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Microbunch Instability Theory and Simulations*

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I. INTRODUCTION

Over the last years there have been several reports of quasiperiodic bursts of coherent synchrotron radiation (CSR) in electron rings in the microwave and far-infrared range. The observations were made on synchrotron radiation light sources which include the Synchrotron Ultraviolet Radiation Facility SURF II [1], the VUV ring at the National Synchrotron Light Source at BNL [2, 3], second generation light sources MAX-I [4], BESSY II [5], and ALS [6]. General features of those observations can be summarized as follows. Above a threshold current, there is a strongly increased radiation of the beam in the range of wavelengths shorter than the bunch length, $\lambda\!\!\!^{-} < \sigma_z$. At large currents, this radiation is observed as a sequence of random bursts. In the bursting regime, intensity of the radiation scales approximately as square of the number of particles in the bunch, indicating a coherent nature of the phenomenon.

It is generally accepted that the source of this radiation is related to the microbunching of the beam arising from development of a microwave instability caused by the coherent synchrotron radiation of the beam. A relativistic electron beam moving in a circular orbit in free space can radiate coherently if the wavelength of the synchrotron radiation exceeds the length of the bunch. In accelerators coherent radiation of the bunch is usually suppressed by the shielding effect of the conducting walls of the vacuum chamber [7–9], which gives an exponential cutoff of wavelengths greater than a certain threshold. However, an initial density fluctuation with a characteristic length much shorter than the shielding threshold would radiate coherently. If the radiation reaction force is such that it results in the growth of the initial fluctuation one can expect an instability that leads to micro-bunching of the beam and an increased coherent radiation at short wavelengths. A possibility of CSR instability was pointed out in Refs. [10, 11].

II. PROPERTIES OF CSR

In application to the CSR instability, we are interested in the synchrotron radiation at wavelengths of the order of a size of microbunches, with a frequency ω typically well below the critical frequency for the synchrotron radiation. For an ultrarelativistic particle with the Lorentz factor $\gamma \gg 1$, in this range of frequencies, the spectrum of the radiation $dP/d\omega$ (per unit length of path) can be written as

$$\frac{dP}{d\omega} = \frac{3^{1/6}}{\pi}\Gamma\left(\frac{2}{3}\right)\left(\frac{\omega}{\omega_H}\right)^{1/3}\frac{e^2\omega_H}{c^2}\,, \tag{1}$$

where $\omega_H = eB/\gamma mc$, with B the magnetic field, e the electron charge, m the electron mass, c the speed of light, and Γ the gamma-function. The characteristic angular spread θ of the radiation with reduced wavelength $\lambda\!\!\!^{-}$ (where $\lambda\!\!\!^{-} = c/\omega = 1/k$) is of order of $\theta \sim (\lambda\!\!\!^{-}/R)^{1/3}$, where R is the bending radius, $R = c/\omega_H$. Another important characteristic of the radiation is the formation length l_f: $l_f \sim \lambda\!\!\!^{-}/\theta^2 \sim (\lambda\!\!\!^{-} R^2)^{1/3}$—this is the length after which the electromagnetic field of the particle moving in a circular orbit "disconnects" from the source and freely propagates away. In a vacuum chamber with perfectly conducting walls, whether this "disconnection" actually occurs depends on another parameter, often called the "transverse coherence size", $l_\perp$. An estimate for $l_\perp$ is: $l_\perp \sim l_f\theta \sim \lambda\!\!\!^{-}/\theta \sim (\lambda\!\!\!^{-2} R)^{1/3}$. One of the physical meanings of $l_\perp$ is that it is equal to the minimal spot size to which the radiation can be focused. Another meaning of this parameter is that it defines a scale for radiation coherence in the transverse direction. Electrons in a transverse cross section of a bunch of size $\sigma_\perp$ would radiate coherently only if $\sigma_\perp \lesssim l_\perp$. We emphasize here that both parameters, $l_\perp$ and l_f, are functions of frequency, with the scalings $l_\perp \propto \omega^{-3/2}$ and $l_f \propto \omega^{-1/2}$.

Closely related to the transverse coherence size is the shielding of the radiation by conducting walls: if the walls are closer than $l_\perp$ to the beam, the field lines during circular motion close onto the conducting walls, rather than disconnect from the charge. This means that the radiation at the wavelengths where $l_\perp \gtrsim a$, where a is the pipe radius, is suppressed, or shielded.

For a bunch with N electrons, the radiation of each electron interferes with others. Assuming full transverse coherence (a one dimensional model of the beam), the total radiation of the bunch is [12]:

$$\left.\frac{dP}{d\omega}\right|_{\mathrm{bunch}} = \frac{dP}{d\omega} N \left(1 + N |\hat{f}(\omega)|^2\right) , \tag{2}$$

where $\hat{f}(\omega) = \int_{-\infty}^{\infty} dz f(z) e^{i\omega z/c}$ is the Fourier transform of the longitudinal distribution function of the beam $f(z)$ (normalized by $\int_{-\infty}^{\infty} f(z)dz = 1$). The first term on the right hand side of Eq. (2) is due to incoherent, and the second one – to coherent radiation. For a smooth distribution function (e.g., Gaussian, with rms bunch length σ_z), the Fourier image $\hat{f}(\omega)$ vanishes for $\lambda\!\!\!^{-} \lesssim \sigma_z$, and the radiation is incoherent. However, beam density modulation with $\lambda\!\!\!^{-} \lesssim \sigma_z$ would contribute to $\hat{f}(c/\lambda\!\!\!^{-})$ and result in coherent radiation, if the amplitude of the perturbation is such that $|\hat{f}(c/\lambda\!\!\!^{-})| \gtrsim N^{-1/2}$.

III. RADIATION REACTION FORCE—CSR WAKE FIELD

The collective force acting on the beam due to its coherent synchrotron radiation is described in terms of the so called CSR longitudinal wake [13–15]. For an ultrarelativistic particle, in one-dimensional approximation, this wake (per unit length of path) is given by the following formula:

$$w(z) = -\frac{2}{3^{4/3} R^{2/3} z^{4/3}} . \tag{3}$$

The wake is valid for distances z such that $R \gg z \gg R/\gamma^3$—a general behavior of the wake function including also distances $z \sim R/\gamma^3$ is shown in Fig. 1. The wake is localized in front of the particle in contrast to "traditional"

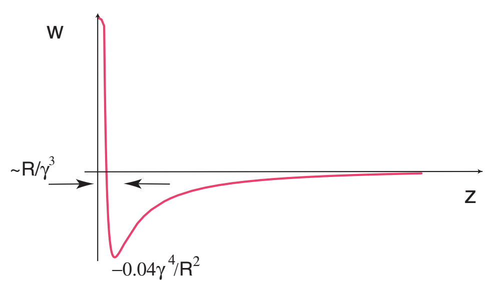

FIG. 1: CSR wake as a function of distance. A simple formula (3) is applicable for not very short distances, to the right of the minimum of the wake. The wake reaches minimum at $z \sim R/\gamma^3$, with the minimum value of $-0.04\gamma^4/R^2$.

wakes in accelerator physics which trail the source charge [16]. This is explained by the fact that the charge follows a circular orbit and the radiation propagates along chords getting ahead of the source. The wake given by Eq. (3) has a strong singularity at $z \to 0$. In calculations, this singularity is eliminated by integration by parts and using the fact that the area under the curve $w(z)$ is equal to zero.

A simple wake Eq. (3) assumes a small transverse beam size [15], $\sigma_\perp \lesssim l_\perp \sim (\lambda\!\!\!^{-2} R)^{1/3}$, and neglects the shielding effect of the conducting walls. It is valid only for long enough magnets, $l_{\mathrm{magnet}} \gg l_f$, when transient effects at the entrance to and exit from the magnet can be neglected. A detailed study of transient effects in a short magnet can be found in Refs. [17, 18].

Using the wake field Eq. (3) one can calculate the CSR longitudinal impedance Z:

$$\begin{aligned} Z(k) &= \frac{1}{c}\int_0^{\infty} dz w(z) e^{-ikz} \\ &= \frac{2}{3^{1/3}} \Gamma\left(\frac{2}{3}\right) e^{i\pi/6} \frac{k^{1/3}}{cR^{2/3}} . \end{aligned} \tag{4}$$

The real part of this impedance is related to the spectrum of the energy loss of a charge due to radiation: $dP/d\omega = (e^2/\pi)\mathrm{Re}\, Z$, see Eq. (1). Plots of a CSR wake for a Gaussian bunch can be found in Refs. [14, 15].

IV. CSR INSTABILITY

Due to the CSR wake, an initial small density perturbation δn induces energy modulation in the beam δE. A finite momentum compaction factor of the ring converts δE into a density modulation. At the same time, the energy spread of the beam tends to smear out the initial density perturbation. Under certain conditions, which depend on the beam current, energy spread, and the wavelength of the modulation, the process can lead to an exponential growth of the perturbation.

A quantitative description of the instability can be obtained if we assume that the wavelength of the perturbation is much shorter than the bunch length, $\lambda \ll \sigma_z$, and use a coasting beam approximation. In this case, the dispersion relation for the frequency ω is given by the Keil-Schnell formula [19]:

$$\frac{inr_0c^2Z(k)}{\gamma}\int_{-\infty}^{\infty}\frac{d\delta\,(df/d\delta)}{\omega + ck\eta\delta} = 1\,, \tag{5}$$

where n is the number of particles per unit length, η is the momentum compaction factor of the ring, $r_0 = e^2/mc^2$, $Z(k)$ is the CSR impedance given by Eq. (4), $f(\delta)$ is the energy distribution function normalized so that $\int f(\delta)d\delta = 1$. To take into account straight sections in the ring, where $R = \infty$ and there is no CSR wake, Z is replaced with a weighted impedance: $Z \to ZR/\langle R\rangle$, where $\langle R\rangle = C/2\pi$. The plot of Re ω and Im ω calculated from Eq. (5) for a

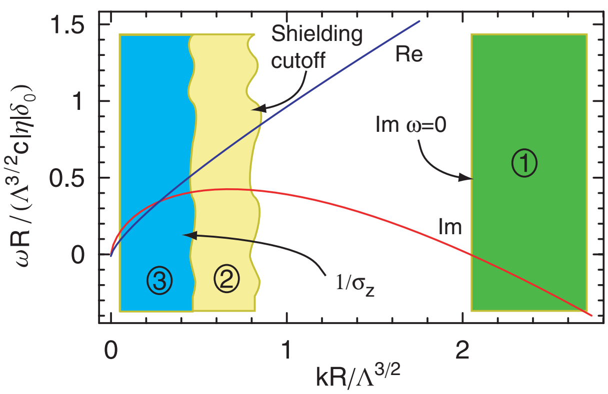

FIG. 2: Plot of real (blue) and imaginary (red) parts of frequency ω as functions of k for positive η. Normalization of the frequency ω and the wavenumber k on the axes involves the parameter Λ defined by Eq. (6).

Gaussian energy distribution with an rms relative energy spread δ_0, and $\eta > 0$ is shown in Fig. 2. It is convenient to introduce the dimensionless parameter Λ:

$$\Lambda = \frac{1}{|\eta|\gamma\delta_0^2}\frac{I}{I_A}\frac{R}{\langle R\rangle}\,, \tag{6}$$

where $I_A = mc^3/e = 17.5$ kA is the Alfven current. The maximum growth rate is reached at $kR = 0.68\Lambda^{3/2}$ and is equal to $(\mathrm{Im}\ \omega)_{\max} = 0.43\Lambda^{3/2}c\eta\delta_0/R$.

Three colored areas in this plot refer to stability regions in the parameter space. In the green area 1, the beam is stable because $\mathrm{Im}\,\omega < 0$ due to Landau damping. This region corresponds to high frequencies, $kR > 2.0\Lambda^{3/2}$. In the yellow region 2, where $k \lesssim R^{1/2}/a^{3/2}$ (a is the transverse size of the vacuum chamber), the instability is suppressed by shielding of the radiation. Finally, at even lower frequencies, in the blue area 3, the wavelength of the instability exceeds the bunch length and the coasting beam theory breaks down. The wavy lines between stability regions indicate fuzziness of the transition boundaries in our model.

There are several effects that are neglected in the simple theory described above. First, a zero transverse emittance of the beam was assumed. Second, the synchrotron damping γ_d due to incoherent radiation was neglected which makes the growth rate of the instability somewhat smaller, Im $\omega \to$ Im $\omega - \gamma_d$ [20]. Finally, the retardation effects were neglected which is valid if the formation time for the radiation is smaller than the instability growth time, $t_f \sim l_f/c \ll 1/\mathrm{Im}\,\omega$. In most cases characteristic for modern rings, those effects are relatively minor.

V. RINGS WITH WIGGLERS

In the damping ring of the Next Linear Collider [21], there will be long magnetic wigglers which introduce an additional contribution to the radiation impedance. The analysis of the CSR instability in such a ring requires knowledge of the impedance of the synchrotron radiation in the wiggler. Based on the earlier study of the coherent radiation from a wiggler [22], in Ref. [23], a steady-state wake averaged over the wiggler period has been derived for the case $K^2/2 \gg 1$ (where K is the wiggler parameter) and $\gamma \gg 1$. The most interesting from the point of view of instability is a low-frequency part of the impedance, given by the following formula (per unit length of path):

$$Z_{\text{wiggler}}(\omega) \approx \frac{Z_0 \omega}{4c} \left[1 - \frac{2i}{\pi} \log \frac{\omega}{\omega_*} \right] , \tag{7}$$

where $\omega_* = 4\gamma^2 c k_w / K^2$ and $Z_0 = 377$ Ohm. Eq. (7) is valid for $\omega \ll \omega_*$.

Results of the analysis of CSR instability in the NLC ring, taking into account the wiggler CSR impedance can be found in Ref. [24].

VI. DISCRETE MODES NEAR SHIELDING THRESHOLD

There are several reasons why the simple theory of CSR microbunching instability developed in Ref. [25] is not applicable near that shielding threshold, $\lambda \sim a^{3/2}/R^{1/2}$. The most important one is that CSR does not have continuous spectrum here, and the modes that can interact with the beam, are discrete. The discreteness of the spectrum has been demonstrated in early papers [9, 26] for toroids of rectangular cross section. A more recent analysis of the shielded CSR impedance [27] extends the previous treatment of the problem and deals with arbitrary shapes of the toroid cross section.

Each synchronous mode in the toroid is characterized by frequency ω_n, a loss factor κ_n (per unit length), and a group velocity $v_{g,n}$. The wake associated with the n-th mode is

$$w_n(z) = 2\kappa_n \cos \left(\frac{\omega_n}{c} z \right) .$$

This wake, for lowest modes, propagates behind the particle. Calculation of ω_n, κ_n, and $v_{g,n}$, in the general case of arbitrary cross section requires numerical solution of two coupled partial differential equations [27]. For a toroid of round cross section of radius a, the lowest mode has been found to have the frequency $\omega_1 = 2.12cR^{1/2}a^{-3/2}$, the loss factor $\kappa_1 = 2.11a^{-2}$ and the group velocity $1 - v_{g,1}/c = 1.1a/R$.

Near the shielding threshold, the CSR instability should be treated as an interaction of the beam with single modes, [28]. When the wavelength of the mode is smaller than the bunch length, one can still use the coasting beam approximation, but one cannot neglect retardation effects. Assuming an ideal toroidal chamber with a constant cross section (no straight sections in the ring), it turns out that the theory of single-mode instability [28] parallels that of SASE FEL (see, e.g., [29]). It gives the maximum growth rate of the instability for nth mode equal $\sqrt{3}\rho_n\omega_n/2$ where

$$\rho_n = \left[\frac{I}{I_A} \frac{c^2 \eta \kappa}{\omega_n^2 \gamma} \left(1 - \frac{v_{g,n}}{c} \right) \right]^{1/3} \tag{8}$$

is an analog of the Pierce parameter in FELs.

The nonlinear regime of the instability in this approximation has been studied in Refs. [28, 30]

VII. SIMULATION OF MULTIPARTICLE DYNAMICS IN THE PRESENCE OF CSR

We consider the dynamics of a bunched electron beam subject to the fields generated by itself, both through synchrotron radiation and through effects that are present even without trajectory curvature: space charge and influence of the vacuum chamber. Numerical simulation of such a system has two aspects: (1) the algorithm for computing the fields given the charge/current density; (2) the algorithm to follow particle motion given the fields.

A Field Computation

In free space the field generated by the bunch is unambiguous in principle, being given in terms of scalar and vector potentials of the form

$$\psi(\mathbf{r},t)=\frac{1}{4\pi}\int d\mathbf{r}'\frac{S(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{9}$$

where the source S is the charge density for the scalar potential or the current density for the vector potential. Evaluation of this formula can be nontrivial, but it has been noticed recently that evaluation is much easier if one takes the second argument of the source (retarded time) as one of the integration variables [31].

Study of models (dating back to the 1940's) indicates that field components of sufficiently long wavelength are exponentially suppressed by the vacuum chamber, the suppression setting in at the so-called "shielding cutoff" which is roughly $\lambda_0 = 2h(h/R)^{1/2}$, where h is the vertical aperture of the chamber and R is the bending radius. If and only if the bunch form has significant Fourier components with wavelength λ somewhat smaller than λ_0, those components will radiate a field that is close to the free-space field of wavelength λ, and in turn be influenced by that field. Thus, if bunch instability is due to CSR itself, a knowledge of the free-space field may be sufficient to find the current threshold of instability, since the cutoff is fairly abrupt. To some approximation, a mode can be considered as either completely shielded or free.

Nevertheless, for a full picture of beam dynamics it usually seems better to include a shielding model in detail. This was judged important for the bursting mode of CSR production [32], where in the course of time the bunch sometimes has substantial components satisfying $\lambda \lesssim \lambda_0$ and sometimes not, with periodic continuous transitions between the two situations during which partial shielding occurs.

Unfortunately, our present ability to calculate effects of shielding is rather limited. One usually relies on a model in which the vacuum chamber is represented by infinite parallel plates with separation h, perfectly conducting, with particles circulating between the plates in planes parallel to the plates. If the orbits are circular one gets the cutoff mentioned above. For general planar orbits, including for instance race track trajectories, the method of images suffices to satisfy the field boundary conditions on the plates. Then Eq.(9) still applies if it is augmented to include an infinite sequence of image bunches [31] in the source S. Only a few images in the sequence are needed in practice. The field from non-circular orbits with image charges can be computed from existing codes [33, 34], but those codes have usually been applied to single-pass systems such as bunch compressors, rather than to storage rings. It is not clear that they would be tractable for a dynamical simulation of a storage ring.

For circular orbits the parallel plate model was solved long ago, in the sense that an impedance or wake potential was determined which gives the voltage induced by a *rigid* bunch in the form

$$\begin{aligned}&V(\theta,t)=-2\pi R\mathcal{E}(\theta,t)=\\&\omega_0Q\sum_n e^{in(\theta-\omega_0t)}Z(n)\lambda_n=-Q\int W(\frac{R}{\sigma_z}(\theta-\omega_0t-\theta'))\lambda(\theta')d\theta'\ ,\end{aligned} \tag{10}$$

where $\mathcal{E}$ is the longitudinal electric field (averaged over the transverse charge distribution), Z is the impedance, W is the corresponding wake potential, and Q is the total charge. The azimuthal location in the lab frame is θ, the orbit radius is R, and the angular revolution frequency $\omega_0 = \beta_0 c/R$. In (10) we use the notation $Z(n)$ that is common when periodicity of fields is emphasized; in the previous notation of (4) this would be written as $Z(n/R)$. The charge density in the beam frame, normalized to unit integral, is $\lambda(\theta)$. The Fourier transform of the charge density is

$$\lambda_n=\frac{1}{2\pi}\int_0^{2\pi}e^{-in\theta}\lambda(\theta)dq\ . \tag{11}$$

Following earlier papers, we make the argument of W dimensionless, using the nominal bunch length σ_z as a scale factor. The formula for Z in terms of Bessel functions is derived in [35], and the equivalent W (modulo certain approximations) is found in [36] .

To apply (11) to the realistic case of a deforming bunch and non-circular orbits we make two reasonable assumptions. First we replace the rigid charge density $\rho(q)$ by the density $\rho(q,t)$ at the current time, obtaining

$$V(\theta,t)\approx\omega_0Q\sum_n e^{in(\theta-\omega_0t)}Z(n)\lambda_n(t)=-Q\int W(\frac{R}{\sigma_z}(\theta-\omega_0t-\theta'))\lambda(\theta',t)d\theta'\ , \tag{12}$$

Second, we identify R with the bending radius (or average bending radius) of the ring, rather than with $C/(2\pi)$ for a ring with circumference C. This identification is made *only* for the computation of the CSR force; otherwise the real ring parameters are used in the equations of motion. Effectively we are neglecting the CSR force in transitions between bends and straight sections, hoping that the neglected force, averaged over a turn, will have a small effect.

The formula (12) is not exact even in the case of circular orbits. Since it depends only on the present value of the charge density it does not account fully for retardation if the density is evolving in time. A light signal emitted from the bunch can catch up with the bunch at a later time when the charge density is altered. This effect is of course embodied in the "retarded potential" (9), and one could go back to (9) in the hope of deriving a tractable formula for the force that could be used in a dynamical simulation.

Alternatively, one can take a frequency domain approach by extending the impedance picture to account for retardation. This has been worked out in detail for the parallel plate model with circular orbits [35]. It involves the so-called *complete impedance* $Z(n,\omega)$, which is a function of the longitudinal mode number n and the complex frequency ω, being analytic in the upper half ω-plane with poles at wave guide cutoff frequencies $\pm\alpha_p c = \pm\pi pc/h$, $p = 1, 2, \cdots$ on the real axis. The *elementary impedance* in (10),(12) is $Z(n) = Z(n, n\omega_0)$. The analysis in [35] gives (12) as a first approximation, plus systematic corrections that account for retardation. The result including the first corrections is

$$\begin{aligned} V(\theta,t) \approx\, & 2\omega_0 Q \mathrm{Re} \sum_{n=1}^{\infty} e^{in(\theta-\omega_0 t)} \Bigg[Z(n,n\omega_0)\lambda_n(t) \\ & + \frac{\partial\tilde{Z}}{\partial\omega}(n,n\omega_0) i\lambda_n'(t) - i\frac{Z_0\pi R}{2\beta_0 h}\sum_p \Lambda_p \\ & \cdot \int_0^t dt'\lambda_n'(t')\Big(e^{iA(p,n)(t'-t)} + e^{iB(p,n)(t'-t)}\Big)\Bigg] , \end{aligned} \tag{13}$$

where $\tilde{Z}(n,\omega)$ is $Z(n,\omega)$ minus its pole terms, $Z_0 = 120\pi\ \Omega$ is the impedance of free space, Λ_p is a factor characterizing the vertical charge distribution [35], and

$$A(p,n) = \alpha_p c - n\omega_0 \ , \quad B(p,n) = -\alpha_p c - n\omega_0 \ . \tag{14}$$

The integrals over past values of $\lambda_n' = \partial\lambda_n/\partial t$ are associated with wave guide cutoffs. The formula (13) is not difficult to apply, since the integrals can be stored and incremented by a small amount at each time step. To date we do not have a similarly efficient method to account for retardation in a space-time approach.

To go beyond the parallel plate model we can consider a toroidal model of the vacuum chamber, again taking only circular orbits of radius R. The case of a circular torus with rectangular cross section and perfectly conducting walls can be solved in terms of Bessel functions [9, 37]. It gives a complete impedance $Z(n,\omega)$ which shares some properties with that for the parallel plates; it gives a shielding cutoff that is roughly the same, and it again has poles in ω at wave guide cutoffs. The physical picture is different, however, since the structure is closed. Energy cannot be radiated to infinity, and there are eigenmodes of the whole structure that are often described as "whispering gallery" modes. These show up as additional poles on the real ω axis. To see the connection with whispering gallery architecture, let the inner torus radius go to zero, so that we have a cylindrical gallery with height h and radius $b > R$, but with $b - R$ small, of order h. The beam excites modes that are concentrated near the cylindrical surface, very similar to corresponding modes of the torus [9]. These modes are well above the wave guide cutoff in frequency, so that they are associated with waves that propagate around the periphery of the gallery.

The question arises as to whether a real vacuum chamber, with corrugations from cavities, bellows, etc., and wall resistance, could support some vestige of these "full structure" modes of the smooth toroidal or cylindrical chamber. If the structure has perfect mechanical stability, it seems likely that vestigial whispering gallery modes do exist. They would of course have less spatial homogeneity than in the ideal case, and would involve coupling of various mode numbers n. Even if they exist in principle, could they be excited and be persistent enough to affect the beam in an observable way? This is an intriguing question that needs further study. On the theoretical side it seems quite possible to study perturbations of the smooth chamber for convenient simple forms of the perturbation, say by a mode matching or integral equation technique. One should also think about possible experimental observation of whispering modes.

It is noteworthy that the induced voltage for the toroidal or cylindrical model has quite a different form in comparison to (12). For those models $Z(n,\omega)$ is analytic in the ω-plane except for poles at waveguide cutoffs and resonance frequencies, and tends to a constant as $\omega \to \infty$ in complex directions. Consequently, one can evaluate the ω-integral in the definition of V by the method of residues. For the cylindrical model the resulting exact formula for the voltage is

$$V(\theta,t) =$$
$$\pi\omega_0 Q Z_0 \frac{R}{h}\sum_p \Lambda_p\Big[\big(\frac{R}{b}\big)^{1/2}\lambda(\theta-\omega_0 t,t) + 2\mathrm{Re}\sum_{n=1}^{\infty} e^{in(\theta-\omega_0 t)}\int_0^t dt'\lambda_n(t')$$
$$\times\big[\sum_s r^E(n,s)\big(e^{iA^E(p,n,s)(t'-t)} + e^{iB^E(p,n,s)(t'-t)}\big)$$
$$\sum_s \frac{r^M(n,s)}{\omega_*(p,n,s)}\big(e^{iA^M(p,n,s)(t'-t)} - e^{iB^M(p,n,s)(t'-t)}\big)$$
$$r(n)\big(A(p,n)e^{iA(p,n)(t'-t)} + B(p,n)e^{iB(p,n)(t'-t)}\big)\big]\Big] \ . \tag{15}$$

The second and third lines of the right hand side arise from transverse electric (TE) and transverse magnetic (TM) modes (transverse to the cylinder axis, not the beam direction), which have frequencies determined by zeros of Bessel functions and their derivatives; namely,

$$TE: \quad \omega_*'(p,n,s) = \frac{c}{b}[(\alpha_p b)^2 + j_{ns}'^{\,2}]^{1/2} \ , \quad J_n'(j_{ns}') = 0 \ ,$$
$$TM: \quad \omega_*(p,n,s) = \frac{c}{b}[(\alpha_p b)^2 + j_{ns}^2]^{1/2} \ , \quad J_n(j_{ns}) = 0 \ . \tag{16}$$

The factors $r^{E,M}$ come from residues of the resonance poles and have the forms

$$r^E(n,s) = \frac{\pi^2 cR}{2b^2}\frac{[j_{ns}'J_n'(j_{ns}'R/b)Y_n'(j_{ns}')]^2}{(n/j_{ns}')^2-1}$$
$$r^M(n,s) = -\frac{\pi^2 n}{2\beta_0}[\alpha_p c J_n(j_{ns}R/b)Y_n(j_{ns})]^2 \ , \tag{17}$$

whereas $r(n)$, from residues of waveguide cutoff poles, is

$$r(n) = \frac{1}{2\beta_0}\big(1-\big(\frac{R}{b}\big)^{2n}\big) \ . \tag{18}$$

The exponents involve A, B as in (14) and also

$$A^E(p,n,s) = \omega_*'(p,n,s) - n\omega_0 \ , \quad B^E(p,n,s) = -\omega_*'(p,n,s) - n\omega_0 \ ,$$
$$A^M(p,n,s) = \omega_*(p,n,s) - n\omega_0 \ , \quad B^M(p,n,s) = -\omega_*(p,n,s) - n\omega_0 \ . \tag{19}$$

For the toroidal model the result is similar to (15), except that one is dealing with poles from zeros of cross products of Bessel functions [9].

The first term in (15), which arises from the constant term in the impedance, is merely proportional to the present value of the charge density in the beam frame. If the total induced voltage were just a multiple of the charge density, that would be the idealized case that is usually called a "purely resistive wake". Besides this resistive term in (15) we have *only* retardation terms, integrals over prior values of the charge distribution. Some of these, especially those corresponding to the lowest TE resonance with $\omega_*'(p,n,s) \approx n\omega_0$, $p = s = 1$, are expected to be dynamically important. If so, we see that a wake potential description of the induced voltage as in (12) is strictly invalid. We are dealing with an unfamiliar situation that needs further exploration.

Oide claims, without a full proof, that a purely resistive wake gives an unstable beam for arbitrarily small current [38]. If that is true, then the first term in (15) has a destabilizing effect that may or may not be compensated by the other terms. In fact, it is hard to see how this model with a closed chamber and no wall resistance could give a stable beam since energy must just build up without bound in resonant modes. Correspondingly, a linearized Vlasov analysis of a coasting beam indicates a zero current threshold for instability, if we retain just a single resonance pole term in $Z(n,\omega)$.

As was shown in [9], one can give a reasonable approximate account of wall resistance in the cylindrical and toroidal models, but then the residue method that led to (15) does not work, since one gets a branch point from the square root frequency dependence of the skin depth. An interesting open question is then to find a good approximate expression for V including wall resistance.

If we account for resistivity in the above mentioned coasting beam analysis, merely by displacing the resonance pole to the lower half-plane by an amount given in the results of [9], then we get a non-zero current threshold of instability, but one (for the resistivity of aluminum) that is much smaller than that of the parallel plate model. Unfortunately, this leaves us in the dark about the quantitative dynamical role of shielding for CSR in rings. Perhaps both wall resistance and corrugations will be necessary in an acceptable model of a closed chamber.

B Numerical Modeling of Phase Space Dynamics

Undoubtedly, the most popular method to simulate multiparticle dynamics is the macroparticle method, sometimes called the particle-in-cell method [39]. For electron rings the macroparticle method must usually be augmented to account for incoherent synchrotron radiation. That is done by adding a damping term and pseudo-random momentum kicks to the single-particle equations of motion, the latter to model random emission of single photons. Another approach, which has aesthetic appeal, is to do a numerical solution of the Vlasov-Fokker-Planck (VFP) equation. This accomplishes the same thing, in principle, as the macroparticle method, but with much less numerical noise. It has been quite successful in the case of a two-dimensional phase space, and efforts are underway to extend this success to higher dimensions. The VFP equation is a nonlinear integro - partial differential equation with at least three independent variables, and one might think that a numerical code to solve it would be very complicated. Fortunately, that is not the case; the coding is no harder than for the macroparticle method, maybe even simpler.

For longitudinal motion in a storage ring above transition with linear RF the VFP equation for the phase space density $f(q,p,\tau)$ has the form [32, 40]

$$\begin{aligned} \frac{\partial f}{\partial \tau} &= A_V(f) + A_{FP}(f)\ , \\ A_V(f) &= -p\frac{\partial f}{\partial q} + \left[q + I_c F(q,f,\tau)\right]\frac{\partial f}{\partial p}\ , \\ A_{FP}(f) &= \frac{2}{\omega_s t_d}\frac{\partial}{\partial p}\left(pf + \frac{\partial f}{\partial p}\right)\ , \end{aligned} \tag{20}$$

where the dimensionless phase space variables are $q = z/\sigma_z$ and $p = -(E - E_0)/\sigma_E$, with z being the distance from the reference particle (positive in front), and $E - E_0$ the displacement from the nominal energy; we normalize by the low-current r.m.s. bunch length and energy spread, σ_z and σ_E. The dimensionless time is $\tau = \omega_s t$, where ω_s is the synchrotron frequency, and t_d is the longitudinal damping time. The collective force is $F = -V/Q$, with V the induced voltage as discussed above, and $I_c = e^2N/(2\pi\nu_s\sigma_E)$ is the normalized current, with ν_s the synchrotron tune. In most work to date the parallel plate model of shielding has been used, with V in the approximation (12).

The Vlasov equation without account of damping and fluctuations from incoherent synchrotron radiation is $\partial f/\partial\tau = A_V(f)$. The Fokker-Planck term A_{FP}, to represent damping and fluctuations, is in some sense small owing to slow damping (the large value of $\omega_s t_d$). We integrate (20) as an initial value problem, proceeding in small time steps $d\tau$. Since the Vlasov and Fokker-Planck operators require entirely different numerical techniques, we use *operator splitting*: for a time step $d\tau$, first advance f by $\partial f/\partial\tau = A_V(f)$, and then by $\partial f/\partial\tau = A_{FP}(f)$. The Vlasov step is unstable if done by a simple finite difference approximation of A_V and any sort of time-stepping, but stable if done by the *method of local characteristics*, equivalent to approximating the Perron-Frobenius operator by discretization. On the other hand, the Fokker-Planck step is easily done by a finite difference formula for A_{FP}, with the elementary Euler method for the time step.

In the method of local characteristics we suppose that the coherent force is nearly independent of time over a sufficiently small time interval $\Delta\tau$, so that we have a single-particle map defined locally in time. This volume preserving map is denoted by $M_{\tau\to\tau+\Delta\tau}(z)\ ,\ z = (q,p)$. Then the Perron-Frobenius (PF) operator $\mathcal{M}$ associated with M gives the time evolution of the distribution function:

$$f(z,\tau+\Delta\tau) = \mathcal{M}f(z,\tau) = f(M^{-1}(z),\tau)\ . \tag{21}$$

This is just another way of stating that the probability of finding a particle in a phase space volume element dz is preserved:

$$f(M(z),\tau+\Delta\tau)d(M(z)) = f(z,\tau)dz\ . \tag{22}$$

A discretization of $\mathcal{M}$ simply consists of choosing a finite-dimensional approximation of f. For instance, f might be described by its values on a grid $\{z_i\}$, with polynomial interpolation to off-grid points. In that case, evaluation of $\mathcal{M}f(z_i,\tau)$ would be done by interpolation, since $M^{-1}(z_i)$ is an off-grid point in general. In the literature the discretized PF method is often called the *semi-Lagrangian method* [41]. In plasma physics it dates back at least to the work of Cheng and Knorr [42] in 1976.

There are myriad possibilities for discretizing $\mathcal{M}$. In a study of longitudinal motion in the SLC damping rings, without CSR, a simple locally quadratic representation of f was used successfully [43]. For recent work on CSR [32, 40] a method of Yabe *et al.* [44] was applied. That method, which was able to deal with the small wavelength bunch perturbations from CSR, was reviewed in an appendix of [40].

We recall some results from [32] and [40], referring to those papers for the choice of parameters. In [32] the VFP integration was used to model the bursting mode of CSR in the VUV ring of NSLS. The impedance for the parallel plate model was used in (12), with a special algorithm to reduce Fourier sums and integrals to FFT evaluations. No contribution of machine impedances was included. Figure 3 shows the computed radiated power, divided by the incoherent power, as a function of time in synchrotron periods. Figure 4 gives the corresponding graph of the normalized bunch length. Here there are fast dipole oscillations with an envelope appearing as the black area of the figure. Figure 5 shows microbunching due to CSR. Each burst is associated with an instability driven by CSR. If the peak current of the bunch is sufficiently high, a small ripple on the bunch, of wavelength shorter than the shielding cutoff, can build up exponentially owing to the large values of the impedance at such wavelengths and the usual feedback mechanism of linear instability. This growth of microbunches gives strong coherent radiation in a range of wavelengths, but it does not last long because the intrinsic nonlinearity of the self-consistent many-particle motion quickly leads to a smoothing of the phase space distribution with attendant bunch lengthening and reduction of peak current. After one or two synchrotron periods a burst is finished, and then damping due to incoherent radiation gradually reduces the bunch length and restores the peak current required for instability, over a time interval that is a substantial fraction of the damping time. Thus, we have "relaxation oscillations" of the bunch length (and energy spread), quite like the well known sawtooth mode [43] that can be induced by machine impedances without CSR. The mechanism of oscillations is essentially the same, even though the source of the instability is different.

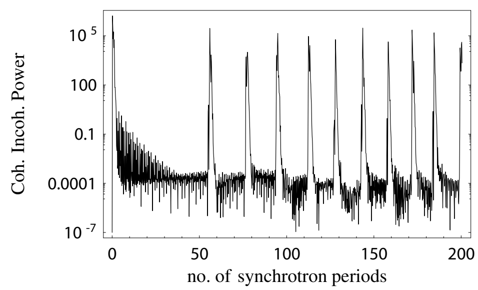

FIG. 3: Coherent over incoherent power

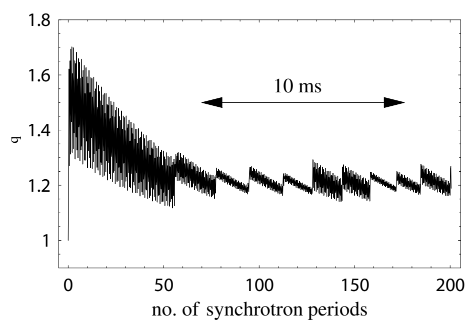

FIG. 4: Bunch length vs. time

[!ht]

One can also study microbunching without radiation damping, and there was a motivation for doing so in the design of a compact low energy storage ring [40, 45] having a damping time that is effectively infinite, being large compared to the storage time. Such a study was carried out in [40] using just the Vlasov equation without Fokker-Planck terms, again with the parallel plate impedance giving the only collective force. This impedance gives the full field

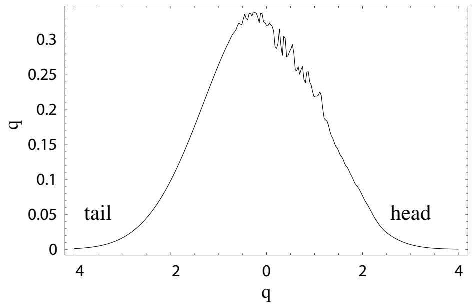

FIG. 5: Bunch density with 'microbunching'

of the model and therefore includes a significant space charge component, owing to the low energy (25 MeV) of the example. Figure 6 shows a grey scale plot of the phase space density, and a graph of the charge density, at three successive times: $\tau = 1.2, 3.2, 9.6$ or $0.19, 0.51, 1.5$ in units of the synchrotron period. Here we see very clearly the evolution described above: initial small ripples quickly amplified in a linear regime, and then a dramatic smoothing and broadening of the distribution through nonlinearity, all within a time comparable to the synchrotron period.

In [40] the time domain simulation was also used to check the validity of the threshold for instability as determined by the linearized Vlasov equation for a coasting beam with current equal to the peak current of the bunch. The agreement between the two determinations was satisfactory, showing that the Boussard criterion works in this case.

In [46] some preliminary work was done to assess the importance of corrections to (12) as stated in (13). Figure 7 shows an evaluation of the three terms in (13), again for the example of the compact storage ring [40]. The calculation was done with a smooth switching on of the current, in accord with the theory of [35]. After initial transients die out the effect of the corrections is fairly small. Further work needs to be done in extending the computation to a longer time interval.

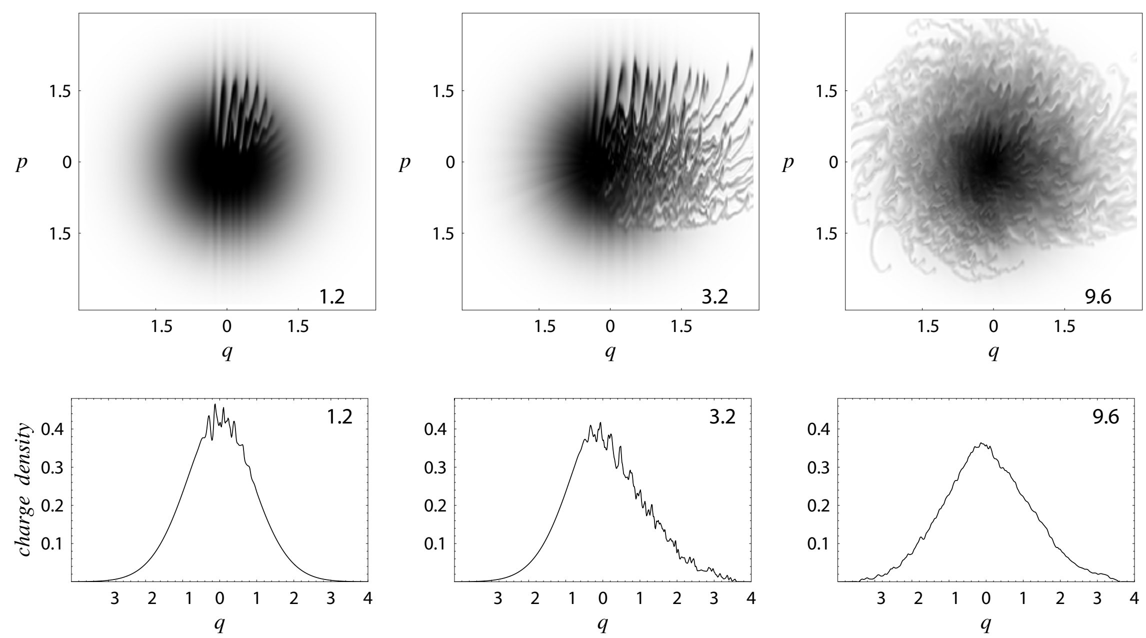

FIG. 6: Time evolution of bunch under effect of CSR in a compact storage ring. Density plots in phase space (top row) and charge density (second row). Pictures are taken at (normalized) time $\tau = 1.2, 3.2$, and 9.6. Instability initiated by a small perturbation with mode number $n = 702$ (wavelength $\lambda = 2.2$ mm). A unit of q corresponds to 1 cm.

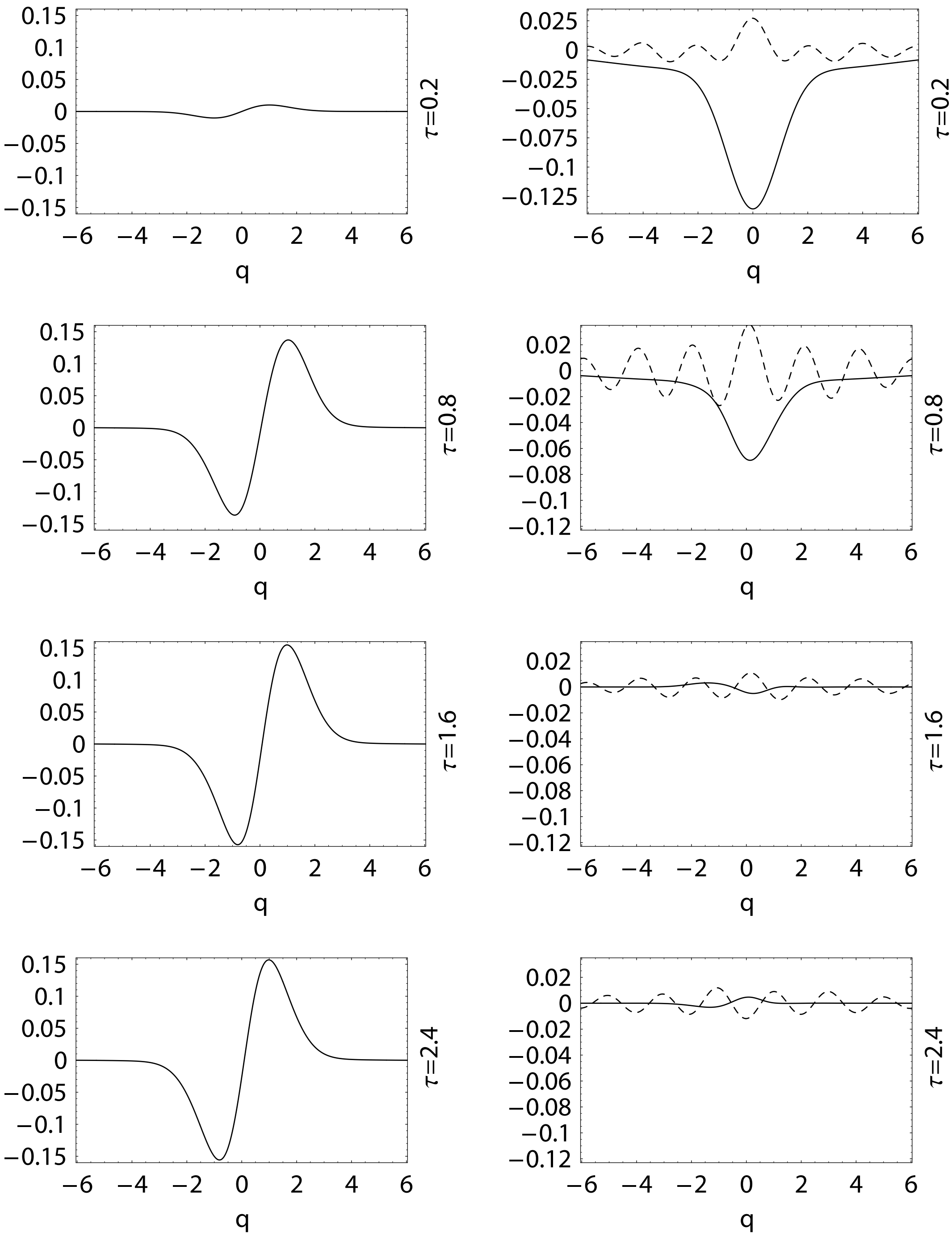

FIG. 7: Collective force $I_c F(q)$ at successive times. First term of Eq.(13) on left graph, second (solid) and third (dashed) on right.

C Possible Improvements in the Vlasov Solution

It would be interesting to explore the effects of the horizontal transverse charge distribution, as has been done for CSR in single-pass systems. If that problem is to be treated by solving the VFP equation, now in 4-dimesional phase space, then more attention should be paid to efficiency of the numerical algorithm to avoid excessively long computations. In principle the Perron-Frobenius operator $\mathcal{M}$ preserves positivity of f and the value of the total probability $\int f(q,p,\tau)dqdp = 1$. After discretization these properties do not hold unless one adopts some special technique. To date in 2-D phase space we have used positivity and probability conservation as criteria for choice of the mesh cell size, requiring that f be positive unless its magnitude is less than some small amount and that its integral be 1 within a small error. In higher dimensions this is not so feasible, since one tries to economize by using a coarse mesh which allows significant negative values of f.

An algorithm that automatically preserves both positivity and total probability may have a better chance of giving satisfactory results with a relatively coarse mesh. One such scheme was suggested in [31]. The idea is to represent the distribution function as a sum of tensor products of B-splines, with the coefficients given by Schönberg's Variation Diminishing Approximation [47]. In this approximation the coefficients are values of the function at averages of spline knot positions, and since B-splines are non-negative this means that positivity is preserved. To ensure probability conservation one can simply divide by the integral of the new function after each time step. That was found not to be a good idea in non-positive schemes, but it may work better in this case.

The B-splines have "small support", meaning that only a few members of a B-spline basis are non-zero at a single point. This means that evaluation of the distribution to compute the collective force will be fast. On the other hand, the tensor product basis has its limitations, as does Schönberg's method which provides only an approximation with error of second order in the cell size (in each dimension).

To overcome the limitations of tensor product bases, and to provide approximations of higher order, it seems interesting to consider *radial basis functions* [48]. Each basis function is associated with a point in phase space (a "site"), and is a function only of the Euclidean distance from the site. The sites can have arbitrary positions. A function is represented as a linear superposition of basis functions with coefficients determined so as to interpolate values of the function at sites. Applied to our case this would not ensure positivity, but one may hope that the good approximation properties of the basis in high dimensions provide an advantage. Because a large number of sites will be required, and because the interpolant must be redetermined at each time step, it seems necessary to use basis functions with small support ([48], Chap.6). Then one has small systems of linear equations to determine the coefficients.

Shepard's method is another idea for interpolation in high dimensions with arbitrary interpolation points, again using functions of the distance from sites [48–50]. It is very simple to implement, requires no solution of linear equations, and would provide a positive interpolation. Its approximation properties leave something to be desired, one peculiarity being that the interpolant has a flat spot (local extremum) at interpolation points. Nevertheless, it might be an interesting candidate in the quest for a Vlasov solver that could compete in speed with the macroparticle method in high dimensions.

Within a particular choice of interpolation method for discretizing the PF operator there are further ideas for improving efficiency. Preliminary work indicated that it might be useful to taper the mesh cell size in such a way as to give roughly equal probabilities of finding a particle in all cells. If the distribution is roughly Gaussian that is easy to do using the inverse error function. Sonnendrücker and collaborators are exploring the possibility of adaptive meshes [41], which change in time to follow the region in which the distribution is appreciable in magnitude. A "one shot" adaptation was proposed in [31], where coordinates were chosen so as to put the Vlasov equation in the interaction picture; namely, the initial conditions of the motion as it would be in the absence of collective effects. When collective effects are turned on these variables have relatively little variation with time. Consequently, an interpolation scheme chosen for good efficiency at time zero should also be good at later times. In this approach the projection to get the charge/current density from the distribution is more costly than usual, because the integrations are not along coordinate axes of the new variables.

Finally, we mention the *finite volume method*, which offers a positive and conserving algorithm that is different in technique from the PF operator discretization, although similar in spirit since it is derived directly from probability conservation [51, 52].

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