SLAC-PUB-12103
September, 2006

Effect of the coupled-bunch modes on the longitudinal feedback system. *

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Abstract

The Pedersen analysis [1, 2] of the low-level rf feedback system assumes that all bunches oscillate in phase what corresponds to the lowest coupled bunch mode. This analysis is extended here to take into account all other coupled-bunch modes what might be important for the strongly detuned cavities in large storage rings such as PEP-II.

1 Introduction

The Pedersen analysis [1, 2] of the low-level rf feedback system gives a solid ground for the design of the low-level rf feedback (FB) system and has been used for analysis and modeling of the FB in PEP-II B-factory. The analysis assumes that all bunches oscillate in phase what corresponds to the lowest coupled bunch mode. This analysis is extended here to take into account all other coupled-bunch modes what might be important for the strongly detuned cavities in large storage rings such as PEP-II.

For completeness and to define notations we, first, reproduce the basic definitions of the steady-state parameters of the rf FB system. In the derivation of the beam response, all coupled-bunch (CB) modes are taken into account and their effect is included in the analysis of the FB stability.

2 Cavity voltage

A rf cavity can be described as an LC contour excited by the power from a klystron and the beam current. Following P. Wilson [3] and G. Kraft [4], we describe the time dependence of the cavity voltage $V_c(t)$ by the equation

*Work supported in part by Department of Energy contract DE-AC02-76SF00515

$$(\frac{d^2}{dt^2} + \frac{\omega_c}{Q_L}\frac{d}{dt} + \omega_c^2)\, V_c(t) = \frac{\omega_c R_L}{Q_L}\frac{dI_{tot}}{dt}. \tag{1}$$

Here $\omega_c/2\pi$ is the fundamental frequency of the cavity, Q_0 and R_0 are the quality factor and shunt impedance of the cavity, and Q_L and R_L are the loaded Q-factor and loaded shunt impedance, depending on the rf coupling parameter β,

$$Q_L = \frac{Q_0}{1+\beta}, \quad R_L = \frac{R_0}{1+\beta}. \tag{2}$$

The cavity is excited by the total current $I_{tot} = I_g(t) - I_B(t)$, where I_g and I_B are the excitation current of the generator and the beam current, respectively.

Consider the rf cavity with the rf feedback system (FB). The FB loop is shown in Fig. (1). The voltage on the cavity is generated by the generator current I_g and the beam current I_B and is defined by the cavity impedance Z_c as it is followed from Eq. (1):

$$\begin{aligned} V_c(\omega) &= Z_C(\omega)(I_g(\omega) - I_B(\omega)), \\ Z_c(\omega) &= \frac{R_L}{1 - iQ_L(\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega})}. \end{aligned} \tag{3}$$

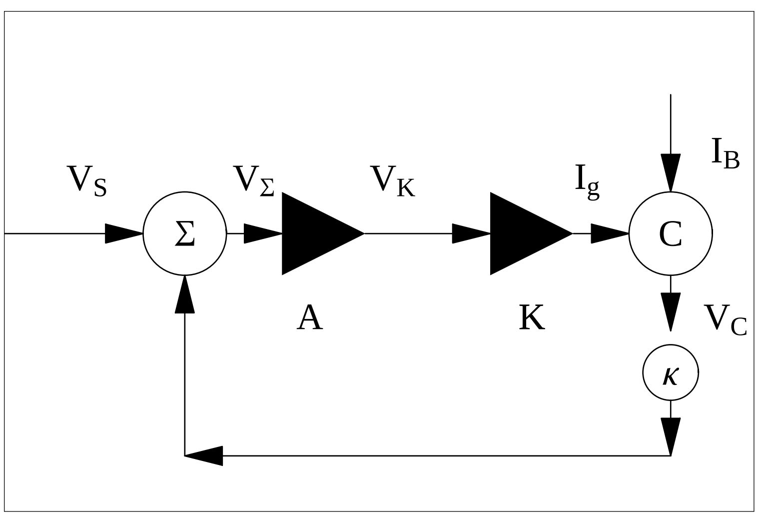

Figure 1: Schematic of the longitudinal rf FB system.

We use the time dependence of all quantities at the frequency ω in the form $e^{-i\omega t}$.

The input voltage on the amplifier is $V_\Sigma = V_s - \kappa V_c$, where V_s is the setup voltage and κ is a real parameter of the attenuator, $\kappa \simeq 10^{-6}$. After the amplifier with the transfer function T_A, the input voltage to the klystron is $V_K = T_A V_\Sigma$. The output generator current feeding the rf cavity is

$$I_g = \frac{V_K}{T_K} = \frac{T_A}{T_K}(V_s - \kappa V_c). \tag{4}$$

Substituting Eq. (4) into Eq. (3) we obtain

$$V_c(\omega) = \frac{(G/\kappa)V_s(\omega) - R_L I_B(\omega)}{1 - 2iQ_L(\frac{\omega-\omega_c}{\omega_c}) + G},$$
$$I_g(\omega) = \frac{G}{\kappa R_L}\, V_s(\omega) - G\,\frac{V_c(\omega)}{R_L}. \tag{5}$$

Here we use notation G for the FB transfer function defined in terms of the electronic gain H and the delay time τ_d as

$$G = \kappa\, \frac{T_A}{T_K}\, R_L = H\, e^{i(\omega-\omega_c)\tau_d}. \tag{6}$$

Parameters H and τ_d are, approximately, independent of frequency. For PEP-II, $H \simeq 6$ and $\tau_d = 450$ ns.

The FB stability requires that the poles of $V_c(\omega)$ as function of complex ω have to be in the lower half plane. In other words, for the FB stability, the imaginary part of zeros of the denominator in Eq. (5) for $V_c(\omega)$ has to be negative. Result of the calculation for the PEP-II coupling $\beta_{RF} = 3.6$, $Q_L = 6956.5$, and delay time $\tau_d = 450$ ns is shown in Fig. (2). The FB system is stable for $H <\simeq 15$. The actual limit is lower and is defined by distortion of the cavity impedance at large H [5], see Fig. (3).

2.1 Steady-state conditions

Consider the rf frequency $\omega_g = h\omega_0$, where $\omega_0/(2\pi)$ is the revolution frequency, and h is the harmonic number. We define quantities

$$\mu = (\omega_g - \omega_c)\tau_d,$$
$$\zeta = 2Q_L\, \frac{\omega_g - \omega_c}{\omega_c},$$
$$R_H = \frac{R_L}{1 + H\cos\mu},$$
$$\tan\psi = \frac{\zeta - H\sin\mu}{1 + H\cos\mu}. \tag{7}$$

Let us define the amplitudes and phases [†]

[†]F. Pedersen uses phases ϕ_L, ϕ_B, and ϕ_z related to our ϕ_c, ϕ_s, and ψ: $\phi_L = -\phi_c$, $\pi/2 - \phi_B = \phi_s$, $\phi_z = \psi$.

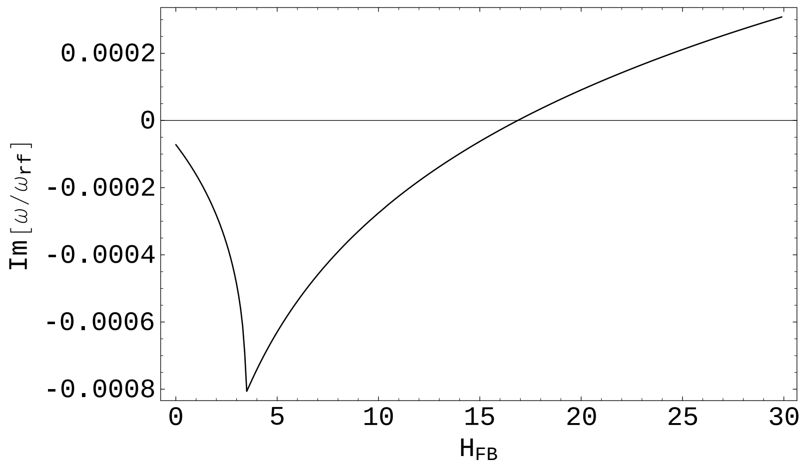

Figure 2: Imaginary part of the poles of $V_c(\omega)$ as function of the FB gain H. The FB is stable if imaginary part shown in the plot is negative.

$$
\begin{aligned}
&V_c(t) = V\ \cos(\omega_g t + \phi_c) = \frac{1}{2}\hat{V}_c\, e^{-i\omega_g t} + c.c., \quad \hat{V}_c = V e^{-i\phi_c}, \\
&I_B(t) = 2I_{dc}\cos(\omega_g t + \phi_c - \phi_s) = \frac{1}{2}\hat{I}_B e^{-i\omega_g t} + c.c.,, \quad \hat{I}_B = 2I_{dc}\, e^{-i(\phi_c - \phi_s)}, \\
&I_g(t) = \frac{1}{2}\hat{I}_g e^{-i\omega_g t} + c.c., \\
&V_s(t) = \frac{1}{2}\hat{V}_s e^{-i\omega_g t} + c.c.
\end{aligned}
\tag{8}
$$

The common phase is arbitrary and we assume that $\hat{I}_g$ is real.
In this notations and at $\omega = \omega_g$, Eq. (5) takes the form

$$
\begin{aligned}
&\hat{V}_c = Z_H(\omega_g)\hat{I}_{tot}, \\
&\hat{I}_{tot} = \frac{H}{\kappa R_L}\hat{V}_s e^{i\mu} - \hat{I}_B.
\end{aligned}
\tag{9}
$$

The impedance $Z_H(\omega_g)$ at the rf frequency is

$$
Z_H(\omega_g) = R_H \cos(\psi)\, e^{i\psi}. \tag{10}
$$

Hence, the detuning angle ψ is the angle between V_c and the total current I_{tot}.
A bunch crosses the cavity at the moments when $I_B(t)$ is maximum and sees the accelerating voltage $V_{acc} = V\ \cos(\phi_s)$. The synchronous phase ϕ_s, therefore, is defined

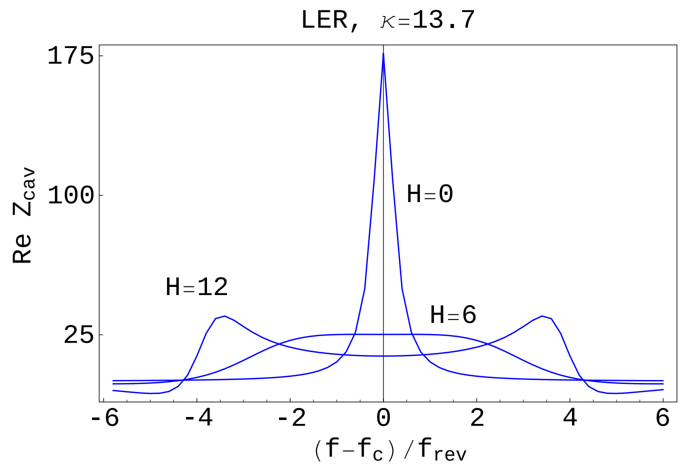

Figure 3: Real part of the impedance for several values of the FB gain H. The impedance starts to grow at large H and can cause instability at the rf sidebands for detuned cavities.

by the energy loss per particle per turn due to synchrotron radiation U_{SR} and the HOM losses $U_{HOM} = N_b e^2 \kappa_l$ proportional to the bunch population N_B and the total loss factor κ_l (except of the loss factor of the fundamental mode).

For the n_{cav} cavities in the ring,

$$n_{cav} V \cos\phi_s = U_{SR} + U_{HOM}. \tag{11}$$

For the beam stability, the accelerating voltage has to have a negative slope, $dV_{acc}/d\phi_s < 0$. In this case, a particle having larger energy and shifted above the transition energy to the tail of the bunch sees the accelerating voltage smaller than the equilibrium particle. The derivative $dV_{acc}/d\phi_s$is taken for fixed parameters $I_{dc}, \omega_g, \omega_c, \kappa, \beta, H, \tau_d$ and the setup voltage V_s. Therefore, R_L, Q_L, R_H, μ and ψ are also constant, while $|\hat{V}_c|$ and ϕ_c vary. Taking derivative of V_c given by the first of Eqs. (9), we get

$$\frac{d|\hat{V}_c|}{d\phi_s} = i|\hat{V}_c|\frac{d\phi_c}{d\phi_s} + 2iI_{dc}R_H \cos\psi\, e^{i(\phi_s+\psi)}\,(\frac{d\phi_c}{d\phi_s} - 1), \quad |\hat{V}_c| = V. \tag{12}$$

The real and imaginary parts give two equations

$$\frac{d\phi_c}{d\phi_s} = \frac{Y_H \cos\psi\, cos(\psi+\phi_s)}{1 + Y_H \cos\psi\, \cos(\psi+\phi_s)},$$
$$\frac{d|\hat{V}_c|}{d\phi_s} = |\hat{V}_c| \tan(\psi+\phi_s)\, \frac{d\phi_c}{d\phi_s}. \tag{13}$$

Here

$$Y_H = \frac{Y_L}{1 + H\cos\mu}, \quad Y_L = \frac{2 I_{dc} R_L}{V}. \tag{14}$$

The Robinson condition of stability takes the form

$$\sin(\phi_s) > \frac{Y_H}{2}\sin(2\psi). \tag{15}$$

Without FB, $H = 0$, it takes the standard form. Limitation on the beam current are shown in Fig. (4). At a fixed voltage, the maximum current increases with the FB on. However, as it was mentioned above, the maximum gain H is limited by the stability of the rf sidebands, see Fig. (3).

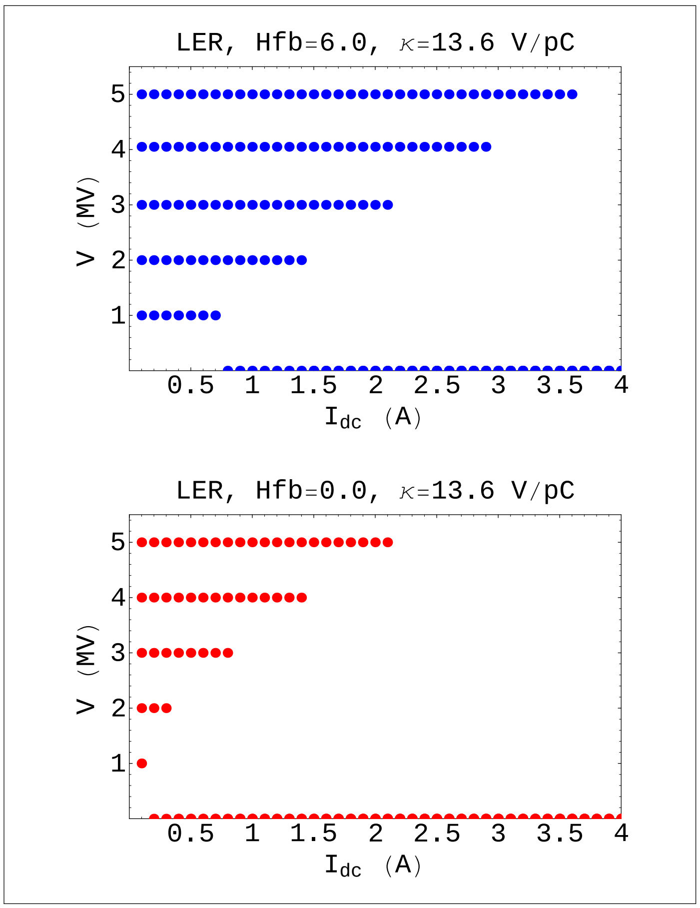

Figure 4: For each rf voltage of the PEP-II LER, the beam currents corresponding to Robinson criterion of stability are shown by dots. Upper plot: with the FB $H_g = 6.0$, bottom plot with the FB turned off, $H_g = 0$.

2.2 Optimum conditions

The optimum conditions correspond to the minimum of reflected power

$$P_r = \frac{\beta}{2(1+\beta)R_L}\,|\hat{V}_c - \frac{R_L(1+\beta)}{2\beta}\,I_g|^2. \tag{16}$$

Substituting

$$\hat{I}_g = \hat{V}_c/Z_c + \hat{I}_B, \tag{17}$$

from Eq. (3), $\hat{V}_c = |\hat{V}_c|e^{-i\phi_c}$, $\hat{I}_B = 2I_{dc}e^{i(\phi_s-\phi_c)}$, and using definition of Y_L, we get

$$P_r = \frac{(1+\beta)|\hat{V}_c|^2}{8\beta R_L}\,|\frac{\beta-1}{\beta+1} - Y_L\cos\phi_s + i(\zeta - Y_L\sin\phi_s)|^2. \tag{18}$$

P_r is minimum if the imaginary part is zero. This defines the optimum detuning

$$\zeta = 2Q_L\frac{\omega_g-\omega_c}{\omega_c} = Y_L\sin\phi_s, \tag{19}$$

and the detuning angle ψ by Eq. (7).

The reflected power in the optimum

$$P_r = \frac{(1+\beta)|\hat{V}_c|^2}{8\beta R_L}\,|\frac{\beta-1}{\beta+1} - Y_L\cos\phi_s|^2. \tag{20}$$

is zero for fixed cavity voltage only at certain current, when

$$Y_L\cos\phi_s = \frac{\beta-1}{\beta+1}. \tag{21}$$

Eqs. (19),(17) give

$$\hat{I}_g = \frac{|\hat{V}_c|}{R_L}e^{-i\phi_c}\,(1+Y_L\cos\phi_s). \tag{22}$$

By definition, $\hat{I}_g$ is real. Hence, ϕ_c in the optimum is zero.

Usually, the FB system operates to produce a given cavity voltage V_c for a beam current I_B. Then Eq. (5) taken at $\omega = \omega_g$ defines the reference voltage V_s,

$$V_s(\omega_g) = \frac{\kappa V_c}{H}e^{-i\mu}\,\{1 - i\zeta + He^{i\mu} + Y_Le^{i\phi_s}\}, \tag{23}$$

where μ and ζ are defined by Eq. (7).

3 Analysis of small perturbations

The steady-state cavity voltage $V^{(0)}(t) = \frac{1}{2}\hat{V}^0 e^{-i\omega_g t} + c.c.$ is the sum of voltages induced by the setup voltage $V_{cav}^{(s)}$, the voltage $V_{cav}^{(B)}$ induced by the beam current I_B, and of the FB $V_{cav}^{(FB)}$,

$$V^0 = V_{cav}^{(s)} - V_{cav}^{(B)} - V_{cav}^{(FB)}. \tag{24}$$

If there is a beam perturbation δI_B, the setup voltage $V_{cav}^{(s)}$ do not change, but the cavity voltage and the total current change.

Following F. Pedersen [1], we write the small variations of the voltage and the total current around the steady-state as

$$\begin{aligned} \delta V_c(t) &= \frac{1}{2}\hat{V}_c e^{-i\omega_g t} U(t) + c.c., \quad U(t) = u[t] - i\Phi[t], \\ \delta I_{tot}(t) &= \frac{1}{2}\hat{I}_{tot} e^{-i\omega_g t} J[t] + c.c., \quad J(t) = a(t) - ib(t). \end{aligned} \tag{25}$$

where the slow functions of time $U(t)$ and $J(t)$ are split in the real and imaginary parts.

$\hat{V}_c$ and $\hat{I}_{tot}$ are related by Eq. (9), $\hat{V}_c = Z_H(\omega_g)\hat{I}_{tot}$. Eq. (25) gives for Fourier harmonics

$$\begin{aligned} \delta V_c(\omega) &= \frac{1}{2}\hat{V}_c U_-(\omega - \omega_g) + \frac{1}{2}\hat{V}_c^* U_+(\omega + \omega_g), \quad U_\pm(\omega) = u(\omega) \pm i\Phi(\omega), \\ \delta I_{tot}(\omega) &= \frac{1}{2}\hat{I}_{tot} J_-(\omega - \omega_g) + \frac{1}{2}\hat{I}_{tot}^* J_+(\omega + \omega_g), \quad J_\pm(\omega) = a(\omega) \pm ib(\omega), \end{aligned} \tag{26}$$

where $U(\omega) = u(\omega) - i\Phi(\omega)$, $J(\omega) = a(\omega) - ib(\omega)$, $u(-\omega) = u^*(\omega)$, $\Phi(-\omega) = \Phi^*(\omega)$, $a(-\omega) = a^*(\omega)$, $b(-\omega) = b^*(\omega)$ and the star indicates complex conjugation.

The perturbation $\delta I_{tot}(\omega) = \delta I_g - \delta I_B$ causes $\delta V_c(\omega) = Z_c(\omega)\delta I_{tot}(\omega)$. Because, the reference voltage V_s is fixed, then $\delta V_\Sigma = -\kappa Z_c \delta I_{tot}$, see Fig.(1). The variation δV_Σ propagates giving $\delta V_K = T_A \delta V_\Sigma$ and $\delta I_g = \delta V_K / T_K$. Hence,

$$\left(1 + \kappa\frac{T_A}{T_K} Z_c(\omega)\right) \delta I_{tot}(\omega) = -\delta I_B(\omega), \tag{27}$$

or

$$\begin{aligned} \delta I_{tot}(\omega) &= -\frac{Z_H(\omega)}{Z_c(\omega)}\, \delta I_B(\omega), \\ \delta V_c(\omega) &= -Z_H(\omega)\delta I_B(\omega) = Z_c(\omega)\delta I_{tot}(\omega). \end{aligned} \tag{28}$$

Combining Eq. (26) and the second Eq. (28) produces the first set of F. Pedersen's relations

$$u(\omega) = G_+(\omega)a(\omega) - iG_-(\omega)b(\omega),$$
$$\Phi(\omega) = iG_-(\omega)a(\omega) + G_+(\omega)b(\omega), \tag{29}$$

where

$$G_\pm = \frac{1}{2}[\pm\frac{Z_c(\omega-\omega_g)}{Z_H^*(\omega_g)} + \frac{Z_c(\omega+\omega_g)}{Z_H(\omega_g)}]. \tag{30}$$

Note that $G_\pm^*(-\omega) = \pm G_\pm(\omega)$.

The beam dynamics relates δV_c to $\delta I_B(\omega)$ and, through the first Eq. (28), to δI_{tot}. This is considered in the next section.

3.1 Beam dynamics

Let us consider a train of n_b ultra-relativistic bunches in the ring with the circumference $2\pi R = cT_0 = 2\pi c/\omega_0$. For the frequencies within the bandwidth of the FB system, bunches can be considered as point-like macro-particles. In an equilibrium, the center of the n-th bunch is at the distance $s_n = (n-1)s_b$ from the head of the train. The position of the bunch in the ring s_n is

$$s_n(t) = c(t-t_0) - (n-1)s_b + \zeta_n(t) \tag{31}$$

where c is velocity of light, $t_0 = const$, and $\zeta_n > 0$ means the displacement of the bunch center to the head of the train. $\zeta_n(t)$ describes the synchrotron oscillations. The amplitude of oscillations may be itself a slow function of time. The constant longitudinal offset of the bunches in the train is included in t_0.

On the k-th turn, the n-th bunch centroid arrives to an rf cavity located at $s = 0$ in the ring at the moment $t_{k,n} = kT_0 + t_0 + (n-1)s_b/c$.

The beam current exciting a cavity located at $s = 0$ around the ring is

$$I_B(t) = \frac{I_{dc}\, c\, T_0}{n_b}\sum_{n=1}^{n_b}\sum_{k=-\infty}^{\infty}\delta[c(t-t_0) - (n-1)s_b + \zeta_n(t) + kcT_0] =$$
$$\frac{I_{dc}T_0}{n_b}\sum_{n=1}^{n_b}\int\frac{d\omega}{2\pi}\sum_{k=-\infty}^{\infty}\exp[-i\omega(t-t_0-(n-1)\frac{s_b}{c} + \frac{\zeta_n(t)}{c} + kT_0)]. \tag{32}$$

Using the identity

$$\sum_{k=-\infty}^{\infty}\exp[-ik\omega T_0] = \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty}\delta[\omega - k\omega_0], \tag{33}$$

Eq. (32) in the linear over ζ approximation can be written in the form

$$I_B(t) = \frac{I_{dc}}{n_b} \sum_{n=1}^{n_b} \sum_{k=-\infty}^{\infty} \exp[ik\omega_0(t - t_0 - (n-1)\frac{s_b}{c})]\,(1 + i\frac{k\omega_0}{c}\zeta_n(t)). \tag{34}$$

We assume the uniform fill of bunches, $\omega_0 s_b/c = 2\pi/n_b$, expanding $\zeta_n(t)$ over the coupled-bunch eigen modes X_n^μ with amplitudes $A_\mu(t)$, $A_\mu^*(t) = A_{-\mu}(t)$, $\mu = 0, 1, .., n_b-1$,

$$\zeta_n(t) = \sum_{\mu=0}^{n_b-1} A_\mu(t) X_n^\mu, \quad X_n^\mu = \frac{1}{\sqrt{n_b}} e^{2\pi i(n-1)\mu/n_b}. \tag{35}$$

Now we can use the identity

$$\sum_{n=1}^{n_b} \exp[-ik\omega_0(n-1)s_b/c] = n_b \sum_{p=-\infty}^{\infty} \delta_{k,pn_b}, \tag{36}$$

and

$$\sum_{n=1}^{n_b} (X_n^\mu) e^{-ik\omega_0(n-1)s_b/c} = \sqrt{n_b} \sum_{p=-\infty}^{\infty} \delta_{k-\mu,pn_b}, \tag{37}$$

where $\delta_{i,k} = 1$ for $i = k$ and $\delta_{i,k} = 0$ otherwise.
Eq. (32) is transformed to

$$I_B(t) = I_{dc} \sum_{p=-\infty}^{\infty} e^{ipn_b\omega_0(t-t_0)}\,\{1 + i\frac{\omega_0}{c\sqrt{n_b}} \sum_\mu (pn_b + \mu) A_\mu(t) e^{i\mu\omega_0(t-t_0)}\}. \tag{38}$$

Here harmonics $p = \pm h/n_b$ corresponds to the steady-state beam current. Hence,

$$\omega_g t_0 = (\phi_s - \phi_c). \tag{39}$$

All other harmonics are separated in frequency by $n_b\omega_0$ and can be ignored for the analysis of the low-level FB with the bandwidth limited to few revolution harmonics.

The perturbation of the beam current is given by the second term in Eq. (38),

$$\delta I_B(t) = i\frac{I_{dc}\omega_g}{c\sqrt{n_b}} \sum_\mu \sum_p (\frac{pn_B + \mu}{h}) e^{i(\frac{pn_b+\mu}{h})(\omega_g t - \phi_s)} A_\mu(t). \tag{40}$$

The Fourier harmonics for the optimum $\phi_c = 0$ are

$$\delta I_B(\omega) = i\frac{I_{dc}\omega_g}{c\sqrt{n_b}} \sum_\mu \sum_p (\frac{pn_B + \mu}{h}) e^{-i(\frac{pn_b+\mu}{h})\phi_s} A_\mu[\omega + (pn_b + \mu)\omega_0]. \tag{41}$$

3.2 Synchrotron motion

The synchrotron oscillations are defined by the rf voltage and the longitudinal wake $W(t)$. The later produces two effects: it changes the synchrotron frequency and generates bunch coupling.

Consider equation of motion for the n-th bunch in the linear approximation over ζ_n taking into account the wake and the perturbation of the rf voltage $\delta V_c(t)$,

$$\frac{d^2\zeta_n(t)}{dt^2} + \omega_{s,0}^2\zeta_n(t) = \sum_{k=-\infty}^{\infty} \delta[t - t_{k,n}]\{-(\frac{\alpha c e n_{cav}}{E})\,\delta V_c(t)\; + \\ (\frac{\alpha N_b e^2}{E}) \sum_{k'=-\infty}^{\infty}\sum_{m=1}^{n_b} W'[t_{k,n} - t_{k',m}](\zeta_m(t_{k',m}) - \zeta_n(t_{k,n}))\}. \tag{42}$$

Here N_B and E are the bunch population and the beam energy, respectively, α is the momentum compaction factor, and $\omega_s^0/2\pi$ is the zero-current synchrotron frequency. The longitudinal wake per turn is defined to be zero for $t < 0$, and the prime in $W'[t]$ means derivative over t. The longitudinal impedance $Z(\omega)$ is the Fourier component of the wake $W[t]$ analytic in the upper half plane of ω.

It is well known [7], that for equidistant bunches and $\delta V_c = 0$, expansion Eq. (35) in the orthogonal modes X_n^μ reduces Eq. (42) to independent equations for the constant in time amplitudes A_μ. The same is true for a slow perturbation $\delta V_c(t) \neq 0$ giving equations for the amplitudes $A_\mu(t)$ slowly varying in time.

The Fourier harmonics $A_\mu(\omega)$,

$$A_\mu(t) = \int d\omega/(2\pi) A_\mu(\omega) e^{-i\omega t} \tag{43}$$

satisfy equations

$$(\omega_{s,0}^2 - \omega^2)\, A_\mu(\omega) = G_\mu(\omega), \tag{44}$$

where

$$G_\mu(\omega) = \int dt e^{i\omega t} \sum_{k=-\infty}^{\infty}\sum_{n=1}^{n_b} [X_n^\mu]^* \delta[t - t_{kn}]\,\{-(\frac{\alpha c e n_{cav}}{E})\,\delta V_c(t) + \\ (\frac{\alpha N_b e^2}{E}) \sum_{k'=-\infty}^{\infty}\sum_{m=1}^{n_b} W'[t_{k,n} - t_{k',m}](\zeta_m(t_{k',m}) - \zeta_n(t_{k,n}))\}. \tag{45}$$

Substituting Fourier expansion for $\delta V(t)$, for the wake, and $\zeta(T)$, we can carry out integration over t. That gives

$$G_\mu(\omega) = -(\frac{\alpha cen_{cav}\omega_0}{2\pi E})\sqrt{n_b}\sum_{p=-\infty}^{\infty}\delta V_c[\omega + (pn_b - \mu)\omega_0]\, e^{-i(pn_b-\mu)\omega_0 t_0}$$

$$-i(\frac{\alpha N_b n_b e^2}{E})\,(\frac{\omega_0}{2\pi})^2 \sum_{p,p'=-\infty}^{\infty}\sum_{\mu'=0}^{n_b-1} A_{\mu'}[\omega - (p+p')n_b\omega_0 - (\mu-\mu')\omega_0]\, e^{i[(p+p')n_b-\mu'+\mu]\omega_0 t_0}$$

$$[(\omega - pn_b\omega_0 - \mu\omega_0)Z(\omega - (pn_b+\mu)\omega_0) - p'n_b\omega_0 Z(p'n_b\omega_0)]. \tag{46}$$

The amplitude $A_\mu(\omega) \neq 0$ for low frequencies, $|\omega| << n_b\omega_0$. Therefore, we can drop all terms with $p' \neq p$. The impedance dependent diagonal term $\mu' = \mu$ can be combined with the term $\omega_{s,0}^2$ in the LHS of Eq. (44) redefining the synchrotron frequency of the mode μ,

$$\omega_{\mu,s}^2 = \omega_{s,0}^2 + \delta\omega_{s,\mu}^2$$

$$\delta\omega_{s,\mu}^2 = i(\frac{\alpha N_b n_b e^2}{E})\,(\frac{\omega_0}{2\pi})^2$$

$$\sum_{p}^{\infty}[(\omega + pn_b\omega_0 - \mu\omega_0)Z(\omega + (pn_b-\mu)\omega_0) - pn_b\omega_0 Z(pn_b\omega_0)] \tag{47}$$

Eq. (46) takes the form

$$G_\mu(\omega) = -(\frac{\alpha cen_{cav}\omega_0}{2\pi E})\sqrt{n_b}\sum_{p=-\infty}^{\infty}\delta V_c[\omega + (pn_b - \mu)\omega_0]\, e^{-i(pn_b-\mu)\omega_0 t_0}$$

$$-\delta\omega_{s,\mu}^2 \sum_{\mu'\neq\mu} A_{\mu'}[\omega - (\mu-\mu')\omega_0]\, e^{-i(\mu'-\mu)\omega_0 t_0}. \tag{48}$$

The last sum (proportional to the synchrotron tune shift $\delta\omega_{s,\mu}^2$) is zero for amplitudes A_μ independent of time. For slow dependence on time it produces coupling of the amplitudes A_μ. Usually, however, $\delta\omega_{s,\mu}$ is small, $\delta\omega_{s,\mu} << \omega_{s,0}$, and we can neglect coupling. In this case,

$$G_\mu(\omega) = -(\frac{\alpha cen_{cav}\omega_0}{2\pi E})\sqrt{n_b}\sum_{p=-\infty}^{\infty}\delta V_c[\omega + (pn_b - \mu)\omega_0]\, e^{-i(pn_b-\mu)\omega_0 t_0}. \tag{49}$$

Eq. (44) defines $A_\mu(\omega)$ in terms of Fourier harmonics of the perturbation $\delta V(\omega)$,

$$A_\mu(\omega) = -(\frac{\alpha cen_{cav}}{E})\,\frac{\omega_0\sqrt{n_b}}{2\pi[\omega_{s\mu}^2 - \omega^2]}\sum_{p=-\infty}^{\infty}\delta V_c[(pn_b-\mu)\omega_0 + \omega]e^{-i(pn_b-\mu)\omega_0 t_0}. \tag{50}$$

The coefficient can be written in terms of $\omega_{s,0}$,

$$(\frac{\alpha cen_{cav}}{E}) = (\frac{\omega_{s,0}^2 cT_0}{\omega_g V_c\sin(\phi_s)}). \tag{51}$$

3.3 Beam transfer function

In the optimum, $\phi_c = 0$ and $\hat{V}_c = \hat{V}_c^*$. Substituting Eq. (50) in Eq. (41) we get for $\delta I_B(\omega)$ expression

$$\delta I_B(\omega) = -i\frac{I_{dc}}{V_c \sin(\phi_s)} \sum_{p=-\infty}^{\infty}\sum_{p'=-\infty}^{\infty} \sigma[\omega, p]\, \delta V_c[\omega + (p+p')n_b\omega_0]\, e^{-i(\frac{(p+p')n_b}{h})\phi_s}, \quad (52)$$

where

$$\sigma[\omega, p] = \sum_{\mu}(\frac{pn_b + \mu}{h})\frac{\omega_{s0}^2}{\omega_{s,\mu}^2 - [\omega + (pn_b+\mu)\omega_0]^2}. \quad (53)$$

Let us define $\delta I_B(t)$ in terms of the slow functions $\kappa_b(t) = a_B(t) - ib_B(t)$ similarly to Eq. (25),

$$\delta I_b(t) = \frac{\hat{I}_B}{2} e^{-i\omega_g t}\kappa_b[t] + c.c. \quad (54)$$

The Fourier harmonics

$$\delta I_b(\omega) = \frac{\hat{I}_B}{2}\kappa_-(\omega - \omega_g) + \frac{\hat{I}_B^*}{2}\kappa_+(\omega + \omega_g), \quad (55)$$

where

$$\kappa_\pm(\omega) = a_B(\omega) \pm ib_B(\omega). \quad (56)$$

Neglecting components $\kappa_\pm(\pm 2\omega_g)$, we get from Eq. (55)

$$\kappa_-(\omega) = \frac{2}{\hat{I}_B}\, \delta I_B(\omega + \omega_g), \quad \kappa_+(\omega) = \frac{2}{\hat{I}_B^*}\, \delta I_B(\omega - \omega_g). \quad (57)$$

Substituting here Eq. (52), using Eq. (26) and

$$\frac{\hat{V}_c}{\hat{I}_B} = \frac{R_L}{Y_L}\, e^{-i\phi_s}, \quad (58)$$

we get

$$\kappa_-(\omega) = -\frac{i}{2\sin\phi_s}\sum_p\sum_{p'}\sigma[\omega+\omega_g, p]e^{-i(\frac{(p+p')n_b}{h})\phi_s - i\phi_s}\{U_-[\omega + (p+p')n_b\omega_0]$$
$$+U_+[\omega + 2\omega_g + (p+p')n_b\omega_0]\},$$
$$\kappa_+(\omega) = -\frac{i}{2\sin\phi_s}\sum_p\sum_{p'}\sigma[\omega-\omega_g, p]e^{-i(\frac{(p+p')n_b}{h})\phi_s + i\phi_s}\{U_-[\omega - 2\omega_g + (p+p')n_b\omega_0]$$
$$+U_+[\omega + (p+p')n_b\omega_0]\}. \quad (59)$$

Because Fourier harmonics of the slow functions $U_{\pm}(\omega)$ are not zero only at low ω, we can simplify Eq. (59) retaining only the main terms in the sum over p':

$$\kappa_{-}(\omega) = -\frac{i}{2\sin\phi_s}\sum_p \sigma[\omega+\omega_g,p]\{U_{-}(\omega)e^{-i\phi_s}+U_{+}(\omega)e^{i\phi_s}\},$$
$$\kappa_{+}(\omega) = -\frac{i}{2\sin\phi_s}\sum_p \sigma[\omega-\omega_g,p]\{U_{-}[\omega]e^{-i\phi_s}+U_{+}[\omega]e^{i\phi_s}\}. \quad (60)$$

In the sum $\sum_p \sigma[\omega \pm \omega_g, p]$ we can retain terms with small denominators. In this approximation,

$$\sum_p \sigma[\omega \pm \omega_g, p] = \mp\sigma_{\pm}(\omega), \quad (61)$$

where

$$\sigma_{\pm}(\omega) = \frac{1}{2}\frac{\omega_{s0}^2}{\omega_{s,\mu}^2-\omega^2} + \frac{1}{2}\sum_{\mu=1-n_b}^{n_b-1}(1\pm\frac{\mu}{h})\,\frac{\omega_{s0}^2}{\omega_{s,\mu}^2-[\omega-\mu\omega_0]^2}. \quad (62)$$

Let us substitute $\kappa_{\pm}(\omega) = a_B(\omega) \pm ib_B(\omega)$ and the definition $U_{pm}(\omega) = u(\omega) \pm i\Phi(\omega)$. Eq. (60) takes the form

$$a_B(\omega) - ib_B(\omega) = \frac{i\sigma_{+}(\omega)}{\sin\phi_s}\{u(\omega)\cos(\phi_s) - \Phi(\omega)\sin(\phi_s)\},$$
$$a_B(\omega) + ib_B(\omega) = -\frac{i\sigma_{-}}{\sin\phi_s}\{u(\omega)\cos(\phi_s) - \Phi(\omega)\sin(\phi_s)\}. \quad (63)$$

With the accuracy of the terms of the order of μ/h, $\sigma_{+}(\omega) = \sigma_{-}(\omega) \equiv \sigma(\omega)$. Then, $a_B(\omega) = 0$ and

$$b_B(\omega) = \sigma(\omega)\,[-u_(\omega)\cot(\phi_s) + \Phi(\omega)]. \quad (64)$$

The phase-to-phase beam transfer function is T_{BV} defined as $b_B(\omega) = T_{BV}\Phi(\omega)$. Therefore, $T_{BV} = \sigma(\omega)$ or

$$T_{BV} = \frac{1}{2}\frac{\omega_{s0}^2}{\omega_{s,\mu}^2-\omega^2} + \frac{1}{2}\sum_{\mu=1-n_b}^{n_b-1}\frac{\omega_{s0}^2}{\omega_{s,\mu}^2-[\omega-\mu\omega_0]^2}. \quad (65)$$

The first term here is the Pedersen's beam transfer function. The sum defines contribution of the $\mu \neq 0$ CB modes.

3.4 Dispersion equation

The perturbation of the total current is related to $\delta I_B(\omega)$ by Eq. (28),

$$\delta I_{tot}(\omega) = -R(\omega)\delta I_B(\omega), \tag{66}$$

where $R(\omega) = Z_H(\omega)/Z_c(\omega)$. Eq. (66) gives after substitution of Eq. (26) and Eq. (55)

$$\hat{I}_{tot}J_-(\omega-\omega_g) + \hat{I}^*_{tot}J_+(\omega+\omega_g) = -R(\omega)[\hat{I}_B\kappa_-(\omega-\omega_g) + \hat{I}^*_B\kappa_+(\omega+\omega_g)]. \tag{67}$$

Because $J_\pm(\omega)$ and $\kappa_\pm(\omega)$ are Fourier harmonics of the slow functions, Eq. (67) can be separated in two for $\omega \simeq \pm\omega_g$:

$$J_-(\omega) = -R(\omega+\omega_g)\frac{\hat{I}_B}{\hat{I}_{tot}}\kappa_-(\omega), \quad J_+(\omega) = -R(\omega-\omega_g)(\frac{\hat{I}_B}{\hat{I}_{tot}})^*\kappa_+(\omega). \tag{68}$$

From Eq. (9), $\hat{I}_{tot} = \hat{V}_c/Z_H(\omega_g)$. Eq. (8) gives $\hat{I}_B = 2I_{dc}\,e^{i\phi_s}$. Therefore, using the definition of Y_L,

$$\frac{\hat{I}_B}{\hat{I}_{tot}} = \frac{Y_L}{R_L}Z_H(\omega_g)e^{i\phi_s}. \tag{69}$$

Substitute Eq. (60) for $\kappa_\pm(\omega)$ and use definition of $J_\pm$, $U_\pm$ of Eq. (26). That gives $a(\omega)$ and $\underline{(}\omega)$ in terms of $u(\omega)$ and $\Phi(\omega)$:

$$\begin{aligned} a(\omega) &= iK_-[u(\omega)cot(\phi_s) - \Phi(\omega)], \\ b(\omega) &= K_+[u(\omega)cot(\phi_s) - \Phi(\omega)], \end{aligned} \tag{70}$$

where

$$K_\pm = \frac{Y_L}{2R_L}[\pm R(\omega+\omega_g)Z_H(\omega_g)\sigma_+(\omega)e^{i\phi_s} + R(\omega-\omega_g)Z^*_H(\omega_g)\sigma_-(\omega)e^{-i\phi_s}]. \tag{71}$$

Combining Eq. (70) and Eq. (66), we obtain the homogeneous system of two linear equations. The nontrivial solution exists if the determinant of the system is equal to zero what gives the dispersion equation for ω,

$$1 + i\frac{Y_L}{2R_L\sin(\phi_s)}\{R(\omega+\omega_g))\sigma_+(\omega)Z_c[\omega_g+\omega] - R(\omega-\omega_g))\sigma_-(\omega)Z_c[\omega-\omega_g]\} = 0. \tag{72}$$

In terms of the Laplace variable $s = -i\omega$, Eq. (72) takes the form

$$1 + i\frac{Y_L}{2R_L \sin(\phi_s)} \{R(s - i\omega_g)\sigma_+(s)Z_c[s - i\omega_g] - R(s + i\omega_g))\sigma_-(s)Z_c[s + i\omega_g]\} = 0. \quad (73)$$

Here $R(s) = Z_H(s)/Z_c(s)$, $Z_H(s)$ is equal to $Z_c(s)$ if $H = 0$, and

$$\sigma_\pm(s) = \frac{1}{2}\frac{\omega_{s0}^2}{\omega_{s0}^2 + s^2} + \frac{1}{2}\sum_{\mu=1-n_b}^{n_b-1}(1 \pm \frac{\mu}{h})\frac{\omega_{s0}^2}{\omega_{s,\mu}^2 + (s + i\mu\omega_0)^2}$$

$$Z_H(s) = R_L[\frac{\sigma_r}{s + i\omega_c + \sigma_r(1 + He^{-(s+i\omega_c)\tau_d})} + \frac{\sigma_r}{s - i\omega_c + \sigma_r(1 + He^{-(s-i\omega_c)\tau_d})}], (74)$$

where $\sigma_r = \omega_c/(2Q_L)$.

If only a single coupled-bunch mode $\mu = 0$ is taken into account, then $\sigma_-(\omega) = \sigma_+(\omega)$ what is the same as the Pedersen's beam transfer function $B(s) = \omega_{s0}^2/(\omega_{s0}^2 + s^2)$.

4 Analysis of stability

The dispersion equation Eq. (72) generalize Pedersen's result including all CB mode. The system is stable if all eigenvalues ω have negative imaginary parts. In terms of $s = -i\omega$, the growth rate $\Gamma_\mu = Re[s]$.

As an example, we analyze stability of the LER of PEP-II B-factory. The main parameters of the system are given in Table 1. Harmonic number $h = 2n_b$. The optimum calculated steady-state parameters are given in Table 2.

Table 1: **Parameters of the LER PEP-II.**

n_{cav}		6.
f_{rf}	MHz	476.
V_c/cavity	MeV	0.85
R_0	MOhm	3.5
Q_0		$3.0\,10^4$
U_{loss}	MeVr	0.77
I_{dc}	Amp	2.25
N_B	10^{10}	5.8
s_b	m	1.24
n_b		1658

The growth rate was calculated by solving numerically the dispersion equation Eq. (72), (73) using MATHEMATICA.

Table 2: **The steady-state parameters.**

β		3.909
$Y/(1+\beta)$		3.774
Q_L		6111.
R_L	MOhm	0.712
ϕ_s	degree	80.96
ψ	degree	74.95
f_s	kHz	4.5
$(\omega_c - \omega_g)/\omega_0$		1.064
τ_d	ns	450

Figs. (5-7) show results of calculations. Fig. (5) gives results with only $\mu = 0$ mode taken into account for several values of the gain H. The upper pane gives the growth rate Γ with the FB off ($H = 0$) and for single mode taken into account ($\mu_{max} = 0$). The result in this case is exactly the same as given by the Pedersen dispersion equation. The next panes show how the growth rate is modified with the increasing gain. The threshold of instability increases with H and instability is completely suppressed at $H = 6$. Fig. (6) shows effect of one additional mode $\mu_{max} = 1$ taken into account for $H = 0$, $H = 3$, and $H = 6$. The instability starts from the zero current as it is suppose to do for the coupled-bunch case. Fig. (7) gives similar result for four CB modes taken into account $\mu_{max} = 3$. In all cases the delay time of the FB is $\tau_d = 450$ ns.

5 Conclusion

F. Pedersen analysis gives the foundation for the design of the low-level rf feedback system. In his analysis the beam response is described by the beam transfer function which takes into account the single $m = 0$ CB mode. We generalize this analysis defining in Eqs. (72),(73) the dispersion equation taking into account all CB modes. The growth rate for the PEP-II was calculated by solving dispersion equation numerically using MATHEMATICA with a single $m = 0$ CB mode ($\mu_{max} = 0$) and with several CB modes. The difference of the results shows effect of the unstable CB modes.

6 Acknowledgement

We appreciate discussions with J. Fox and C. Riretta.

Work supported by Department of Energy contract DE–AC03–76SF00515.

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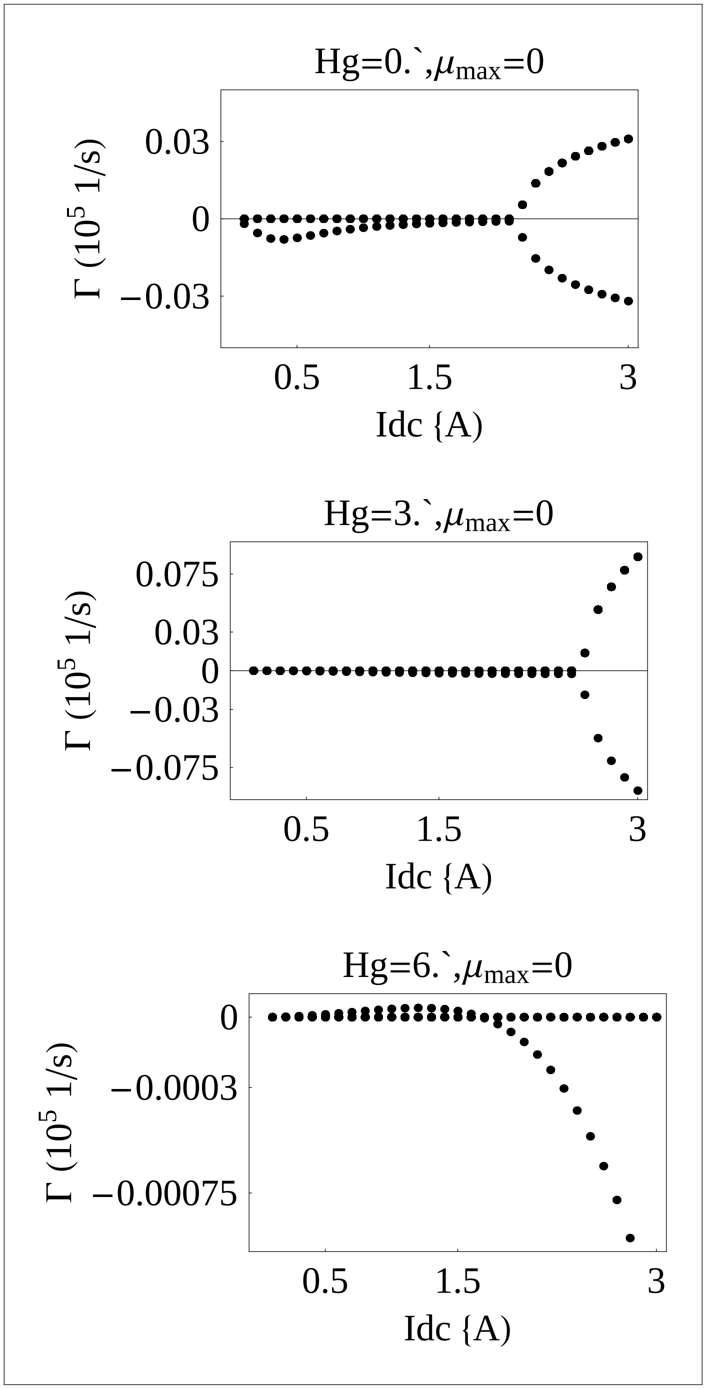

Hg=0.`,$\mu_{\max}$=0
Γ (10^5 1/s)
0.03
0
−0.03
0.5
1.5
3
Idc {A)
Hg=3.`,$\mu_{\max}$=0
Γ (10^5 1/s)
0.075
0.03
0
−0.03
−0.075
0.5
1.5
3
Idc {A)
Hg=6.`,$\mu_{\max}$=0
Γ (10^5 1/s)
0
−0.0003
−0.00075
0.5
1.5
3
Idc {A)

Figure 5:

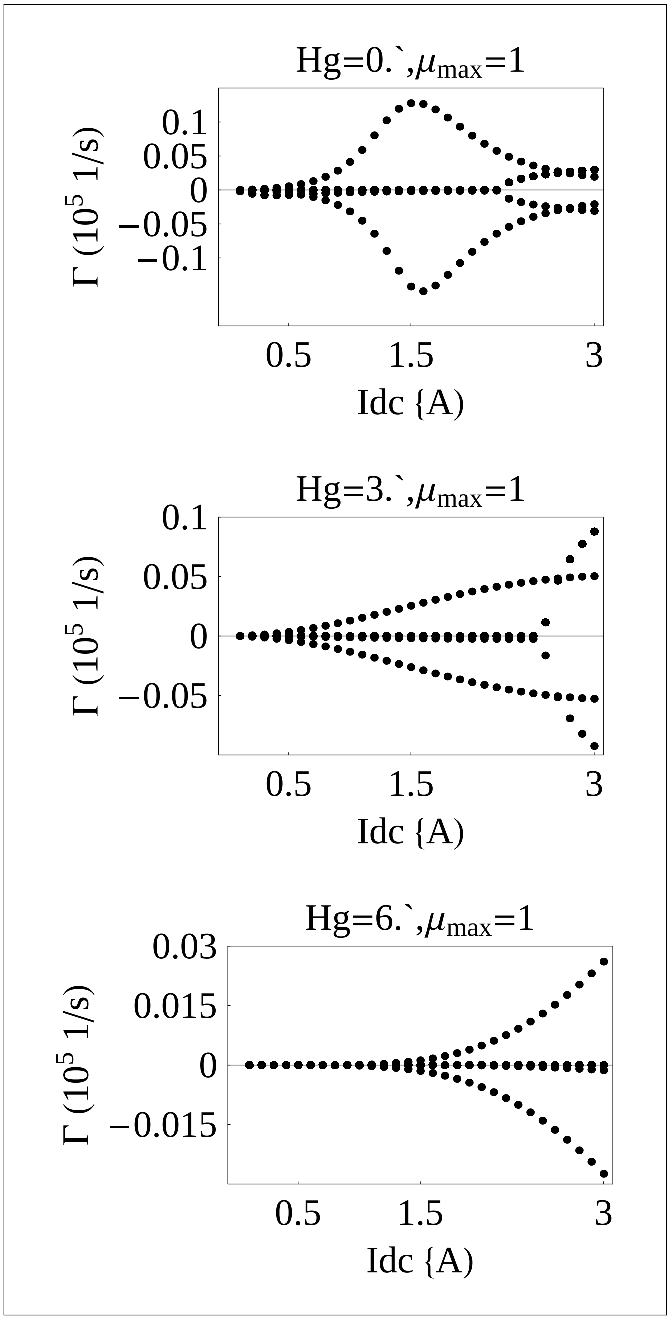
Hg=0.`,μ_{max}=1
0.1
0.05
0
−0.05
−0.1
Γ (10^5 1/s)
0.5
1.5
3
Idc {A}
Hg=3.`,μ_{max}=1
0.1
0.05
0
−0.05
Γ (10^5 1/s)
0.5
1.5
3
Idc {A}
Hg=6.`,μ_{max}=1
0.03
0.015
0
−0.015
Γ (10^5 1/s)
0.5
1.5
3
Idc {A}

Figure 6:

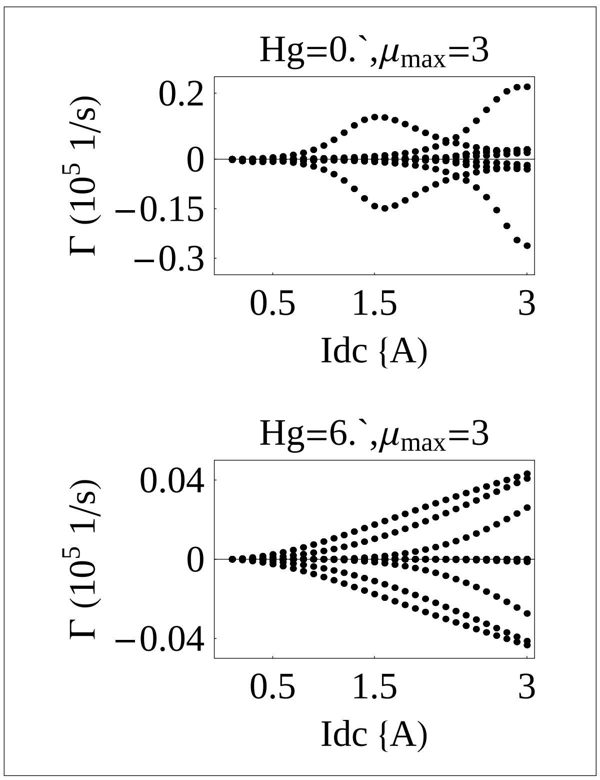
Hg=0.`,μ_{max}=3
0.2
0
−0.15
−0.3
Γ (10^5 1/s)
0.5
1.5
3
Idc {A}
Hg=6.`,μ_{max}=3
0.04
0
−0.04
Γ (10^5 1/s)
0.5
1.5
3
Idc {A}

Figure 7: