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Cosmokinetics

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Abstract. Our fundamental lack of understanding of the acceleration of the Universe suggests that we consider a kinematic description. The simplest formalism involves the third derivative of the scale factor through a jerk parameter. A new approach is presented for describing the results of astronomical observations in terms of the contemporary jerk parameter and this is related to the equation of state approach. Simple perturbative expansions about ΛCDM are given.

1. Approaches to Dark Energy

The Universe is spatially flat, accelerating and has a subcritical matter density. The evidence for this comes from:

- observation of acoustic peaks in the microwave background radiation spectrum which effectively measures the angular diameter distance d_A to the last scattering surface (e.g. Spergel et al. 2003).
- magnitudes of Type Ia supernovae which provide the luminosity distance d_L as a function of redshift (e.g. Riess et al. 2001).
- X-ray observations of rich clusters of galaxies which measure the mean density of dark matter and also, with less confidence, the distance–redshift relation (e.g. Allen *et al.* 2004).

Although the possibility that the Universe contains a repulsive entity counteracting the attractive effects of gravity was entertained by Einstein in the first, relativistic cosmology, the discovery of acceleration was mostly a surprise. There have been several approaches to rationalizing this discovery from a contemporary perspective. The simplest, and probably still the favorite description, is that the field equation is supplemented by a term proportional to the metric tensor, which guarantees covariance. The coefficient of proportionality (the cosmological constant) is the single free parameter in the theory.

If we regard this term as an augmentation of the stress energy tensor of cold matter and adopt a fluid description, then the enthalpy must vanish and the pressure is the negative of the energy density (roughly 0.7 nJ m^{-3}). Alternatively, we can regard this term as a necessary component of the geometrical response of a spacetime manifold to a material source that only becomes detectable on cosmological scales. Either choice leads to the flat "ΛCDM" cosmology in which the cosmic time varies with the scale factor $a = 1/(1+z)$ according to the

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solution of Lemaître (1927) equation

$$H_0 t(a) = \frac{2}{3(1-\Omega_m)^{1/2}} \sinh^{-1}\left(\frac{a^3(1-\Omega_m)}{\Omega_m}\right)^{1/2} \tag{1}$$

(Bondi 1952), where Ω_m is the density parameter of matter ($\rho = \Omega\rho_c$ where the critical density $\rho_c = 3H_0^2/8\pi G$) and a is normalized to unity at the present time[1]. Following Spergel *et al.* (2003), we adopt $H_0 = 72$ km s^{-1} Mpc^{-1}.

In order to test this description, it is necessary to introduce a framework which embraces a larger class of models labeled by measurable parameters. The overwhelmingly favourite choice of model for "dark energy" is to suppose that the acceleration is caused by a fluid which expands according to a modified adiabatic law, so that PV^{1+w} is equal to a (negative) constant (Steinhardt 1997; Turner & White 1997). (This is reminiscent of a polytropic description of a star with the important distinction that it is the same material that undergoes the change as the Universe expands as opposed to matter with different thermal histories as is the case with a star.) The parameter w ($= -1$ under ΛCDM) can be allowed to vary but we shall treat it as constant for the moment. Given this prescription a satisfies a Friedmann differential equation (neglecting curvature)

$$\dot{a}^2 \equiv -2H_0^2 V(a) = H_0^2\left[\Omega_m a^{-1} + (1-\Omega_m)a^{-(1+3w)}\right] \tag{2}$$

This approach can be confusing because, if we were to treat w as a pure number then small scale perturbations to the dark energy fluid would grow exponentially with time for $w < 0$. This is clearly untenable. Instead w is usually associated with a scalar field theory with a more complex rule for handling perturbations. Under these circumstances, it would seem preferable to parameterize the field theory itself which will lead to a different functional departure of the models from ΛCDM. An analogy is that one invariably solves problems in electromagnetic theory working from Maxwell's field equations rather than his stress tensor. In practice small scale dark energy perturbations propagate with the speed of light in most, though not all, models (e.g. Liu 2004).

A huge variety of scalar field models have been discussed (e.g.Caldwell *et.al.* 1998; Peebles & Ratra 1988). However, there is no guarantee that the acceleration of the Universe has anything to do with a scalar field. Other explanations include the notion that the Universe has extra dimensions (e.g. Dvali & Tye 1999). The scalar field is probably the best bet although it should be remembered that no such fundamental fields have been detected experimentally as yet. Electromagnetic and gravitational fields were thought to be scalar, but turned out not to be. To make this point, consider a blind astrophysicist practising her craft during the radiation era. She might have measured that the average spacing of protons increased as the square root of time and have inferred that the expansion was driven by a scalar field with an exponential potential. She would have been hopelessly wrong.

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[1] Eddington (1935) strongly advocated a ΛCDM model on the grounds that it would prolong the life of the Universe and that the existence of a cosmological constant was necessary to explain the disparity between the atomic and cosmological scales.

2. Kinematic Approach

2.1. Jerk Parameter

In view of the huge diversity of dynamical models on offer, there may be some merit in the observational cosmologist taking an unprejudiced kinematical approach. This has been historically productive. Galileo, who originally argued theoretically that the speed of a falling body increased in proportion to distance, determined the true law by empirical, kinematic measurement (Dugas 1955). Modern cosmology began with the discovery of the expansion of the universe, quantified by the Hubble parameter $H(t) = \dot{a}/a$. In more recent times the pursuit of observational cosmology was dominated by the measurement of the contemporary deceleration parameter q_0 where $q(t) = -\ddot{a}a/\dot{a}^2$ (Sandage 1975). In terms of the scale factor and Hubble parameter $q = -d\ \ln aH/d\ \ln a$.

As the Universe was once decelerating and is now accelerating, it is useful to consider the third derivative of a. A convenient quantity to use is the dimensionless jerk (Blandford 2004)

$$j = \frac{\dddot{a}\ a^2}{\dot{a}^3} \tag{3}$$

with its current value denoted by j_0. One reason why this is convenient is that $j = 1$ for ΛCDM, the baseline model about which we are perturbing. In terms of the Hubble parameter $j = (a^2H^2)''/2H^2$. Different formalisms involving the third derivative have been discussed independently by (Sahni et al. 2003; Visser 2004).

2.2. Dynamical considerations

Although, we eschew dynamics, j can be interpreted in fluid terms, through the easily derived relation $j = 1 - 4\pi\dot{P}/H^3 = 1 + (a^4\rho')'/2\rho a^2$, where P is the pressure, ρ the total energy density while we ignore spatial curvature (analagously $-q = 1 + a\rho'/2\rho$). The expressions involving H rather than ρ above are valid even considering curvature. For a slow rolling scalar field ϕ and potential $U(\phi)$ we get $j \approx 1 - 4\pi U'(\phi)^2/3H^4$.

2.3. Hubble Parameter

We solve Eq. 3 by rewriting it in the form

$$V'' = \frac{2jV}{a^2} \tag{4}$$

where $V(a) = -\dot{a}^2/2H_0^2 = -H^2(a)a^2/2H_0^2$ and prime denotes differentiation with respect to a. This equation has power law solutions for constant j, but the only one that is of interest is the ΛCDM solution with $j = 1$. This is because we seek solutions that approximate the Einstein-De Sitter case for early times. We therefore take for our simplest, one parameter extension of ΛCDM, the variation

$$j(a) = 1 + j'a = 1 + (j_0 - 1)a \tag{5}$$

In general, if we specify an expression for $j(a)$ and require that the solution match an Einstein-De Sitter solution at early time, then we can solve for the

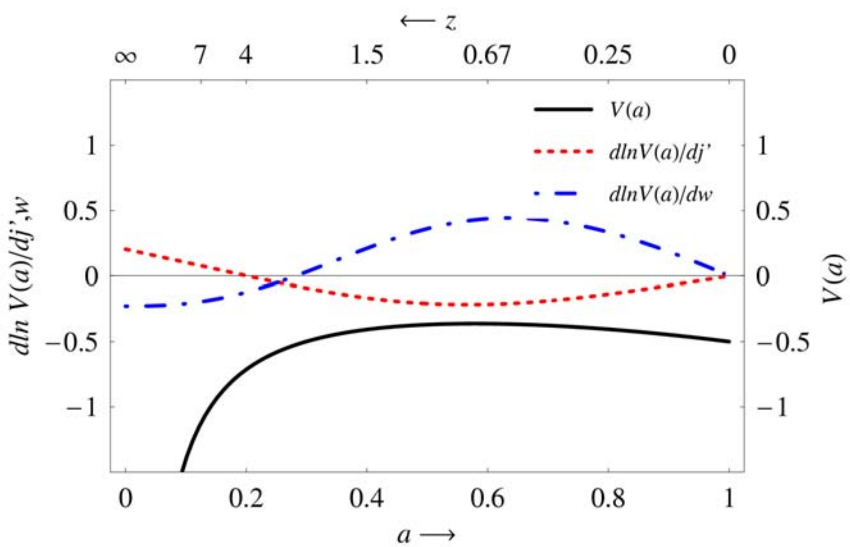

Figure 1. The effective potential $V(a)$, and the derivative $d\ln V(a)/dj', dw$, evaluated at $j' = 0$ ($w = -1$).

scale factor, $a(t)$, the Hubble parameter, $H(t)$, and the deceleration parameter, $q(t)$ for all time. (Specifically all we need to know is $j(a)$ and $\lim_{a\to 0}(a\dot{a}^2/H_0^2)$.) As is the case with the w formalism, it is possible to introduce extra parameters through a Taylor expansion $j(a) = 1 + j'a + j''a^2/2 + \ldots$ However, we shall not explore this generalization here, and instead restrict ourselves to a jerk that is linear in the scale factor. Eq. 4 has the solution

$$V(a) = -\frac{1}{2}\left[c_1 f_1(8aj')/a + c_2 f_2(8aj')a^2\right] \tag{6}$$

where

$$\begin{aligned} f_1(x) &= x^{3/2}K_3(x^{1/2})/8 = 1 - x/8 + x^2/64 + \ldots \\ f_2(x) &= 48x^{-3/2}I_3(x^{1/2}) = 1 + x/16 + x^2/640 + \ldots \end{aligned} \tag{7}$$

and $c_{1,2}$ are constants.

2.4. *j* vs. *w*

In order to compare the j and w prescriptions, we must compare world models computed adopting similar boundary conditions. We choose to fix the co-moving distance $H_0 d(a_r) = H_0 \int_{a_r}^1 dt/a$ to the recombination surface($a_r \approx 0$). Note that the distance to the recombination surface, $H_0 d(0)$ is in practice measured quite accurately by the microwave background fluctuation spectra. In everything that follows, we shall assume $H_0 d(0) = 3.4$. This condition allows us to solve for Ω_m as a function of j' (or w). If we linearize in j' and assume that matter evolves in the usual way then in the j model, $c_1 = \Omega_m$ and c_2 can be chosen such

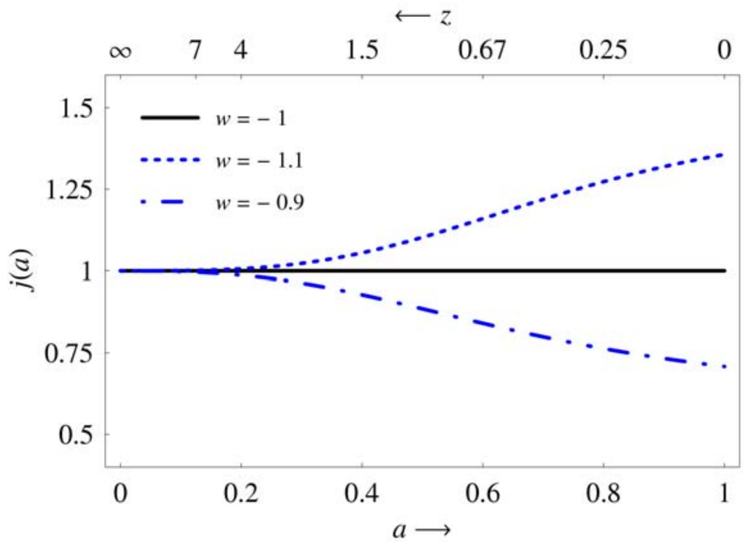

Figure 2. The kinematic jerk parameter $j(a)$ shown for three different values of the dark energy equation of state parameter w.

that $V(1) = -1/2$. The linearized expressions are shown below for illustrative purposes.

$$V(a) = -\frac{1}{2}\left[\Omega_m/a + (1-\Omega_m)a^2 + j'\{(1-\Omega_m)a^3/2 - (1-3\Omega_m)a^2/2 - \Omega_m\}\right]$$
$$V(a) = -\frac{1}{2}\left[\Omega_m/a + (1-\Omega_m)a^2 - 3(w+1)(1-\Omega_m)a^2 \ln a\right] \quad (8)$$

In Fig. 1 we plot the exact solution for $V(a)$ along with the derivative $d\ln V(a)/dj', dw$ evaluated at $j' = 0$ $(w = -1)$ to show the difference between the w and j formalisms.

In general $d\ln X/dj'$ characterizes the deviation of an observable X from the ΛCDM model. This can be seen by considering the following expansion:

$$X(a, j') \approx X(a, 0)\left(1 + j'\frac{d\ln X}{dj'}\bigg|_{j'=0}\right) \quad (9)$$

Note that this provides a simple route to the generation of fitting formulae. Substituting w for j' and expanding about the ΛCDM value of $w = -1$ gives the corresponding expression in the dynamical formalism.

We can also compute $j(a)$ for a given w model.

$$j(a) = 1 + \frac{9w(1+w)(1-\Omega_m)}{2\{1-\Omega_m(1-a^{3w})\}} \quad (10)$$

This relationship is plotted in Fig. 2.

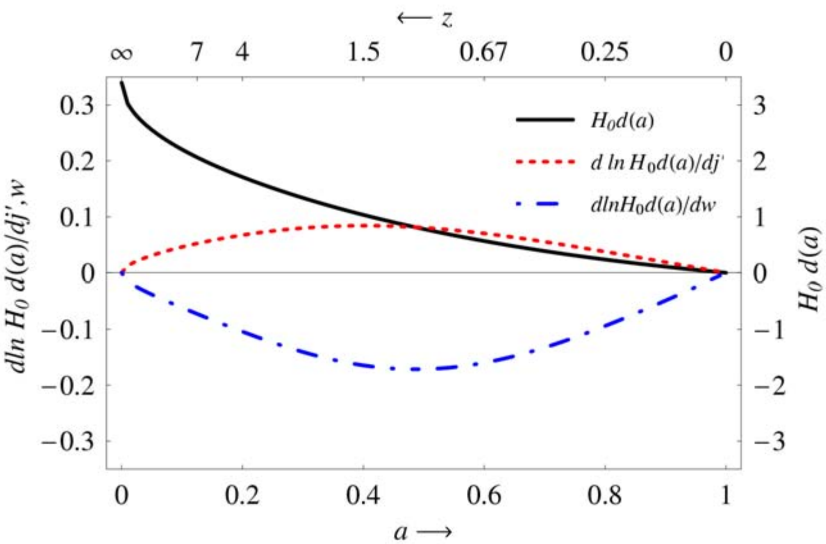

Figure 3. Co-moving distance $H_0d(a)$ and the derivative $d\ln H_0d(a)/dj', dw$ evaluated at $j' = 0$ ($w = -1$).

2.5. Distance

Many techniques for exploring the expansion of the Universe utilize distance measurement through the angular diameter $d_A \equiv ad$ or luminosity distance $d_L \equiv a^{-1}d$ or the comoving volume $\mathcal{V} = 4\pi d^3/3$. For ΛCDM,

$$H_0d(a) = H_0d(0) - \frac{B[1/6, 1/3; (1-\Omega_m)a^3/(\Omega_m + (1-\Omega_m)a^3)]}{3\Omega_m^{1/3}(1-\Omega_m)^{1/6}} \tag{11}$$

where $B[a,b;x] = \int_0^x dt\, t^{a-1}(1-t)^{b-1}$ is the incomplete Beta function. The distances can be computed in the perturbed models with the the boundary condition $H_0d(0) = 3.4$. We contrast the j and w formalisms in Fig. 3 where we plot H_0d and $d\ln H_0d/dj', dw$ evaluated at $j' = 0(w = -1)$ as functions of a. The fractional changes in the co-moving volume are three times these functions. A simple fitting formula for H_0d is given by combining Eq. 9 with $d\ln H_0d/dj' \approx \sum_{n=1}^{4} c_n a^n$, where the coefficients $c_n = \{0.54, -1.1, 0.81, -0.26\}$.

2.6. Curvature

The WMAP observations have demonstrated that the Universe is quite flat; any curvature can surely be treated as a perturbation and will be very hard to detect in the face of the above effects. We therefore contrast the deviation expected from pure curvature in the distance relation. Curvature with radius R appears in standard Friedmann cosmology in two related places. The first is in the relation between the comoving radial coordinate $r = \int dt/a$

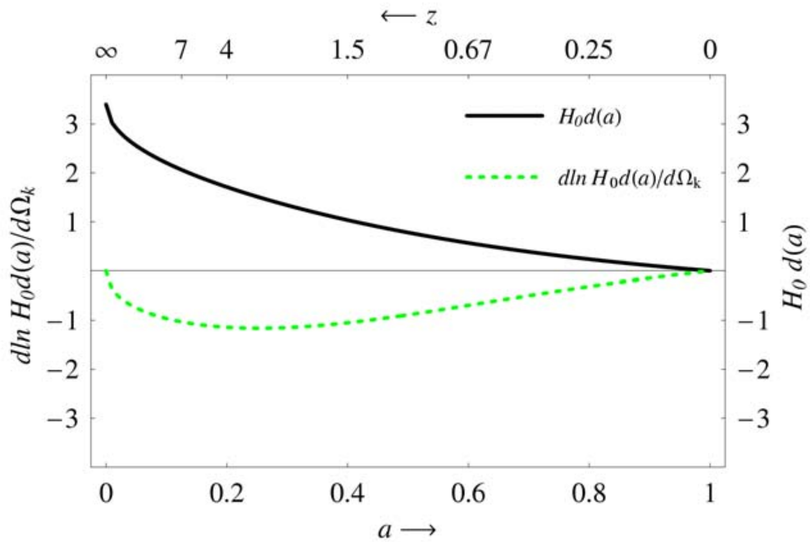

Figure 4. Co-moving distance $H_0d(a)$, and the derivative $d\ln H_0d(a)/d\Omega_k$, evaluated at $\Omega_k = 0, j' = 0(w = -1)$.

and the distance $d = r + \Omega_k r^3/6 + \ldots$, and $|\Omega_k| = 1/R^2$, where the sign indicates positive or negative spatial curvature (Ω_k negative is positive curvature). Note that this effect is purely geometric, making no reference to dynamics. The second is in the dynamical equation, Eq. 2, which changes so that $V = -\left[\Omega_m a^{-1} + (1 - \Omega_m - \Omega_k)a^{-(1+3w)} + \Omega_k\right]/2$. The jerk parameter can be derived from $j = 1 + (a^4\rho')'/2a^2\rho$ by defining $\rho_k = \Omega_k\rho_c a^{-2}$ and taking $\rho \rightarrow \rho + \rho_k$. For the pure w model, $j_0 = 1 - \Omega_k + 9(1 - \Omega_m - \Omega_k)w(1 + w)/2$. In Fig. 4 we show $d\ln H_0d/d\Omega_k$ evaluated at $\Omega_k = 0, j' = 0(w = -1)$.

2.7. Time

At present we do not have very accurate age measurements but it is still useful to compute the relative changes in the chronology induced by our models. We obtain the time under ΛCDM from Eq. 1. The times for the w, j modifications are obtained by integrating Eq. 8 subject to a third boundary condition that $t(0) = 0$. $d\ln H_0t/dj', dw$ for $j' = 0(w = -1)$ are plotted in Fig. 5.

2.8. Growth of Perturbations

The rate at which the perturbations that are observed directly in the microwave background fluctuation spectra grow under the action of Newtonian gravitational forces is quite sensitive to the expansion rate of the Universe. If we ignore radiation, pressure and the distinction between cold and baryonic matter, then a relative matter density perturbation on a fixed co-moving scale, denoted by δ, grows according to a second order differential equation which can be cast in

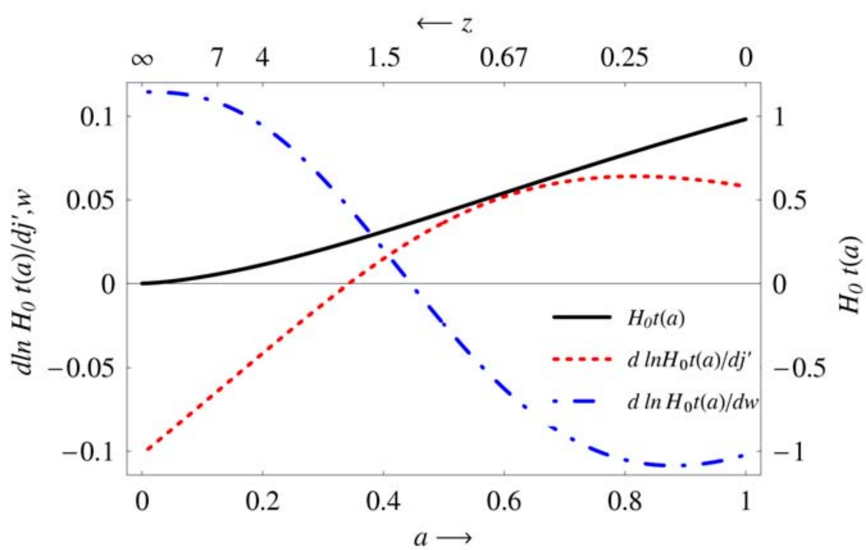

Figure 5. Cosmic time $H_0t(a)$, and the derivative $d\ln H_0t(a)/dj', dw$, evaluated at $j' = 0 (w = -1)$.

Sturm-Liouville form

$$(a^2V^{1/2}\delta')' + \frac{3\Omega_m\delta}{4aV^{1/2}} = 0 \tag{12}$$

We solve this differential equation evolving the linear growth transfer function for the w and j models from the epoch of recombination to the present day assuming $\delta(a_r) = 1$ and $\delta'(a_r) = 1/a_r$. The difference between the w and j models is shown using $d\ln\delta/dj', dw$.

3. Discussion

The above calculations suffice to contrast the j and w formalisms. The procedure that is actually followed in fitting various data sets is to explore the likelihood of a set of cosmological parameters subject to prior assumptions about the nature of the solution. It is clear that sensitivity of a given analysis to a specific parameter depends upon the source redshift distribution and such choices can either flatter or disparage a given approach! The two choices contrasted here are, in the absence of any compelling theoretical guidance, essentially arbitrary and it is not our purpose to advocate one over the other. In particular, it seems to be a mistake to suppose that because the w formulation produces maximal deviation from ΛCDM at $a \sim 0.6$ then observations should be concentrated around $z \sim 0.7$. Rather, we seek to raise awareness of alternative parameterizations of the cosmological model which may help increase understanding, and prevent misunderstanding, in the face of forthcoming data.

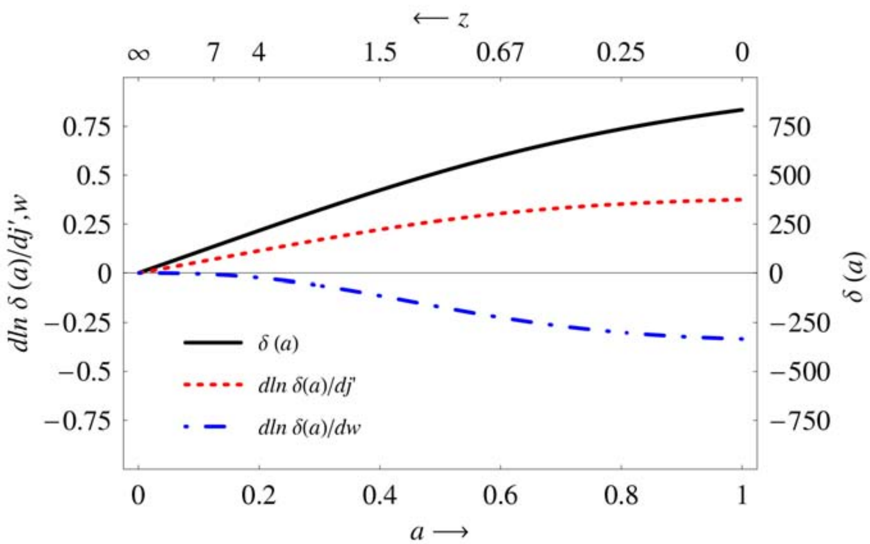

Figure 6. Perturbation $\delta(a)$, and the derivative $d\ln\delta(a)/dj', dw$, evaluated at $j' = 0 (w = -1)$.

Of course, just as one should consider performing dark energy observations over the whole available redshift range, one should also consider a range of models when analysing these data. Attempts to measure functions $d(a)$ directly from the data via some necessary interpolation scheme form one such class of models. The much simpler jerk and dynamical parameterisations described above trade flexibility for ease of interpretation, but are as valuable and valid as analytical tools. The problem of which model provides the most appropriate description of these data has a well-defined solution that depends on both the goodness-of-fit (via the likelihood) and the model complexity (via the prior). The jerk parameterisation is a natural way to perturb the model that we know to fit the current data very well, and so may be expected to come with some sensible prior for the parameter j'. Indeed, noting that it is a straight line gradient, insisting on rotational invariance in the plotting plane leads to a Lorentzian prior pdf of unit width (Dose 2003). This distribution is broad enough to allow the data to speak for themselves, and featureless for as long as we ignore any dynamical considerations. The situation in the dynamical modelling scenario is somewhat different: there, the onus is on the theorists to provide prior distributions for physically-motivated parameters such as w, a task that has not been undertaken in any great detail as yet.

Our main objective in this work has been to provide an alternative approach to the problem of characterising the cosmological world model in the presence of a poorly understood dark energy component. We may expect simple parameterisations to be of great use in interpreting the data arriving from the many current and future dark energy experiments; this is especially true in the early stages when the signal-to-noise is low. The jerk formalism worked through here has a number of advantages; we hope that it will be considered by observers as a useful parallel analysis tool, and as such we have provided an example fitting formula for the co-moving distance. A more comprehensive jerk toolkit, and a more detailed discussion of points made earlier in this talk, will be given elsewhere.

4. Acknowledgments

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References

Allen, S. W., Schmidt, R. W., Ebeling, H., Fabian, A. C. and van Speybroeck, L 2004, arXiv:astro-ph/0405340.
Blandford, R. D. 2004, in Carnegie Observatories Astrophysics Series, Vol. 2: Measuring and Modeling the Universe, ed. W. L. Freedman (Cambridge: Cambridge Univ. Press)
Bondi, H. 1952, Cosmology, (Cambridge: Cambridge University Press)
Caldwell, R. R., Dave, R. and Steinhardt, P. 1998, Phys. Rev. Lett. 80, 1582
Dose, V. 2003 in AIP Conference Proceedings, 659(1), 350
Dugas, R. 1955, A History of Mechanics, (London: Taylor Francis & Routledge)
Dvali, G. R. and Tye, S. H. H. 1999, Phys. Lett. B 450, 72
Eddington, A. S. 1935, New Pathways in Science, (Cambridge: Cambridge University Press)
Lemaître, G. E., 1927, Ann. Soc. Sci. Brux A, 57, 49
Liu, J. S. 2004 Phys. Rev. D, 69, 083504
Peebles, P. J. E and Ratra, B. 1988, Astrophys. J. Lett., 325, L17
Riess, A. G. et al. 2004, Astrophys. J., 607, 665
Sahni, V., Saini, T. D., Starobinsky, A. A. and U. Alam 2003, JETP Lett. 77, 201 [Pisma Zh. Eksp. Teor. Fiz. 77, 249 (2003)]
Sandage, A. 1975, Astrophys. J. 202, 563
Spergel, D. N. et al. 2003, Astrophys. J. Suppl. 148, 175
Steinhardt, P.J. 1997, in Critical Dialogues in Cosmology, ed. Turok, N. (Singapore: World Scientific)
Turner, M. S. and White, M. 1997, Phys. Rev. D, 56, R4439
Visser, M. 2004, Class. Quant. Grav. 21, 2603