

# Black Hole Entropy, Marginal Stability and Mirror Symmetry

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## Abstract

We consider the superconformal quantum mechanics associated to BPS black holes in type IIB Calabi–Yau compactifications. This quantum mechanics describes the dynamics of D-branes in the near-horizon attractor geometry of the black hole. In many cases, the black hole entropy can be found by counting the number of chiral primaries in this quantum mechanics. Both the attractor mechanism and notions of marginal stability play important roles in generating the large number of microstates required to explain this entropy. We compute the microscopic entropy explicitly in a few different cases, where the theory reduces to quantum mechanics on the moduli space of special Lagrangians. Under certain assumptions, the problem may be solved by implementing mirror symmetry as three T-dualities: this is essentially the mirror of a calculation by Gaiotto, Strominger and Yin. In some simple cases, the calculation may be done in greater generality without resorting to conjectures about mirror symmetry. For example, the  $K3 \times T^2$  case may be studied precisely using the Fourier-Mukai transform.

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# 1 Introduction

The study of quantum mechanical black holes continues to provide new insights into the basic structure of string theory and quantum gravity. In this paper we will focus on supersymmetric black holes arising in Calabi–Yau compactifications of type II string theory, which are composed of BPS D-branes wrapping various cycles in the Calabi–Yau.

The most straightforward computations of black hole entropy involve a direct enumeration of supersymmetric ground states of particular D-brane system [1]. In general this computation is quite difficult, but considerable progress has been made in a variety of special cases, often by using duality, as in [2]. Recently, however, an alternative approach to black hole entropy has been proposed [3, 4]. Rather than study the complete brane configuration, one instead investigates the near horizon quantum mechanics of a collection of D-branes moving in the supergravity background geometry sourced by the remainder of the branes comprising the black hole. By separating the system into “probe” and “background” branes in this way, many of the mathematical problems become considerably more tractable. One can then, at least in some cases, reproduce entropy by enumerating the supersymmetric ground states of the near horizon probe quantum mechanics.

Actually, as we will argue below, this picture of “probe” and “background” is essentially unavoidable once one realizes that the attractor mechanism forces BPS black holes to be marginally stable.

In this paper we will adopt the strategy of [3, 4]. These papers focused on type IIA, which contains a variety of BPS states: the B-branes, which — very roughly speaking — correspond to holomorphic sub-manifolds of the Calabi–Yau. The authors of [3, 4] successfully reproduced the entropy of black holes whose charge is comprised mainly of 0-branes. In doing so, they used the fact that the moduli space of a probe 0-brane on the Calabi–Yau is just the Calabi–Yau itself. So the near horizon theory of a collection of probe 0-branes reduces to a non-linear sigma model whose target space is the Calabi–Yau. The general case — which involves the quantum mechanics of various higher dimensional branes — remains unsolved, although some progress has been made [5]. In this paper we will focus on type IIB, where we are presented with only one type of BPS state: the A-branes, which wrap special Lagrangian 3-cycles of the Calabi–Yau. One might therefore hope that a IIB description would allow us to compute in one fell swoop the black hole entropy for all possible BPS configurations. In this paper we will report only partial progress towards this ambitious goal.

To start, we must first understand the quantum mechanics on the moduli space of A-branes. The analysis of the moduli space is somewhat subtle. This is because as the Calabi–Yau moduli are varied, it is possible for these D-branes to decay and form bound states with other D-branes. In fact, the attractor mechanism forces the moduli to take on special values which happen to make the D-brane marginally stable against a large number of decays. This leads one to consider the moduli space of decay products. In general, one can then enumerate the quantum states of a D-brane black hole by the following procedure

1. Enumerate the possible marginal decay products.
2. Enumerate the marginal bindings “at threshold” of these products.

3. Calculate the cohomology of the moduli space of the resulting objects.

In terms of string coupling, this first step is classical while the second and third steps are quantum mechanical. For the first step one may use concepts such as  $\Pi$ -stability or the existence of special Lagrangians. The second step involves the use of the Myers effect [6], following [3, 4].

Most of this paper will focus on the third step, where we compute cohomology on the moduli space of special Lagrangian 3-brane probes. The crucial point is that the RR charge of a black hole background interacts with the D-brane probe, effectively producing a “magnetic field” on the moduli space of the D-branes probe. That is, the cohomology computation is bundle-valued for some  $U(1)$ -bundle. We will argue that  $c_1$  of this bundle is proportional to the Kähler form on moduli space, and that moreover this Kähler form is fixed by the attractor mechanism. This is very similar in flavor to the computation of [4]. These authors focused on D0-D4 systems but we will attempt, with modest success, to be more general.

The primary problem is then to determine the structure of the A-brane moduli space. In general this is quite complicated, but in some cases one can use mirror symmetry to turn this back into a problem in type IIA. This relies on the conjecture of [7] that any Calabi–Yau manifold can be written as a (possibly degenerate)  $T^3$  fibration. If we consider black holes whose charge comes mostly from D3 branes wrapping this  $T^3$  fiber, then we may use the techniques of [7] to study this moduli space. This argument reproduces the correct entropy, and is essentially the mirror of the computation of [4].

This argument suffers from the same problems as the original SYZ construction. In particular, any proper argument must account for the degeneracies of the  $T^3$  fibration. The remainder of the paper describes the more precise  $K3 \times T^2$  construction, which uses the Fourier-Mukai transform to implement mirror symmetry.

An outline of this paper is as follows. In section 2 we discuss attractive Calabi–Yau threefolds, and demonstrate that they have special properties when it comes to D-brane decay. In section 3 we analyze the quantum mechanics of D-branes wrapping special Lagrangian cycles. In section 4 we perform an entropy calculation by implementing mirror symmetry as T-duality. As an illustration of this technique, we describe the special case of type IIB on  $T^6$ . In section 5 we describe an analogous computation for compactifications on  $K3 \times T^2$ . In this case we can do the computation exactly, without resorting to conjectures about the action of mirror symmetry. We end with a few concluding remarks.

## 2 Black Hole Attractors and D-Brane Decay

We start by reviewing the black hole attractor mechanism in type IIB, before describing its relation to marginal stability and D-brane decay.

### 2.1 Review: IIB Attractors

Consider type IIB string theory compactified on a Calabi–Yau 3-fold  $Y$ . In the perturbative string description, a supersymmetric state of this theory is given by a D3 brane wrapping a

special Lagrangian 3-cycle of  $Y$ . Such a wrapped D3 brane looks like a charged point-like object in four dimensions, whose charge depends on the choice of 3-cycle. We will denote by  $F^{(3)}$  the element of  $H^3(Y, \mathbb{Z})$  characterizing this charge:  $F^{(3)}$  is Poincaré dual to the homology cycle of the D3 brane.

We can also describe this supersymmetric object as a charged BPS black hole solution of  $\mathcal{N} = 2$  supergravity in four dimensions [8–10]. This solution exhibits a curious feature known as the attractor mechanism: at the horizon of the black hole the vector multiplet moduli approach fixed values, which are determined only by the charge  $F^{(3)}$  and not by the asymptotic values of the moduli. In type IIB, these moduli describe complex structure deformations of  $Y$ . At the horizon, the moduli are fixed by the condition that  $F^{(3)}$  lies in  $H^{3,0}(Y) \oplus H^{0,3}(Y)$  [11]. The holomorphic three form  $\Omega$  on  $Y$  is a basis element of  $H^{3,0}(Y)$ , so this condition can be written as

$$F^{(3)} = \text{Im}(C\Omega) \tag{1}$$

for some complex constant  $C$ . By introducing a symplectic basis for  $H^3(Y, \mathbb{Z})$ , one can write this as an equation for the periods of the holomorphic 3-form. However, (1) is sufficient for our purposes.

This attractor equation (1) is an equation for  $2h_{2,1} + 2$  unknowns — the  $2h_{2,1}$  complex structure moduli and the complex constant  $C$  — in terms of the  $b_3(Y) = 2h_{2,1} + 2$  charges. So it is natural to expect that solutions of (1) are isolated as points in moduli space. However, it has been shown that although the solutions are isolated they are not always unique [11].

To describe the structure of these solutions further, note that  $\mathcal{N} = 2$  supergravity contains a gauge boson in the gravity multiplet — the graviphoton — in addition to gauge bosons in vector multiplets. The charge measured by the graviphoton plays a special role in the solution, since it is the central charge appearing in the supersymmetry algebra. It is this charge that appears in all BPS-type relations describing the black hole solutions. In terms of  $F^{(3)}$ , this charge is

$$Z = ie^{K/2} \int \Omega \wedge F^{(3)}, \tag{2}$$

where

$$e^{-K} = i \int \Omega \wedge \bar{\Omega} \tag{3}$$

is the Kähler potential on the vector multiplet moduli space. The constant appearing in (1) can be fixed by wedging both sides with  $\Omega$  and integrating over  $Y$ . It is  $C = 2\bar{Z}e^{K/2}$ .

The near horizon geometry of the black hole is  $AdS_2 \times S^2 \times Y$ , where the geometry of  $Y$  is constrained by (1). The  $AdS_2$  and  $S^2$  factors both have radius  $|Z|$  in four dimensional Planck units. So the Bekenstein-Hawking entropy, which is proportional to the area of the  $S^2$ , is

$$S = \pi|Z|^2. \tag{4}$$

In this formula the central charge  $Z$  is evaluated at the attractor fixed point (1).

In addition, the D3 brane sources a 5-form field strength, which at the horizon is

$$F^{(5)} = \omega_{AdS_2} \wedge F^{(3)} + \omega_{S^2} \wedge \star_6 F^{(3)}. \tag{5}$$

Here  $\omega_{AdS_2}$  and  $\omega_{S^2}$  are volume forms on  $AdS_2$  and  $S^2$ , and  $\star_6$  is the Hodge star on  $Y$ .

The attractor solutions we have just described are valid only when certain conditions are met. First, in order for the supergravity approximation to be good the characteristic length scale of  $AdS_2 \times S^2$  must be large. That is, the area of the event horizon must be large compared to the string scale. So we must take  $|Z| \gg 1$ , which can be accomplished, for example, by taking all of the charges to be large.

The second condition is slightly more subtle. We are assuming the degrees of freedom observed are that of a compactification on  $Y$ . That is, we are ignoring any massive excitations of the Calabi–Yau threefold (although in principle these may be accounted for in the context of the attractor mechanism, see e.g. [12]). This requires all the characteristic sizes of  $Y$  to be small compared to the size of  $AdS_2 \times S^2$ . Normally one thinks of “size” as being associated with the complexified Kähler form  $B + iJ$  of a threefold, while the deformations of complex structure are associated purely to the “shape”. This is a little naïve, however, as we now discuss.

Let  $X$  be mirror to  $Y$  and consider deformations of the complex structure of  $Y$  and the mirror deformations of  $B + iJ$  of  $X$ . There is a partition function of string states associated to these spaces which will vary with the moduli and respect mirror symmetry. If a characteristic length in the Calabi–Yau gets large one would expect the partition function to contain light states. The areas of holomorphic curves in  $X$  are determined by  $B + iJ$  and are insensitive to deformations of complex structure. Thus, if  $B + iJ$  has any large component one would expect the appearance of light states irrespective of the complex structure.

Mirror to this statement, one expects that there must be complex structures for which  $Y$  exhibits light states irrespective of the Kähler form. This seems counterintuitive at first, as one can rescale the metric on  $Y$  (which is a deformation of the Kähler form) to make all lengths small and thus remove any light (non-massless) states. However, this argument is too classical. If  $Y$  is at “large complex structure” the characteristic length scales within  $Y$  will differ wildly. The canonical example is that of a  $T^2$  with one very long and one very short 1-cycle. If we try to shrink the metric to shorten the longer scales, we will “run out of moduli space” before the offending light modes can be brought under control. That is, the shorter lengths would be made so short that they would violate any “minimum distance” constraint as in [13].

So, our second constraint is that  $Y$  should not have large  $B + iJ$  (with respect to the horizon area) and that it should not be mirror to space with large  $B + iJ$ . The first of these conditions is not relevant to black hole solutions, as the  $B + iJ$  moduli are contained in hypermultiplets which are constants for these solutions. So we are free to choose their values to be whatever we like. The second condition is a constraint on complex structure moduli, and limits the charges that we may consider. In particular, as we will see later in sections 4 and 5, it will force us to take certain *ratios* of charges to be large. This constraint is easiest to understand in the mirror IIA language, where it is just the requirement that volumes of two cycles on  $X$  must be small compared to the characteristic length scale of the four dimensional black hole geometry.

## 2.2 Marginal Stability at Attractor Points

A central concept when discussing entropy is the notion of the moduli space of a D-brane. When constructing a moduli space, the notion of *stability* is very important.

In general, many moduli space constructions run as follows: one looks for the moduli space of some object by constructing the moduli space of more easily defined objects which satisfy an appropriate stability criterion. For example, the moduli space of vector bundles with an Hermitian–Yang–Mills connection is studied by starting with holomorphic vector bundles and imposing  $\mu$ -stability. In discussions of this form, a special case must always be made for the marginally stable object. A marginally stable object must be viewed as a “direct-sum” of its constituents in order to obtain the correct moduli space. There might be other configurations of the constituents which are not equivalent to a direct sum, but they are considered to be “S-equivalent” to the direct sum and are not counted as different states. We refer to [14, 15], for example, for a discussion of S-equivalence. The important point is that we need to know if a D-brane is unstable, marginally stable, or truly stable in order to compute the moduli space correctly.

The attractor behavior described above has several striking consequences, which we will exploit in our computation of black hole entropy. The most crucial of these involves D-brane decay and marginal stability, as we will now describe. We will continue to work with type IIB on a Calabi–Yau  $Y$ . For other discussions of special Lagrangians at attractor points, see [16, 17].

To understand D-brane decay, consider what happens to a supersymmetric D3 brane as one varies the complex structure of  $Y$ . This brane wraps a special Lagrangian submanifold, which is defined as a Lagrangian submanifold  $L \subset Y$  with

$$dV_L = R \exp(-i\pi\xi)\Omega|_L. \quad (6)$$

Here  $dV_L$  is the volume form on  $L$ ,  $\Omega$  is the holomorphic 3-form on  $Y$  and  $R$  and  $\xi$  are real numbers. The phase  $\xi$  of the special Lagrangian is constant over  $L$ . Since  $\Omega$  is defined only up to multiplication by an overall constant, one can set  $\xi = 0$  for a given brane. However, we will need to compare values of  $\xi$  for different branes, so will leave  $\xi$  unfixed. Two D3 branes, wrapping different special Lagrangians, are mutually BPS only if their respective values of  $\xi$  are equal.

Now, as one varies the complex structure of  $Y$  it is possible for the special Lagrangian  $L$  to become “pinched”; that is, at a particular value of the complex structure moduli  $L$  will become the union of two submanifolds touching at a point [18]. At this point the D-brane is marginally stable. As one deforms the complex structure past this point,  $L$  splits up into two distinct components  $L \rightarrow L_1 + L_2$ . In general the phases of the two components will become distinct, and so the union  $L_1 \cup L_2$  is no longer itself special Lagrangian. The resulting pair of D-branes is no longer mutually BPS. In this way, a single A-brane can “decay”  $L \rightarrow L_1 + L_2$  as one deforms the complex structure.

The key idea in looking for D-brane decays is to find sub-branes<sup>1</sup>  $L_1$  into which  $L$  can

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<sup>1</sup>The idea of a sub-brane is actually poorly-defined but we will use this language here. More correctly one should use the language of triangulated categories as explained in [19].

decay. The decay can occur, i.e.  $L$  will be marginally unstable, at the point in moduli space where  $\xi = \xi_1$ . That is, when the phases of two periods of the holomorphic 3-form become equal:

$$\arg \int_L \Omega = \arg \int_{L_1} \Omega. \quad (7)$$

At a generic point in moduli space we expect the periods of  $\Omega$  to be transcendental complex numbers. So any two A-branes whose charges are not proportional will typically have different phases. Thus at a generic point there are no marginally stable D-branes: all branes are either properly stable or properly unstable.

The attractor fixed points described above are very special, however, in that they admit many marginally stable branes. In fact, they admit the *maximal* number of marginally stable branes.

To see this, consider the 3-brane black hole described above. The D-brane under consideration, with charge  $F^{(3)}$ , wraps a special Lagrangian with phase

$$\arg \int_Y \Omega \wedge F^{(3)} = \arg Z. \quad (8)$$

Consider a second “probe” 3-brane with charge  $v \in H^3(Y, \mathbf{Z})$ . The phase of this A-brane will be aligned (or anti-aligned) with the phase of the black hole only if

$$\arg(\pm Z) = \arg \int_Y \Omega \wedge v. \quad (9)$$

Since  $v$  is real, this can be written as

$$0 = \bar{Z} \int_Y \Omega \wedge v - Z \int_Y \bar{\Omega} \wedge v = -ie^{-K/2} \int_Y F^{(3)} \wedge v. \quad (10)$$

The final expression is the natural symplectic inner product on  $H^3(Y, \mathbf{Z})$ . Our condition is just that  $v$  is perpendicular to  $F^{(3)}$  with respect to this inner product:  $\langle F^{(3)}, v \rangle = \int_Y F^{(3)} \wedge v = 0$ .

This is a very striking result. It shows that a D-brane in an attractor background is far more likely to be marginally stable than a generic D-brane. In particular, the whole codimension-one sublattice of  $H^3(X, \mathbf{Z})$  orthogonal to  $F^{(3)}$  gives sub-branes with respect to which  $L$  can be marginally unstable. This is in stark contrast to the generic values of complex structure discussed above.

It is easy to show that all possible D-brane charges cannot correspond to mutually BPS states, and so, in this sense, this codimension one sublattice is the maximal set of states that can be mutually BPS. In other words, *the attractor equations force the D-brane to be maximally marginally-stable*.

Thus the attractor mechanism forces us to consider the D-brane as a “direct sum” of constituent objects when we consider moduli spaces. So at the level of moduli spaces we regard the constituent D-branes as completely non-interacting. As we will see, however, the

RR-fields *do* produce interactions between the constituent D-branes and it is very important to take this into account in order to correctly compute the entropy of the system.

Although we have focused on A-branes of type IIB, one can make analogous comments concerning the B-branes in type IIA string theory on the mirror Calabi–Yau  $X$ . As B-brane computations tend to be more tractable, any entropy calculation will almost certainly be easier in the IIA language. However, as the mathematical machinery is more abstract we will make only a few comments here.

A B-brane  $E$  may be regarded, in ascending order of honesty, as a vector bundle over a holomorphic submanifold, as a coherent sheaf, or as an object in the derived category of coherent sheaves [20] (see [19] for a review). Its charge is

$$ch(E) \wedge \sqrt{td(T_X)} \in H^{\text{even}}(X, \mathbb{Q}). \quad (11)$$

The natural inner product between B-brane charges, which is mirror to the intersection form on 3-cycles given above, is

$$\langle E, F \rangle = \int_X ch(E)^\vee \wedge ch(F) \wedge td(T_X), \quad (12)$$

where  $^\vee$  reverses the sign of  $(4n+2)$ -forms for all  $n$ . The notion of a stable special Lagrangians is replaced by  $\Pi$ -stability and distinguished triangles.

In the discussion below we will use mirror symmetry to cast some of the computations in type IIA language, but we will do so only for simple cases where we can evade subtle issues of  $\Pi$ -stability.

### 3 The Quantum Mechanics of Special Lagrangians

In this section we will describe the moduli space of BPS D3 branes moving in the attractor geometry described above. We will study the moduli space of probe D3 branes in the geometry produced by a fixed “background” D3-brane whose entropy we wish to calculate. The probe D3-branes will be taken to be mutually supersymmetric with the background D3 brane — they may be thought of as candidate decay products formed out of the background D3 branes making up the black hole. Much of the material in the first part of this section is a straightforward generalization of [7].

Consider a stack of  $N$  probe D3 branes in the near-horizon  $AdS_2 \times S^2 \times Y$  attractor geometry. The probe D3 branes are taken to wrap a special Lagrangian  $L \subset Y$ , and are point-like in the  $S^2$  and  $AdS_2$  spatial directions. Since the  $L$  directions are compact, one can integrate over the  $L$  directions to obtain a one-dimensional world volume quantum mechanics. Because of the  $AdS_2$  factor, the theory has an  $SU(1,1|2)$  superconformal symmetry. Conformal quantum mechanics systems of this type were described in [3], which considered D0 branes moving in IIA attractor geometries. In fact, since our D3 branes are point-like in the  $AdS_2 \times S^2$  directions, these spatial components of the quantum mechanics are identical to those described in [3]. We will therefore focus on the Calabi–Yau component of the moduli space.



Before describing a stack of  $N$  D3 branes, first consider a single A-brane wrapping  $L$ . First assume we have a *smooth embedding*  $f : L \rightarrow Y$ . The special Lagrangian condition is [21]

$$f^*\omega = 0, \quad f^*(\text{Im}(e^{-i\xi}\Omega)) = 0, \quad (13)$$

where  $\omega$  is the Kähler form on  $Y$ . In addition, the D3 brane comes equipped with a world-volume gauge field  $A$ .<sup>2</sup> Supersymmetry implies that  $A$  describes a flat connection on a  $U(1)$  bundle<sup>3</sup> over  $L$ . We will now do a local analysis of the moduli space of these supersymmetric D3 branes, following [7, 22] (see also [17, 23] for a review).

We can imagine deforming a D3 brane in two ways, either by changing  $f$  or by changing  $A$ . Infinitesimal deformations of  $f$  are in one-to-one correspondence with harmonic one forms on  $L$ . To see this, consider a one-parameter family of embeddings  $f_t : \mathbb{R} \times L \rightarrow Y$  which preserve the special Lagrangian condition. Here  $t$  is a coordinate on  $\mathbb{R}$ . It is straightforward to show that

$$f_t^*(\omega) = \theta \wedge dt, \quad f_t^*(\text{Im}(e^{-i\xi}\Omega)) = e^{-K/2} \frac{1}{2k} * \theta \wedge dt \quad (14)$$

where  $*$  is the Hodge star on  $L$  and the constant  $k = \sqrt{8 \text{Vol}(Y)}$ . Both of these forms are necessarily closed, so  $\theta$  is harmonic. If we denote by  $\theta^a$ ,  $a = 1, \dots, b_1(L)$ , a basis of harmonic one forms on  $L$ , this provides us with a family of infinitesimal deformations  $dt^a$  of  $f$ . It was shown in [22] that one can integrate these infinitesimal deformations to find a good set of local coordinates  $t^a$  on the moduli space of special Lagrangians.

The space of flat connections  $A$  is also of dimension  $b_1(L)$ . This is because by a judicious choice of gauge we may always put the world-volume gauge field in the form  $A = \sum_a s^a \theta^a$ . The constants  $s^a$  form a set of coordinates on the moduli space of flat connections.

It is important to note that the moduli space of A-branes is not necessarily equal to this moduli space of bundles and special Lagrangians. Quantum corrections coming from holomorphic disks with boundary on  $L$  can lead to obstructions. Thus, in general, the moduli space of A-branes can be less than  $2b_1(L)$ . We refer to [24] for an example of this and [19] for further discussion.

We will ignore such obstructions here. We conclude that locally the moduli space of BPS D3 branes is a product of the form  $\mathcal{M} = H^1(L) \times H^1(L)$ , with coordinates  $(t^a, s^a)$ . There is a natural metric on  $\mathcal{M}$ , of the form  $ds^2 = g_{ab} dt^a dt^b + g_{ab} ds^a ds^b$ , where

$$g_{ab} = \frac{1}{2k} \int_L \theta^a \wedge * \theta^b. \quad (15)$$

Here, as above, the Hodge star on  $L$  is defined using the metric induced on  $L$  by the embedding  $f$  of  $L$  into  $Y$ . Thus  $g_{ab}$  is a function of the embedding  $f$ , and hence of  $t^a$  but not  $s^a$ . In fact, the metric can be shown to obey  $g_{ab,c} = g_{ac,b}$ . This implies that the natural Kähler two form on moduli space,  $\mathcal{J} = g_{ab} dt^a \wedge ds^b$ , is closed and defines an integrable complex structure on  $\mathcal{M}$ .

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<sup>2</sup>This world-volume gauge field  $A$  should not be confused with the connection  $\mathcal{A}$  on moduli space that we will discover below.

<sup>3</sup>Assuming  $B = 0$  on  $Y$ . A nonzero  $B$ -field results in a “twisted” line bundle.

We are interested in the dynamics of BPS D3 branes, which are described by a superconformal quantum mechanics on  $\mathcal{M}$ . The kinetic term is found by taking the moduli  $(t^a, s^a)$  to depend on time, and expanding the Dirac-Born-Infeld world-volume action to quadratic order in  $(\dot{t}^a, \dot{s}^a)$ . The bosonic part of the action becomes

$$\begin{aligned} S_{DBI} &= \int dt \int_L \sqrt{\det(G + B - 2\pi\alpha' F)} \\ &= 2k \int dt g_{ab} (\dot{t}^a \dot{t}^b + \dot{s}^a \dot{s}^b) + \dots \end{aligned} \tag{16}$$

This is a non-linear sigma model on  $\mathcal{M}$ . There are of course also fermion terms, as well as various terms involving motion in the  $AdS_2$  and  $S^2$  directions, which are just as in [3]. The  $2k$  prefactor follows from our choice of normalization of the metric  $g_{ab}$  defined above. It's importance will become apparent below, where we discuss Chern-Simons terms which are quantized in units of  $k$ .

Now, let us consider what happens for  $N$  D3 branes. The coordinates  $(t^a, s^a)$  described above are promoted to matrices, and the world-volume action will include terms involving the commutators of these matrices. For example, there is now a Chern-Simons type term of the form [6, 25]

$$\int dt \int_L f_t^*(F^{(5)})[\phi^a, \phi^b] \tag{17}$$

where  $\phi^a$  is an  $N \times N$  matrix and  $f_t^*(F^{(5)})$  is the pullback of the Ramond-Ramond field strength (5) sourced by the background D3 branes.

Matrix systems of this form admit a large number of possible ground states, including both commuting and non-commuting configurations of matrices. According to the Myers effect [6], the various non-commuting configurations should be interpreted as D5 or D7 branes wrapping cycles in the  $S^2 \times X$ . However, a non-commuting configuration wrapping a cycle in  $X$  couples to Ramond-Ramond fields in the same way as the associated D5 or D7 brane. It will therefore contribute to the overall charge of the black hole as measured at infinity. In evaluating the entropy we should sum only over configurations with the correct asymptotic charges. So we should include states where the D3 branes are allowed to form D5 branes wrapping the  $S^2$ , but not where they wrap an internal direction. In this case the matrices  $\phi^a$  describing the D3 brane positions form an  $N$  dimensional representation of  $SU(2)$ . This representation can be written as a sum of irreducible representations, each of which corresponds to a D5 brane wrapping the  $S^2$  horizon. So, the total number of different ways our  $N$  D3 branes may puff up into a collection of D5 branes is equal to the number of partitions of the integer  $N$ . In fact, configurations where the D3 branes form a D5 branes in this way give the dominant contribution to the entropy. This observation was made in [4], which studied configurations of D0 branes in IIA that formed a spherical D2 brane wrapping the horizon  $S^2$ .<sup>4</sup>

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<sup>4</sup>We should emphasize that the non-commuting configurations considered here are supersymmetric, as in [3], so they are genuine zero energy ground states of the system. This is in contrast with the dielectric configurations originally considered in [6], which were non-supersymmetric and had positive energy.

For this configuration, where  $N$  D3 branes form a D5 wrapping the horizon, (17) becomes the usual Chern-Simons interaction term on the D5 world volume

$$S_{CS} = \int dt \int_{L \times S^2} A \wedge f_t^*(F^{(5)}). \quad (18)$$

From (14), the pullback of the associated RR 3-form  $F^{(3)}$  onto the brane world-volume is

$$f_t^*(F^{(3)}) = f_t^*(\text{Im}(C\Omega)) = \frac{|Z|}{\sqrt{8V}} \sum_a * \theta^a t^a dt. \quad (19)$$

In writing the second equality we have used the fact that  $\xi = \arg Z$ , since our probe D3-brane is marginally bound to the background D3-brane. This allows us to compute the pullback using (14). Up to a total derivative, (18) may be written as

$$\begin{aligned} \int dt \int_L F \wedge f_t^*(F^{(3)}) &= \frac{|Z|}{\sqrt{8V}} \int dt \sum_{ab} s^a t^b \int_L \theta^a \wedge * \theta^b \\ &= |Z| \int dt g_{ab} s^a t^b. \end{aligned} \quad (20)$$

A “magnetic field” coming from a  $U(1)$ -bundle with connection one-form  $\mathcal{A}$  contributes a term to the action of the form

$$S_{\mathcal{A}} = \int_{\gamma} \mathcal{A}, \quad (21)$$

for a path  $\gamma$  in  $\mathcal{M}$ . Thus we may interpret the Chern–Simons term as producing a magnetic field with gauge potential

$$\mathcal{A} = |Z| g_{ab} s^a dt^b. \quad (22)$$

Now, since  $g_{ab,c} dt^b dt^c = 0$  we can evaluate the field-strength

$$\mathcal{F} = d\mathcal{A} = |Z| g_{ab} ds^a \wedge dt^b = |Z| \mathcal{J}. \quad (23)$$

## 4 Black Hole Entropy from Mirror Symmetry

We will now use the results of the previous section to compute the black hole entropy.

We have demonstrated that the dynamics of BPS probe D3 branes is governed by the world volume quantum mechanics of the form

$$\int dt \mathcal{G}_{ab} \dot{z}^a \dot{z}^b + \mathcal{A}_a \dot{z}^a + \text{fermions} \quad (24)$$

where  $z^a$  and  $\mathcal{G}_{ab}$  are the coordinates and metric on moduli space  $\mathcal{M}$  of special Lagrangians, and  $\mathcal{A}$  is the connection with curvature  $\mathcal{F} \sim \mathcal{J}$  described above. As this is a superconformal

quantum mechanics, the number of ground states in this system is encoded in the number of chiral primaries. The chiral primary conditions can be written as

$$\bar{D}h = \bar{D}^*h = 0, \tag{25}$$

where  $h$  is a  $(p, q)$  form on  $\mathcal{M}$  and  $D$  is the holomorphic covariant derivative with connection  $\mathcal{A}_a$ . Solutions of (25) are in one-to-one correspondence with elements of  $H^{0,q}(X, \Omega^p \otimes \mathcal{L})$ . Here  $\mathcal{L}$  is the line bundle over  $X$  with first Chern class  $c_1(\mathcal{L}) = [\mathcal{F}]$ . So the black hole entropy counting can be reduced to a cohomology problem.

We should emphasize that in order to render the supergravity approximation implicit in the previous discussion valid, we must consider black holes with large charge. This makes the general computation of the cohomology an exceedingly formidable task. Even relatively simple D-branes such as certain ones on the quintic threefold are hard to study in terms of  $\Pi$ -stability [26]. Added to this is the complication that we need to examine an enormous number of possible decay paths as discussed in the previous section.

However, as we describe in the next section, in some cases one can compute the dimension of  $H^q(X, \Omega^p \otimes \mathcal{L})$  using mirror symmetry. As an illustration, we will show how this works for the simple case  $Y = T^6$ .

## 4.1 Mirror Computation

There is, of course, one 3-brane for which the moduli space is easy to compute. If  $X$  is mirror to  $Y$ , then we know from homological mirror symmetry (or, less rigorously but more transparently, by using [7]) that a 0-brane on  $X$  is mirror to a 3-torus on  $Y$ . The moduli space of a D3-brane wrapping such a  $T^3$  on  $Y$  is just  $X$ . Thus one can compute  $H^q(X, \Omega^p \otimes \mathcal{L})$  — and hence the black hole entropy — using mirror symmetry. We will consider a black hole whose charge is dominated by such 3-branes.

Consider a B-brane on a Calabi–Yau threefold  $X$  whose charge is large and made up almost entirely of 0-branes. Let us also assume that a 0-brane is a possible decay product among the multitude of possible marginal decays. This means that the inner product under (12) of a 0-brane with our black hole brane  $L$  must be zero. If  $E$  is a 0-brane, then  $ch(E)$  is a pure 6-form. Thus, using (12) we require that  $ch(L)$  has no 0-form part. The 0-form part of a Chern character measures the rank of a vector bundle. Thus, our big D-brane must correspond to a vector bundle of rank 0, i.e., it is supported over a proper holomorphic subspace of  $X$ . To put it another way, *it must have no 6-brane charge*.

The probe quantum mechanics of the special Lagrangian 3-brane described in the previous section can now be recast as the quantum mechanics of D0 branes on  $X$ . This D0 brane theory was studied in [3], and used to compute the black hole entropy in [4]. The rest of this subsection is essentially a review of this work, which we include here to make the discussion self contained.

We should emphasize that we cannot consider black holes made entirely of D0 branes, however, as in the leading supergravity approximation these black holes have zero area. So we need to include some 4-brane charge. This means that there will be a moduli space of

states involving 4-branes. However, we should note that the number of 0-branes must be much greater than the number four branes wrapping any given cycle in  $X$ . To see this, note that the Kähler form on  $X$  (in ten dimensional units) is determined by the IIA attractor equation to be

$$J = \sqrt{\frac{q_0}{D}} p^A \omega_A, \quad D = D_{ABC} p^A p^B p^C. \quad (26)$$

Here we have chosen a basis  $\omega_A$  of  $H^2(X, \mathbf{Z})$  and denoted by  $p^A$  the number of 4-branes wrapping the 4-cycle Poincaré dual to  $\omega_A$ . Here  $D_{ABC} = \frac{1}{6} \int \omega_A \wedge \omega_B \wedge \omega_C$  is a triple intersection number and  $q_0$  is the number of 0-branes. As discussed in section 2.1, the supergravity approximation is valid only when the size of any two cycle in  $X$  is much smaller than the black hole horizon area (although it must still be large in string units). From (26), this implies that we must have  $q_0 \gg p^A$ . As long as the 0-brane charge dominates, the contributions from the moduli space of 4-branes should be negligible in determining the entropy.<sup>5</sup> As discussed above, while it is in principle possible to add D2 brane charge to such a black hole, one can not add D6 branes and maintain supersymmetry.

Now, the moduli space of a 0-brane moving on  $X$  is just  $X$  itself. This means that the metric  $\mathcal{G}_{ab}$  described in section 3 is just the metric on  $X$ . In section 3 we demonstrated that there is magnetic field on moduli space whose field strength is equal to the central charge  $|Z|$  times the Kähler form on  $X$  given above. In the IIA language, it is easy to see where this magnetic field comes from. The IIA supergravity solution for the black hole includes a Ramond-Ramond four form fieldstrength  $\omega_{S^2} \wedge p^A \omega_A$ , which is sourced by the background D4 branes. For configurations of fuzzy D0 branes wrapping the  $S^2$  horizon, this four form fieldstrength couples to the D0 brane worldvolume fields via the Myers effect, in a manner similar to that describe in section 3. The result is a magnetic fieldstrength  $\mathcal{F} = p^A \omega_A$  on moduli space [4].<sup>6</sup>

To compute the cohomology, note that the bundle  $\mathcal{L}$  is very ample (as its first Chern class is the Kähler form). Thus  $H^{p,q}(X, \mathcal{L}) = 0$  if  $p > 0$ . We can then use an index theorem to compute  $H^{0,q}(X, \mathcal{L})$  just as in [4]. To leading order in the charges, the answer is

$$\dim H^{0,q}(X, \mathcal{L}) = \begin{cases} D & q = 0, 3, \\ 3D & q = 1, 2. \end{cases} \quad (28)$$

We now need to count the total number of supersymmetric ground states of the theory, using the chiral primary degeneracies  $h^q = \dim(H^{0,q}(X, \mathcal{L}))$  computed above. As usual, it is

<sup>5</sup>However, there has been recent progress in understanding these 4-brane contributions in some cases [5].

<sup>6</sup>This expression for  $\mathcal{F}$  also follows from equations (23) and (26). However, one must be careful because the usual expression for central charge  $|Z| = (q_0 D)^{1/4}$  is written in four dimensional Planck units rather than in ten dimensional units. To rewrite it in ten dimensional units we must use the conversion factor

$$\frac{l_4}{l_{10}} = l_{10}^3 \text{Vol}(X)^{-1/2}. \quad (27)$$

where  $\text{Vol}(X) = \int_X J^3 = q_0^{3/2} D^{-1/2}$ .

convenient to package the answer as a partition function

$$Z(q) = \sum_N p(N)q^N. \quad (29)$$

Here  $p(N)$  is the number of chiral primary states that may be formed out of  $N$  D0 branes. As described in section 3, according to the Myers effect these  $N$  D0 branes may form a collection of D2 branes wrapping the horizon  $S^2$ . There is one such collection of D2 branes for each partition of the integer  $N$ . In addition, each D2 brane so formed may occupy any one of the chiral primary states on the Calabi–Yau counted above. The only restriction is that chiral primaries with even  $q$  obey bosonic statistics, while states with odd  $p$  obey fermionic statistics. Putting this together, it turns out that  $Z(q)$  is precisely the partition function of a conformal field theory with  $h^0 + h^2$  free bosons and  $h^1 + h^3$  free fermions:

$$Z(q) = \prod_n \frac{(1 + q^n)^{h_1+h_3}}{(1 - q^n)^{h_0+h_2}}. \quad (30)$$

Using the large  $N$  expansion for this formula gives the entropy

$$S = \log p(N) \sim \pi \sqrt{\frac{1}{3}N (h^0 + h^2 + \frac{1}{2}(h^1 + h^3))}. \sim 2\pi\sqrt{ND} \quad (31)$$

As a word of warning, one should probably view the computation in this section and the previous section as somewhat heuristic, as it is prone to the same objections as the general SYZ argument. We have certainly not taken into account the fact that special Lagrangians typically degenerate for certain points in the moduli space. However, we will see that we can reconstruct this result more rigorously for  $K3 \times T^2$  in section 5.

## 4.2 An $\mathcal{N} = 8$ Example

In this section we will describe the computation for the case  $Y = T^6$ , which provides a useful explicit illustration of these methods. Here the moduli space of special Lagrangians takes a particularly simple form and the calculations can be done explicitly without using the mirror symmetry conjecture of [7].

It is straightforward to write down and solve the attractor equations on  $T^6$  (see e.g. [11]). We will take coordinates on the  $T^6$  to be  $x^i \sim x^i + 1, y^i \sim y^i + 1, i = 1, 2, 3$ . A choice of holomorphic one forms  $dz^i = dx^i + \tau^{ij} dy^j$  fixes the period matrix  $\tau$ . We will suppress  $i$  indices when possible. The metric is

$$dz \cdot d\bar{z} = dx \cdot dx + 2dx \cdot \text{Re}\tau \cdot dy + dy \cdot \tau^\dagger \tau \cdot dy. \quad (32)$$

We will take the following symplectic basis for  $H^3(T^6, \mathbb{Z})$ ,

$$\begin{aligned} \alpha_0 &= dx^1 dx^2 dx^3, & \alpha_{ij} &= \frac{1}{2} \epsilon_{ilm} dx^l dx^m dy^j \\ \beta^0 &= -dy^1 dy^2 dy^3, & \beta^{ij} &= \frac{1}{2} \epsilon_{jlm} dx^i dy^l dy^m \end{aligned} \quad (33)$$

so that  $\int \alpha_I \wedge \beta^J = \delta_I^J$ , where  $I = (0, ij)$ . The charge of the black hole is parameterized by an element of  $H^3(T^6)$ , which can be decomposed in this basis as

$$F^{(3)} = p^0 \alpha_0 + P^{ij} \alpha_{ij} + q_0 \beta^0 + Q_{ij} \beta^{ij}. \quad (34)$$

We will focus on the case where  $p^0$  and  $Q_{ij}$  vanish and  $P$  is symmetric. In this case the attractor equations fix the holomorphic three form to be (up to an overall constant)

$$\Omega^{3,0} = dz^1 \wedge dz^2 \wedge dz^3 = \alpha_0 + \alpha_{ij} \tau^{ij} + \beta^{ij} (\text{Cof } \tau)_{ij} - \beta^0 (\det \tau) \quad (35)$$

where  $(\text{Cof } \tau)_{ij}$  is the matrix of cofactors of  $\tau^{ij}$  and

$$\tau = \frac{i}{2} \sqrt{\frac{q_0}{\det P}} P. \quad (36)$$

The entropy of the black hole is

$$S = 2\pi \sqrt{|q_0 \det P|}. \quad (37)$$

Now, consider a probe three brane wrapping  $\alpha_0$  (i.e. the  $x^i$  directions). The induced world-volume metric is flat. The moduli space is parameterized by a position  $y^i(t)$  and world-volume gauge field  $A = a_i(t) dx^i$ , which depend on time. The kinetic terms for  $y^i$  and  $a_i$  come from the DBI action

$$S_{DBI} = \int dt (\dot{y} \cdot \tau^\dagger \tau \cdot \dot{y} + \dot{a} \cdot \dot{a}) + \dots \quad (38)$$

The world volume Chern-Simons term depends only on  $P^{ij}$ , and is

$$S_{CS} = \int A \wedge f^* F_3 = \int dt (a \cdot P \cdot \dot{y}). \quad (39)$$

The action  $S_{DBI} + S_{CS}$  describes a particle in a magnetic field, with moduli space metric and magnetic field strength

$$ds^2 = dy \cdot \tau^\dagger \tau \cdot dy + da \cdot da, \quad \mathcal{F} = dy \cdot P \cdot da. \quad (40)$$

Note that both the  $a$  and  $y$  coordinates are identified with periodicity one. As an aside, we should note that the two form  $\mathcal{F}$  defines a complex structure on the moduli space. The associated holomorphic coordinates are

$$w = a + \tau \cdot y, \quad (41)$$

in terms of which the metric and field strength are

$$ds^2 = dw \cdot d\bar{w}, \quad \mathcal{F} = -i \sqrt{\frac{\det P}{q_0}} dw \cdot \wedge d\bar{w}. \quad (42)$$

We can now count the number of chiral primaries, by computing  $H^{0,p}(T^6, \Omega^q \otimes \mathcal{L})$  where  $\mathcal{L}$  is a line bundle with curvature  $c_1(\mathcal{L}) = \mathcal{F}$ . By a Kodaira vanishing theorem, these vanish unless  $p = 0$  (at large charge) so we may use an index theorem for the twisted Dolbeault complex:

$$h^{0,q} = \text{Ind}_{\bar{\partial}}(T^6, \Omega^q \otimes \mathcal{L}) = \binom{3}{q} \int \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = \binom{3}{q} \det P. \quad (43)$$

We can then count the number of chiral primaries using the combinatorics of the previous section, reproducing the black hole entropy (37).

## 5 An $\mathcal{N} = 4$ Example

In this section we will consider compactifications with  $\mathcal{N} = 4$  supersymmetry, where  $Y$  is of the form  $T^2 \times S$  for some K3 surface  $S$ . Although in this case the moduli space of special Lagrangians is more complicated than the  $\mathcal{N} = 8$  example described above, we may still perform the computation without relying on the analysis of section 3. One may view this section as evidence that (23) is correct even when the special Lagrangian fibration in section 3 has degenerate fibers. Some of what we discuss below is related to the ‘‘Donaldson–Mukai map’’ as described in [27].

The attractor equations for  $T^2 \times K3$  were studied in [11]; we will simply quote the results here. We will denote the coordinates on  $T^2$  as  $x$  and  $y$ , with  $x \sim x + 1$  and  $y \sim y + 1$ . The charge of the black hole is

$$W = p dx + q dy \quad (44)$$

where  $p$  and  $q$  live in  $H^2(S, \mathbb{Z})$ . We will write the holomorphic 3-form on  $Y$  as

$$\begin{aligned} \Omega^{3,0} &= dz \wedge \Omega \\ &= (dx + \tau dy) \wedge \Omega, \end{aligned} \quad (45)$$

where  $\tau$  is the complex modulus of  $T^2$  and  $\Omega$  is a holomorphic 2-form on  $S$ . The attractor equation (1) then forces (for a suitable choice of normalization)

$$\Omega = q - \bar{\tau}p. \quad (46)$$

Furthermore,  $\tau$  is determined by  $p$  and  $q$  to be

$$\tau = \frac{p \cdot q + i\sqrt{p^2 q^2 - (p \cdot q)^2}}{p^2}, \quad (47)$$

where the dot product is given by the usual intersection form on  $S$ :

$$p \cdot q = \int_S p \wedge q. \quad (48)$$

We will now consider a probe 3-brane on  $Y$ , and interpret the Chern–Simons contribution to the world volume action as an effective magnetic field on moduli space, which in this case will be the mirror,  $X$ , of  $Y$ .



We will take the probe to wrap the special Lagrangian cycle in  $T^2 \times S$  that consists of the circle Poincaré dual to  $dy$  times a special Lagrangian 2-cycle  $L \subset S$ . As above, this probe D-brane must have its phase aligned with that of the background black hole, so from (10)

$$\int_{Y^2 \times S} (p dx + y dy) \wedge l dy = l \cdot p = 0, \quad (49)$$

where  $l$  is the 2-form Poincaré dual to  $L$ . Since  $L$  is Lagrangian,  $l \cdot J = 0$  where  $J$  is the cohomology class of the Kähler form on  $S$ .

To write down the Chern–Simons term, we need to consider a one parameter family of D-brane probes. For now let us focus on just the K3 factor and consider a one-parameter family of maps  $L \rightarrow S$ . We will take this one-parameter family to form a loop — i.e., the final D-brane is the same as the initial D-brane. Denote by  $T$  the 3-dimensional subspace swept out in  $S$  by this family; note that  $T$  has no boundary, since the one parameter family is a loop. Each special Lagrangian D-brane comes equipped with a line bundle and a connection  $A$ , which we will extend to a connection over  $T$ . The Chern–Simons term is just<sup>7</sup>

$$S_{CS} = \int_T A \wedge p. \quad (50)$$

Now, note that the third homology of  $S$  is zero, so there must be a 4-dimensional subspace  $U \subset S$  such that  $T = \partial U$ . Stokes’ theorem yields

$$S_{CS} = \int_U F \wedge p. \quad (51)$$

Let  $\mathcal{M}_S$  be the moduli space of probe D2 branes on  $S$ , i.e., the moduli space of special Lagrangians with bundle data. Recall that  $iS_{CS}$  takes values in  $\mathbb{R}/2\pi\mathbb{Z}$ . It therefore defines a map

$$S_{CS} : \Omega(\mathcal{M}_S) \rightarrow S^1, \quad (52)$$

where  $\Omega(\mathcal{M}_S)$  is the loop-space of  $\mathcal{M}_S$ . Given any space  $M$ ,  $\pi_2(M)$  is defined as  $\pi_1(\Omega(M))$ . So, applying  $\pi_1$  to (52) yields a map

$$\eta_{CS} : \pi_2(\mathcal{M}_S) \rightarrow \mathbb{Z}. \quad (53)$$

The moduli space  $\mathcal{M}_S$  is expected to break up into many disconnected components, corresponding to different types of probe D-branes. To proceed, we must therefore choose the type of probe brane under consideration. As above we will assert that  $L$  is a  $T^2$ -fiber of  $S$  so that it is mirror to a 0-brane. According to the SYZ conjecture [7], which is easily proven for K3 (as we discuss next), this component of  $\mathcal{M}_S$  is itself a K3 surface. We denote this component  $\hat{S}$ . In a sense,  $\hat{S}$  is “the” mirror of  $S$ .

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<sup>7</sup>We are assuming this family produces an *embedding* of  $T$  into  $S$ . This need not be the case in general but is true for the case we consider here.

The rigorous way to prove that  $\hat{S}$  is a K3 surface is as follows. The K3 surface  $S$  is hyperkähler, and so admits a whole  $S^2$  of complex structures compatible with a given metric. By choosing a different complex structure, we may turn a special Lagrangian into a holomorphic curve. To be precise, we may turn the special Lagrangian 2-torus with a flat line bundle into an elliptic curve in  $S$  with a flat line bundle. This is a sheaf supported on the elliptic curve. One may then compute the moduli space  $\hat{S}$  of such sheaves. This was done by Mukai [28], who showed that  $\hat{S}$  is a K3 surface. In fact,  $\hat{S}$  is isomorphic to  $S$  as a complex variety.

It is worth emphasizing that attractive K3 surfaces have many special properties. Because of their large Picard number, they contain many algebraic curves and thus homologically distinct elliptic curves. Any elliptic curve leads to an elliptic fibration so a typical attractive K3 surface can probably be elliptically fibered in many inequivalent ways. It is always possible to choose an elliptic fibration of an attractive K3 surface such that it admits a *section*. The explicit form of such a fibration was given in [29]. We will assume that we have chosen the fibration so that this is the case.

Consider this section  $\sigma$  of the elliptic fibration  $\hat{S}$ . This corresponds to an element of  $\pi_2(\hat{S})$ . The precise family of elliptic curves (or special Lagrangians, depending on the chosen complex structure) in  $S$  corresponding to this section may be determined by following Mukai's construction. Indeed, a Fourier–Mukai transform may be applied to this section to yield the family of elliptic curves on  $S$ . For an account of this transform in this context we refer to [30].

In our case, this family of elliptic curves will sweep out a line bundle  $E$  over the whole of  $S$ . This line bundle is, of course, the bundle whose curvature  $F$  appears in (51). It follows that

$$\eta_{CS}(\sigma) = \frac{1}{2\pi i} \int_S F \wedge p. \quad (54)$$

We can now use the Fourier–Mukai transform to express this Chern–Simons term as a gauge connection on the moduli space  $\hat{S}$ . To do this, we will first write (54) in a more convenient form. To start, recall from section 2 that there is a natural inner product on  $H^*(S, \mathbb{Z})$ ,

$$\langle \alpha, \beta \rangle = \int_S \alpha^\vee \wedge \beta \quad (55)$$

for any  $\alpha$  and  $\beta$  in  $H^*(S, \mathbb{Z})$ . Here  $\alpha^\vee$  is just  $\alpha$  with the sign of the 2-form component reversed. Furthermore, to any bundle (or sheaf) we may associate its D-brane charge, or Mukai vector, defined as

$$v(E) = ch(E) \wedge \sqrt{td(T_S)} \in H^*(S, \mathbb{Z}). \quad (56)$$

We may therefore write

$$\eta_{CS}(\sigma) = -\langle v(E), p \rangle. \quad (57)$$

Mukai's mirror symmetry construction can now be applied using the Fourier–Mukai transform. This has some known action on  $H^*$ , which we denote  $\mu$ :

$$\mu : H^*(S, \mathbb{R}) \rightarrow H^*(\hat{S}, \mathbb{R}). \quad (58)$$

This action has the nice feature that it preserves the inner product  $\langle \alpha, \beta \rangle$ . Thus

$$\eta_{CS} = -\langle v(\sigma), \mu(p) \rangle. \quad (59)$$

We now just have to evaluate  $\mu(p)$ . This will require a few more facts about K3 surfaces.

As is standard in the construction of string theories on K3 (see, for example, [31]) the moduli of  $S$  are determined by a space-like 4-plane  $\Pi \subset \mathbb{R}^{4,20}$ , where  $\mathbb{R}^{4,20} = H^*(S, \mathbb{R})$ . To put a geometric interpretation on this 4-plane one proceeds as follows:

1. First one chooses a vector  $w$  in the lattice  $H^*(S, \mathbb{Z})$  which generates  $H^4(S, \mathbb{Z})$ , and a vector  $w^\vee$  which generates  $H^0(S, \mathbb{Z})$ . We then identify  $H^2(S, \mathbb{Z})$  as the orthogonal complement of the span of  $w$  and  $w^\vee$ . Note that

$$\langle w, w \rangle = 0, \quad \langle w, w^\vee \rangle = 1, \quad \langle w^\vee, w^\vee \rangle = 0. \quad (60)$$

2. Define  $\Sigma' = \Pi \cap w^\perp$ .
3. Define the vector  $x$  such that  $\Pi$  is the span of  $\Sigma'$  and  $x$ ,  $x$  is orthogonal to  $\Sigma'$  and  $\langle x, w \rangle = 1$ .
4. Project  $\Sigma'$  into  $H^2(S, \mathbb{R})$  to obtain  $\Sigma$ .
5. Decompose

$$x = \alpha w + w^\vee + B, \quad (61)$$

where  $B \in H^2(S, \mathbb{R})$ .

$\Sigma$ , a space-like 3-plane in  $H^2(S, \mathbb{R})$ , is then spanned by  $\text{Re}(\Omega)$ ,  $\text{Im}(\Omega)$  and  $J$ . In fact, the 2-plane spanned by  $\text{Re}(\Omega)$  and  $\text{Im}(\Omega)$  is spanned by  $p$  and  $q$ , since we have assumed that  $S$  is an attractive K3 surface. So  $\Sigma$  is spanned by  $p$ ,  $q$  and  $J$ .

The  $B$ -field is then given by  $B$ ; and  $\langle J, J \rangle$ , the volume of the K3 surface, is given by  $\langle x, x \rangle = 2\alpha + \langle B, B \rangle$ .

Now,  $l$  — the Poincaré dual of our special Lagrangian fiber — is perpendicular to both  $p$  (from (49)) and  $J$  (since it is Lagrangian). We can now apply mirror symmetry, which consists of the hyperkähler rotation of the complex structure required to make the fiber holomorphic, followed by the Fourier–Mukai transform  $\mu$ . Our special Lagrangian fiber turns into a 0-brane, so  $\mu(l)$  must be a 4-form, and thus  $w$ . Now, since  $l$  is perpendicular to  $p$ ,  $\mu(p)$  is perpendicular to  $w$ .

To fix  $\mu(p)$  completely we must deal with one subtlety. Mirror symmetry is generally thought of as exchanging deformations of complex structure with deformations of  $B + iJ$ . This is somewhat ambiguous for K3 surfaces, because the hyperkähler structure provides many possible maps that can be interpreted as mirror symmetry. This ambiguity may be fixed as follows. We will take the space-like 4-plane  $\Pi$  to be spanned by two 2-planes,  $\Omega$  and  $\mathcal{U}$ , where  $\Omega$  is spanned by  $p$  and  $q$  and  $\mathcal{U}$  is spanned by  $J'$  and  $x$ .<sup>8</sup>  $J'$  is projected into

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<sup>8</sup>We are implicitly assuming that  $B$  lies in the span of *algebraic* 2-cycles. We are free to make this assumption for attractive K3 surfaces.

$H^2(S, \mathbb{R})$  to obtain  $J$ .  $\Omega$  and  $\mathfrak{U}$  should be thought of as encoding complex structure and  $B + iJ$  deformations, respectively. We now require that mirror symmetry interchanges  $\Omega$  and  $\mathfrak{U}$ .

We will use hats to refer to quantities for  $\hat{S}$ . Thus

$$\mu(\Omega) = \hat{\mathfrak{U}}, \quad \mu(\mathfrak{U}) = \hat{\Omega}. \quad (62)$$

$\mu(p)$  must lie in the plane spanned by  $\hat{J}'$  and  $\hat{x}$ . Since  $l$  is perpendicular to  $p$  and not to  $q$ ,  $\hat{w} = \mu(l)$  is perpendicular to  $\mu(p)$  and not to  $\mu(q)$ . But  $\hat{w}$  is perpendicular to  $\hat{J}'$  and not  $\hat{x}$ , so

$$\mu(p) = \hat{\kappa} \hat{J}', \quad (63)$$

for some real number  $\kappa$ . We may assume that we have chosen our mirror symmetry transform such that  $\kappa > 0$ .

$\kappa$  may be computed from above by knowing the volume of  $\hat{S}$ . We compute

$$\hat{x} = \frac{1}{(q \cdot l)} \left( q - \frac{(p \cdot q)}{p^2} p \right). \quad (64)$$

It follows from above that

$$\begin{aligned} \hat{J}^2 &= \hat{x}^2 \\ &= \frac{p^2 q^2 - (p \cdot q)^2}{p^2 (q \cdot l)^2} \end{aligned} \quad (65)$$

Thus, using (47), we have

$$\kappa = \frac{(q \cdot l)}{\text{Im}(\tau)}. \quad (66)$$

We conclude that the Chern-Simons contribution (54) is simply related to  $\hat{J}'$ :

$$\begin{aligned} \eta_{CS} &= -\langle v(\sigma), \mu(p) \rangle \\ &= \int_S \check{\sigma} \wedge \kappa \hat{J}' \\ &= \kappa \int_{\sigma} \hat{J}', \end{aligned} \quad (67)$$

where  $\check{\sigma}$  is Poincaré dual to  $\sigma$ . Note that  $\hat{J}' = \hat{J} + r\hat{w}$  for some real number  $r$ . The  $r\hat{w}$  component of  $\hat{J}'$  corresponds to a 4-form and therefore has no contribution in the above. We can therefore assert that

$$\eta_{CS} = \kappa \int_{\sigma} \hat{J}. \quad (68)$$

In fact, this is precisely the contribution of a magnetic field on moduli space, with curvature equal to  $\kappa \hat{J}$ . To see this, consider a connection  $\mathcal{A}$  on  $\hat{S}$ , and let  $\gamma$  be a loop in  $\hat{S}$ .

Since  $\pi_1(\hat{S}) = 0$ , there is a disk  $D$  such that  $\partial D = \gamma$ . The Wilson line associated to this loop contributes to the path integral as

$$S_W = \int_{\gamma} \mathcal{A} = \int_D \mathcal{F}, \quad (69)$$

where  $\mathcal{F}$  is the curvature of  $\mathcal{A}$ . This time  $iS_W$  is valued in  $\mathbb{R}/2\pi\mathbb{Z}$  and so we may repeat the argument given above to get a map

$$\eta_W : \pi_2(\hat{S}) \rightarrow \mathbb{Z}, \quad (70)$$

where, for our section  $\sigma$  we have

$$\eta_W(\sigma) = \frac{1}{2\pi i} \int_{\sigma} \mathcal{F}. \quad (71)$$

Comparing this to (68) we see that the effect of the Chern–Simons term on  $S$  is mirror to the Wilson line contribution of a connection on a line bundle on  $\hat{S}$  whose curvature is given by  $2\pi i \kappa \hat{J}$ .

We may also analyze the contribution to the magnetic field from the  $T^2$  part of  $Y$ . This computation can be done in a way similar to section 4.2. The result is that the cohomology class of the curvature obeys

$$\left[ \frac{1}{2\pi i} \mathcal{F} \right] = \mu(p) + (q.l)e, \quad (72)$$

where  $e$  generates  $H^2(T^2, \mathbb{Z})$  and the Kähler form is given by (at least in cohomology)

$$\hat{J} = \frac{\text{Im}(\tau)}{(q.l)} \left[ \frac{1}{2\pi i} \mathcal{F} \right]. \quad (73)$$

This is the analogue of equation (23) subject to the same scale change as (27).

The area of the event horizon can be computed [11] to be  $4\pi\Delta$ , where

$$\Delta = \sqrt{p^2 q^2 - (p.q)^2}. \quad (74)$$

So far everything we have said in this section is exact. In order to compute the entropy we need to start making some approximations. Let us decompose  $p$  and  $q$  as follows:

$$\begin{aligned} p &= sl + \tilde{p} \\ q &= Nl + ml^\vee + \tilde{q}, \end{aligned} \quad (75)$$

where  $l^\vee$  is Poincaré dual to the sum of the section and fiber of the elliptic fibration (i.e.,  $\mu(l^\vee) = w^\vee$ ),  $\tilde{q}$  is perpendicular to  $l$  and  $l^\vee$  and similarly for  $\tilde{p}$ .

Following the mirror symmetry construction above one finds that on  $X = \hat{S} \times T^2$  we have the following interpretation of these charges:

- $N$  counts the 0-brane charge.

- $\tilde{q}$  counts 2-branes wrapping 2-cycles in  $\hat{S}$ .
- $s$  counts 2-branes wrapping the  $T^2$  factor.
- $\tilde{p}$  counts 4-branes wrapping the  $T^2$  factor and 2-cycles in  $\hat{S}$ .
- $m$  counts 4-branes wrapping  $\hat{S}$ .
- As promised in section 4 there can be no 6-brane charge.

Since we are assuming 0-brane charge dominates, we have  $q^2 \simeq 2Nm$  and  $p^2q^2 \gg (p.q)^2$ . This gives

$$\text{Area} = 4\pi\sqrt{2Nmp^2}. \quad (76)$$

Now we can compute the entropy using the method of section 4. As in [4] we may use an index theorem to compute the Hodge numbers  $h^{p,0}$  in terms of  $\mathcal{F}$ . We obtain

$$S = \pi\sqrt{2Nm(p^2 - \frac{2}{3})}. \quad (77)$$

Thus we have agreement between the area and entropy so long as  $p^2 \gg 1$ . This condition is explained by the discussion at the end of section 2.1. It is easy to compute the area of the  $T^2$  factor:

$$\begin{aligned} \text{Area}(T^2) &= \int_{T^2} \hat{j} \\ &= \frac{\Delta}{p^2}. \end{aligned} \quad (78)$$

Thus the ratio of the area of event horizon to the area of the  $T^2$  factor, which must be large, is  $4\pi p^2$ . If we view this as a type IIB compactification, then the complex structure of  $T^2$  is constrained to not be too large as expected in section 2.1.

Note that the 2-brane charge,  $\tilde{q}$  and  $s$ , plays no rôle in either the entropy or the area so long as the D-brane is dominated by 0-brane charge. If  $Nm$  is not much greater than  $\tilde{q}^2$ , for example, then we will have a non-negligible 2-brane contribution to the area and (76) will not longer be valid. However, in this case we will have other important decay modes involving 2-branes that we have not accounted for and so (77) will be modified too.

## 6 Conclusion

We have described a procedure for computing the entropy of Calabi–Yau black holes in type IIB string theory, at least for large charges. This procedure is simple to state, but in practice is technically challenging. It relies on an ability to analyze the stability of the given D-brane and enumerate the resulting constituent D-branes. In addition, one requires an understanding of the moduli space of special Lagrangians, which we can claim only for a small subsector. Within that context, we have provided evidence that it works for some

simple examples. It should also be noted that this procedure highlights the special nature of Calabi–Yau spaces at attractor points. The analysis relies crucially on the near-horizon superconformal quantum mechanics developed by [3, 4].

Clearly it would be nice to go beyond the examples described in this paper, and compute the entropy of D-branes dominated by some charge other than 0-branes (or their mirrors). That is, we would like to consider probes that are not mirror to 0-branes. Section 3 indicates that this is possible. We argued that the moduli space of probe branes is valued in a line bundle  $\mathcal{L}$  over the moduli space of special Lagrangians, with  $c_1(\mathcal{L})$  given by the Kähler form associated to the natural metric on this moduli space. Although the methods in section 3 cannot be considered rigorous, it is certainly tempting to conjecture that this beautifully simple result is true in general. Counting the probe quantum states is then given by cohomology with values in this bundle.

An obvious place to look for other examples is  $T^2 \times K3$ . The probe 3-brane will be of the form  $S^1 \times C$  for some 2-cycle  $C$  in  $K3$ . In section 5 we described the case where  $C$  is an elliptic curve. One might think the simplest case to consider is  $C \cong S^2$ . The moduli space of such a curve is a point, so the analysis starts to look rather trivial. In fact, for black holes whose charge is dominated by such a D-branes it is easy to prove that  $p^2q^2 - (p.q)^2 < 0$ . This means that the attractor mechanism breaks down and the solution is not a spherically symmetric black hole.

The next case to consider would be a curve  $C$  of genus  $g > 1$ . The moduli space of such curves is understood to an extent following Mukai’s work [32]. In particular it is known that the moduli space is hyperkähler and of complex dimension  $2g$ . This was studied in [27]. It would be interesting to see if some examples in this case could be computed.

A few caveats remain. We have not explained the insight of [4] that the dominant contribution to the entropy comes from states which wrap the horizon a single time. We also have not provided a guess as to the states which give the subleading terms to the entropy (as an expansion in inverse charge). The answers to these questions will surely lead to new insights into the nature of black holes in string theory.

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## References

- [1] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, Phys. Lett. **B379** (1996) 99–104, hep-th/9601029.
- [2] J. M. Maldacena, A. Strominger, and E. Witten, *Black Hole Entropy in M-theory*, JHEP **12** (1997) 002, hep-th/9711053.
- [3] D. Gaiotto, A. Simons, A. Strominger, and X. Yin, *D0-branes in Black Hole Attractors*, hep-th/0412179.
- [4] D. Gaiotto, A. Strominger, and X. Yin, *Superconformal Black Hole Quantum Mechanics*, JHEP **11** (2005) 017, hep-th/0412322.
- [5] D. Gaiotto et al., *D4-D0 Branes on the Quintic*, JHEP **03** (2006) 019, hep-th/0509168.
- [6] R. C. Myers, *Dielectric-branes*, JHEP **12** (1999) 022, hep-th/9910053.
- [7] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror Symmetry is T-Duality*, Nucl. Phys. **B479** (1996) 243–259, hep-th/9606040.
- [8] S. Ferrara, R. Kallosh, and A. Strominger, *N=2 Extremal Black Holes*, Phys. Rev. **D52** (1995) 5412–5416, hep-th/9508072.
- [9] A. Strominger, *Macroscopic Entropy of N = 2 Extremal Black Holes*, Phys. Lett. **B383** (1996) 39–43, hep-th/9602111.
- [10] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54** (1996) 1514–1524, hep-th/9602136.
- [11] G. W. Moore, *Arithmetic and Attractors*, hep-th/9807087.
- [12] J. P. Hsu, A. Maloney, and A. Tomasiello, *Black Hole Attractors and Pure Spinors*, hep-th/0602142.
- [13] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *Measuring Small Distances in N = 2 Sigma Models*, Nucl. Phys. **B420** (1994) 184–242, hep-th/9311042.
- [14] E. R. Sharpe, *Kaehler Cone Substructure*, Adv. Theor. Math. Phys. **2** (1999) 1441–1462, hep-th/9810064.
- [15] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics **31**, Friedrick Vieweg & Son, 1997.
- [16] F. Denef, *Supergravity Flows and D-brane Stability*, JHEP **08** (2000) 050, hep-th/0005049.
- [17] F. Denef, *(Dis)assembling special Lagrangians*, hep-th/0107152.



- [18] D. Joyce, *Special Lagrangian Submanifolds with Isolated Conical Singularities. V. Survey and Applications*, math.DG/0303272.
- [19] P. S. Aspinwall, *D-Branes on Calabi–Yau Manifolds*, in J. M. Maldacena, editor, “Progress in String Theory. TASI 2003 Lecture Notes”, pages 1–152, World Scientific, 2005, hep-th/0403166.
- [20] M. R. Douglas, *D-Branes, Categories and  $N=1$  Supersymmetry*, J. Math. Phys. **42** (2001) 2818–2843, hep-th/0011017.
- [21] K. Becker, M. Becker, and A. Strominger, *Five-branes, Membranes and Nonperturbative String Theory*, Nucl. Phys. **B456** (1995) 130–152, hep-th/9507158.
- [22] R. C. McLean, *Deformations of Calibrated Submanifolds*, Comm. Anal. Geom. **6** (1998) 705–747.
- [23] D. Joyce, *Lectures on Calabi–Yau and Special Lagrangian Geometry*, in “Global Theory of Minimal Surfaces”, Clay Math. Proc. **2**, Amer. Math. Soc., 2005, math.DG/0108088.
- [24] S. Kachru, S. Katz, A. E. Lawrence, and J. McGreevy, *Open String Instantons and Superpotentials*, Phys. Rev. **D62** (2000) 026001, hep-th/9912151.
- [25] W. Taylor and M. Van Raamsdonk, *Multiple D0-Branes in Weakly Curved Backgrounds*, Nucl. Phys. **B558** (1999) 63–95, hep-th/9904095.
- [26] P. S. Aspinwall and M. R. Douglas, *D-Brane Stability and Monodromy*, JHEP **05** (2002) 031, hep-th/0110071.
- [27] R. Dijkgraaf, *Instanton Strings and HyperKähler Geometry*, Nucl. Phys. **B543** (1999) 545–571, hep-th/9810210.
- [28] S. Mukai, *Moduli of Vector Bundles on  $K3$  Surfaces and Symplectic Manifolds*, Sugaku Expositions **1** (1988) 139–173.
- [29] T. Shioda and H. Inose, *On Singular  $K3$  Surfaces*, in W. L. Baily and T. Shioda, editors, “Complex Analysis and Algebraic Geometry”, pages 119–136, Cambridge, 1977.
- [30] P. S. Aspinwall and R. Y. Donagi, *The Heterotic String, the Tangent Bundle, and Derived Categories*, Adv. Theor. Math. Phys. **2** (1998) 1041–1074, hep-th/9806094.
- [31] P. S. Aspinwall,  *$K3$  Surfaces and String Duality*, in C. Efthimiou and B. Greene, editors, “Fields, Strings and Duality, TASI 1996”, pages 421–540, World Scientific, 1997, hep-th/9611137.
- [32] S. Mukai, *Symplectic Structure of the Moduli Space of Sheaves on an Abelian or  $K3$  Surface*, Invent. Math. **77** (1984) 101–116.