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On-Shell Unitarity Bootstrap for QCD Amplitudes

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1. Introduction

Seeking and measuring new physics at the imminent Large Hadron Collider (LHC) will require extensive calculations of high-multiplicity backgrounds in perturbative QCD to next-to-leading order (NLO). The Les Houches 2005 workshop defined a target list, reproduced in table 1, for theorists to attack. In addition to the processes in the table, one would also like to compute processes such as $W, Z + 4$ jets, which are important backgrounds to searches for supersymmetry and other models of new electroweak physics. Such computations require one-loop amplitudes with seven external particles, including the vector boson, as depicted in fig. 1. These are challenging calculations and Feynman-diagrammatic computations have only recently reached six-point amplitudes [1]. (Some of this progress has been described in this conference [2].)

Table 1
The NLO target list. (From ref. [3].)

process ($V \in \{Z, W, \gamma\}$)	relevant for
1. $pp \to V\,V$ jet	$t\bar{t}H$, new physics
2. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
3. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
4. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
5. $pp \to V\,V + 2$ jets	VBF$\to H \to VV$
6. $pp \to V + 3$ jets	various new physics signatures
7. $pp \to V\,V\,V$	SUSY trilepton

The last two decades have produced an increasing collection of explicit expressions for amplitudes in gauge theory. Many of these results are dramatically simpler in their analytic forms than would have been expected based on counting Feynman diagrams and terms therein. This is especially true in the maximally ($\mathcal{N} = 4$) supersymmetric theory, but is also true of amplitudes in QCD, directly relevant to collider experiments.

Many of these results were not obtained using Feynman-diagram techniques, and some (the one-loop all-multiplicity results, in particular) are not accessible to calculations done using these traditional techniques. The traditional approach

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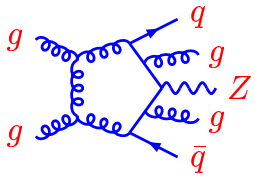

Figure 1. An important background to searches for missing energy signals of supersymmetry and other models of physics beyond the Standard Model is $pp \to Z+$ jets. This calculation involves seven-point one-loop amplitudes.

makes the Lagrangian manifestly symmetric under Lorentz and local gauge symmetries, and hence simple in form. The price we pay is the introduction of many non-physical degrees of freedom. To remove the redundant degrees of freedom in calculations of scattering amplitudes requires fixing a gauge. This makes computations more complicated, because individual diagrams do not preserve gauge invariance, which is recovered only at the end of a long calculation. We end up calculating a lot of unphysical and redundant information which is thrown away at the end. From a practical point of view, however, what matters more than a simple Lagrangian is simplicity and efficiency of calculation, where we calculate no more than what is really needed for the result. What we want is a calculational formalism that involves *only* (perturbative) physical states. Light-cone gauge is a first step towards this goal, as it removes the unphysical helicities propagating inside diagrams. Like all diagrammatic formalisms, however, it still involves off-shell states; in a gauge theory, off-shell formulations are inherently non-gauge invariant.

What we are seeking is a way of doing field theory in terms of gauge-invariant, on-shell states. The possibility of doing this flies in the face of existing graduate education and a great deal of lore. Nonetheless, for massless theories we now understand how to do this explicitly for tree and one-loop amplitudes; and there is every reason to believe the procedure will work to all orders in perturbation theory. A theme that runs through the technologies underlying on-shell methods is the transformation of general properties of amplitudes into practical tools for computing them.

There are three basic technologies that underlie the on-shell approach to calculations in gauge theories. The first is the spinor helicity method [4], which gives efficient representations for the physical content of external states. The second to be developed was the unitarity method [5] for obtaining loop amplitudes with multiple kinematic variables. This method makes use of the basic unitarity property of field theory to provide a systematic procedure for constructing amplitudes with any number of kinematic invariants.

With the four-dimensional version of the unitarity method one can compute complete one-loop amplitudes in supersymmetric theories, and all terms containing branch cuts in QCD or supersymmetric theories to all loop orders. Curiously, the on-shell method for tree amplitudes, which is the third technology underlying the on-shell bootstrap approach, was developed well after the on-shell technique for loop amplitudes. The tree-level technique had to await new inspiration from the twistor-space picture for amplitudes [6]. The approach at tree level makes use of factorization, another basic property of field-theoretic amplitudes, to obtain on-shell recursion relations [7,8]. The most recent development, on which we report here, also gives a method for computing the rational parts of gauge-theory loop amplitudes, that is precisely those terms not accessible to four-dimensional unitarity (but requiring the computationally more awkward full D-dimensional unitarity).

A practical method should exhibit only modest growth in its complexity as the number of external legs increases. It should lead to numerically stable results, allowing for straightforward numerical integration over all experimentally-accessible phase space. Conventional methods have neither of these features; they tend to have numerical stability issues due to large numbers of high degree spurious singularities. (Solutions are being explored, as reported at the conference [1,2].) The on-shell bootstrap that we describe in this report does; and it is applicable to a wide class of processes in a manner that likely

allows for automation.

2. Hints from Twistor Space

The recent suggestion, that a twistor-space string theory is dual to the maximally supersymmetric gauge theory [6], points at additional structure in scattering amplitudes. It meshes nicely with the techniques we describe and may in the future offer a new formal framework for these developments. An important inspiration for the recent advances comes from the surprising simplicity amplitudes exhibit in twistor space.

In QCD color-ordering [9] and spinor helicity [4] are widely used at tree and loop level to provide simplified descriptions of amplitudes. QCD amplitudes can be expressed entirely in terms of spinors by representing gluon polarizations in the spinor helicity basis,

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\langle q\, k\rangle},$$
$$\varepsilon_\mu^-(k;q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}[k\, q]},$$

where

$$\epsilon_{ab}\lambda_j^a\lambda_l^b = \langle j\, l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l),$$
$$\epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_l^{\dot{a}}\tilde{\lambda}_j^{\dot{b}} = [j\, l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l),$$

and $u_\pm(k)$ is a massless Weyl spinor with momentum k and positive or negative chirality respectively. Lorentz inner products of momenta can also be expressed in terms of spinors via

$$s_{ij} = 2k_j\cdot k_l = \langle j\, l\rangle[l\, j].$$

A twistor-space description arises from performing an asymmetric Fourier transform, one with respect to negative- (but *not* positive-) helicity spinors,

$$\tilde{A}(\lambda_i,\mu_i) = \int\prod_i \frac{d^2\tilde{\lambda}_i}{(2\pi)^2}\exp\Big(i\sum_i \mu_i^{\dot{a}}\tilde{\lambda}_{i\dot{a}}\Big)A(\lambda_i,\tilde{\lambda}_i),$$

where the λ_i are positive-helicity spinors and μ_i are the conjugate variables to the negative-helicity ones.

Witten conjectured that in twistor space, gauge-theory amplitudes have delta-function support on curves of degree

$$d = q - 1 + L,$$

where q is the number of negative-helicity legs and L the number of loops.

Surprisingly, there are multiple descriptions of the amplitudes in terms of non-degenerate and degenerate curves [10,11], two of which are displayed in fig. 2. The degenerate description in terms of intersecting degree-one curves (straight 'lines') has been the most useful for practical calculations. This description led to the MHV rules [10] of Cachazo, Svrček and Witten. They effectively compute amplitudes in terms of a sum over all multi-particle factorizations. The rules provided a concrete demonstration that scattering amplitudes have a simple underlying structure not understood previously.

Does this simplicity underlie loop amplitudes as well? We have ample evidence that it does. In particular, from quadruple cuts one can demonstrate that in twistor space the coefficients of all box integrals in any massless gauge theory are have delta-function support on intersecting lines forming a closed loop [12,13]. This corresponds to the possibility of computing the coefficients of any box integral in a four-dimensional theory from a product of four tree amplitudes, by solving four on-shell constraints [14]. The recent computation of n-point one-loop QCD amplitudes in refs. [15–17] also points at the presence of simple underlying structures.

Witten also made a simple observation that has proven of great practical utility. Ordinarily, one works with real momenta, in which case three-point amplitudes vanish identically. This results from the vanishing of all momentum invariants s_{ij}, which in turn forces the vanishing of all spinor products $\langle i\, j\rangle$ and $[i\, j]$. If one were to use complex momenta, however, the vanishing of momentum invariants only requires one flavor of spinor product to vanish, either $\langle i\, j\rangle$ or $[i\, j]$. The other spinor product can be used to define a non-vanishing three-point amplitude. All other amplitudes can be built out of this basic amplitude.

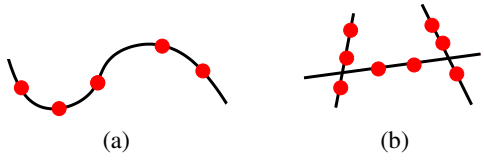

Figure 2. In twistor space amplitudes have delta-function support on algebraic curves. The dots represent the external points. The curve (a) represents a non-degenerate cubic curve. In (b) the curve degenerates to intersecting straight lines.

3. Unitarity Method

Our approach is based on unitarity, which has been a fundamental concept in quantum field theory since since its inception [18]. In the 60's most attempts to describe the strong interactions relied on the unitarity and analyticity of the S-matrix. But with the advent of QCD in the 70's as the description of the strong interactions, Feynman diagrams became the primary tool for describing scattering at large transverse momentum.

Although an approach based on unitarity offers advantages because one can avoid using unphysical (off-shell) states, a number of difficulties prevented its use as a practical tool. The primary difficulty was the inability to obtain amplitudes depending on more than two kinematic amplitudes via multiple dispersion relations. Other difficulties include technical issues in applying unitarity to massless theories, as well as non-convergence of dispersion relations, which require subtractions for well-defined results. A resolution of these difficulties occurred with the advent of the 'unitarity method' [5].

In the unitarity method one systematically constructs amplitudes by merging the various unitarity cuts as exemplified in fig. 3 into Feynman-like integrals which give the correct cuts in all channels. In this approach both dispersive and absorptive parts are obtained simultaneously, bypassing the need for dispersion integrals. Furthermore, by making appropriate use [19] of dimensional regularization within the method one can easily avoid many of the earlier technical complications.

The unitarity approach has proven to be a powerful method for determining amplitudes, especially in supersymmetric theories where complete one-loop amplitudes may be obtained using only tree-level four-dimensional helicity amplitudes as input [5]. In non-supersymmetric theories complete amplitudes may be obtained using unitarity in D dimensions. The required tree amplitudes in the latter case are more complicated, so only a limited number of computations have been performed with this approach [20]. After performing a series expansion in $\epsilon = (4-D)/2$, the difference between using four-dimensional or D-dimensional states and momenta in the cuts gives rise to rational functions of spinor invariants.

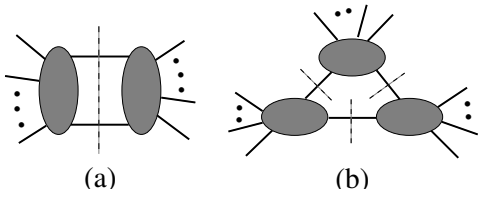

Figure 3. A two-particle cut (a) as well as a generalized triple cut (b).

Because the scalar bubble, triangle and box integrals form a complete basis of cut-containing functions for dimensionally-regularized one-loop amplitudes in a four-dimensional theory [21], all a computation needs to determine are the coefficients of these integrals. Recent improvements to the four-dimensional unitarity method [14] use generalized unitarity [18,22,23] to allow for a direct algebraic determination of box integral coefficients. The efficiency of extracting the coefficients of bubble and triangle integrals has also been improved recently [24], and has been applied to six-gluon amplitudes. Combined with the previously-computed results for the cut-containing pieces [5, 25], these results give a complete analytic solution for the cut-containing terms in all six-gluon amplitudes.

4. On-Shell Recursion Relations

This leaves us to compute the rational terms efficiently. We again use analytic properties, but instead of branch cuts we use the poles, in the guise of on-shell recursion relations. Our focus here will be on obtaining results with large numbers of external legs. Our interest in constructing all-multiplicity amplitudes stems partly from the desire to study the growth in complexity of the amplitudes as the number of external partons increases. Because the explosive growth in complexity with each additional leg has been a stumbling block in previous methods, it is important to understand this behavior with any new method. Furthermore, experience has shown that analytic all-n expressions provide a wealth of information about the general structure of scattering amplitudes.

The on-shell unitarity bootstrap [15,17,26] has its origins in an early approach taken to compute the $Z \to 4$ parton one-loop matrix elements [22] (or equivalently, by crossing, for $pp \to W, Z +$ 2 jets). In this more primitive version of the on-shell bootstrap approach, the cut-containing (poly)logarithmic terms were obtained using the unitarity method while purely rational terms were obtained using on-shell factorization properties, writing down an ansatz and constraining its form. It proved difficult to turn the approach into a general and systematic one. On-shell recursion for the rational terms provides such a general and systematic method.

In special cases, when certain criteria are satisfied by the cuts, one may even use on-shell recursion to obtain the cut-containing terms of amplitudes [27]. That is, one may use the kinematic poles appearing in the coefficients of integral functions to construct them. This technique was used to obtain the cut-containing parts of all one-loop n-gluon amplitudes with the helicities arranged in a 'split helicity' configuration.

Very recently, Xiao, Yang and Zhu have presented a different method for obtaining rational function terms by applying spinor simplifications together with integrations that target only the rational terms [28]. They have used this to obtain all the rational terms in the one-loop six-gluon amplitudes.

4.1. Tree-level Recursion Relations

On-shell recursion relations have a curious history which did not foreshadow their widespread applicability. Motivated by Witten's conjecture that $\mathcal{N} = 4$ super-Yang-Mills gauge theory amplitudes should have a simple structure in twistor space, and by Brandhuber, Spence, and Travaglini's observation [29] that this simplicity indeed held beyond tree-level, for the simplest maximally-helicity-violating class of one-loop amplitudes, Del Duca and several of the authors computed the seven-point next-to-maximally-helicity violating (NMHV) amplitudes [30], one of which was also computed by Britto, Cachazo and Feng [31]. These amplitudes have three negative helicities, and were expected to lie on a genus-one, degree-three curve. Subsequently, amplitudes with three negative helicities and an arbitrary number of positive-helicity gluons were computed [13,14].

The compact forms of seven- and higher-point tree amplitudes [30,13] that emerged from studying one-loop infrared singularity consistency equations, together with the observations that one-loop $\mathcal{N} = 4$ super-Yang-Mills amplitudes are composed solely of box integrals [5] whose coefficients may be algebraically determined from products of tree amplitudes [14], led Roiban, Spradlin and Volovich to suggest [32] the existence of tree-level on-shell recursion relations. These recursion relations were constructed explicitly by Britto, Cachazo and Feng [7].

Because of the indirect way in which on-shell recursion relations were obtained, at first it was not clear how widespread their applicability could be. However, a simple proof of the tree-level recursion relations by Britto, Cachazo, Feng and Witten [8] followed, based on general factorization properties of tree-level amplitudes as well as elementary complex variable theory. The remarkable generality and simplicity of the proof allowed widespread application [33], including to theories with massive particles [34,35] and gravity [36].

The proof of the tree-level relations employs a parameter-dependent shift of two of the external

momenta

$$k_j^\mu \;\to\; k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle,$$
$$k_l^\mu \;\to\; k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle\,, \qquad (1)$$

where z is a complex parameter. Under this shift, the momenta remain massless, $k_j^2(z) = k_l^2(z) = 0$, and overall momentum conservation is maintained. The z dependence of the momenta makes the on-shell amplitude, $A(z)$, z-dependent as well.

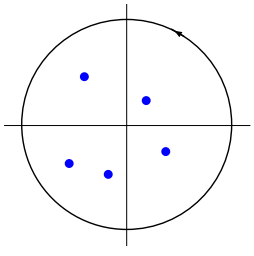

Figure 4. The contour at infinity used for deriving tree-level recursion relations. The dots represent poles in $A(z)$.

At tree level, the amplitude, $A(z)$, is a meromorphic rational function of z, so we may exploit Cauchy's theorem to construct it from its residues. Assuming $A(z) \to 0$ as $z \to \infty$, the contour integral around the circle at infinity, depicted in fig. 4, must vanish,

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,A(z) = 0\,.$$

Using Cauchy's theorem we may evaluate the integral as a sum of residues which allows us to solve for the physical amplitude $A(0)$ in terms of residues on each pole,

$$A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}\,.$$

At tree level, there are many shifts for which $A(z)$ vanishes as $z \to \infty$ [7,8,34].

Each residue comes from factorization in a shifted momentum. Summing over all residues gives us the tree-level on-shell recursion relation,

$$A(0) = \sum_P \sum_h \frac{A_L^h(z_P)\,A_R^{-h}(z_P)}{K_P^2}\,, \qquad (2)$$

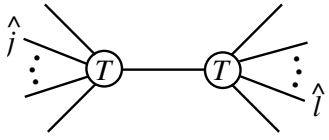

Figure 5. The recursive diagrams at tree level. The 'T' on each vertex signifies an on-shell tree amplitude.

where P is the set of ordered partitions of the legs, separating the two shifted legs j and l, and h is the helicity of the intermediate state. The lower-point amplitudes $A_L^h(z_P)$ and $A_R^{-h}(z_P)$ are shifted but with z frozen at the pole,

$$z_P = \frac{K_P^2}{\langle j^-|\,\not{K}_P\,|l^-\rangle}\,.$$

As illustrated in fig. 5, each of the terms in eq. (2) may be given a diagrammatic interpretation, where the vertices represent lower-point on-shell tree amplitudes. In them, we must consider states carrying complex four-momentum, but otherwise on-shell; transversality conditions as well as overall four-momentum conservation remain unchanged.

4.2. Loop-Level Recursion Relations

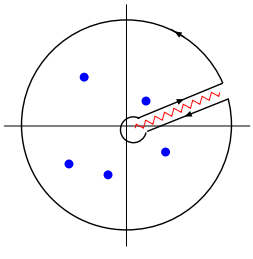

Figure 6. A schematic of the contour used for deriving one-loop recursion relations.

At loop level, we face several issues in constructing such recursion relations. The most obvious one is the appearance of branch cuts, so in

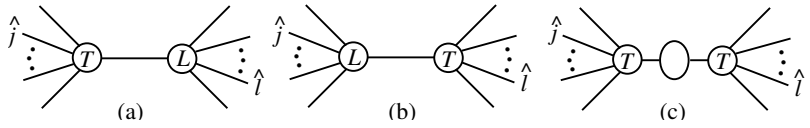

Figure 7. The recursive diagrams at one loop. A 'T' signifies a tree amplitude and an 'L' a loop amplitude.

addition to the contour used to derive the tree-level recursion, we will need contours of the form shown in fig. 6. We must also deal with spurious singularities, and in some cases, the non-standard nature of factorization in complex momenta (differing from 'ordinary' factorization in real momenta). In non-standard factorization channels (always two-particle ones with like-helicity gluons), double poles and 'unreal' poles not present with real momenta may appear [37,38]. It is best, and fortunately possible, to avoid these channels in constructing recursion relations.

This avoidance comes at a price: in general, when choosing shifts to avoid non-standard factorizations, the shifted amplitude $A(z)$ may not vanish as $z \to \infty$. The contour integration in fig. 6 makes it clear that additional 'boundary' contributions arise in this case. The approach taken in ref. [17], is to allow for such contributions, and to determine them using an auxiliary shift and recursion relation. Choices for shift momenta with the required properties may be found in ref. [17].

After applying the shift (1), a loop amplitude is of the schematic form,

$$A(z) \;=\; \sum \text{polylog terms} + \sum_b \frac{\text{Res}_i}{z - z_i} + \sum_i a_i z^i \,.$$

Our approach to determining this function uses the four-dimensional unitarity method to obtain the polylogarithmic and logarithmic terms, on-shell recursion to determine the residues and an auxiliary recursion relation (when needed) using a different shift to obtain the coefficients a_i.

In the on-shell bootstrap one first computes the cut-containing terms. These will usually contain unphysical spurious singularities that cancel against the rational functions. For reasons of numerical stability it is useful to absorb most of these into functions that are free of these singularities. We can construct such functions by adding appropriate rational functions to the polylogarithmic terms. For example, we may complete

$$\frac{\ln(r)}{(1-r)^2} \to \frac{\ln(r)}{(1-r)^2} + \frac{1}{1-r}\,,$$

so that it is singularity-free as $r \to 1$ (r is a ratio of kinematic invariants). The use of a 'cut completion' also aids the construction of an on-shell recursion, since we do not need to compute residues at these unphysical poles.

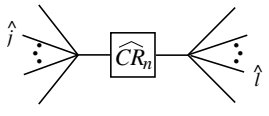

Figure 8. The overlap diagrams corresponding the to different physical channels.

The recursive contributions, illustrated in fig. 7, are similar to tree-level ones, except that they involve loop vertices created from loop amplitudes by setting all cut-containing terms to zero. In addition, we have 'overlap contributions' coming from the appearance of physical poles in the completed-cut terms. We need to subtract off this overlap; we simply perform the shift (1) and extract the residues of the poles in all physical channels. The correspondence of these contributions to physical channels again allows us to give them a diagrammatic interpretation, as illustrated in fig. 8. When the amplitude does not vanish at large values of the shift parameter, the extra contributions may be computed using an auxiliary recursion relation as described in ref. [17].

5. Results

The bootstrap approach has already been used to obtain the rational terms in a variety of new amplitudes:

1. The finite two-quark $(n-2)$-gluon amplitudes, with gluons all of identical helicity [38].
2. All one-loop corrections to MHV n-gluon amplitudes [15,16,26].
3. All one-loop n-gluon amplitudes with three color-adjacent negative-helicity gluons and the rest of positive helicity [17].

A key feature of our construction of these amplitudes is the moderate increase in computational complexity as the number of external legs increases, in contrast to the explosive growth encountered with more traditional methods.

As one example, the rational parts of the six-gluon QCD amplitude with three color-adjacent negative-helicity gluons may be expressed as,

$$A_{6;1}^{QCD}(1^-,2^-,3^-,4^+,5^+,6^+) = c_\Gamma \left[\widehat{C}_6 + \widehat{R}_6\right].$$

where c_Γ is the constant prefactor that appears in all one-loop amplitudes, $\widehat{C}_6$ is the completed cut, which may be obtained using the four-dimensional unitarity method, and $\widehat{R}_6$ contains all the remaining rational terms that we are interested in obtaining. In ref. [17] these terms were obtained following the methods outlined above. They are given by a remarkably compact formula,

$$\widehat{R}_6 = \widehat{R}_6^a + \widehat{R}_6^a\Big|_{\text{flip 1}},$$

where flip 1 is the flip operation,

$$X(1,2,3,4,5,6)\Big|_{\text{flip 1}} \equiv X(3,2,1,6,5,4)\,, \text{ and}$$

$$\begin{aligned}\widehat{R}_6^a \;=\; & \frac{i}{6}\frac{1}{[2\,3]\langle 5\,6\rangle\,\langle 5^-|\,(3+4)\,|2^-\rangle}\Bigg\{ \\ & - \frac{[4\,6]^3[2\,5]\langle 5\,6\rangle}{[1\,2][3\,4][6\,1]} - \frac{\langle 1\,3\rangle^3\langle 2\,5\rangle[2\,3]}{\langle 3\,4\rangle\langle 4\,5\rangle\langle 6\,1\rangle} \\ & + \frac{\langle 1^-|\,(2+3)\,|4^-\rangle^2}{[3\,4]\langle 6\,1\rangle} \\ & \times \left(\frac{\langle 1^-|\,(2-5)\,|4^-\rangle}{s_{234}} + \frac{\langle 1\,3\rangle}{\langle 3\,4\rangle} - \frac{[4\,6]}{[6\,1]}\right) \\ & - \frac{\langle 1\,3\rangle^2(3\,\langle 1^-|\,2\,|4^-\rangle + \langle 1^-|\,3\,|4^-\rangle)}{\langle 3\,4\rangle\langle 6\,1\rangle} \\ & + \frac{[4\,6]^2(3\,\langle 1^-|\,5\,|4^-\rangle + \langle 1^-|\,6\,|4^-\rangle)}{[3\,4][6\,1]}\Bigg\}.\end{aligned}$$

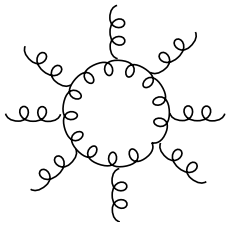

Figure 9. One of the three million Feynman diagrams for describing eight-gluon scattering

In addition to providing analytic expressions, refs. [17,26] also record numerical values at sample kinematic points for up to eight external gluons. A brute-force computation of eight-gluon amplitudes using standard Feynman diagrams would require over three million diagrams, one of which is displayed in fig. 5. Because of the relatively compact nature of the analytic expressions for the final amplitudes, the numerical evaluation of these amplitudes is fast. We do not anticipate any significant complications arising from round-off error in numerical evaluation, because of the mild degree of spurious singularities appearing in the amplitudes.

6. Outlook

The developments described above open several directions that would be interesting to pursue. For massive particles inside loops, suitable extensions should be possible but remain to be developed. It would be helpful to have a first-principles derivation of the complex factorization properties, as well as of the behavior of loop amplitudes at large values of the shift parameter. In this regard, recent papers [39] linking tree-level on-shell recursion with gauge-theory Lagrangians

in particular gauges may prove useful. The unitarity method with D-dimensional cuts [20] may also be useful for developing a formal derivation of properties of gauge-theory amplitudes.

The on-shell bootstrap approach we have described here has already established a track record in providing new one-loop amplitudes. The techniques we have presented in these talks are systematic and thus should lend themselves to automation, which will be helpful for dealing with large numbers of subprocesses. The method should carry over to amplitudes with external vector bosons or Higgs particles, as well as quarks. It offers a promising approach for attacking the processes needed for LHC physics, and we expect that it will see widespread application towards that goal.

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