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Why π and mrad?

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Today it is generally accepted in the ion source and accelerator communities to communicate emittance data with units of π mm mrad or π cm mrad. The emittance of a beam with 2 cm diameter or width and an opening angle of 200 mrad then is written as $\boldsymbol{E}=100$ π cm mrad. This example shows that π has not been involved when calculating the half-axis' product which is the numerical definition for the emittance. Why should we add π to the dimensions? We will explain that this convention is not at all a logical one and is confusing to all those who are entering the field as freshmen or just do not want to become experts by adjusting to strange rules. We therefore propose to skip the "π" in the dimension and to add characters as sub- or superscripts to the variable $\boldsymbol{E}$, which describe the specific kind of emittance formulation used, e.g., $\boldsymbol{E}_{r,r'}$, $\boldsymbol{E}_{x,x'}$, $\boldsymbol{E}_{\mathrm{rms}}$, $\boldsymbol{E}_{4^*\mathrm{rms}}$, $\boldsymbol{E}_{\mathrm{norm}}$, $\boldsymbol{E}_{\mathrm{area}}$, and $\boldsymbol{E}_{\mathrm{ellipse}}$. In a real case, such an emittance naming could appear as $E^{\mathrm{norm}}_{x,x',4\mathrm{rms}}$. Additionally—to be consistent with cleaning up—the dimension of emittances should be given in centimeters or meters, because rad are ratios and free of dimension. Therefore, instead of using cm mrad it is much more logical to communicate emittance data by writing ε m, where ε is the product of the maximum value for half of the beam thickness in meters and its opening angle in rad.

I. THE ORIGIN OF THE CONFUSION WITH π

In the six-dimensional phase space of coordinates and velocities the volume occupied by noninteracting particles is conserved (Liouville's theorem). Without coupling between the two-dimensional subphase spaces, each phase volume is conserved as well. Sometimes the contour of these two-dimensional areas are ellipses, almost never having a constant density. Under these rare conditions, it could make sense to define the elliptically bounded two-dimensional phase space as an emittance. Generally, however, ion source measurements do not result in an elliptical shape, and the resulting shape is not uniformly populated. A definition which can take into account both the nonelliptical shape as well as a nonuniform distribution has been proposed by Lapostolle.[1] This is a rms-averaging procedure, which has found wide acceptance in the accelerator community mainly because the rms emittance directly enters into the KV equation[2] describing beams with space charge in periodic focusing channels. The rms emittance for the two-dimensional phase space x, x' is given by Eq. (1):

$$E_{xx',\mathrm{rms}}=\frac{\sum_i I_i\sqrt{\overline{x_i^2x_i'^2}-\overline{(x_ix_i')}^2}}{\sum_i I_i}, \tag{1}$$

where i stands for the particle number i in simulations or the area i of the emittance slit and I_i is the current given to particle i or the current measured in slit i. In the case that the emittance pattern is symmetric with respect to both axes, the second term in the square root disappears. Then the rms emittance will be given by the product of the average absolute x value and the average absolute x' value of all particles, if I_i has the same value for all i. This is easy to provide in simulations, but almost never happens in measurements.

Historically there has been a transition from elliptically bounded phase spaces in the mid of the last century to rms-weighted half-axis products later. Naturally this has caused a lot of confusion and always needed the additional information, how the given emittance value should be understood. In asking "good old friends" who were active in that period, we have found the following explanation for the decorating of the dimensions by π: "If π is written, we know that it has not entered into the number given," hence the number is the half-axis product (see Fig. 1); omitting the π then means that it is not the half-axis product, but the ellipse area. Clearly, the π is an additional encryption signal, telling the reader how to understand the numbers. Therefore it should not appear as a factor to the dimensions, nor as a factor to the number given. Everywhere in science where mathematical expressions are used, sub- and superscripts have been introduced for this purpose. We should do the same for distinguishing

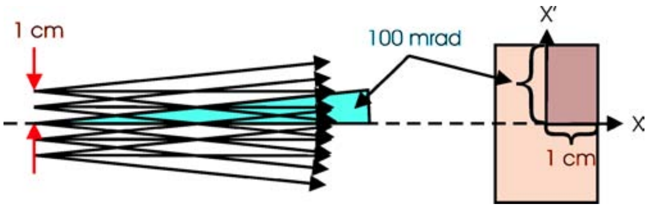

FIG. 1. Today's definition of a "100 π cm mrad" emittance and the phase-space area for it.

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between an emittance designating an ellipse area or giving the half-axis product by writing E_{ellipse} or $E_{xx'}$. Once we have started in this way, we can place all other flavors, such as rms, the normalization, etc., also into sub- and superscripts.

II. HOW ABOUT MRAD?

At first glance, adding mrad or rad to the dimension of an emittance value looks like a common procedure in physics: Whenever we multiply physical quantities, the dimension becomes the product of the dimensions of the quantities. The difference here is, however, that the beam divergence expressed in rad is found as the ratio of two lengths, e.g., the transverse shift of a second slit with respect to a first one divided by the distance of the slits (see Fig. 2). By this division the dimension then drops out.

Now we should ask if any feature will be missing if "mrad" or "rad" will not appear in the dimension of the emittance. In the case that the standard dimension appendix would have been rad, we probably had dropped it a long time before. It is the mrad which seems to have appeal: It drives our imagination directly to a small angle and reading " 100 cm mrad" tells us that a beam of 1 cm radius has a divergence of 100 mrad with respect to its axis. Seriously, this "appeal" is not worth to break with general rules for communicating physical quantities. It is much more convenient to write for the elementary charge $e=1.602\times10^{-19}$ A s than $e=1.602\times10^{-16}$ mA s or $e=1.602\times10^{-10}$ nA s.

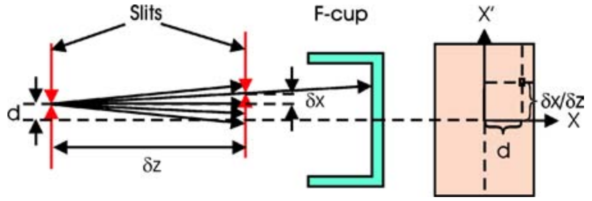

FIG. 2. Emittance measurement with two slits: The ion current in the Faraday cup determines the density in position d (transverse displacement of the first slit) and angle $\delta x/\delta z$ of the phase space. δx is the transverse displacement of the second slit with respect to the first one and δz is the distance between the slits.

III. DISCUSSION

π and mrad should be eliminated from the dimension of emittance data. Instead of writing $E_{\text{rms}}=100\ \pi$ cm mrad for the two-dimensional emittance for the x,x' phase space we should use $E_{xx',\text{rms}}=10^{-3}$ m. This proposal for a new writing rule is much more in agreement with normal scientific syntax and less confusing. It also agrees with the common syntax used in the high-energy accelerator community.

Noted added in proof: An appealing proposal for the change of emittance dimensions has been made recently by Rick Baartman, TRIUMF, who suggested to replace mm mrad by μ (microns). This leaves the given numbers for emittances unchanged or makes them just 10 times larger, if cm mrad had been used before.

[1] P. Lapostolle, IEEE Trans. Nucl. Sci. **18**, 1101 (1971).
[2] I. Kapchinski and V. Vladimirskij, Proceedings of the International Conference on High Energy Accelerators, CERN, 1959 (unpublished), p. 274.