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$N = 1$ Supersymmetric One-loop Amplitudes and the Holomorphic Anomaly of Unitarity Cuts

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Abstract

Recently, it has been shown that the holomorphic anomaly of unitarity cuts can be used as a tool in determining the one-loop amplitudes in $N = 4$ super Yang-Mills theory. It is interesting to examine whether this method can be applied to more general cases. We present results for a non-MHV $N = 1$ supersymmetric one-loop amplitude. We show that the holomorphic anomaly of each unitarity cut correctly reproduces the action on the amplitude's imaginary part of the differential operators corresponding to collinearity in twistor space. We find that the use of the holomorphic anomaly to evaluate the amplitude requires the solution of differential rather than algebraic equations.

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1 Introduction

It has been proposed recently that a "weak-weak" duality exists between $N = 4$ supersymmetric gauge theory and topological string theory [1]. This relationship becomes manifest by transforming amplitudes into twistor space where they are supported on simple curves. A consequence of this picture is that tree amplitudes, when expressed in terms of spinor variables $k_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, are annihilated by various differential operators corresponding to the localization of points upon lines and planes in twistor space.

In particular the operator corresponding to collinearity of points i, j, k in twistor space,

$$[F_{ijk}, \eta] = \langle i\, j\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_k}, \eta\right] + \langle j\, k\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_i}, \eta\right] + \langle k\, i\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_j}, \eta\right] \tag{1}$$

annihilates the "maximally helicity violating" (MHV) n-gluon amplitudes,

$$[F_{ijk}, \eta] A_n^{\text{MHV-tree}}(1, 2, \cdots n) = 0. \tag{2}$$

These MHV colour-ordered amplitudes, where exactly two of the gluons have negative helicity, have a remarkably simple form, conjectured by Parke and Taylor [2] and proven by Berends and Giele [3],

$$A_n^{\text{MHV-tree}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+) = i \frac{\langle p\, q\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle 3\, 4\rangle \cdots \langle n-1,\, n\rangle \langle n\, 1\rangle}\,. \tag{3}$$

Using "cut-constructibility" and collinear limits, the one-loop MHV amplitudes have also been constructed for $N = 4$ [4] and $N = 1$ supersymmetric theories [5]. "Cut-constructible" implies that the entire amplitude can be reconstructed from a knowledge of its four-dimensional cuts — cuts evaluated with the intermediate states labelled by four-dimensional helicities [4, 5].

The MHV tree amplitudes appear to play a key role in gauge theories and recently Cachazo, Svrček and Witten conjectured that Yang-Mills amplitudes could be calculated using off-shell MHV vertices [6]. Points localised on a single line become attached to a MHV vertex. There have already been multiple applications of the CSW construction to the calculation of tree amplitudes [7].

Brandhuber *et al.* [8] emphasised the usefulness of the twistor picture beyond tree level by computing with MHV vertices and reproducing the one-loop $N = 4$ MHV amplitudes. The steps in their computation closely paralleled the cut-based computation [4] (though the MHV vertices are off-shell), stressing the key role of unitarity. Despite this success, application of the collinear and planar operators to one-loop MHV amplitudes does not result in their annihilation [9], even though naively it should. More specifically, consider the operator $F_{i\,i+1\,i+2}$ acting upon the cut $C_{i,\ldots,j}$ — the imaginary part, or 1/2 the discontinuity, in the channel $(k_i + k_{i+1} + \cdots + k_j)^2 > 0$ — of a one-loop MHV amplitude,

$$C_{i,\ldots,j} \equiv \frac{i}{2} \int d\text{LIPS}\left[A^{\text{MHV-tree}}(\ell_1, i, i+1, \ldots, j, \ell_2) \times A^{\text{MHV-tree}}(-\ell_2, j+1, j+2, \ldots, i-1, -\ell_1)\right]. \tag{4}$$

Because $F_{i\,i+1\,i+2}$ annihilates both tree amplitudes, we might naively expect it to annihilate the imaginary part of the amplitude — but it explicitly does not. (This computation has recently been extended to the special case of $N = 1$ MHV one-loop amplitudes [10].)

This apparent paradox was resolved [11] by observing that differential operators acting within the loop-momentum integral yield δ functions. This "holomorphic anomaly of the unitarity cut" produces a rational function as a result even though the tree amplitudes within the cut are localized on lines [11, 12, 13]. As a spin-off of this resolution, it was observed that acting with F_{ijk} upon both the cut and the imaginary part of the amplitude, and demanding consistency, leads to algebraic equations for the coefficients of the integral functions which appear in the amplitude. These algebraic equations are helpful in computing the entire amplitude [13], as demonstrated in the calculation of one of the four seven-point non-MHV $N = 4$ amplitudes [14]. The result agrees with that of ref. [15], where this and the remaining three amplitudes were evaluated directly from the cuts.

In this letter we present the results for a six-gluon non-MHV one-loop amplitude, where an $N = 1$ chiral multiplet propagates in the loop. This amplitude, $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$, was calculated by conventional (cut-based) means. We examine the action of the F_{ijk} operator upon the cuts of this amplitude, and demonstrate the consistency of the holomorphic anomaly with the result of F_{ijk} acting upon its imaginary part. We also investigate how the holomorphic anomaly could be used to calculate $N = 1$ one-loop amplitudes. We find in general, that the coefficients of the integral functions must now satisfy differential rather than algebraic equations. We also examine how coefficients of integral functions may be determined by these differential equations, together with other constraints, *e.g.,* collinear limits, by considering the n-point amplitude, $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$.

2 Organisation of $N = 1$ Supersymmetric Amplitudes

The computation of (unrenormalized) one-loop amplitudes may be simplified by carefully considering the integral functions I_i that may appear, and by realizing that the full amplitude is a linear combination of such functions with rational (in the variables λ^i_a and $\tilde{\lambda}^j_{\dot{a}}$) coefficients c_i,

$$A = \sum_i c_i I_i \,. \tag{5}$$

For supersymmetric amplitudes the summation is over a restricted set of functions. In $N = 4$ theories the functions that appear are only scalar box functions; whereas for $N = 1$ theories we are limited to scalar boxes, scalar triangles and scalar bubbles. In [4, 5] it was demonstrated that both amplitudes are "cut-constructible", *i.e.,* the coefficients c_i can be entirely determined by knowledge of the four-dimensional cuts of the amplitude.

For $N = 1$ super Yang-Mills with external gluons there are two possible supermultiplets contributing to the loop amplitude — the vector and the chiral matter multiplets. For simplicity we consider the leading-in-colour components of colour-ordered one-loop amplitudes. These can be decomposed into the contributions from single particle spins,

$$\begin{aligned} A_n^{N=1 \text{ vector}} &\equiv A_n^{[1]} + A_n^{[1/2]} \,, \\ A_n^{N=1 \text{ chiral}} &\equiv A_n^{[1/2]} + A_n^{[0]} \,, \end{aligned} \tag{6}$$

where $A_n^{[J]}$ is the one-loop amplitude with n external gluons and particles of spin-J circulating in the loop. (For spin-0 we mean a complex scalar.) For $N = 4$ super Yang-Mills theory there is a single multiplet which is given by

$$A_n^{N=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]} \,. \tag{7}$$

The contributions from these three multiplets are not independent but satisfy

$$A_n^{N=1 \text{ vector}} \equiv A_n^{N=4} - 3A_n^{N=1 \text{ chiral}} \,. \tag{8}$$

Thus, provided the $N = 4$ amplitude is known, one must only calculate one of the two possibilities for $N = 1$. The amplitude $A^{N=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ has been calculated [5], so in this letter we choose to examine the $N = 1$ chiral matter multiplet contribution.

3 The Six-point Amplitude $A^{N=1,\mathrm{chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

For a six-point Yang-Mills amplitude there are a relatively small number of independent colour-ordered helicity configurations. The non-vanishing supersymmetric amplitudes are either MHV, the conjugate of MHV (googly), or have three negative and three positive helicities.

The MHV amplitudes are rather special cases and indeed the holomorphic anomaly of the three particle cuts of $A^{N=1,\mathrm{chiral}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ is zero and is a rather uninteresting case. Consequently, we consider an amplitude with three negative helicities which has a richer structure. There are three possible such colour-ordered configurations: $A(- - - + ++)$, $A(- - + - ++)$ and $A(- + - + -+)$. We shall consider the effect of the holomorphic anomaly on the first of these. The amplitude $A(- - - + ++)$ has not been published previously. (It can be viewed as a component of $A_n^{[1]}$ and $A_n^{[1/2]}$, and thereby contributes to a six-gluon QCD amplitude required for the next-to-leading order production of four jets at hadron colliders.)

This amplitude is fairly simple in that it contains no box integral functions, but only L_0 and K_0 functions. The amplitude is

$$\begin{aligned} A_6^{N=1\ \mathrm{chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) &= a_1 K_0[s_{61}] + a_2 K_0[s_{34}] \\ &\quad - \frac{i}{2}\left[b_1 \frac{L_0[s_{345}/s_{61}]}{s_{61}} + b_2 \frac{L_0[s_{234}/s_{34}]}{s_{34}} + b_3 \frac{L_0[s_{234}/s_{61}]}{s_{61}} + b_4 \frac{L_0[s_{345}/s_{34}]}{s_{34}} \right] \end{aligned} \tag{9}$$

where the coefficients are

$$a_1 = a_2 = \frac{1}{2} A_6^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+), \tag{10}$$

and

$$\begin{aligned} b_1 &= \frac{\langle 6|\not{P}|3\rangle^2 \langle 6^+|(\not{2}\not{P}\not{P} - \not{P}\not{2}\not{P})|3^+\rangle}{\langle 2|\not{P}|5\rangle\, [6\,1]\, [1\,2]\, \langle 3\,4\rangle\, \langle 4\,5\rangle\, P^2}, & P &= P_{345} \equiv k_3 + k_4 + k_5, \\ b_2 &= \frac{\langle 4|\not{P}|1\rangle^2 \langle 4^+|(\not{P}\not{2}\not{P} - \not{2}\not{P}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\, [2\,3]\, [3\,4]\, \langle 5\,6\rangle\, \langle 6\,1\rangle\, P^2}, & P &= P_{234} \equiv k_2 + k_3 + k_4, \\ b_3 &= \frac{\langle 4|\not{P}|1\rangle^2 \langle 4^+|(\not{P}\not{P}\not{5} - \not{P}\not{5}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\, [2\,3]\, [3\,4]\, \langle 5\,6\rangle\, \langle 6\,1\rangle\, P^2}, & P &= P_{234}, \\ b_4 &= \frac{\langle 6|\not{P}|3\rangle^2 \langle 6^+|(\not{P}\not{5}\not{P} - \not{P}\not{P}\not{5})|3^+\rangle}{\langle 2|\not{P}|5\rangle\, [6\,1]\, [1\,2]\, \langle 3\,4\rangle\, \langle 4\,5\rangle\, P^2}, & P &= P_{345}. \end{aligned} \tag{11}$$

The six-point tree amplitudes were calculated in ref. [16]. The amplitude has an overall factor in dimensional regularisation of $(\mu^2)^\epsilon c_\Gamma$, where

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{12}$$

which we do not write explicitly. We define $s_{ij} \equiv [i\,j]\,\langle j\,i\rangle$, $s_{ijk} \equiv P_{ijk}^2 \equiv [i\,j]\,\langle j\,i\rangle + [j\,k]\,\langle k\,j\rangle + [k\,i]\,\langle i\,k\rangle \equiv (k_i + k_j + k_k)^2$ and $\langle a|\not{b}|c\rangle \equiv \langle a^+|\not{b}|c^+\rangle \equiv [ab]\langle bc\rangle$, where $\langle i\,j\rangle$ and $[i\,j]$ are the usual spinor helicity inner products [17]. The L_0 and K_0 functions are defined by

$$
\begin{aligned}
\mathrm{K}_0[s] &= \frac{1}{\epsilon(1-2\epsilon)}(-s)^{-\epsilon} = \frac{1}{\epsilon} - \ln(-s) + 2 + \mathcal{O}(\epsilon)\ , \\
\mathrm{L}_0[r] &= \frac{\ln(r)}{1-r} + \mathcal{O}(\epsilon)\ .
\end{aligned}
\tag{13}
$$

The function $\mathrm{K}_0[s]$ is simply proportional to the scalar bubble function. The function $\mathrm{L}_0[r]$ has several representations; it can be expressed as a linear combination of bubble functions or as a Feynman parameter integral for a two-mass triangle integral [5].

This amplitude was constructed by calculating the three-particle cuts together with an analysis of the infra-red poles. We shall be revisiting the three-particle cuts when we consider the action of the holomorphic anomaly. We exhibit the limit where legs 1 and 2 become collinear to illustrate the consistency of the amplitude at this infra-red pole. The other limits are analogous, although not identical.

Collinear limit $1-2$

The collinear limits of the (colour-ordered) one-loop partial amplitudes have the following form:

$$
\begin{aligned}
A_n^{\mathrm{loop}} \stackrel{a\|b}{\longrightarrow} \sum_{\lambda=\pm} \Bigg(& \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A^{\mathrm{loop}}_{n-1}(\ldots (a+b)^{\lambda} \ldots) \\
& + \mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A^{\mathrm{tree}}_{n-1}(\ldots (a+b)^{\lambda} \ldots) \Bigg),
\end{aligned}
\tag{14}
$$

where $k_a \to z k_P$, $k_b \to (1-z)k_P$, and λ is the helicity of the state P. The splitting amplitudes are universal and may be derived, for example, from the five-gluon amplitudes [18, 4].

For supersymmetric theories the loop splitting amplitudes $\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})$ are proportional to the tree splitting amplitudes,

$$
\mathrm{Split}^{\mathrm{loop}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \;=\; c_\Gamma \times \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b}) \times r_S^{\mathrm{SUSY}}(z, s_{ab}). \tag{15}
$$

For the $N=1$ matter multiplet (but not the vector) the collinear limit is simplified since

$$
r_S^{N=1\ \mathrm{chiral}}(z, s_{ab}) = 0 \tag{16}
$$

for collinear gluons. This reduces the collinear condition to

$$
A_n^{\mathrm{loop}} \stackrel{a\|b}{\longrightarrow} \sum_{\lambda=\pm} \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A^{\mathrm{loop}}_{n-1}(\ldots (a+b)^{\lambda} \ldots)\,. \tag{17}
$$

The 'target' of the $1-2$ collinear limit will be the five-point $N=1$ matter amplitude [18],

$$
\begin{aligned}
A_5^{N=1\ \mathrm{chiral}}(1^-, 3^-, 4^+, 5^+, 6^+) = \frac{A_5^{\mathrm{tree}}}{2} \Bigg[& K_0[s_{34}] + K_0[s_{61}] \\
& - \frac{L_0[s_{345}/s_{34}]}{s_{34}} \frac{\mathrm{tr}_+[135(1+6)] - \mathrm{tr}_+[13(1+6)5]}{s_{13}} \Bigg].
\end{aligned}
\tag{18}
$$

In the limit $k_1 \to z k_1, k_2 \to (1-z)k_1$, the coefficients b_2 and b_3 are non-singular, where $\mathrm{tr}_+[ijkl] = [i\,j]\,\langle j\,k\rangle\,[k\,l]\,\langle l\,i\rangle$. Using the collinear properties of tree amplitudes, the coefficient a_2 becomes

$$
a_2 = \frac{A_6^{\mathrm{tree}}}{2} \;\longrightarrow\; S_+^{--} \frac{A_5^{\mathrm{tree}}(1,3,4,5,6)}{2}\,, \tag{19}
$$

where $S_+^{--} \equiv \mathrm{Split}_+^{\mathrm{tree}}(1^-, 2^-)$. So the combination $a_2 K_0[s_{34}]$ behaves as,

$$a_2 K_0[s_{34}] \;\longrightarrow\; S_+^{--} \frac{A_5^{\mathrm{tree}}(1,3,4,5,6)}{2} K_0[s_{34}]\,, \tag{20}$$

which is one of the required terms.

The integral function multiplying b_4 trivially goes to the function $\mathrm{L}_0[s_{345}/s_{34}]$. The coefficient b_4 approaches

$$\begin{aligned} b_4 \;\longrightarrow\; & \frac{-1}{\sqrt{z(1-z)}\,[1\,2]} \times \frac{\langle 6|\not{1}|3\rangle^2 \langle 6^+|((\not{1}+\not{6})\not{5}(\not{1}+\not{6}) - (\not{1}+\not{6})(\not{1}+\not{6})\not{5})|3^+\rangle}{\langle 1|\not{6}|5\rangle\,[6\,1]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{61}} \\ =\; & S_+^{--} \times \frac{[6\,1]^2\,\langle 1\,3\rangle^2}{[1\,6]\,\langle 6\,5\rangle\,[6\,1]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{61}} \times \langle 6^+|[\not{1}\not{5}(\not{1}+\not{6}) - \not{1}\not{6}\not{5}]|3^+\rangle \\ =\; & -\,S_+^{--} \frac{A_5^{\mathrm{tree}}}{i} \times \frac{\mathrm{tr}_+[13((6+1)5-5(6+1))]}{s_{13}}\,, \end{aligned} \tag{21}$$

after some rearrangement, which is as required.

The remaining two integrals must combine in the collinear limit. An identity similar to eq. (III.10) of ref. [5] can be used,

$$(1-z)\frac{\mathrm{L}_0[s_{612}/s_{61}]}{s_{61}} + z\frac{\mathrm{K}_0[s_{61}]}{s_{61}} \;\longrightarrow\; \frac{\mathrm{K}_0[s_{61}]}{s_{61}}\,. \tag{22}$$

The limits of the combination $a_1 s_{61}$ and of b_1 are both proportional to the five-point tree amplitude:

$$a_1 s_{61} \longrightarrow S_+^{--} \times \frac{A_5^{\mathrm{tree}}(1,3,4,5,6)}{2} \times s_{61} \times z, \tag{23}$$

and

$$\begin{aligned} b_1 \;\longrightarrow\; & \frac{-(1-z)}{\sqrt{z(1-z)}\,[1\,2]} \times \frac{\langle 6|\not{1}|3\rangle^2 \langle 6^+|(\not{1}\not{6}\not{1} - \not{1}\not{1}(\not{6}+\not{1}))|3^+\rangle}{\langle 1|\not{6}|5\rangle\,[6\,1]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{61}} \\ =\; & S_+^{--} \times (1-z) \times \frac{([6\,1]\,\langle 1\,3\rangle)^2 s_{61} \langle 6|1|3\rangle}{[1\,6]\,\langle 6\,5\rangle\,[6\,1]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{61}} \\ =\; & -\,S_+^{--} \times (1-z) \times \frac{A_5^{\mathrm{tree}}}{i} \times s_{61}\,. \end{aligned} \tag{24}$$

Thus, using eq. (22), we have

$$a_1 K_0[s_{61}] - \frac{i}{2} b_1 \frac{L_0[s_{612}/s_{61}]}{s_{61}} \longrightarrow S_+^{--} \times \frac{A_5^{\mathrm{tree}}(1,3,4,5,6)}{2} \times \mathrm{K}_0[s_{61}]\,. \tag{25}$$

Adding all the pieces together, we find that the amplitude has the correct collinear limit.

The other collinear limits are similar. The limits $2-3$, $4-5$ and $5-6$ are related by symmetry; they follow from the $1-2$ limit by relabelling and conjugation. The $6-1$ and $3-4$ limits are different but can be shown to have the correct limit in an analogous manner. The amplitude also has the correct multi-particle poles [19] when $s_{234} \to 0$ or $s_{345} \to 0$.

4 Holomorphic Anomaly of the Unitary Cuts

The amplitude we are considering has three potential three-particle cuts: $s_{123} > 0$, $s_{234} > 0$ and $s_{345} > 0$. The first of these vanishes identically for $N = 1$ matter: $A_5^{\mathrm{tree}}(\ell_1^{h_1}, 1^-, 2^-, 3^-, \ell_2^{h_2}) = 0$

unless $h_1 = h_2 = 1$, which requires the states crossing the cut to be gluons, not fermions or scalars. The two non-vanishing cuts are not independent but may be obtained from one another by the symmetry $1 \leftrightarrow 3, 4 \leftrightarrow 6$.

In order to examine the holomorphic anomaly, we compute the action of F_{561} on the cut C_{561} (which is equal to C_{234}). The cut for $s_{561} > 0$ (the imaginary part, or 1/2 the discontinuity) is defined as

$$C_{561} = \frac{i}{2} \int d\mathrm{LIPS} \sum_{h \in \{-1/2, 0, 1/2\}} A_5^{\mathrm{tree}}(\ell_1^h, 5^+, 6^+, 1^-, \ell_2^{-h}) A_5^{\mathrm{tree}}((-\ell_2)^h, 2^-, 3^-, 4^+, (-\ell_1)^{-h}), \quad (26)$$

where $\ell_1 + \ell_2 = P_{234} \equiv P$. Writing out all amplitudes in this expression and summing over the supersymmetric multiplet we obtain

$$C_{561} = \frac{i}{2} \int d\mathrm{LIPS} \frac{\langle 1\,\ell_1\rangle^2 \langle 1\,\ell_2\rangle^2}{\langle 5\,6\rangle \langle 6\,1\rangle \langle 1\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle \langle \ell_1\,5\rangle} \times \frac{[4\,\ell_1]^2\,[4\,\ell_2]^2}{[2\,3]\,[3\,4]\,[4\,\ell_1]\,[\ell_1\,\ell_2]\,[\ell_2\,2]} \times \rho_{N=1}\,. \quad (27)$$

The factor $\rho_{N=1}$ may be obtained using supersymmetric Ward identities [20], giving

$$\rho_{N=1} = \frac{\langle 4|\not{P}|1\rangle^2}{\langle 1\,\ell_1\rangle\,[\ell_1\,4]\,\langle 1\,\ell_2\rangle\,[\ell_2\,4]}\,. \quad (28)$$

Simplifying the expression, we can write C_{561} in a compact form

$$C_{561} = i\frac{K}{2} \int d\mathrm{LIPS} \frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle}\,, \quad (29)$$

where we define K as

$$K = \frac{\langle 4|\not{P}_{234}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,s_{234}}\,. \quad (30)$$

Next we act with the collinear operator $[F_{561}, \eta]$ on this expression. It is clear that we will only pick up the contribution from the term with $\partial/\partial\tilde{\lambda}_{5\dot{a}}$, so that

$$[F_{561}, \eta] C_{561} = \frac{iK}{2} \int d\mathrm{LIPS} \frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]} \langle 6\,1\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_5}, \eta\right] \frac{1}{\langle 5\,\ell_1\rangle}\,. \quad (31)$$

The parametrization of the Lorentz-invariant phase-space measure dLIPS is the same as that employed in [6, 12], *i.e.,*

$$\begin{aligned} \int d\mathrm{LIPS}(\bullet) &\equiv \int d^4\ell_1\ \delta^{(+)}(\ell_1^2) \int d^4\ell_2 \delta^{(+)}(\ell_2^2) \delta^{(4)}(\ell_1 + \ell_2 - P)(\bullet) \\ &= \int_0^\infty t dt \int \langle \lambda_{\ell_1}, d\lambda_{\ell_1}\rangle [\tilde{\lambda}_{\ell_1}, d\tilde{\lambda}_{\ell_1}] \int d^4\ell_2 \delta^{(+)}(\ell_2^2) \delta^{(4)}(\ell_1 + \ell_2 - P)(\bullet), \end{aligned} \quad (32)$$

and we change coordinates, $\lambda \to \lambda'$ and $\tilde{\lambda} \to t\tilde{\lambda}'$, then drop the primes. Hence the integral becomes

$$\begin{aligned} [F_{561}, \eta] C_{561} = i\frac{K}{2} &\int_0^\infty t dt \int \langle \lambda_{\ell_1}, d\lambda_{\ell_1}\rangle [\tilde{\lambda}_{\ell_1}, d\tilde{\lambda}_{\ell_1}] \\ &\int d^4\ell_2 \delta^{(+)}(\ell_2^2) \delta^{(4)}(\ell_1 + \ell_2 - P) \frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle\,\langle 6\,1\rangle}{[2\,\ell_2]} \left[\frac{\partial}{\partial\tilde{\lambda}_5}, \eta\right] \frac{1}{\langle 5\,\ell_1\rangle}\,. \end{aligned} \quad (33)$$

We follow the prescription of ref. [13] and use the identity,

$$\left[\frac{\partial}{\partial\tilde{\lambda}_5}, \eta\right] \frac{1}{\langle \ell_1\,5\rangle} = -\left[\frac{\partial}{\partial\tilde{\lambda}_{\ell_1}}, \eta\right] \frac{1}{\langle \ell_1\,5\rangle}\,, \quad (34)$$

which can be rewritten using the Schouten identity $[A\,B]\,[C\,D] = [A\,C]\,[B\,D] - [A\,D]\,[B\,C]$, so that

$$[\tilde\lambda_{\ell_1}, d\tilde\lambda_{\ell_1}]\left[\frac{\partial}{\partial\tilde\lambda_{\ell_1}},\eta\right] = \left[\tilde\lambda_{\ell_1},\frac{\partial}{\partial\tilde\lambda_{\ell_1}}\right][d\tilde\lambda_{\ell_1},\eta] - [\tilde\lambda_{\ell_1},\eta]\left[d\tilde\lambda_{\ell_1},\frac{\partial}{\partial\tilde\lambda_{\ell_1}}\right], \tag{35}$$

where the first term does not contribute to the integral. Hence inside the integral we can rewrite

$$[\tilde\lambda_{\ell_1}, d\tilde\lambda_{\ell_1}]\left[\frac{\partial}{\partial\tilde\lambda_{5}},\eta\right]\frac{1}{\langle\ell_1\,5\rangle} = [\tilde\lambda_{\ell_1},\eta]\left[d\tilde\lambda_{\ell_1},\frac{\partial}{\partial\tilde\lambda_{\ell_1}}\right]\frac{1}{\langle\ell_1\,5\rangle} = [\tilde\lambda_{\ell_1},\eta]2\pi\bar\delta(\langle\lambda_{\ell_1},\lambda_5\rangle)\,, \tag{36}$$

and the integral becomes

$$\begin{aligned}[F_{561},\eta]C_{561} = -i\pi K\int_0^\infty t dt\int\langle\lambda_{\ell_1}, d\lambda_{\ell_1}\rangle \\ \int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle\,\langle 6\,1\rangle\,[\tilde\lambda_{\ell_1},\eta]}{[2\,\ell_2]}\bar\delta(\langle\lambda_{\ell_1},\lambda_5\rangle)\,.\end{aligned} \tag{37}$$

The δ function in $\langle\lambda_{\ell_1},\lambda_5\rangle$ reduces the integral to

$$[F_{561},\eta]C_{561} = -i\pi K\int_0^\infty t dt\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{[4\,\ell_2]\,\langle\ell_2\,a\rangle\,\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]}{[2\,\ell_2]\,\langle\ell_2\,a\rangle}\,. \tag{38}$$

We have introduced a factor of $\langle\ell_2\,a\rangle\,/\,\langle\ell_2\,a\rangle$, which makes applying the δ function in ℓ_2 more transparent. Doing the integral in ℓ_2 using $\delta^{(4)}(\ell_1+\ell_2-P)$ we end up with

$$[F_{561},\eta]C_{561} = -i\pi K\int_0^\infty t dt\delta^{(+)}(\ell_2^2)\frac{[4\,\ell_2]\,\langle\ell_2\,a\rangle\,\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]}{[2\,\ell_2]\,\langle\ell_2\,a\rangle}\,, \tag{39}$$

where now $\ell_2^\mu = P^\mu - t k_5^\mu$, and hence $\ell_2^2 = P^2 - 2tk_5\cdot P$, where $t = \frac{P^2}{2k_5\cdot P}$. Doing the t-integral yields

$$\begin{aligned}[F_{561},\eta]C_{561} &= i\pi K\frac{\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]P^2}{(2k_5\cdot P)^2}\frac{(2k_5\cdot P)\langle 4|\not{P}|a\rangle - P^2\langle 4|\not{5}|a\rangle}{(2k_5\cdot P)\langle 2|\not{P}|a\rangle - P^2\langle 2|\not{5}|a\rangle} \\ &= i\pi\frac{\langle 4|\not{P}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle}\frac{\langle 1\,5\rangle\,[5,\eta]}{(2k_5\cdot P)^2}\frac{\langle 4|\not{P}|5\rangle}{\langle 2|\not{P}|5\rangle}\,,\end{aligned} \tag{40}$$

after reinstating the definition of K and choosing $a = 3$, for example.

From the optical theorem, the cut C_{561} is equal to the imaginary part of the amplitude in the kinematic region $s_{561} > 0$ [21]. For our amplitude (9), using $\mathrm{Im}\ln(-s)|_{s>0} = -\pi$, this is

$$-\frac{1}{\pi}\mathrm{Im}A_{s_{561}>0} = -\frac{i}{2}\left[\frac{b_3}{2k_5\cdot P} - \frac{b_2}{2k_2\cdot P}\right]. \tag{41}$$

Operating on eq. (41) with the collinear operator $[F_{561},\eta]$ we have

$$[F_{561},\eta]\left(-\frac{1}{\pi}\mathrm{Im}A_{s_{561}>0}\right) = -\frac{i}{2}\left[\frac{[F_{561},\eta](b_3)}{2k_5\cdot P} - \frac{[F_{561},\eta](b_2)}{2k_2\cdot P} - \frac{b_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2}\right]. \tag{42}$$

Using the solutions for b_2 and b_3 we have

$$b_3 = \frac{\langle 4|\not{P}|1\rangle^2}{\langle 2|\not{P}|5\rangle}\frac{\langle 4^+|\not{P}\not{P}\not{5} - \not{P}\not{5}\not{P}|1^+\rangle}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2} = \frac{K}{\langle 2|\not{P}|5\rangle}\langle 4^+|\not{P}\not{P}\not{5} - \not{P}\not{5}\not{P}|1^+\rangle \equiv K'\hat b_3\,, \tag{43}$$

where

$$K' \equiv \frac{K}{\langle 2|\not{P}|5\rangle} \tag{44}$$

is annihilated by F_{561}, and where

$$\hat{b}_3 \equiv 2P^2\langle 4|\not{5}|1\rangle - (2k_5\cdot P)\langle 4|\not{P}|1\rangle \,.$$

Also,

$$b_2 = \frac{\langle 4|\not{P}|1\rangle^2\langle 4^+|(\not{P}\not{2}\not{P} - \not{2}\not{P}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2} = \frac{K}{\langle 2|\not{P}|5\rangle}\langle 4^+|(\not{P}\not{2}\not{P} - \not{2}\not{P}\not{P})|1^+\rangle \equiv K'\hat{b}_2\,, \tag{45}$$

where

$$\hat{b}_2 \equiv -2P^2\langle 4|\not{2}|1\rangle + (2k_2\cdot P)\langle 4|\not{P}|1\rangle \,. \tag{46}$$

Using $[F_{561},\eta](2k_5\cdot P) = \langle\eta|\not{P}|5\rangle\langle 16\rangle$, we have

$$\begin{aligned}[F_{561},\eta]\hat{b}_3 &= 2P^2\langle 51\rangle[F_{561},\eta][45] - \langle 4|\not{P}|1\rangle[F_{561},\eta](2k_5\cdot P)\\ &= -2P^2[\eta,4]\,\langle 1\,6\rangle\,\langle 5\,1\rangle - \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\,,\end{aligned} \tag{47}$$

and

$$[F_{561},\eta]\hat{b}_2 = 0. \tag{48}$$

Inserting eqs. (47) and (48) into eq. (42), we find,

$$\begin{aligned}-\frac{1}{\pi}[F_{561},\eta]\mathrm{Im}A &= -\frac{i}{2}\frac{K'}{2k_5\cdot P}\Big(-2P^2[\eta,4]\,\langle 1\,6\rangle\,\langle 5\,1\rangle - \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\\ &\qquad - \frac{2P^2\langle 4|\not{5}|1\rangle\langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle}{2k_5\cdot P} + \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\Big)\\ &= -\frac{i}{2}\frac{2K'P^2\langle 16\rangle\langle 15\rangle}{(2k_5\cdot P)^2}\Big[[\eta,4](2k_5\cdot P) + \langle\eta|\not{P}|5\rangle\,[4\,5]\Big]\,.\end{aligned} \tag{49}$$

Combining $[\eta,4][P,5] - [\eta,P][45] = [\eta,5][P,4]$ using the Schouten identity, we obtain

$$-\frac{1}{\pi}[F_{561},\eta]\mathrm{Im}A = -i\frac{K'P^2\,\langle 1\,6\rangle\,\langle 1\,5\rangle\,\langle 5,P\rangle}{(2k_5\cdot P)^2}\Big[[\eta,5][P,4]\Big] = -i\frac{\langle 4|\not{P}|1\rangle^2\,\langle 1\,5\rangle}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle}\frac{[5,\eta]}{(2k_5\cdot P)^2}\frac{\langle 4|\not{P}|5\rangle}{\langle 2|\not{P}|5\rangle}\,, \tag{50}$$

which matches the expression in eq. (40). Thus we have shown that the holomorphic anomaly of the unitarity cuts correctly reproduces the action of F_{ijk} upon the imaginary part of the amplitude.

5 Reconstructing Amplitudes from Differential Equations

In $N=4$ one-loop amplitudes, appropriate collinear operators F_{ijk} annihilate the coefficients of the scalar box integral functions which span the amplitude [13]. This has the implication that the coefficients may be reconstructed by solving algebraic equations resulting from the action of the F_{ijk} operator upon the cuts equation. For $N=1$ we have a more delicate situation as the collinear operator F_{ijk} in this case acts non-trivially on the coefficients b_i in the amplitude. This means that to reconstruct the amplitude we will generally have to solve differential equations for the coefficients b_i. In this section we explore the possibility of reconstructing the amplitude using the holomorphic anomaly of the cuts. In general $N=1$ amplitudes contain integral functions derived from box, triangle and bubble integrals. As for the $N = 4$ case, we expect that the appropriate F_{ijk} operators should annihilate the coefficients of the box integral functions. However, F_{ijk} need not annihilate the coefficients of bubble and triangle functions. Instead, the action of F_{ijk} produces differential equations which these coefficients must satisfy.

To clarify the situation, consider the amplitude $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ which contains only triangle and bubble integrals. Consider the action of F_{561} on the C_{561} cutting equation,

$$[F_{561}, \eta] \mathrm{Im} A_{s_{561}>0} = [F_{561}, \eta] C_{561} \, . \tag{51}$$

Expanding the amplitude according to eq. (5) and keeping only those coefficients which have non-vanishing cuts in this channel, namely b_2 and b_3 in eq. (9), we have

$$\frac{i\pi}{2} [F_{561}, \eta] \left(\frac{b_3}{2k_5 \cdot P} - \frac{b_2}{2k_2 \cdot P} \right) = [F_{561}, \eta] C_{561} \, . \tag{52}$$

The right-hand side of this equation is a rational function of λ_i and $\tilde{\lambda}_j$, determined *via* the holomorphic anomaly to be the expression given in eq. (40). In eq. (52) the functions multiplying the b_i are rational functions — in contrast to the $N = 4$ situation where logarithms appear. Although the left-hand side is required to be rational this does not imply that F_{ijk} annihilate the b_i. The b_i must satisfy the linear differential equation

$$\frac{i\pi}{2} \left[\frac{[F_{561}, \eta] b_3}{2k_5 \cdot P} - \frac{b_3 [F_{561}, \eta] (2k_5 \cdot P)}{(2k_5 \cdot P)^2} - \frac{[F_{561}, \eta] b_2}{2k_2 \cdot P} \right] = [F_{561}, \eta] C_{561} \, . \tag{53}$$

We can also act with the operator

$$\langle \bar{F}_{ijk}, \bar{\eta} \rangle = [i\, j] \left\langle \frac{\partial}{\partial \lambda_k}, \bar{\eta} \right\rangle + [j\, k] \left\langle \frac{\partial}{\partial \lambda_i}, \bar{\eta} \right\rangle + [k\, i] \left\langle \frac{\partial}{\partial \lambda_j}, \bar{\eta} \right\rangle , \tag{54}$$

which produces an "anti-holomorphic anomaly" upon the same cut to yield

$$\frac{i\pi}{2} \left[\frac{\langle \bar{F}_{234}, \bar{\eta} \rangle b_2}{2k_2 \cdot P} - \frac{b_2 \langle \bar{F}_{234}, \bar{\eta} \rangle (2k_2 \cdot P)}{(2k_2 \cdot P)^2} - \frac{\langle \bar{F}_{234}, \bar{\eta} \rangle b_3}{2k_5 \cdot P} \right] = \langle \bar{F}_{234}, \bar{\eta} \rangle C_{561} \, . \tag{55}$$

As a function of $\tilde{\lambda}_5$, $\tilde{\lambda}_6$,and $\tilde{\lambda}_1$, we find explicitly that $[F_{561}, \eta] C_{561}$ is a function of $\tilde{\lambda}_5$ only. Similarly $\langle \bar{F}_{234}, \bar{\eta} \rangle C_{561}$ is a function of λ_2 only. The coefficients b_2 and b_3 are related by the symmetry of the amplitude to satisfy $b_2(123456) = \bar{b}_3(456123)$. Also note that $\langle \bar{F}_{234}, \bar{\eta} \rangle [F_{561}, \eta] C_{561} = 0$. This motivates us to separate the equations, by assuming that $\langle \bar{F}_{234}, \bar{\eta} \rangle b_3 = 0$ and $[F_{561}, \eta] b_2 = 0$, to obtain the equation for b_3,

$$\frac{i\pi}{2} \left[\frac{[F_{561}, \eta] b_3}{2k_5 \cdot P} - \frac{b_3 [F_{561}, \eta] (2k_5 \cdot P)}{(2k_5 \cdot P)^2} \right] = [F_{561}, \eta] C_{561} \tag{56}$$

(with the equation for b_2 obtained by relabelling). To solve this equation, it is convenient to define

$$b_3 = K' \hat{b}_3 \tag{57}$$

as in eq. (43). Note that K' is independent of $\tilde{\lambda}_i$, $i = 5, 6, 1$. Since eq. (56) is independent of $\tilde{\lambda}_i$, $i = 6, 1$, we deduce that b_3 depends only on $\tilde{\lambda}_5$. The right-hand side of eq. (56), from eq. (40),

$$\frac{[F_{561}, \eta] C_{561}}{K'} = i\pi \frac{P^2 \langle 16 \rangle \langle 15 \rangle \langle 5, P \rangle}{(2k_5 \cdot P)^2} \Big[[\eta, 5][P, 4] \Big] \, , \tag{58}$$

is of the form $[X,5]$. So we make a trial solution for $\hat{b}_3$

$$\hat{b}_3 = [5,\mathcal{C}]\,, \tag{59}$$

which implies

$$\begin{aligned}\frac{[F_{561},\eta]\hat{b}_3}{(2k_5\cdot P)} - \frac{\hat{b}_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2} &= -\frac{[5,P]\langle P,5\rangle[\eta,\mathcal{C}]\,\langle 6\,1\rangle}{(2k_5\cdot P)^2} + \frac{[5,\mathcal{C}]\,\langle 6\,1\rangle\,[\eta,P]\langle P,5\rangle}{(2k_5\cdot P)^2}\\ &= \frac{\langle 1\,6\rangle\,\langle 5,P\rangle}{(2k_5\cdot P)^2}\Big[[\eta,5][P,\mathcal{C}]\Big]\,.\end{aligned} \tag{60}$$

Thus eq. (56) is solved by

$$\mathcal{C}_{\dot a} = 2P^2\langle 15\rangle\tilde\lambda_{4\dot a}\,, \tag{61}$$

giving

$$\hat{b}_3 = 2P^2\langle 4|\not{5}|1\rangle \tag{62}$$

as a specific solution to eq. (56). However, this solution is not unique, as

$$\hat{b}_3 = 2P^2\langle 4|\not{5}|1\rangle + (2k_5\cdot P)\times A \tag{63}$$

is also a solution, for any rational function A not involving $\tilde\lambda_i$, $i = 5,6,1$. To also satisfy $\langle \bar{F}_{234},\bar\eta\rangle(b_3/(2k_5\cdot P)) = 0$, we must have,

$$\langle \bar{F}_{234},\bar\eta\rangle A = 0. \tag{64}$$

This relation is not sufficient to fix A. Indeed, any function of $P_{a\dot a} = \sum_{i=5,6,1}(\lambda_i)_a(\tilde\lambda_i)_{\dot a}$ will satisfy eq. (64). We have used the action of all F_{ijk} functions which give rational functions acting upon the cut. The information in other cut channels is equivalent to this cut by relabelling. Thus we are led to conclude that the action of the F_{ijk} operators upon the cuts does not uniquely fix the coefficients without the input of further information. In some sense, acting upon the cut with differential operators is destroying information which must be covered by examining boundary conditions or other constraints. Examples of the constraints that $\hat{b}_3$ must satisfy are: dimensionality, spinor weight, collinear limits, multi-particle poles, *etc.* For example, the coefficient $\hat{b}_3$ must have dimension 2 and the spinor weight of $+1$ with respect to leg 4, -1 with respect to leg 1, and 0 for other legs. (Spinor weight is an additive assignment of $+r$ for each $(\tilde\lambda_i)^r$ and $-r$ for each $(\lambda_i)^r$ in a product of terms.) The simplest solution to this condition is a quartic polynomial in the $\tilde\lambda_i, \lambda_i$, linear in $\tilde\lambda_4$ and λ_1, with others appearing in the combination $\tilde\lambda_i\lambda_i$. The differential equation then forces a solution of the form

$$\hat{b}_3 = 2P^2\langle 4|\not{5}|1\rangle + \alpha(2k_5\cdot P)\langle 4|\not{P}|1\rangle\,. \tag{65}$$

The arbitrary coefficient α can easily be fixed to be -1 by considering the collinear limit $2-3$.

Thus we have demonstrated how the action of the holomorphic anomaly on the cuts can be used to provide information about $N=1$ supersymmetric amplitudes. In general, we obtain differential equations; hence fixing the coefficients unambiguously does require the input of suitable physical information, such as the collinear limits.

6 A term in $A^{N=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,\ldots,n^+)$

As a further example let us consider the n-point amplitude $A^{N=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,\ldots,n^+)$ and deduce some of its integral function coefficients. Consider the cut analogous to the previous case

$C_{5\cdots n1}$ which is

$$C_{5\cdots n1} = \frac{iK}{2}\int d\mathrm{LIPS}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle}\,, \tag{66}$$

where now

$$K = \frac{\langle 4|\not{P}_{234}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\cdots\langle n\,1\rangle\, s_{234}} \tag{67}$$

Notice that on the cut the integrand is

$$\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle} = \frac{\langle 4^+|\not{\ell}_2|2^+\rangle\langle 5^+|\not{\ell}_1|1^+\rangle}{\langle 2^+|\not{\ell}_2|2^+\rangle\langle 5^+|\not{\ell}_1|5^+\rangle} = -\frac{\langle 4^+|\not{\ell}_2\not{2}\not{P}_{234}\not{5}\not{\ell}_1|1^+\rangle}{\langle 2^+|\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2(\ell_1+k_5)^2}\,. \tag{68}$$

The two propagators in eq. (68), plus the two cut propagators, make up a cut box integral. However, in the numerator of eq. (68), we can anticommute $\not{\ell}_2$ and $\not{\ell}_1$ toward each other, to get

$$\begin{aligned}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle} \quad &= \quad \frac{\langle 4^+|\not{P}_{234}\not{5}\not{\ell}_1|1^+\rangle}{\langle 2^+|\not{P}_{234}|5^+\rangle(\ell_1+k_5)^2} + \frac{\langle 4^+|\not{2}\not{\ell}_2\not{P}_{234}|1^+\rangle}{\langle 2^+|\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2} \\ &\quad -\frac{\langle 4^+|\not{2}\not{\ell}_2(\not{\ell}_1+\not{\ell}_2)\not{\ell}_1\not{5}|1^+\rangle}{\langle 2^+|\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2(\ell_1+k_5)^2}\,,\end{aligned} \tag{69}$$

where we used $P_{234} = \ell_1+\ell_2$ in the last term, making it clear that it vanishes. Thus the cut reduces to a sum of two cut linear triangles, or in other words,

$$\begin{aligned}&A^{N=1,\mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+,\ldots,n^+) \\ &\qquad = -\frac{i}{2}\left[b_2\frac{L_0[s_{234}/s_{34}]}{s_{34}} + b_3\frac{L_0[s_{234}/s_{6\ldots1}]}{s_{6\ldots1}}\right] + \text{terms not contributing to the } C_{5\cdots n1}\text{ cut,}\end{aligned} \tag{70}$$

where $s_{6\cdots1} \equiv (k_6+k_7+\cdots+k_n+k_1)^2$. Acting upon $C_{5\cdots n1}$ as before with $\langle\bar{F}_{234},\bar{\eta}\rangle$, we obtain

$$\langle\bar{F}_{234},\bar{\eta}\rangle C_{5\cdots n1} = -i\pi K\frac{P^2\,[2\,4]\,[3\,4]\,\langle 2,\bar{\eta}\rangle}{(2k_2\cdot P)^2}\frac{\langle 2|\not{P}|1\rangle}{\langle 2|\not{P}|5\rangle}\,. \tag{71}$$

Applying exactly the same steps as before we have a trial solution

$$\hat{b}_2 = -2P^2\langle 4|\not{2}|1\rangle + \alpha(2k_2\cdot P)\langle 4|\not{P}|1\rangle\,, \tag{72}$$

where we can fix $\alpha = 1$ using collinear limits.

7 Conclusions

We have examined how the holomorphic anomaly acts upon the cuts of $N=1$ supersymmetric one-loop amplitudes, focusing upon a six-gluon non-MHV amplitude (calculated by independent methods). We have shown that one must take into account the anomaly when acting with the collinear differential operators on the cuts, in order to match the effect of the operator acting upon the imaginary part of the amplitude — as required by the optical theorem. As a calculational tool to evaluate amplitudes, application of the holomorphic anomaly gives differential equations for the coefficients of the integral functions, unlike the $N=4$ case where algebraic equations arose. Since the equations are differential, their general solution contains homogeneous parts which can be fixed by the boundary conditions or physical constraints such as collinear limits.

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