omega+\omega_s) &= -i\frac{\lambda_y\omega_0 cN_b}{2\pi}\sum_{M,k} Z_y(\omega)G(\kappa)\frac{\partial G^*(\kappa)}{\partial\kappa}\,e^{-i\omega\tau_b(N-M)} \\
&\times \left\{\tilde{A}^{0}_{M}(\Omega)-\frac{\kappa}{2}\left[Q^{+}_{M}(\Omega+\omega_s)-\tilde{Q}^{-}_{M}(\Omega-\omega_s)\right]\right\}, \\
(\Omega-\omega_s)\tilde{Q}^{-}_{N}(\Omega-\omega_s) &= i\frac{\lambda_y\omega_0 cN_b}{2\pi}\sum_{M,k} Z_y(\omega)G(\kappa)\frac{\partial G^*(\kappa)}{\partial\kappa}\,e^{-i\omega\tau_b(N-M)} \\
&\times \left\{\tilde{A}^{0}_{M}(\Omega)-\frac{\kappa}{2}\left[Q^{+}_{M}(\Omega+\omega_s)-\tilde{Q}^{-}_{M}(\Omega-\omega_s)\right]\right\}, \qquad (29)
\end{aligned}
$$

where again $\omega=\omega_y+\Omega+k\omega_0$.

3.1 Eigen-mode expansion

The coupled-bunch motion can be analyzed by expanding oscillations of individual bunches over the eigen-modes. For the uniform distribution of n_b bunches in the ring, the normalized eigen-modes are

$$
X^{\mu}_{M}=\frac{1}{\sqrt{n_b}}\,e^{\frac{2\pi i}{n_b}(M-1)\mu}, \quad M=1,2,..n_b, \quad \mu=1,..,n_b-1. \qquad (30)
$$

The form of the expansion over eigen-modes is defined by the condition of periodicity which can be explained by the following arguments. The average of the transverse offset of the M-th bunch

$$
y_{j,M}(s)=A_{j,M}(s)e^{-\frac{i}{c}(\omega_y s-\omega_\xi z_{j,M}(s))}+c.c \qquad (31)
$$

takes the form

$$
\langle y_M(s)\rangle=\left[A^{0}_{M}(s)+\frac{\omega_\xi}{2c}(Q^{+}_{M}(s)e^{\frac{i\omega_s}{c}s}-Q^{-}_{M}(s)e^{\frac{i\omega_s}{c}s})\right]G^*(\omega_\xi)e^{-i\frac{\omega_y s}{c}}+c.c. \qquad (32)
$$

For a fixed time t, moving from a bunch number M to $M+n_b$ brings us to the same bunch while $s=ct-(M-1)s_b$ changes to $s-C$. Therefore, the transverse offset of the M-th bunch has to satisfy the condition of periodicity,

$$
\langle y_{M+n_b}(s-C)\rangle=\langle y_M(s)\rangle. \qquad (33)
$$

Hence, the periodicity conditions for the amplitudes are

$$
\begin{aligned}
A^{0}_{M+n_b}(s-C)\,e^{i\omega_y T_0} &= A^{0}_{M}(s), \\
Q^{\pm}_{M+N_b}(s-C)\,e^{i(\omega_y\mp\omega)T_0} &= Q^{\pm}_{M}(s). \qquad (34)
\end{aligned}
$$

For Fourier amplitudes the conditions take the form

$$
\begin{aligned}
\tilde{A}^{0}_{N+n_b}(\Omega)\,e^{i(\Omega+\omega_y)T_0} &= \tilde{A}^{0}_{N}(\Omega), \\
\tilde{Q}^{\pm}_{M+n_b}(\Omega)\,e^{i(\Omega+\omega_y\mp\omega_s)T_0} &= \tilde{Q}^{\pm}_{M}(\Omega). \qquad (35)
\end{aligned}
$$

Equations (35) define the phase factor in the expansion over the eigen-modes,

$$\begin{aligned}\tilde{A}^0_M(\Omega) &= e^{-i(\omega_y+\Omega)\tau_b(M-1)}\sum_\mu g^0_\mu(\Omega)X^\mu_M,\\ \tilde{Q}^\pm_M(\Omega) &= e^{-i(\omega_y+\Omega\mp\omega_s)\tau_b(M-1)}\sum_\mu g^\pm_\mu(\Omega)X^\mu_M.\end{aligned} \tag{36}$$

Substituting Eqs. (36) into Eq. (28) and using orthogonality of the eigen-functions X^μ_M, we get

$$\begin{aligned}\Omega g^0_\mu(\Omega) &= -i\frac{\lambda_y\omega_0 cN_b}{2\pi}\sum_{N,M,\nu,k}[X^\mu_N]^* X^\nu_M\, e^{ik\omega_0\tau_b(N-M)}\, Z_y(\omega^0_y+\Omega+k\omega_0)\,|G(\kappa)|^2\\ &\times\ \left\{g^0_\nu(\Omega)-\frac{\kappa}{2}\left[g^+_\nu(\Omega+\omega_s)-g^-_\nu(\Omega-\omega_s)\right]\right\},\end{aligned} \tag{37}$$

where $\kappa=(\omega^0_y+\Omega+k\omega_0-\omega_\xi)/c$. Summing over N, M gives

$$\sum_{N,M}[X^\mu_N]^* X^\nu_M\, e^{ik\omega_0\tau_b(N-M)} = n_b\,\delta_{\mu,\nu}\sum_p \delta_{k,pn_b+\mu}. \tag{38}$$

Equation (37) is then simplified to

$$\Omega g^0_\mu(\Omega) = -i\frac{\lambda_y\omega_0 cN_b n_b}{2\pi}\sum_p Z_y(\omega_p)\,|G(\kappa_\mu)|^2\left\{g^0_\mu(\Omega)-\frac{\kappa_\mu}{2}\left[g^+_\mu(\Omega+\omega_s)-g^-_\mu(\Omega-\omega_s)\right]\right\}, \tag{39}$$

where

$$\omega_p=\omega^0_y+\Omega+(pn_b+\mu)\omega_0,\quad \kappa_\mu=\frac{\omega_p-\omega_\xi}{c}. \tag{40}$$

Similarly, Eq. (29) is transformed to

$$\begin{aligned}(\Omega\pm\omega_s)g^\pm_\mu(\Omega\pm\omega_s) &= \mp i\frac{\lambda_y\omega_0 cN_b n_b}{2\pi}\sum_p Z_y[\omega_p]\,G(\kappa_\mu)\frac{\partial G^*(\kappa_\mu)}{\partial\kappa_\mu}\\ &\times\ \left\{g^0_\mu(\Omega)-\frac{\kappa_\mu}{2}\left[g^+_\mu(\Omega+\omega_s)-g^-_\mu(\Omega-\omega_s)\right]\right\}.\end{aligned} \tag{41}$$

Equations (39) and (41) are the system of linear equations for the amplitudes $g^0_\mu(\Omega)$, and $g^\pm_\mu(\Omega\pm\omega_s)$. For Gaussian longitudinal distribution function with the rms bunch length σ,

$$|G(\kappa_\mu)|^2=e^{-(\kappa_\mu\sigma)^2},\quad G(\kappa_\mu)\frac{\partial G^*(\kappa_\mu)}{\partial\kappa_\mu}=-\kappa_\mu\sigma^2\,e^{-(\kappa\sigma)^2}. \tag{42}$$

The bunch-by-bunch feedback system adds damping to each bunch proportional to the bunch centroid velocity $\langle\dot{y}_N\rangle=(1/N_b)\sum_i dy_{i,N}/dt$. The action of the feedback can be described by replacing $d^2y_{i,N}/dt^2+(\omega^y_{i,N})^2 y_{i,N}$ in the equation of motion by $d^2y_{i,N}/dt^2+$

$2\gamma_{FB}\langle dy_N/dt\rangle + (\omega^y_{i,N})^2 y_{i,N}$. Eqs. (39) and (41) are then modified by changing the factor Ω to $\Omega + 2i\gamma_{FB}$ in the left-hand-side.

In the following four sections, we apply the results of analysis obtained above to reproduce the well known results for the transverse dipole coupled-bunch (dipole CB) [6] and the head-tail (HT) [5] instabilities of the rigid bunches. Using the same set of equations, we also obtain results for transverse head-tail coupled-bunch (head-tail CB) and coupled-mode coupled-bunch (CMCB) [2] instabilities.

4 Transverse dipole coupled-bunch instability

In this section, let us consider only Eq. (39) neglecting terms involving $g^{(\pm)}_\mu$ which, as we will see later, are related to the head-tail transverse CB modes. That leaves us with a single homogeneous equation,

$$\Omega g^0_\mu(\Omega) = -i\frac{\lambda_y\omega_0 cN_b n_b}{2\pi}\sum_p Z_y(\omega_p)\,|G(\kappa_\mu)|^2\, g^0_\mu(\Omega), \tag{43}$$

which in turn yields the dispersion equation for the frequency Ω_μ of the μ-th transverse dipole CB mode,

$$\Omega_\mu = -i\frac{\lambda_y\omega_0 cN_b n_b}{2\pi}\sum_p Z_y(\omega_p)\,|G(\kappa_\mu)|^2. \tag{44}$$

The real part of Ω_μ gives the frequency shift of the coherent mode, while its imaginary part gives the instability growth rate of the mode. If Ω is small, we can neglect it in the argument of the impedance and obtain the approximate but explicit solution of the dispersion equation. For a Gaussian bunch that gives

$$\Omega_\mu = -i\frac{\lambda_y\omega_0 c}{2\pi}N_B n_b\sum_{p=-\infty}^{\infty} Z_y[(pn_b+\mu)\omega_0+\omega_y]e^{-(\frac{\sigma}{c})^2((pn_b+\mu)\omega_0+\omega_y-\omega_\xi)^2}. \tag{45}$$

Note that one can relate this to the dc beam current I^{dc}_{beam}, the beam energy E, and nominal betatron tune ν_y by

$$\frac{\lambda_y\omega_0}{2\pi}N_b n_b = \frac{I^{dc}_{beam}}{4\pi(E/e)\nu_y}. \tag{46}$$

Also note that it would be more accurate to replace the tune ν_y in the last formula by R/β_y with the β-function taken at the location of the impedance-generating element, and note that Eq. (45) agrees with the result [7].

5 Transverse head-tail coupled-bunch instability

Here we consider Eqs. (41) neglecting for a while the reverse effect of the amplitudes $g^{(\pm)}$ on $g^{(0)}$, and considering the latter as an external excitation. That gives us the system of two coupled equations. Using notations

$$\hat{g}^{\pm} = g_{\mu}^{\pm}(\Omega \pm \omega_s), \tag{47}$$

where ω_p and κ (we drop the index μ) are defined in Eq. (40), and

$$\begin{aligned} h_{\mu}(l) &= i\frac{\lambda_y \omega_0 c N_b n_b}{2\pi} \sum_p Z_y[\omega_p]\,|G(\kappa)|^2\,(\frac{\kappa}{2})^l, \\ h'_{\mu}(l) &= i\frac{\lambda_y \omega_0 c N_b n_b}{2\pi} \sum_p Z_y[\omega_p]\,G(\kappa_{\mu})\frac{\partial G^*(\kappa_{\mu})}{\partial \kappa_{\mu}}\,(\frac{\kappa}{2})^l, \end{aligned} \tag{48}$$

we get

$$\begin{aligned} [\Omega + \omega_s - h'_{\mu}(1)]\,\hat{g}^+ + h'_{\mu}(1)\,\hat{g}^- &= -h'_{\mu}(0)\,g_{\mu}^0(\Omega), \\ h'_{\mu}(1)\,\hat{g}^+ + [\Omega - \omega_s - h'_{\mu}(1)]\,\hat{g}^- &= h'_{\mu}(0)\,g_{\mu}^0(\Omega). \end{aligned} \tag{49}$$

The response to the excitation by the bunch centroid is infinite at the eigen-frequencies Ω given by the zeros of the determinant

$$\det \begin{bmatrix} \Omega + \omega_s - h'_{\mu}(1) & h'_{\mu}(1) \\ h'_{\mu}(1) & \Omega - \omega_s - h'_{\mu}(1) \end{bmatrix} = 0. \tag{50}$$

Equation (50) shows that $|\Omega| \simeq \omega_s$. Approximate solution for a moderate current where $|\kappa_{\mu}^{(2)}| \ll 1$ is

$$\Omega_{\mu}^{(\pm)} = \pm\omega_s + h'_{\mu}(1). \tag{51}$$

For a Gaussian bunch,

$$\begin{aligned} \Omega_{\mu}^{(\pm)} &= \pm\omega_s - i\frac{\lambda_y \omega_0}{4\pi} N_B n_b \sum_{p=-\infty}^{\infty} \left(\frac{\sigma}{c}\right)^2 [(pn_b + \mu)\omega_0 + \omega_y - \omega_{\xi} \pm \omega_s]^2 \\ &\times \; Z_y[(pn_b + \mu)\omega_0 + \omega_y \pm \omega_s] e^{-(\frac{\sigma}{c})^2((pn_b+\mu)\omega_0 + \omega_y - \omega_{\xi} \pm \omega_s)^2}. \end{aligned} \tag{52}$$

The result gives the coherent frequency shift and the growth rate of the CB head-tail modes.

Eq. (49) defines the structure of the excited eigen-modes,

$$\begin{aligned} \hat{g}^{(+)} &\simeq -\frac{h'_{\mu}(1)}{2\omega_s}\,\hat{g}^{(-)}, \quad \Omega \simeq \omega_{\rm s} \\ \hat{g}^{(-)} &\simeq \frac{h'_{\mu}(1)}{2\omega_s}\,\hat{g}^{(+)}, \quad \Omega \simeq -\omega_{\rm s}. \end{aligned} \tag{53}$$

The amplitudes $\hat{g}^{\pm}$ describe correlation of the transverse and longitudinal oscillations (so called head-tail modes). Let us consider the case of zero chromaticity, $\omega_{\xi} = 0$. The correlated transverse/longitudinal motion for two head-tail modes with $\Omega \simeq \pm\omega_s$ is described by the terms

$$\langle z_N(s) y_N(s) \rangle \,\propto\, \pm\frac{1}{2i}\,X_N^{\mu}\,e^{-i\omega_y s/c - i(\omega_y \pm \omega_s)\tau_b(N-1)}\,\hat{g}_{\mu}^{\pm} + c.c. \tag{54}$$

corresponding to the periodic tilts with frequency ω_s of the bunch in the moving frame of the bunch centroid.

6 Mode coupling in multibunch system (CMCB instability)

Now we can take into account effect of the head-tail modes $g_\mu^{(\pm)}$ on the motion of the bunch centroid. The full system of Eqs. (39), (41) is the system of linear equations $M(\Omega)V = 0$ for the vector $V = \{g_\mu^{(0)}(\Omega), g_\mu^{(+)}(\Omega+\omega_s), g_\mu^{(-)}(\Omega-\omega_s)\}$. The system has a nontrivial solution at frequencies Ω given by the zeros of the determinant of the matrix $M(\Omega)$,

$$M(\Omega) = \begin{bmatrix} \Omega + h_\mu(0) & -h_\mu(1) & h_\mu(1) \\ h'_\mu(0) & \Omega+\omega_s - h'_\mu(1) & h'_\mu(1) \\ -h'_\mu(0) & h'_\mu(1) & \Omega-\omega_s-h'_\mu(1) \end{bmatrix}. \tag{55}$$

A nontrivial situation arises when the coherent tune shift is of the order of ω_s. Then, the equations can not be considered separately as it is done in the previous sections. The solution describes the CMCB instability in the multibunch system [2]. The instability is essentially the mode coupling (CM) instability but takes into account that, due to long-range wake, the frequency shift for different CB modes may be different. Hence, the crossing of the CM modes for the CB system may take place at lower currents and the CMCB instability may have lower threshold than CM single bunch instability.

For illustration, we considered the PEP-II Low Energy Ring impedance shown in Fig. 1. Fig. 2 shows comparison of the calculations of CB modes and CMCB modes for the beam current $I_{beam}^{dc} = 1$ A and the number of bunches $n_b = 1616$. The maximum growth rate for rigid-dipole CB modes is 0.75 1/ms, and for dipole CMCB modes is 0.88 1/ms. For the head-tail modes, the growth rates for CB and CMCB modes are 0.0057 1/ms and 0.0067 1/ms, respectively.

7 Head-tail instability

Let us apply these results to a single bunch, putting $n_b = N = M = 1$, $\mu = 0$ in Eqs. (28), (29). Let us use notations

$$\begin{aligned}
\hat{A}^0 &= \tilde{A}_N^0(\Omega), \\
\hat{Q}^\pm &= \tilde{Q}^{(\pm)}(\Omega \pm \omega_s), \\
h(l) &= i\frac{\lambda_y \omega_0 c N_b}{2\pi} \sum_k Z_y[\omega_k]\, |G(\kappa)|^2 \left(\frac{\kappa}{2}\right)^l, \\
h'(l) &= i\frac{\lambda_y \omega_0 c N_b}{2\pi} \sum_k Z_y[\omega_k]\, G(\kappa) \frac{\partial G^*(\kappa)}{\partial \kappa} \left(\frac{\kappa}{2}\right)^l,
\end{aligned} \tag{56}$$

where $\omega_k = \omega_y + k\omega_0 + \Omega$, $\kappa = (\omega_k - \omega_\xi)/c$. The system of linear equations takes the form

$$\Omega \hat{A}^0 = -h(0)\, \hat{A}^0 + h(1)\, [\hat{Q}^+ - \hat{Q}^-],$$

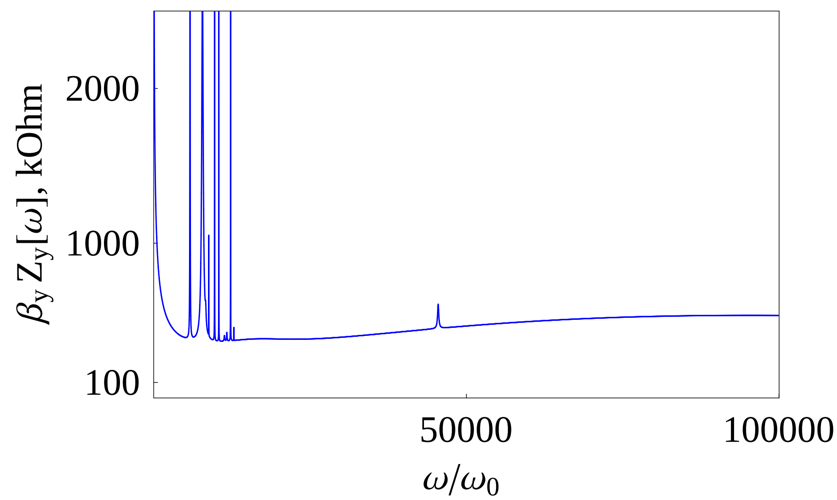

Figure 1: Impedance of the PEP-II LER given by the contributions from 6 cavities, resistive wall, BPMs, and mostly inductive vacuum components.

$$\begin{aligned}(\Omega+\omega_s)\,\hat{Q}^+ &= -h'(0)\,\hat{A}^0 + h'(1)\,[\hat{Q}^+ - \hat{Q}^-],\\ (\Omega-\omega_s)\,\hat{Q}^- &= h'(0)\,\hat{A}^0 - h'(1)\,[\hat{Q}^+ - \hat{Q}^-].\end{aligned} \tag{57}$$

Solutions exist if

$$\det\begin{bmatrix}\Omega+h(0) & -h(1) & h(1)\\ h'(0) & \Omega+\omega_s-h'(1) & h'(1)\\ -h'(0) & h'(1) & \Omega-\omega_s-h'(1)\end{bmatrix}=0. \tag{58}$$

In the lowest order in the beam current, the roots are

$$\Omega = -h(0), \quad \Omega = \pm\omega_s + h'(1). \tag{59}$$

We can compare this result with the Satoh-Chin theory of the head-tail instability for a Gaussian bunch. See Appendix. The Satoh-Chin theory gives solution in terms of an infinite matrix $M_{h,l}$, $(h,l)=0,1,2,....$ To get the solution, the matrix is truncated to finite rank. The rank of the matrix defines how many synchrotron modes are taken into account. The threshold, usually, corresponds to the crossing of the modes with mode indices $m=0$ and $m=-1$. Therefore, it can be defined with a good accuracy truncating the matrix to the rank $r=1$, taking only components $(h,l)=0$ and $(h,l)=1$. In this approximation and in the lowest order in the bunch current, the roots of the Satoh-Chin theory are

$$\Omega = -iK_{SCh}\omega_s M_{0,0}, \quad \Omega = \pm\omega_s - iK_{SCh}\,\omega_s\,M_{1,1}. \tag{60}$$

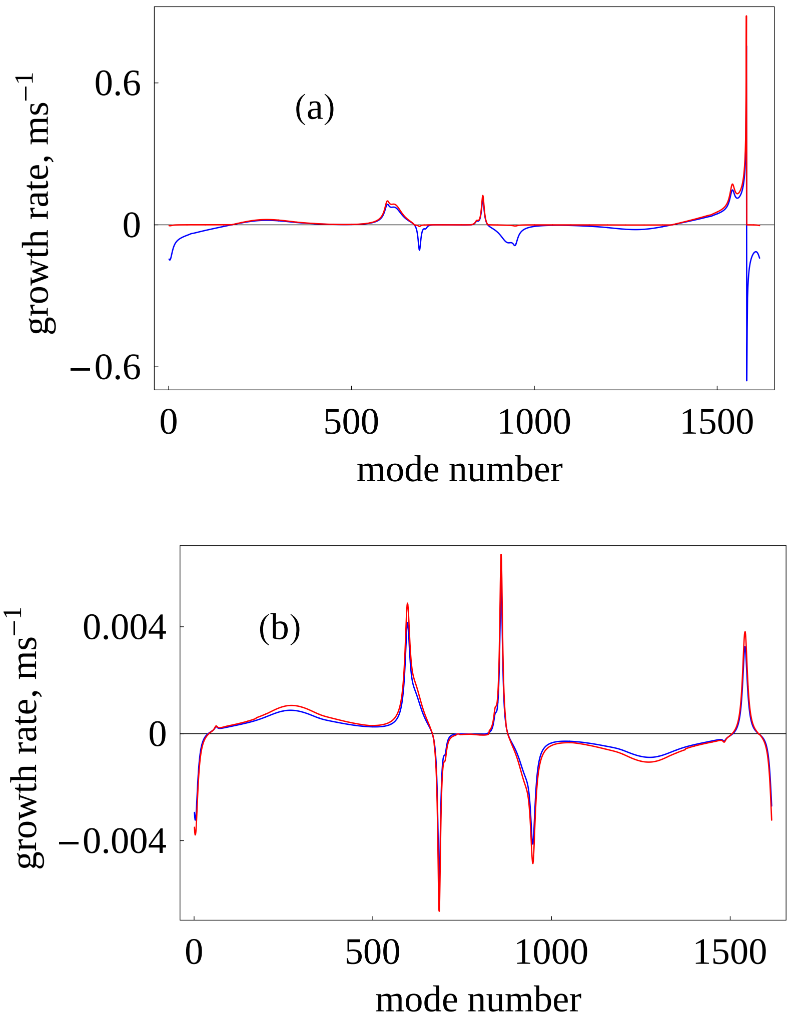

Figure 2: The growth rate for CB (blue) and CMCB modes (red) for rigid-dipole modes $m = 0$ (a) and head-tail $m = 1$ modes (b).

The Satoh-Chin coefficient K_{SCh} is

$$K_{SCh} = \frac{\lambda_y \omega_0 N_b c}{2\pi\omega_s}, \tag{61}$$

and the matrix $M_{h,l}(\lambda)$ in the Satoh-Chin formalism gives

$$h(0) = iK_{SCh}\omega_s M_{0,0}, \quad h'(1) = -iK_{SCh}\omega_s M_{1,1}. \tag{62}$$

Therefore, the roots Eq. (60) are exactly the same as given by the Satoh-Chin theory in this approximation. We also compared the result of Eq. (58) numerically with the

Satoh-Chin formalism with the rank of the truncated matrix equal to 2. Result is shown in Fig. 3. Parameters are the same as in [8], see Appendix. The chromaticity has been set to $\xi = 0$. The threshold of instability is defined by the crossings of the modes $m = 0$ and $m = -1$. Agreement of the results is quite good and confirms that the higher synchrotron modes give only small correction to the threshold.

The advantage of our approach is that Eqs. (58) do not require the Gaussian bunch profile, thus allowing us to take into account the potential well distortion (PWD). Fig.(4) shows the threshold of the head-tail instability for a Gaussian bunch (blue dots) and for the bunch with profile given by the current dependent Haissinski distribution. The effect on the threshold is negligible. More accurately, to be consistent, taking into account the PWD we should also take into account the synchrotron frequency spread $\omega_s(a)$ calculating the $G(\kappa)$ factor. Such generalization of our formalism does not contradict to single harmonic approximation Eq. (2).

8 Summary

The transverse instabilities of coupled bunches are studied considering correlation of the transverse and longitudinal motion. We show that the results for dipole and head-tail coupled bunch instabilities can be obtained in this way as well as the results for the coupled bunch coupled mode instability. All of that can be obtained as limiting cases of the same framework of equations. Applying these equations for a single bunch gives results which agree quite well with the Satoh-Chin theory of the head-tail instability for a Gaussian bunch. The equations derived here for the head-tail instability allow to take into account also the potential wake distortion of the bunch profile. The statements are illustrated by numerical examples. We believe that the correlation-moment analysis presented here provides a way to describe the collective beam instability mechanism as an alternative, and in some aspects an improvement, compared with the usual analysis based on linearization of the Vlasov equation.

9 Acknowledgement

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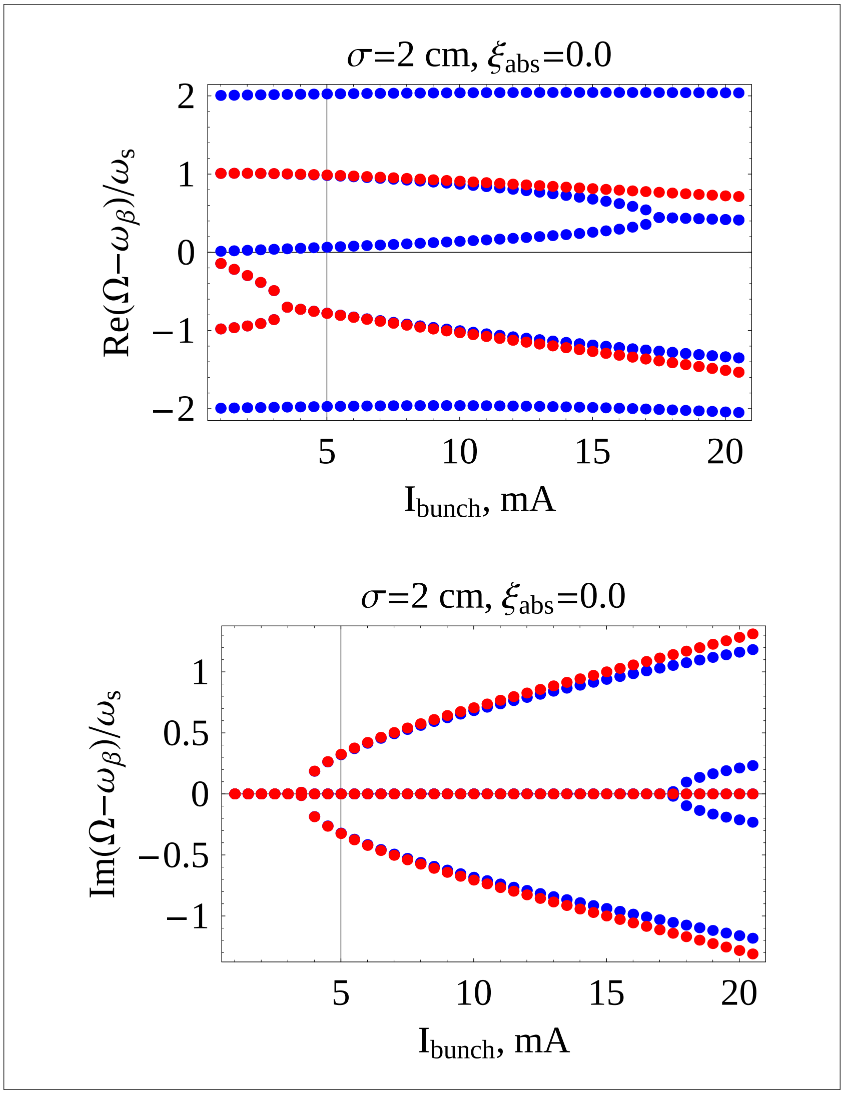

Figure 3: Head-Tail instability for a Gaussian bunch and the broad-band $Q = 1$ impedance; $I_{\rm bnch}$ is the dc beam current I^{dc}_{beam} when the beam has a single bunch. The blue dots are calculated using the Satoh-Chin formalism where the synchro-betatron modes $m = 0, \pm 1, \pm 2$ are taken into account. The threshold of instability is defined by the crossings of the modes $m = 0$ and $m = -1$. The red dots show result of calculations based on Eq. (58) for the same Gaussian bunch. The threshold in the last case is in a good agreement with the more accurate first method.

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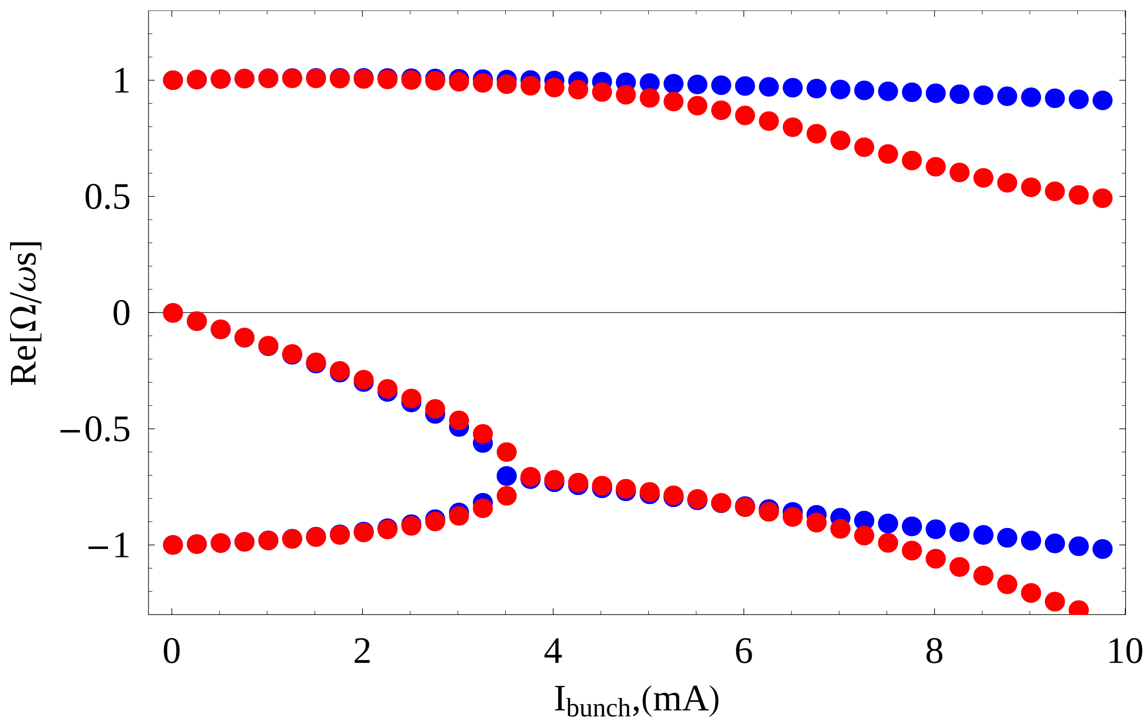

Figure 4: Head-tail instability for a single bunch and a broad-band $Q = 1$ impedance. The blue dots are calculated using the Satoh-Chin formalism for Gaussian bunch, where the synchro-betatron modes $m = 0, \pm 1, \pm 2$ are taken into account (the rank of the matrix $r = 2$). The red dots show result of calculations based on Eq. (58) taking into account the potential well distortion. The thresholds in both cases are practically the same, although the eigen-mode frequencies differ at higher bunch currents.

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10 Appendix: Satoh-Chin formalism for head-tail instability

The accurate consideration of the head-tail instability for a Gaussian bunch was given by Satoh and Chin [8]. The result is formulated as a matrix equation

$$\det[\delta_{h,l} + iK_{SCh} b_h(\lambda)\, M_{h,l}] \;=\; 0, \tag{63}$$

for the parameter $\lambda = \Omega/\omega_s$ where Ω is the coherent shift from the zero-current betatron tune $\nu_\perp$, the instability takes place when the growth rate $Im[\Omega] > 0$, and

$$K_{SCh} = \frac{I_{bunch}\beta_\perp}{4\pi(E/e)\nu_s} \tag{64}$$

Here $M_{h,l}$, $h, l = 0, 1, 2, ..$ is the matrix element

$$\begin{aligned} M_{h,l} \;&=\; \sum_{p=-\infty}^{\infty} Z_\perp[(p + \nu_\perp + \lambda\nu_s)\omega_0] \\ &\times\;\; C_h\left[(p + \nu_\perp + \lambda\nu_s - \nu_\perp\frac{\xi}{\alpha})\frac{\sigma}{R}\right] C_l\left[(p + \nu_\perp + \lambda\nu_s - \nu_\perp\frac{\xi}{\alpha})\frac{\sigma}{R}\right], \end{aligned} \tag{65}$$

where $Z_\perp(\omega)$ is transverse impedance (dimension Ω/m,) ξ is the relative chromaticity, R is the average machine radius, α is the momentum compaction, and

$$C_h(x) = \frac{1}{\sqrt{h!}}\left(\frac{x}{\sqrt{2}}\right)^h e^{-\frac{x^2}{2}}, \tag{66}$$

The coefficients $b_h(\lambda)$ are

$$b_h(\lambda) = \sum_{k=0}^{[h/2]} \frac{h!}{k!(h-k)!}\,\frac{\lambda}{\lambda^2 - (h-2k)^2} P[h,k], \tag{67}$$

$$b_0(\lambda) = \frac{1}{\lambda}, \quad b_1(\lambda) = \frac{2\lambda}{\lambda^2 - 1}. \tag{68}$$

The upper limit of summation is given by the integer part of $h/2$ and $P(h,k) = 1$ if $2k = h$ and $P(h,k) = 2$ otherwise. In actual calculations the matrix is truncated to a finite rank which is approximately equal to the number of azimuthal modes taken into account. Usually, the threshold of instability is given by the lowest modes. An example of calculations based on the Satoh-Chin formalism is given in the text. For illustration in the text above, we took parameters of the PEP-II broad-band $Q = 1$ wake used by Satoh-Chin: the resonance frequency $f_{res} = 1.3$ GHz, the shunt impedance $R_s = 0.68$ MΩ/m, energy $E = 14.5$ GeV, the synchrotron tune $\nu_s = 0.044$, the betatron tune $\nu_y = 21.25$, the momentum compaction $\alpha = 1.2 \times 10^{-3}$, the revolution frequency $f_{rev} = 136.4$ kHz, rms bunch length $\sigma = 2$ cm, and β-function $\beta_y = 160$ m.