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Analysis of Laser-Driven Particle Acceleration from Planar Transparent Boundaries*

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Abstract

This article explores the interaction between a monochromatic plane wave laser beam and a relativistic electron in the presence of a thin dielectric transparent boundary. It is found that the sign of the interaction between the laser and the electron in the downstream space is determined by the optical phase delay of the laser caused by the boundary, and that it can add to or cancel the interaction in the upstream space. Both the inverse-transition radiation picture and the electric field path integral method show this result.

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Introduction

The thin infinite boundary geometry for laser-driven particle acceleration is an effective and simple method to satisfy the Lawson-Woodward theorem, and was successfully employed in the first proof-of-principle experiment for laser acceleration in vacuum [1]. Although the interaction between the laser and the electron in this geometry is poor, due to its simplicity the infinite thin boundary scheme is ideal for studying the basic physical mechanisms for laser-driven particle acceleration in vacuum. In essence, different materials for the thin boundary allow us to explore three different boundary conditions for the electromagnetic field:

- Reflective
- Absorptive
- Transparent

Figure 1 illustrates the three cases of interest. For the proof-of-principle experiment a reflective gold-coated Kapton tape was employed as a boundary, corresponding to Figure 1(a).

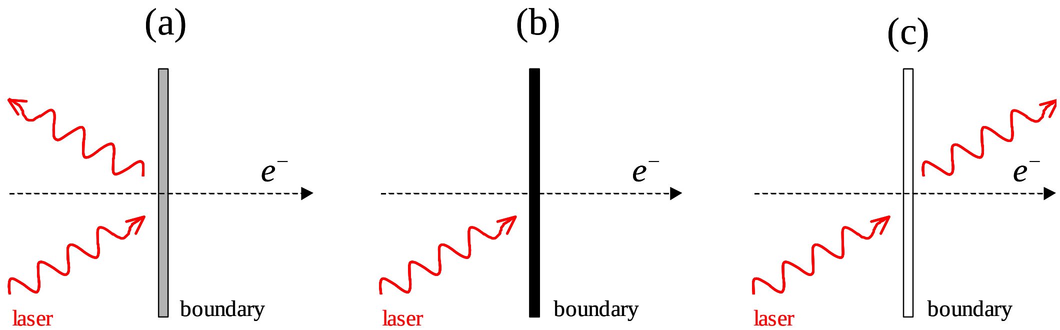

Fig. 1: laser acceleration from a (a) reflective boundary, (b) absorptive "black" boundary, and (c) a lossless transparent boundary

Cases (a) and (b) have been explored in a previous article [2], and it was found that for relativistic electrons the expected energy gain from the laser is

$$\Delta U = \frac{qE_0\lambda}{\pi}\frac{\alpha}{\alpha^2 + 1/\gamma^2}\cos\rho\sin\varphi_L \qquad 1$$

where q is the electric charge of the particle, E_0 is the amplitude of the electric field of the laser plane wave having the form $\vec{E}\,(\vec{r},t) = E_0\hat{P}_L(\rho)\cos(\vec{k}\cdot\vec{r} - \omega_0 t + \varphi_L)$, λ is the laser wavelength, α is the laser-electron beam crossing angle, ρ is the polarization angle of the laser, and φ_L is the initial optical phase of the laser.

It was found that for the reflective boundary the Inverse Radiation method and the Path Integral method predict the same electron energy gain of equation 1, regardless of the boundary orientation. For the case of the absorbing boundary however, the Inverse Radiation picture for calculating energy gain does not apply.

In this article we analyze the expected energy gain for relativistic electrons from a transparent boundary by both the Inverse Radiation method and the Path Integral method. For simplicity we will focus at boundaries oriented at near-Brewster angle for the laser plane wave, such that there is no reflected laser beam.

The Path Integral Method

For the analysis we will use the geometry depicted in Figure 2.

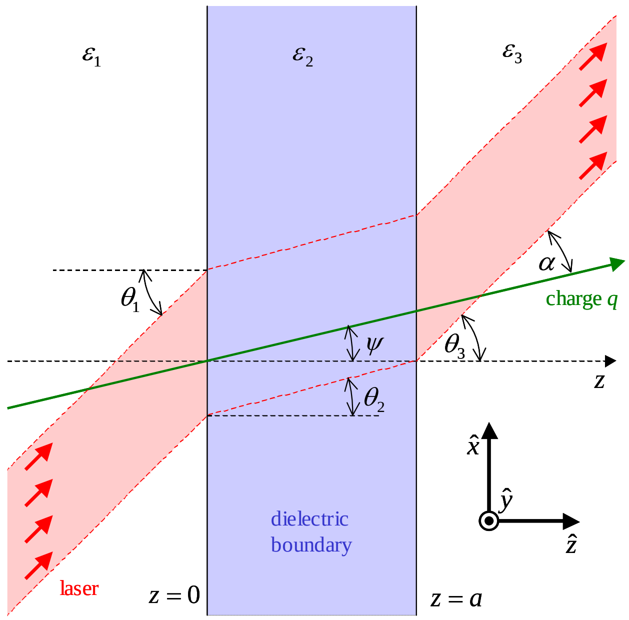

Fig. 2: The laser-electron beam interaction in the presence of three different dielectric layers

The three laser plane waves in media 1,2 and 3 are given by

$$\begin{aligned}
E_1(x,z,t) &= (E_0/\varepsilon_1)\cos(kn_1(\sin\theta_1 x + \cos\theta_1 z) - \omega t + \phi_1) \\
E_2(x,z,t) &= (E_0/\varepsilon_2)\cos(kn_2(\sin\theta_2 x + \cos\theta_2 z) - \omega t + \phi_2) \\
E_3(x,z,t) &= (E_0/\varepsilon_3)\cos(kn_3(\sin\theta_3 x + \cos\theta_3 z) - \omega t + \phi_3)
\end{aligned} \qquad 2$$

where E_0 is the amplitude of the electric field in vacuum and $\varepsilon_j = 1 + \chi_j$ are the relative dielectric permittivities. Here the indices of refraction are $n_j \sim \sqrt{\varepsilon_j}$ (as commonly done, I assume that the magnetic permeability of the dielectric material is no different from that in vacuum; $\mu_j \sim \mu_0$). As shown in Figure 2, θ_j is the angle of the plane wave in medium j with respect to the z-axis. k is defined by $k = 2\pi/\lambda = \omega/c$ where λ is the wavelength in vacuum. For continuity of the fields at the boundary z=0 we require that $\phi_2 = \phi_1$ and at

$z=a$ we require that $ka\cdot n_2\cos\theta_2+\phi_2=ka\cdot n_3\cos\theta_3+\phi_3$. Let the initial laser phase be $\phi_1=\phi_L$. The phase of the plane wave in medium 3 becomes

$$\phi_3=ka\left(n_2\cos\theta_2-n_3\cos\theta_3\right)+\phi_1 \qquad 3$$

In our case we will assume vacuum both upstream and downstream of the plate. Hence $n_1=n_3=1$, $\varepsilon_1=\varepsilon_3=1$, $\theta_1=\theta_3\equiv\theta_L$, and define $n_2\equiv n$, $\varepsilon_2\equiv\varepsilon$, $\theta_2\equiv\theta_m$. Therefore the fields can be expressed as

$$\begin{aligned}E_1(x,z,t)&=E_0\cos\left(k\left(\sin\theta_L x+\cos\theta_L z\right)-\omega t+\phi_L\right)\\ E_2(x,z,t)&=\left(E_0/\varepsilon\right)\cos\left(kn\left(\sin\theta_m x+\cos\theta_m z\right)-\omega t+\phi_L\right)\\ E_3(x,z,t)&=E_0\cos\left(k\left(\sin\theta_3 x+\cos\theta_3 z\right)-\omega t+\phi_3\right)\end{aligned} \qquad 4$$

Interaction in the upstream space

From Figure 2 we can see that the electric field component parallel to the electron beam trajectory is $E_{z'}=-E_1\sin\alpha$ where $\alpha=\theta_L-\psi$. Therefore

$$E_{z'}(x(z),z,t)=-\sin\alpha E_0\cos\left(k\left(\sin\theta x+\cos\theta z\right)-\omega t+\phi_L\right) \qquad 5$$

The electron beam orbit is described by $x(z)=z\tan\psi$. Therefore the parallel electric field component is described by

$$E_{z'}(x(z),z,t)=-\sin\alpha E_0\cos\left(k\left(\sin\theta_L\tan\psi+\cos\theta_L\right)z-\omega t+\phi_L\right) \qquad 6$$

Assuming uniform electron velocity the time t can be expressed as $t(z)=z/\beta c\cos\psi$. This gives

$$E_{z'}(x(z),z,t)=-\sin\alpha E_0\cos\left(kz\left(\frac{\cos\alpha-1/\beta}{\cos\psi}\right)+\phi_L\right) \qquad 7$$

Therefore the work done on the particle by this field is

$$\begin{aligned}\Delta U_1&=\int_{-\infty}^{0}qE_{z'}(x(z),z,t)ds(z)=\int_{-\infty}^{0}qE_{z'}(z,t)dz/\cos\psi\\ &=-q\sin\alpha E_0\int_{-\infty}^{0}\cos\left(kz\left(\frac{\cos\alpha-1/\beta}{\cos\psi}\right)+\phi_L\right)\frac{dz}{\cos\psi}\end{aligned} \qquad 8$$

Using $\lim_{\kappa\to 0}\int_0^\infty \cos(u+\phi)e^{-\kappa u}du = -\sin\phi$ we find

$$\Delta U_1 = -\frac{\sin\alpha\sin\phi_L qE_0}{k(\cos\alpha - 1/\beta)} \qquad 9$$

In the relativistic limit where $\beta \sim 1 - 1/2\gamma^2$ and $|\alpha| << 1$ ΔU becomes

$$\Delta U_1 = \frac{qE_0\lambda}{\pi}\frac{\alpha}{\alpha^2 + 1/\gamma^2}\sin\phi_L \qquad 10$$

which is, not surprisingly, the same energy gain as the one calculated from the reflective or absorptive boundary. At the optimum crossing angle of $\alpha = \pm 1/\gamma$ the energy gain has a maximum value of

$$\Delta U_{1,\max} = \frac{qE_0\lambda\gamma}{2\pi} \qquad 11$$

Interaction in the medium

Inside the medium the electric field component parallel to the electron beam is

$$E_{z'}(x(z), z, t) = -\sin\alpha_m\, E_0/\varepsilon\cos(kn(\sin\theta_m x + \cos\theta_m z) - \omega t + \phi_L) \qquad 12$$

where $\alpha_m = \theta_m - \psi$, and θ_m is related to θ by Snell's law. By analogy to the upstream case the work done on the particle in this region is

$$\begin{aligned}\Delta U_2 &= \int_0^a qE_{z'}(z,t)\,dz/\cos\psi \\ &= -q\sin\alpha_m\, E_0/\varepsilon\int_0^a \cos\left(kz\left(\frac{n\cos\alpha_m - 1/\beta}{\cos\psi}\right) + \phi_L\right)\frac{dz}{\cos\psi}\end{aligned} \qquad 13$$

which simplifies to

$$\Delta U_2 = \frac{q\lambda\sin\alpha_m\, E_0/\varepsilon}{2\pi(n\cos\alpha_m - 1/\beta)}\left(\sin\phi_L - \sin\left(ka\left(\frac{n\cos\alpha_m - 1/\beta}{\cos\psi}\right) + \phi_L\right)\right) \qquad 14$$

We assume that the laser crossing angle is optimized for the interaction in the vacuum $\alpha = 1/\gamma$. Then the interaction in the medium ΔU_2 has a maximum value of

$$\Delta U_{2,\max} = \frac{q\lambda E_0}{\pi}\left|\frac{1}{\varepsilon}\left(\frac{\sin\alpha_m}{n\cos\alpha_m - 1/\beta}\right)_{\max}\right| = \frac{q\lambda E_0}{\pi} f(n) \qquad 15$$

As shown in Figure 3 *f(n)* is smaller than 1 and drops with increasing index *n*. As can be seen from the plots for γ =50, γ =100 and γ =1000 *f(n)* shows only a very weak dependence on γ.

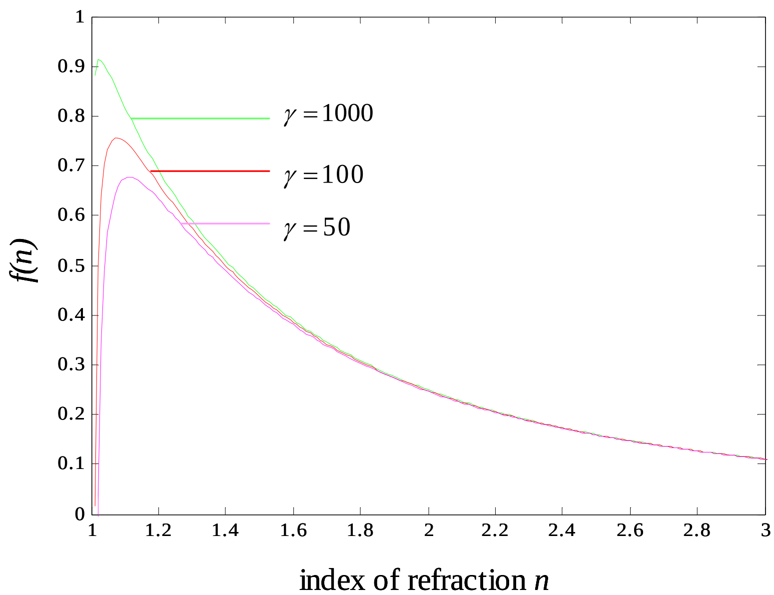

Fig. 3: *f(n)* versus *n* for various values of γ

Therefore under these circumstances $\Delta U_{2,\max} \sim \Delta U_{1,\max}/\gamma$ and the maximum interaction in the medium is much smaller than the maximum interaction in the vacuum. Notice that this value for the energy gain is independent of the thickness of the dielectric plate.

Interaction in the downstream space

Here the parallel electric field component is given by

$$E_{z'}(x(z), z, t) = -\sin\alpha E_0 \cos\left(k\left(\sin\theta_L x + \cos\theta_L z\right) - \omega t + \phi_3\right) \qquad 16$$

and the work done on the particle is

$$
\begin{aligned}
\Delta U_3 &= \int_a^\infty qE_{z'}(x(z),z,t)ds(z) \\
&= -q\sin\alpha E_0 \int_a^\infty \cos\left(kz\left(\frac{\cos\alpha - 1/\beta}{\cos\psi}\right)+\phi_3\right)\frac{dz}{\cos\psi} \\
&= -\frac{q\sin\alpha E_0}{k(\cos\alpha - 1/\beta)}\int_A^\infty \cos(u+\phi_3)du, \qquad A = ka\left(\frac{\cos\alpha - 1/\beta}{\cos\psi}\right) \\
&= \frac{q\sin\alpha E_0}{k(\cos\alpha - 1/\beta)}\sin(A+\phi_3) \\
&\sim -\frac{qE_0\lambda}{\pi}\frac{\alpha}{\alpha^2+1/\gamma^2}\sin\left(ka\left(\frac{\cos\alpha - 1/\beta}{\cos\psi}+n\cos\theta_m - \cos\theta\right)+\phi_L\right)
\end{aligned} \tag{17}
$$

The total energy gain is the sum of the interactions in the three regions $\Delta U = \Delta U_1 + \Delta U_2 + \Delta U_3$. However as observed before $\Delta U_{2,\max} << \Delta U_{1,\max}$ and the contribution from the energy gain in the medium can be neglected. Hence the total energy gain is approximately $\Delta U \sim \Delta U_1 + \Delta U_3$.

$$
\Delta U \sim \frac{qE_0\lambda}{\pi}\frac{\alpha}{\alpha^2+1/\gamma^2}\left\{\sin\phi_L - \sin\left(ka\left(\frac{\cos\alpha - 1/\beta}{\cos\psi}+n\cos\theta_m - \cos\theta\right)+\phi_L\right)\right\} \tag{18}
$$

Notice that the energy gain still has the same linear dependence on the wavelength λ and on the electric field amplitude E_0, and shows the same dependence on the laser-crossing angle as the energy gain from the reflective or absorptive boundary of equation 1. The difference lies in the term shown in brackets in equation 18. This term shows the interference of two phase terms, one of which depends on the optical phase retardation caused by the boundary. To illustrate this it is convenient to rewrite equation 17 as

$$
\Delta U \sim \frac{qE_0\lambda}{\pi}\frac{\alpha}{\alpha^2+1/\gamma^2}\{\sin\phi_L - \sin(\phi_{\text{ret}}+\phi_L)\} \tag{19}
$$

where the optical phase retardation term ϕ_{ret} is

$$
\phi_{\text{ret}} = ka\left(\frac{\cos\alpha - 1/\beta}{\cos\psi}+n\cos\theta_m - \cos\theta_L\right) \tag{20}
$$

Notice that ϕ_{ret} is proportional to the relative plate thickness with respect to the wavelength $\phi_{\text{ret}} \propto a/\lambda$. When $\alpha \sim 1/\gamma$ the first term is very small and can be neglected.

$$
\phi_{\text{ret}} \sim ka(n\cos\theta_m - \cos\theta_L) \tag{21}
$$

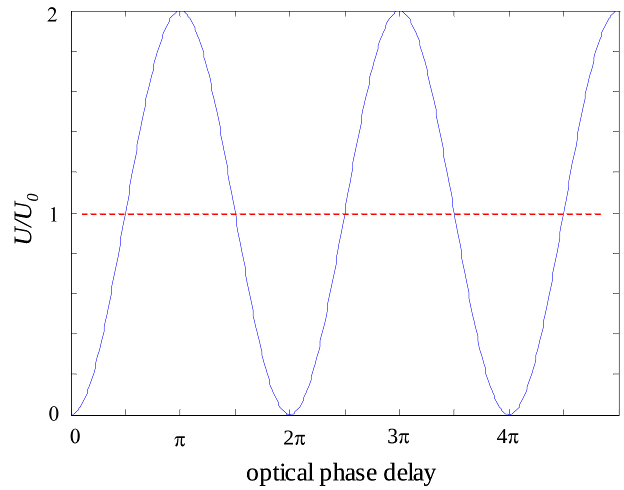

Fig. 4: Comparison between the energy gain from a reflective boundary (dashed line) and the energy gain from a transparent boundary (solid line) as a function of the optical phase delay

Figure 4 illustrates the effect of the optical phase retardation on the energy gain from transparent boundaries. Whenever the sign of the field is reversed the energy gain in the upstream and in the downstream region add constructively. The maximum value for the total energy gain is

$$\Delta U(\alpha = 1/\gamma) \sim \frac{q\gamma E_0 \lambda}{\pi} \qquad 22$$

The Inverse-Radiation method

As discussed elsewhere [2] this method is based on Poynting's Theorem and assumes no transfer of energy from or to the medium itself. Under these circumstances the energy gain can be shown to be [3,4]

$$\Delta U = -\frac{1}{\pi Z_0} \int_{-\infty}^{\infty} \oint_S \left(\vec{E}_L(\omega) \cdot \vec{E}^*{}_{TR}(\omega) \right) ds\, d\omega \qquad 23$$

where $\vec{E}_L$ is the laser field and $\vec{E}_{TR}$ is the particle's wake field. In this instance $\vec{E}_{TR}$ corresponds to the transition radiation from a dielectric plate. This particular transition radiation problem has been analyzed by various authors in the past [5]. In our particular case we are not interested in the most general expression of $\vec{E}_{TR}$ and can derive a simplified form for the field. Using the same approach as in [5] by performing a plane wave decomposition of the fields to find the transition radiation field. First we find the plane wave spectrum of the field of a uniformly moving particle of charge q. Inside a medium the potentials have to satisfy

$$\nabla \times \vec{A} - \mu\varepsilon d_t^2 \vec{A} = -\mu\vec{J} = -\mu q\vec{v}\,\delta(\vec{r} - \vec{v}t)$$
$$\nabla^2\Phi - -\mu\varepsilon d_t^2\Phi = -\rho/\varepsilon = -q/\varepsilon\,\delta(\vec{r} - \vec{v}t) \qquad 24$$

We perform a Fourier transformation on equation 24 and find that

$$\tilde{A} = \frac{2\pi q\mu\vec{v}}{k^2 - \omega^2\mu\varepsilon}\delta\left(\omega - \vec{v}\cdot\vec{k}\right)$$
$$\tilde{\Phi} = \frac{2\pi\, q/\varepsilon}{k^2 - \omega^2\mu\varepsilon}\delta\left(\omega - \vec{v}\cdot\vec{k}\right) \qquad 25$$

Using $\tilde{E} = i\omega\tilde{A} - i\vec{k}\tilde{\Phi}$ we find for the plane wave spectrum of the particle's electric field

$$\tilde{E}_{p,j} = 2\pi i q Z_0 c\frac{\omega\vec{v}/c^2 - \vec{k}/\varepsilon_j}{k^2 - \omega^2/c^2\,\varepsilon_j}\delta\left(\omega - \vec{v}\cdot\vec{k}\right) \qquad 26$$

where $\varepsilon_j = 1 + \chi_j$ and $Z_0 = \mu c$ is the vacuum impedance. $\tilde{E}_{p,j}$ represents the plane wave spectrum of the particle's field in the medium j. To match the boundary conditions at the front and back interfaces of the dielectric plate we add a set of plane waves (the homogeneous solutions to equation (24) as shown in Figure 5

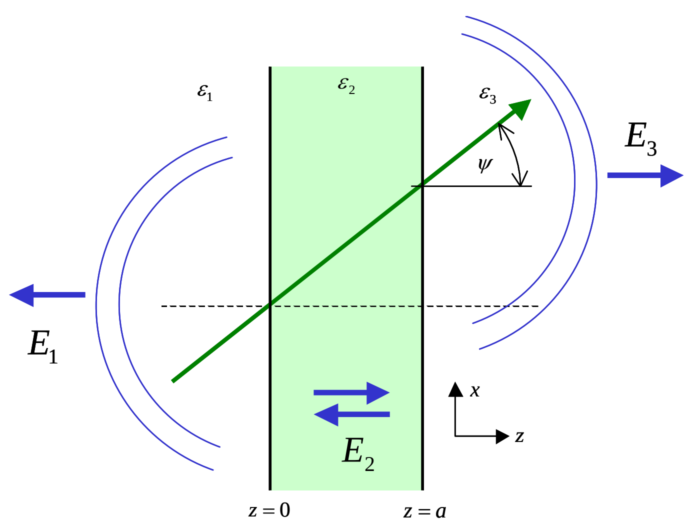

Fig.5: Transition radiation from a plate; Plane wave field components

The plane waves satisfy $\nabla\cdot\vec{E} = 0$ or alternatively $\vec{k}\cdot\tilde{E} = 0$. Furthermore their amplitudes will be chosen such that the electric field boundary conditions at z=0 and at z=a are satisfied. These are

$$
\begin{aligned}
\varepsilon_1\left(\vec{E}_1(\vec{r},t)+\vec{E}_{p,1}(\vec{r},t)\right)_\perp\Big|_{z=0} &= \varepsilon_2\left(\vec{E}_2(\vec{r},t)+\vec{E}_{p,2}(\vec{r},t)\right)_\perp\Big|_{z=0} \\
\left(\vec{E}_1(\vec{r},t)+\vec{E}_{p,1}(\vec{r},t)\right)_{\|}\Big|_{z=0} &= \left(\vec{E}_2(\vec{r},t)+\vec{E}_{p,2}(\vec{r},t)\right)_{\|}\Big|_{z=0} \\
\varepsilon_3\left(\vec{E}_3(\vec{r},t)+\vec{E}_{p,3}(\vec{r},t)\right)_\perp\Big|_{z=a} &= \varepsilon_2\left(\vec{E}_2(\vec{r},t)+\vec{E}_{p,2}(\vec{r},t)\right)_\perp\Big|_{z=a} \\
\left(\vec{E}_3(\vec{r},t)+\vec{E}_{p,3}(\vec{r},t)\right)_{\|}\Big|_{z=a} &= \left(\vec{E}_2(\vec{r},t)+\vec{E}_{p,2}(\vec{r},t)\right)_{\|}\Big|_{z=a}
\end{aligned} \qquad 27
$$

In terms of the plane wave components these boundary conditions become

$$
\begin{aligned}
\int \varepsilon_1\left(\tilde{E}_{1z}+\tilde{E}_{p,1z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} &= \int \varepsilon_2\left(\tilde{E}_{2z}+\tilde{E}_{p,2z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} \\
\int \left(\tilde{E}_{1x}+\tilde{E}_{p,1x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} &= \int \left(\tilde{E}_{2x}+\tilde{E}_{p,2x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} \\
\int \left(\tilde{E}_{1y}+\tilde{E}_{p,1y}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} &= \int \left(\tilde{E}_{2y}+\tilde{E}_{p,2y}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} \\
\int \varepsilon_3\left(\tilde{E}_{3z}+\tilde{E}_{p,3z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=a} &= \int \varepsilon_2\left(\tilde{E}_{2z}+\tilde{E}_{p,2z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=a} \\
\int \left(\tilde{E}_{3x}+\tilde{E}_{p,3x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=a} &= \int \left(\tilde{E}_{2x}+\tilde{E}_{p,2x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=a} \\
\int \left(\tilde{E}_{3y}+\tilde{E}_{p,3y}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=0} &= \int \left(\tilde{E}_{2y}+\tilde{E}_{p,2y}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}d^3k\,d\omega\Big|_{z=a}
\end{aligned} \qquad 28
$$

We are interested in the plane waves that will overlap with the laser field that is horizontally polarized and traveling in the x-z plane. Hence we are not interested in the values of $\tilde{E}_{1y}$ and $\tilde{E}_{3y}$. Furthermore only the horizontally traveling plane wave components, for which $k_y = 0$, can overlap with the laser in the far field. Therefore for this special set of plane waves the boundary conditions simplify to

$$
\begin{aligned}
\int \varepsilon_1\left(\tilde{E}_{1z}+\tilde{E}_{p,1z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=0} &= \int \varepsilon_2\left(\tilde{E}_{2z}+\tilde{E}_{p,2z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=0} \\
\int \left(\tilde{E}_{1x}+\tilde{E}_{p,1x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=0} &= \int \left(\tilde{E}_{2x}+\tilde{E}_{p,2x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=0} \\
\int \varepsilon_3\left(\tilde{E}_{3z}+\tilde{E}_{p,3z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=a} &= \int \varepsilon_2\left(\tilde{E}_{2z}+\tilde{E}_{p,2z}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=a} \\
\int \left(\tilde{E}_{3x}+\tilde{E}_{p,3x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=a} &= \int \left(\tilde{E}_{2x}+\tilde{E}_{p,2x}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}dk_x dk_z d\omega\Big|_{z=a}
\end{aligned} \qquad 29
$$

The plane waves are of the form $\tilde{E} = f(\vec{k},\omega)\delta\left(\vec{k}\cdot\vec{k} - \omega^2/c^2\varepsilon_j\right)$. Hence integrating over k_z eliminates the delta functions appearing in the field components and we obtain

$$
\begin{aligned}
\int \varepsilon_1\left(\tilde{E}_{1z}(-\kappa_1)+\tilde{E}_{p,1z}(r)\right)e^{i(k_x x-\omega t)}dk_x d\omega &= \int \varepsilon_2\left(\tilde{E}_{2z}(r)+\tilde{E}_{p,2z}(\pm\kappa_2)\right)e^{i(k_x x-\omega t)}dk_x d\omega \\
\int\left(\tilde{E}_{1x}(-\kappa_1)+\tilde{E}_{p,1x}(r)\right)e^{i(k_x x-\omega t)}dk_x d\omega &= \int\left(\tilde{E}_{2x}(r)+\tilde{E}_{p,2x}(\pm\kappa_2)\right)e^{i(k_x x-\omega t)}dk_x d\omega \\
\int \varepsilon_3\left(\tilde{E}_{3z}(\kappa_3)e^{ia\kappa_3}+\tilde{E}_{p,3z}(r)e^{iar}\right)e^{i(k_x x-\omega t)}dk_x d\omega &= \int \varepsilon_2\left(\tilde{E}_{2z}(\pm\kappa_2)e^{\pm ia\kappa_2}+\tilde{E}_{p,2z}(r)e^{iar}\right)e^{i(k_x x-\omega t)}dk_x d\omega \\
\int\left(\tilde{E}_{3x}(\kappa_3)e^{ia\kappa_3}+\tilde{E}_{p,3x}(r)e^{iar}\right)e^{i(k_x x-\omega t)}dk_x d\omega &= \int\left(\tilde{E}_{2x}(\pm\kappa_2)e^{\pm ia\kappa_2}+\tilde{E}_{p,2x}(r)e^{iar}\right)e^{i(k_x x-\omega t)}dk_x d\omega
\end{aligned}
\qquad 30
$$

where $r=\omega/v_z - v_x k_x/v_z$ and $\kappa_j=\sqrt{\omega^2/c^2\varepsilon_j - k_x{}^2}$ is the k_z value where the delta function $\delta\left(\vec{k}\cdot\vec{k}-\omega^2/c^2\varepsilon_j\right)$ is zero. Since we assume that the plane wave fields $\tilde{E}_{1z}$ and $\tilde{E}_{1x}$ are moving to the left (see Figure 5) we take only the negative root $k_z=-\kappa_1$ while for the fields in the downstream space $\tilde{E}_{3z}$ and $\tilde{E}_{3x}$ that are moving to the right we take the positive root $k_z=+\kappa_3$. For the fields inside the plate we have to assume that there are components traveling in both directions. Therefore it will be convenient to label the plane wave components in medium 2 separately

$$
\begin{aligned}
\tilde{E}_{2x}(k_x,\omega;k_z=+\kappa_2) &\equiv \tilde{E}_{2px}(k_x,\omega) \\
\tilde{E}_{2z}(k_x,\omega;k_z=+\kappa_2) &\equiv \tilde{E}_{2pz}(k_x,\omega) \\
\tilde{E}_{2x}(k_x,\omega;k_z=-\kappa_2) &\equiv \tilde{E}_{2nx}(k_x,\omega) \\
\tilde{E}_{2z}(k_x,\omega;k_z=-\kappa_2) &\equiv \tilde{E}_{2nz}(k_x,\omega)
\end{aligned}
\qquad 31
$$

The set of equations in 30 has to be valid for all *x* and *t*. This is satisfied when the arguments of the Fourier transformations are equal for all k_x and ω. Hence

$$
\begin{aligned}
\varepsilon_1\left(\tilde{E}_{1z}(-\kappa_1)+\tilde{E}_{p,1z}(r)\right) &= \varepsilon_2\left(\tilde{E}_{2pz}(\kappa_2)+\tilde{E}_{2nz}(-\kappa_2)+\tilde{E}_{p,2z}(r)\right) \\
\tilde{E}_{1x}(-\kappa_1)+\tilde{E}_{p,1x}(r) &= \tilde{E}_{2px}(\kappa_2)+\tilde{E}_{2nx}(-\kappa_2)+\tilde{E}_{p,2x}(r) \\
\varepsilon_3\left(\tilde{E}_{3z}(\kappa_3)e^{ia\kappa_3}+\tilde{E}_{p,3z}(r)e^{iar}\right) &= \varepsilon_2\left(\tilde{E}_{2pz}(\kappa_2)e^{+ia\kappa_2}+\tilde{E}_{2nz}(-\kappa_2)e^{-ia\kappa_2}+\tilde{E}_{p,2z}(r)e^{iar}\right) \\
\tilde{E}_{3x}(\kappa_3)e^{ia\kappa_3}+\tilde{E}_{p,3x}(r)e^{iar} &= \tilde{E}_{2px}(\kappa_2)e^{+ia\kappa_2}+\tilde{E}_{2nx}(-\kappa_2)e^{-ia\kappa_2}+\tilde{E}_{p,2x}(r)e^{iar}
\end{aligned}
\qquad 32
$$

Since the homogeneous plane wave solutions satisfy $\vec{k}\cdot\tilde{E}=0$ we can relate $\tilde{E}_{jz}$ to $\tilde{E}_{jx}$ by

$$
\tilde{E}_{jz} = -\frac{k_x}{k_z}\tilde{E}_{jx} \qquad 33
$$

Now we use the specific values of $k_z = \pm\kappa_j$ to express the planar field components and keep track of the sign of k_z manually, depending on the direction of the plane wave component. We have

$$\begin{aligned}
\tilde{E}_{1z} &= +k_x/\kappa_1\,\tilde{E}_{1x}\\
\tilde{E}_{3z} &= -k_x/\kappa_3\,\tilde{E}_{3x}\\
\tilde{E}_{2nz} &= +k_x/\kappa_2\,\tilde{E}_{2nx}\\
\tilde{E}_{2pz} &= -k_x/\kappa_2\,\tilde{E}_{2px}
\end{aligned} \qquad 34$$

It is convenient to define the following

$$\begin{aligned}
b_j &= \frac{2\pi iqZ_0c}{v_z}\frac{1}{k_x^{\ 2} + r^2 - \omega^2/c^2\,\varepsilon_j}\\
h_j &= \frac{\omega v_x}{c^2} - \frac{k_x}{\varepsilon_j}\\
s_j &= \frac{\omega v_z}{c^2} - \frac{\omega}{v_z\varepsilon_j} + \frac{v_xk_x}{\varepsilon_j v_z}
\end{aligned} \qquad 35$$

where $r = \omega/v_z - v_xk_x/v_z$ (as defined before). With these definitions and the relation of the components of equation 34 the boundary conditions of equation 32 become

$$\begin{aligned}
s_1b_1\varepsilon_1 + \left(\frac{\varepsilon_1k_x}{2\kappa_1^{\ 2}}\right)\tilde{E}_{1x} &= s_2b_2\varepsilon_2 + \left(\frac{\varepsilon_2k_x}{2\kappa_2^{\ 2}}\right)\left(-\tilde{E}_{2px} + \tilde{E}_{2nx}\right)\\
h_1b_1 + \left(\frac{1}{2\kappa_1}\right)\tilde{E}_{1x} &= h_2b_2 + \left(\frac{1}{2\kappa_2}\right)\left(+\tilde{E}_{2px} + \tilde{E}_{2nx}\right)\\
s_3b_3\varepsilon_3e^{iar} - \left(\frac{\varepsilon_3k_x}{2\kappa_3^{\ 2}}\right)\tilde{E}_{3x}e^{ia\kappa_3} &= s_2b_2\varepsilon_2e^{iar} + \left(\frac{\varepsilon_2k_x}{2\kappa_2^{\ 2}}\right)\left(-\tilde{E}_{2px}e^{ia\kappa_2} + \tilde{E}_{2nx}e^{-ia\kappa_2}\right)\\
h_3b_3e^{iar} + \left(\frac{1}{2\kappa_3}\right)\tilde{E}_{3x}e^{ia\kappa_3} &= h_2b_2e^{iar} + \left(\frac{1}{2\kappa_2}\right)\left(+\tilde{E}_{2px}e^{ia\kappa_2} + \tilde{E}_{2nx}e^{-ia\kappa_2}\right)
\end{aligned} \qquad 36$$

We now have a set of four linear equations and four unknowns. Using the first pair we can express $\tilde{E}_{2px}$ and $\tilde{E}_{2nx}$ in terms of $\tilde{E}_{1x}$

$$\begin{aligned}
\tilde{E}_{2px}\left(\frac{\varepsilon_2k_x}{\kappa_2^{\ 2}}\right) &= -\left(s_1b_1\varepsilon_1 - s_2b_2\varepsilon_2\right) + \left(\frac{\varepsilon_2k_x}{2\kappa_2}\right)\left(h_1b_1 - h_2b_2\right) + \frac{k_x}{2\kappa_1}\left(\frac{\varepsilon_2}{\kappa_2} - \frac{\varepsilon_1}{\kappa_1}\right)\tilde{E}_{1x}\\
\tilde{E}_{2nx}\left(\frac{\varepsilon_2k_x}{\kappa_2^{\ 2}}\right) &= +\left(s_1b_1\varepsilon_1 - s_2b_2\varepsilon_2\right) + \left(\frac{\varepsilon_2k_x}{2\kappa_2}\right)\left(h_1b_1 - h_2b_2\right) + \frac{k_x}{2\kappa_1}\left(\frac{\varepsilon_2}{\kappa_2} + \frac{\varepsilon_1}{\kappa_1}\right)\tilde{E}_{1x}
\end{aligned} \qquad 37$$

We can do a similar thing with the second pair of boundary conditions and express $\tilde{E}_{2px}$ and $\tilde{E}_{2nx}$ in terms of $\tilde{E}_{3x}$.

$$\begin{aligned}\tilde{E}_{2px}\left(\frac{\varepsilon_2 k_x}{\kappa_2{}^2}\right) &= \left(-\left(s_3 b_3 \varepsilon_3 - s_2 b_2 \varepsilon_2\right) + \left(\frac{\varepsilon_2 k_x}{2\kappa_2}\right)\left(h_3 b_3 - h_2 b_2\right)\right) e^{ia(r-\kappa_2)} + \frac{k_x}{2\kappa_3}\left(\frac{\varepsilon_2}{\kappa_2} + \frac{\varepsilon_3}{\kappa_3}\right)\tilde{E}_{3x} e^{ia(\kappa_3-\kappa_2)} \\ \tilde{E}_{2nx}\left(\frac{\varepsilon_2 k_x}{\kappa_2{}^2}\right) &= \left(+\left(s_3 b_3 \varepsilon_3 - s_2 b_2 \varepsilon_2\right) + \left(\frac{\varepsilon_2 k_x}{2\kappa_2}\right)\left(h_3 b_3 - h_2 b_2\right)\right) e^{ia(r+\kappa_2)} + \frac{k_x}{2\kappa_3}\left(\frac{\varepsilon_2}{\kappa_2} - \frac{\varepsilon_3}{\kappa_3}\right)\tilde{E}_{3x} e^{ia(\kappa_3+\kappa_2)}\end{aligned} \qquad 38$$

Combining equations 36 and 37 and eliminating E_{2px} and E_{2nx} we can find the amplitudes E_{1x} and E_{3x}. For E_{3x} we find

$$\tilde{E}_{3x} = \frac{2\kappa_1 e^{-ia\kappa_1}}{k_x} \frac{E_-\left(p+q\right)\left\{e^{ia(r+\kappa_2)} - 1\right\} - E_+\left(-p+q\right)\left\{e^{ia(r-\kappa_2)} - 1\right\}}{E_+{}^2 e^{-ia\kappa_2} - E_-{}^2 e^{+ia\kappa_2}} \qquad 39$$

where $E_\pm = \varepsilon_2/\kappa_2 \pm \varepsilon_1/\kappa_1$, $p = s_1 b_1 \varepsilon_1 - s_2 b_2 \varepsilon_2$, and $q = \dfrac{\varepsilon_2 k_x}{2\kappa_2}\left(h_1 b_1 - h_2 b_2\right)$.

Special case

We are interested in a special case where the angle of the laser and the electron trajectory are near Brewster's angle. This ensures full transmission of the laser field to the downstream space and hence maximizes the energy gain of the electron beam within the suitable phase conditions. Also, although equation 38 is general we will focus on the relativistic limit for the particle's velocity and will assume a small angle difference between the laser propagation direction and the electron's trajectory.

We use the same notation for the angles of interest as in Fig. 2. At Brewster's angle, and assuming that mediums 1 and 3 are vacuum $\tan\theta = n$ and $\tan\theta_m = 1/n$. We introduce a new angle $\alpha' = \theta - \psi$ and since we are interested in the field overlapping with the laser we are interested in the region where $|\alpha'| << 1$. Since we defined $k = \omega/c$ the x-components of the k-vectors in the different media are $k_{x,j} = kn_j \sin\theta_j$ where $n_j = \sqrt{\varepsilon_j}$ is the index of refraction of the medium in question. With these definitions we also find the z-components $\kappa_j = kn_j \cos\theta_j$.

$$\begin{aligned}\tilde{E}_{1z} &= +\tan\theta\, \tilde{E}_{1x} \\ \tilde{E}_{3z} &= -\tan\theta\, \tilde{E}_{3x} \\ \tilde{E}_{2nz} &= +\tan\theta_m\, \tilde{E}_{2nx} \\ \tilde{E}_{2pz} &= -\tan\theta_m\, \tilde{E}_{2px}\end{aligned} \qquad 40$$

With these definitions we find that

$$\begin{aligned}&(\varepsilon_m \tan\theta_m - \tan\theta)\tilde{E}_{1x} - (\varepsilon_m \tan\theta_m + \tan\theta)e^{ia\kappa_1}\tilde{E}_{3x} = 2\kappa_1(-p+\varepsilon_2 q)\left\{e^{ia(r-\kappa_2)}-1\right\}\\ &(\varepsilon_m \tan\theta_m + \tan\theta)\tilde{E}_{1x} - (\varepsilon_m \tan\theta_m - \tan\theta)e^{ia\kappa_1}\tilde{E}_{3x} = 2\kappa_1(+p+\varepsilon_2 q)\left\{e^{ia(r+\kappa_2)}-1\right\}\end{aligned} \quad 41$$

We can find small-angle approximations for p and q. These are

$$\begin{aligned}&p = s_1 b_1 \varepsilon_1 - s_2 b_2 \varepsilon_2 \quad \sim \quad \frac{2\pi i Z_0}{k\beta}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}\tan\theta\\ &\varepsilon_2 q = \frac{\varepsilon_2 k_x}{2\kappa_2}(h_1 b_1 - h_2 b_2) \sim -\frac{2\pi i Z_0}{k\beta}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}\varepsilon_2 \tan\theta_m\end{aligned} \quad 42$$

where $\beta c = v$ is the speed of the particle and Z_0 is the vacuum impedance. Using the Brewster angle conditions we can substitute $\varepsilon_m \tan\theta_m \pm \tan\theta = n \pm n$ and the values of p and q and $\kappa_1 = k\cos\theta$ we obtain for equation 41

$$\begin{aligned}&0\tilde{E}_{1x} - 2ne^{ia(\kappa_1-\kappa_2)}E_{3x} \sim 2\kappa_1 \frac{2\pi i Z_0}{k\beta}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}(-2n)\left\{e^{ia(r-\kappa_2)}-1\right\}\\ &2n\tilde{E}_{1x} - 0E_{3x} \quad \sim \quad 2\kappa_1(+0)\left\{e^{ia(r+\kappa_2)}-1\right\}\end{aligned} \quad 43$$

which simplifies to

$$\begin{aligned}&\tilde{E}_{3x} \sim 2\cos\theta \frac{2\pi i Z_0}{\beta}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}\left\{e^{ia(r-\kappa_2)}-1\right\}e^{-ia(\kappa_1-\kappa_2)}\\ &\tilde{E}_{1x} \sim 0\end{aligned} \quad 44$$

For relativistic particles where $\beta \sim 1$ and for small laser crossing angle $\theta = \psi + \alpha$, $|\alpha| << 1$ the coefficient r becomes

$$r = \frac{\omega}{v_z} - \frac{v_x}{v_z}k_x = k\left(\frac{1}{\beta\cos\psi} - \tan\psi\cos\theta\right) \sim k\cos\theta \sim \kappa_1 \quad 45$$

where, as assumed elsewhere in this paper $k = \omega/c$ is the wavenumber in vacuum. Note that in the medium 1, vacuum, $\kappa_1 = \sqrt{k^2 \cdot 1 - k^2\cos\theta} = k\sin\theta$. Since the total plane wave amplitude at angle θ is $E_3 = E_{3x}/\cos\theta$ equation 44 becomes

$$\tilde{E}_3 \sim 2\frac{2\pi i Z_0}{\beta}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}\left\{1 - e^{-ia(\kappa_1-\kappa_2)}\right\} \qquad 46$$
$$\tilde{E}_1 \sim 0$$

This is a curious result that says that at Brewster's angle there is no reflected transition radiation component. Intuitively one would expect this since the p-polarized plane wave components from the particle's field should suffer no reflection from the boundaries at Brewster's angle.

In the far field, at a location $(R \to \infty, \alpha', \phi)$ the electric field is related to the plane wave spectrum $\tilde{E}_3$ of equation 46 by

$$\tilde{E}_{TR}(\vec{r}, \omega) = -\frac{ie^{ikR}}{2(2\pi)^2 R}\tilde{E}_3(k, \alpha', \phi, \omega) \qquad 47$$

Evaluating equation 47 we get

$$\tilde{E}_{TR}(\vec{r}, y = 0, \omega) \sim -\frac{e^{ikR}}{R}\frac{qZ_0}{2\pi}\frac{\alpha'}{\alpha'^2 + 1/\gamma^2}\left\{1 - e^{-ia(\kappa_1-\kappa_2)}\right\} \qquad 48$$

As in the previous section we define the laser-crossing angle as $\alpha = \theta_L - \psi$ and assume that $|\alpha| << 1$. For the plane wave laser field in the downstream region given by equation 2 the far-field is

$$\vec{E}_L(\vec{r}, \omega) = -\frac{2\pi\lambda E_0}{R}\delta(\alpha - \alpha', \xi)\frac{i}{2}\left(\delta(\omega - \omega_0)e^{i(kR'+\varphi_L)} - \delta(\omega + \omega_0)e^{i(kR'-\varphi_L)}\right) \qquad 49$$

Now we can use the overlap integral of equation 23 to find the energy gain predicted by the inverse-radiation picture and find that

$$\Delta U = -\frac{1}{\pi Z_0}(-2\pi\lambda E_0)\left(-\frac{qZ_0}{2\pi}\right)\frac{\alpha}{\alpha^2 + 1/\gamma^2}\frac{i}{2}\left\{e^{-i\phi_3} - e^{i(-\phi_3 - a(\kappa_1-\kappa_2))} - e^{+i\phi_3} - e^{i(\phi_3 + a(\kappa_1-\kappa_2))}\right\} \qquad 50$$

but since $a(\kappa_2 - \kappa_1) = ka(n\cos\theta_m - \cos\theta_L) \sim \phi_{\mathrm{ret}}$ (see equation 21) and $\phi_3 = \phi_{ret} + \phi_L$ (see equation 3)

$$\Delta U = -\frac{\lambda q E_0}{\pi}\frac{\alpha}{\alpha^2 + 1/\gamma^2}\frac{i}{2}\left\{e^{-i(\phi_{ret}+\phi_L)} - e^{i(-\phi_{ret}-\phi_L+\phi_{ret})} - e^{+i(\phi_{ret}+\phi_L)} - e^{-i(-\phi_{ret}-\phi_L+\phi_{ret})}\right\} \qquad 51$$

which becomes

$$\Delta U = \frac{\lambda q E_0}{\pi}\frac{\alpha}{\alpha^2 + 1/\gamma^2}\left\{\sin\phi_L - \sin\left(\phi_{ret} + \phi_L\right)\right\} \qquad 52$$

which is in agreement with the energy gain formula of equation 19 derived from the path integral method.

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