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Resistive-wall wake effect in the beam delivery system*

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General formulae for resistive-wall induced beam dilution are presented and then applied to the final beam delivery system of linear colliders. Criteria for the design of final beam delivery systems are discussed.

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A. Equation and Solution

Recently, the beam breakup (BBU) problem due to the resistive-wall impedance was studied for uniform single bunch and also point-like bunch train [1,2]. However for linear collider, the beam at the interaction point normally has some microstructure. This is evidenced by start-to-end simulation. Hence, in this paper, we study the resistive wall BBU problem for the case of arbitrary beam current profile.

We denote the location along the beamline by the variable s. The beam travels in the positive s direction, and the entrance to the beamline is located at $s = 0$. We assume in this paper that the accelerator is uniform and that there is no acceleration. This is not unduly restrictive [3]. In a continuum approximation, the transverse motion $y(\tau, s)$ of a beam in a misaligned beamline under the combined influence of focusing and wake field can be modelled by [4]

$$\frac{\partial^2 y(\sigma,\zeta)}{\partial \sigma^2} + \kappa^2 \left[y(\sigma,\zeta) - d_f(\sigma)\right] = \varepsilon \int_0^\zeta w(\zeta-\zeta_1) F(\zeta_1) \left[y(\sigma,\zeta_1) - d_c(\sigma)\right] d\zeta_1 \,, \qquad (1)$$

where $\sigma = s/\mathcal{L}$, with $\mathcal{L}$ being the length of the element where wakefield is generated; $\zeta = \omega_0 \tau$, with ω_0 being a reference angular frequency, and $\tau = t - s/v$ describes the relative longitudinal position of the particle inside the bunch, v is the particle velocity; $\kappa = k_y \mathcal{L}$, is the betatron phase advance with k_y being the betatron focusing strength; $F(\zeta) = I(\zeta)/\bar{I}$, the current form factor, is the instantaneous current $I(\zeta)$ divided by the average current $\bar{I}$; $d_f(\sigma)$ and $d_c(\sigma)$ are the lateral displacement of the focusing elements and element where the wakefield is generated. The right hand side of Eq. (1) represents the effects due to the wakefield $\mathcal{W}(\tau)$, which is introduced via

$$(\mathcal{L}^2/\omega_0)\, \mathcal{W}(\tau) = \varepsilon\, w(\zeta) \,. \qquad (2)$$

The exact meaning of ε and ω_0 will be made clear in the following. The general solution of Eq. (1) is [3]

$$\begin{aligned} y(\sigma,\zeta) =& \sum_{n=0}^{\infty} \varepsilon^n [y_0\, h_n(\zeta)\, j_n(\kappa,\sigma) + y_0'\, g_n(\zeta)\, i_n(\kappa,\sigma)] \\ &- \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta)\, i_n(\kappa,\sigma) * d_c(\sigma) \\ &+ \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta)\, i_n(\kappa,\sigma) * d_f(\sigma) \end{aligned} \qquad (3)$$

where $i_n(\kappa,\sigma) * d(\sigma) = \int_0^\sigma du\, i_n(\kappa,u)\, d(\sigma-u)$. Via inverse Laplace transform, $i_n(\kappa,\sigma)$ and $j_n(\kappa,\sigma)$ are

$$\begin{aligned} i_n(\kappa,\sigma) &= \mathbf{L}_\sigma^{-1}\left[\frac{1}{(p^2+\kappa^2)^{n+1}}\right] \\ &= \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \frac{1}{\kappa}\sqrt{\frac{\pi\kappa\sigma}{2}} J_{n+(1/2)}(\kappa\sigma)\,, \end{aligned} \qquad (4)$$

$$\begin{aligned} j_n(\kappa,\sigma) &= \mathbf{L}_\sigma^{-1}\left[\frac{p}{(p^2+\kappa^2)^{n+1}}\right] \\ &= \frac{d}{d\sigma} i_n(\kappa,\sigma) = \frac{\sigma}{2n} i_{n-1}(\kappa,\sigma) \\ &= \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}} J_{n-(1/2)}(\kappa\sigma)\,. \end{aligned} \qquad (5)$$

In the absence of focusing $i_n(\kappa,\sigma)$ and $j_n(\kappa,\sigma)$ reduce to $i_n(0,\sigma) = \sigma^{2n+1}/(2n+1)!$, and $j_n(0,\sigma) = \sigma^{2n}/(2n)!$.

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The functions $f_n(\zeta)$, $g_n(\zeta)$, and $h_n(\zeta)$ are defined as

$$\begin{Bmatrix} f_{n+1}(\zeta) \\ g_{n+1}(\zeta) \\ h_{n+1}(\zeta) \end{Bmatrix} = \int_{-\infty}^{\zeta} \begin{Bmatrix} f_n(\zeta_1) \\ g_n(\zeta_1) \\ h_n(\zeta_1) \end{Bmatrix} w(\zeta - \zeta_1)\, F(\zeta_1)\, d\zeta_1 \,, \tag{6}$$

where

$$f_0(\zeta) = 1 \,, \tag{7}$$

$$y_0'\, g_0(\zeta) = y_0'(\zeta) = \frac{\partial}{\partial\sigma} y(\sigma,\zeta)\bigg|_{\sigma=0} \,, \tag{8}$$

$$y_0\, h_0(\zeta) = y_0(\zeta) = y(\sigma = 0, \zeta) \,. \tag{9}$$

B. Resistive-Wall Wake

If the wakefield source is the resistive-wall of a circularly cylindrical pipe, then the long-range wakefield is [5]

$$\mathcal{W}(\tau) = \frac{A}{\sqrt{\tau}} \quad \text{for} \quad \tau > 0 \,, \tag{10}$$

where

$$A = (4c^2 \bar{I})/(v\gamma b^3 I_{\text{Alfvèn}})\sqrt{\epsilon_0/\pi\sigma_c} \,. \tag{11}$$

In the above expression, c is the speed of light in vacuum, γ is the Lorentz factor, b is the radius of the pipe, $I_{\text{Alfvèn}} = 4\pi\epsilon_0 mc^3/e \approx 17,045$ Amp is the Alfvèn current, $\epsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C}^2/(\mathrm{N\ m}^2)$ is the vacuum permittivity, and σ_c is the pipe conductivity. According to Eq. (2), we have

$$\varepsilon = (\mathcal{L}^2 A)/\sqrt{\omega_0}, \tag{12}$$

so that

$$w(\zeta) = 1/\sqrt{\zeta}. \tag{13}$$

Hence, the series solution in Eq. (3) would converge quickly if $\varepsilon \ll 1$.

Note that Eq. (10) is only an approximation [6,7] for

$$\tau_s \equiv \left(\frac{b^2}{Z_0\sigma_c c^3}\right)^{1/3} \ll \tau \ll \min\left[\frac{Z_0\sigma_c b^2}{c}, \frac{Z_0\sigma_c \Delta r^2}{c}\right] \equiv \tau_l, \tag{14}$$

with $Z_0 \approx 376.7\ \Omega$ being the vacuum impedance; and Δr the thickness of the wall. In particular $\mathcal{W}(0) = 0$, and $\mathcal{W}(\zeta)$ decays faster than $\zeta^{-1/2}$ when $\zeta \to \infty$ since $\int_0^\infty \mathcal{W}(\zeta)\, d\zeta$ — the dc impedance — is finite [8].

C. Single Bunch

For a bunch of uniform current distribution — $F(\zeta) = 1$— we get from Eq. (6)

$$f_n(\zeta) = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}+1\right)} \zeta^{\frac{n}{2}}. \tag{15}$$

We further assume that $y_0(\zeta) = y_0$ and $y_0'(\zeta) = y_0'\mathcal{L}$, so that $g_n(\zeta) = h_n(\zeta)\mathcal{L} = f_n(\zeta)\mathcal{L}$. For arbitrary current profile $F(\zeta)$, Eqs. (3-9) and (13) set up the calculation frame.

D. Periodic Bunch Train

For a steady-state periodic bunch train, the current form factor $F(\zeta)$ is given by

$$F(\zeta) = \sum_{k=-\infty}^{\infty} F_k e^{i(2\pi/\omega_0\tau_b)k\zeta}, \tag{16}$$

where τ_b is the laboratory-frame period of the longitudinal beam modulation, or the bunch separation in a bunch train. The meaning of previously introduced ω_0 is clear now. We could set it to be $\omega_0 = 2\pi/\tau_b$, even though not have to. The corresponding general solution is

$$\begin{aligned} y(\sigma,\zeta) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}}\,\Big[y_0 J_{n-(1/2)}(\kappa\sigma) \\ &+ \frac{y_0'\mathcal{L}}{\kappa} J_{n+(1/2)}(\kappa\sigma)\Big]\, f_n(\zeta), \end{aligned} \tag{17}$$

where

$$f_n(\zeta) = \sum_k \exp\left[i\frac{2\pi}{\omega_0\tau_b}k\zeta\right] f_{n,k}, \tag{18}$$

with the following recursion relation $f_{0,k} = \delta_{0,k}$, and $f_{n+1,k} = \tilde{w}_k \sum_{k_1} F_{k_1} f_{n,k-k_1}$; where, $\tilde{w}_k = \tilde{w}[(2\pi/\omega_0\tau_b)k]$, and $\tilde{w}(Z) = \int_{-\infty}^{\infty} d\zeta w(\zeta) e^{-iZ\zeta}$ is the Fourier transform of the wake $w(\zeta)$, i.e., the impedance.

Now, suppose a beam is composed of bunches of constant current density, separated by $\omega_0\tau_b$, of length $\alpha\omega_0\tau_b$. The parameter α allows a continuous transition from a dc beam ($\alpha = 1$) to a beam composed of $\delta-$function bunches separated by $\omega_0\tau_b(\alpha = 0)$. Choosing $\zeta = 0$ as being in the middle of a bunch, the Fourier coefficients of the current form factor are $F_k = [\sin(k\alpha\pi)/(k\alpha\pi)]$.

For the resistive-wall wake, the impedance is

$$\tilde{w}(Z) = \sqrt{\frac{\pi}{Z}} e^{-i\pi/4}. \tag{19}$$

Since $Z > 0$, the Fourier integral contour is chosen at the lower-right quarter in the complex $\zeta-$plane. Hence,

$$\tilde{w}_k = \tilde{w}\left[\frac{2\pi}{\omega\tau}k\right] = \sqrt{\frac{\omega\tau}{2k}} e^{-i\pi/4}. \tag{20}$$

Notice that there is a singularity at $k = 0$ or $Z = 0$. This is artificial, since the wakefield in Eq. (10) is an oversimplified form. Detailed calculation shows [8]

$$\tilde{w}_0 = \frac{B\sqrt{\omega_0}}{A}, \tag{21}$$

where A and B are defined in Eqs. (11) and (**??**).

dc beam: For a dc beam, the general solution is

$$y(\sigma,\zeta) = y_0 \cos\left[\sigma\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}\right] + y_0'\mathcal{L}\frac{\sin\left[\sigma\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}\right]}{\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}}. \tag{22}$$

For the resistive-wall wake, $\tilde{w}_0$ is given in Eq. (21).

$\delta-$**function beam**: In the case of a bunch train comprised of $\delta-$function bunches, for $\zeta = M\omega_0\tau_b$, i.e., for bunch M, the displacement becomes

$$y(\sigma, M\omega_0\tau_b) = y_0 \cos\left[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}\right] + y_0'\mathcal{L}\frac{\sin\left[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}\right]}{\sqrt{\kappa^2-\varepsilon\tilde{W}_0}}. \tag{23}$$

For the resistive-wall wake, we have $\tilde{W}_0 = \sqrt{\omega_0\tau_b}\mathrm{Zeta}(1/2) + B\sqrt{\omega_0}/A$.

E. Transient Periodic Beam

Let us analyze a periodic bunch train that was turned on at $\zeta = 0$. For the case of $y_0(\zeta) = y_0$ and $y_0'(\zeta) = 0$, the general solution is [3]

$$y(\sigma,\zeta) = y_0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n\sqrt{\frac{\pi\kappa\sigma}{2}}J_{n-(1/2)}(\kappa\sigma)h_n(\zeta), \tag{24}$$

where $h_0(\zeta) = H(\zeta)$, with $H(\zeta)$ being the Heaviside function. The recursion relation is $h_{n+1}(\zeta) = \int_0^\zeta h_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1$.

dc beam: For a dc beam, in the case of resistive-wall wake, we have for $n = 0, 1, 2, \cdots$

$$h_n(\zeta) = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}+1\right)}\zeta^{\frac{n}{2}}. \tag{25}$$

This together with Eq. (24) defines completely the transverse displacement at an arbitrary location σ and time ζ.

$\delta-$**function beam**: For a bunch train comprised of point-like bunches turned on at $\zeta = 0$, the displacement of bunch M at location σ is given by [3]

$$y_M(\sigma) = y_0\sum_{n=0}^{M}\frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n\sqrt{\frac{\pi\kappa\sigma}{2}}h_n(M\omega_0\tau_b)J_{n-(1/2)}(\kappa\sigma).$$

Here $h_n(M\omega_0\tau_b)$ is defined as $h_n(M\omega_0\tau_b) = \frac{1}{2\pi i}\oint z^{M-1}\check{h}_n(z)dz$, where $\check{h}_n(z) = \frac{z}{z-1}[\omega_0\tau_b\check{w}(z)]^n$, and $\check{w}(z) = \sum_{k=0}^{\infty} z^{-k}w(k\omega_0\tau_b)$. In the case of the resistive-wall wake, $w(\zeta) = 1/\sqrt{\zeta}$, so that $\check{w}(z) = \mathrm{PolyLog}\left(1/2, z^{-1}\right)$. Notice that, this is only an approximation for $\check{w}(z)$, since $w(\zeta) = 1/\sqrt{\zeta}$ is not valid for $\zeta \to \infty$, though $w(0) = 0$ has been used.

Finite train of finite bunches: For a bunch train of finite but identical bunches turned on at $\zeta = 0$, the current form factor is $F(\zeta) = H(\zeta)\sum_{k=-\infty}^{\infty} F_k \exp\left(i\frac{2\pi}{\omega_0\tau_b}k\zeta\right)$. The first-order term is [3]

$$\begin{aligned} h_1(\zeta) &= \int_{-\infty}^{\zeta} H(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1 = \int_0^{\zeta} F(\zeta-\zeta_1)w(\zeta_1)d\zeta_1 \\ &= \sum_k F_k\tilde{w}_k e^{i\frac{2\pi k\zeta}{\omega_0\tau_b}} - \sum_k F_k e^{i\frac{2\pi k\zeta}{\omega_0\tau_b}}\int_{\zeta}^{\infty} e^{-i\frac{2\pi k\zeta_1}{\omega_0\tau_b}} w(\zeta_1)d\zeta_1. \end{aligned} \tag{26}$$

Notice that the first term is the steady state obtained previously, while the second term is the transient that decays when $\zeta \to +\infty$. For the resistive-wall wake, we obtain

$$\begin{aligned} h_1(\zeta) &= \sqrt{\omega_0\tau_b}\sum_k \frac{F_k}{\sqrt{k}}e^{i\frac{2\pi k\zeta}{\omega_0\tau_b}}\left\{\mathrm{FresnelC}\left(2\sqrt{\frac{k\zeta}{\omega_0\tau_b}}\right)\right. \\ &+ \left. i\left[-1+\mathrm{FresnelS}\left(2\sqrt{\frac{k\zeta}{\omega_0\tau_b}}\right)\right]\right\}. \end{aligned} \tag{27}$$

F. Application and Discussion

Now let us study the USWarm and USCold linear collider design [9]. According to the design, there will be about 300 meter long transformer with large $\beta-$function in the final beam delivery system. There is essentially no focusing, hence the resistive-wall effect need be studied. Typical parameters are given in Table I. Notice that $\kappa\sigma \approx 0$, according to Eqs. (3-5), and (15), in the case of $d_c = 0$ and $d_f = 0$, we have

$$\begin{aligned} y(\sigma,\zeta) &= \sum_{n=0}^{\infty}\varepsilon^n\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}+1\right)}\zeta^{n/2}\frac{1}{n!}\left(\frac{\sigma}{2}\right)^{2n}\frac{\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)} \\ &\times \left\{y_0 + y_0'\mathcal{L}\frac{\sigma}{2}\frac{1}{n+\frac{1}{2}}\right\}; \end{aligned} \tag{28}$$

and $y'(\sigma,\zeta)$ similarly. It is interesting to observe that $\varepsilon \propto 1/\sqrt{\omega_0}$, while $\zeta \propto \omega_0$, hence the parameter ω_0 is gone.

	USWarm	USCold
Bunch charge (nC)	1.2	3.2
Single bunch rms length (μm)	110	300
Bunch separation τ_b (ns)	1.4	337
Bunch number	192	2820
Pipe radius b (cm)	2	2
Pipe length $\mathcal{L}$ (m)	300	300
Conductivity σ_c ($10^7\Omega^{-1}\mathrm{m}^{-1}$)	3.47	3.47
k_y (m^{-1})	1/50000	1/12500
Beam energy (GeV)	250	250
$\Delta\sigma_y/\sigma_y$ (Single) (%)	0.6	2.5
$\Delta\sigma_{y'}/\sigma_{y'}$ (Single) (%)	19.3	7.8
$\Delta\sigma_y/\sigma_y$ (Multi) (%)	1.3	4.9
$\Delta\sigma_{y'}/\sigma_{y'}$ (Multi) (%)	26.1	9.0

TABLE I: Parameters for the USWarm and USCold design.

Given the parameters in Table I, we compute the bunch spot size increase $\Delta\sigma_y/\sigma_y$, and the angular divergence increase $\Delta\sigma_{y'}/\sigma_{y'}$. The calculation indicates that to maintain a relatively small increase, we need to use Aluminum and keep the pipe radius to be larger than 2 cm. Given these, the increase at a single bunch tail due to the wakefield of a single bunch, and that at the bunch train tail due to the wakefield of the entire bunch train are given in Table I. We find that the majority contribution of $\Delta\sigma_{y'}/\sigma_{y'}$ comes from the single bunch effect; while for $\Delta\sigma_y/\sigma_y$, contribution of single bunch and multi bunch effect are almost equal.

Investigation in this paper indicates that the resistive-wall effect in the final beam delivery system needs to be considered in design. As we pointed out in Eq. (14), the wake given in Eq. (10) is only an approximation. Suppose that the wall thickness is $\Delta r = 3$ mm, we have $\tau_l \approx 0.39$ ms, since $b >> \Delta r$. Therefore, the wake given in Eq. (10) fails at the bunch train tail in the USCold. The long-range wake decays even faster [8] than that in Eq. (10). However, since the single bunch effect dominates, especially for the angular divergence, the long-range wake will only introduce a small correction. For the other limit, with $b = 2$ cm, and Aluminum pipe, $\tau_s \approx 100$ fs, hence a more accurate calculation utilizing the short-range wake [7] should be considered, whenever it is needed.

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H. References

[1] J.-M. Wang *et al.*, *EPAC'00*, p. 1179.

[2] J.-M. Wang and J. Wu, Phys. Rev. ST Accel. Beams **7**, 034402 (2004).

[3] J.R. Delayen, Phys. Rev. ST Accel. Beams **6**, 084402 (2003).

[4] A.W. Chao *et al.*, Nucl. Instrum. Methods **178**, 1 (1980).

[5] P.L. Morton *et al.*, J. Appl. Phys. **37**, 3875 (1966).

[6] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, (Wiley-Interscience, 1993).

[7] K.L.F. Bane and M. Sands, SLAC-PUB-7074, 1995.

[8] A. Burov and V. Lebedev, *EPAC'02*, p. 1452.

[9] http://www.slac.stanford.edu/xorg/accelops/.