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All Non-Maximally-Helicity-Violating One-Loop Seven-Gluon Amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills Theory*

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Abstract

We compute the non-MHV one-loop seven-gluon amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory, which contain three negative-helicity gluons and four positive-helicity gluons. There are four independent color-ordered amplitudes, $(---++++)$, $(--+-+++)$, $(--++-++)$ and $(-+-+-++)$. The MHV amplitudes containing two negative-helicity and five positive-helicity gluons were computed previously, so all independent one-loop seven-gluon helicity amplitudes are now known for this theory. We present partial information about an infinite sequence of next-to-MHV one-loop helicity amplitudes, with three negative-helicity and $n-3$ positive-helicity gluons, and the color ordering $(---++\cdots++)$; we give a new coefficient of one class of integral functions entering this amplitude. We discuss the twistor-space properties of the box-integral-function coefficients in the amplitudes, which are quite simple and suggestive.

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I. INTRODUCTION

Gauge theories play a central role in modern theoretical physics. They form the backbone of the Standard Model of particle interactions. They also play an increasingly important role in our understanding of string theories via the AdS/CFT correspondence. That correspondence is a strong–weak coupling duality between type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ background and four-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. In a recent paper, Witten [1] proposed a *weak–weak* coupling duality between $\mathcal{N} = 4$ supersymmetric gauge theory and the topological open-string B model on $\mathbb{CP}^{3|4}$ using D-instanton contributions. This proposal generalizes Nair's earlier description [2] of the simplest gauge-theory amplitudes as correlation functions on $\mathbb{CP}^1$. Berkovits [3, 4], Neitzke and Vafa [5], and Siegel [6] have given alternative descriptions of such a possible topological dual to the $\mathcal{N} = 4$ theory. Both of these dualities motivate the computation of perturbative amplitudes in gauge theory.

Indeed, the availability of an extensive literature of explicit results for both tree-level and loop amplitudes in gauge theories has provided an important stimulus to and guide in building the topological constructions. Most notable are the series of helicity amplitudes with arbitrarily many external gluons, the maximally helicity violating (MHV) amplitudes at tree level [7, 8], and at one loop in supersymmetric theories [9, 10]. These amplitudes display a remarkable simplicity. The entire amplitude at a given multiplicity is much simpler than the typical Feynman diagram, chosen from the great number that are assembled into the amplitude in a traditional approach. There are indications that this simplicity continues to higher loops [11–13].

The forthcoming generation of experiments probing beyond the Standard Model at the LHC provide another important motivation for computing amplitudes in perturbative gauge theory. Precision measurements at SLC and LEP have proven to be a powerful means of advancing our understanding of the Standard Model. A drive towards precision physics at hadron colliders will require a corresponding theoretical effort in precision calculations in QCD, to higher loops and multiplicities. While a traditional field-theoretic point of view would view the pure glue contributions to QCD amplitudes as a small part of the $\mathcal{N} = 4$ result, it is more natural to view matters the other way around. The $\mathcal{N} = 4$ results are much simpler than the QCD ones and can be regarded as a building block for the latter [9]. For example, the gluon (or quark) circulating in loop in a one-loop n-gluon QCD amplitude can be decomposed into a linear combination of an $\mathcal{N} = 4$ super-multiplet (absent in the quark case), an $\mathcal{N} = 1$ chiral super-multiplet, and a scalar in the loop. Each piece has a different analytic structure, so it makes sense to separate the components. The $\mathcal{N} = 4$ amplitudes can always be written as a sum over a more limited class of integral functions than the other components, as they require only scalar box integrals. These four-point integrals have no loop momenta in the numerator of the integral [9]. The remaining computational problem is to determine the coefficient with which each scalar box integral appears in the amplitude. This simplicity can be traced back to the much-improved ultraviolet behavior of the $\mathcal{N} = 4$ theory. At the same time, the $\mathcal{N} = 4$ component captures the leading infrared singularities present in the QCD amplitude.

Calculations in QCD and in $\mathcal{N} = 4$ super-Yang-Mills theory are closely related to each other. In general, an $\mathcal{N} = 4$ amplitude may be extracted from a corresponding QCD amplitude, if the number of gluonic and fermionic states is tracked in the computation. The conversion of a QCD amplitude to an $\mathcal{N} = 4$ super-Yang-Mills amplitude is accomplished by assigning 8 spin degrees of freedom to gluons circulating in the loops, and modifying the multiplicity of the fermions to also carry a total of 8 spin degrees of freedom. The color algebra also needs to be modified to account for the gluinos being in the adjoint color representation, instead of the quarks' fundamental color representation. With these modifications, the theory has the number of degrees of freedom of ten-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory compactified to four dimensions, which is another name for $\mathcal{N} = 4$ super-Yang-Mills theory. This simple conversion holds at any loop order. As a non-trivial example, after carrying out this conversion on the two-loop four-gluon amplitudes given in ref. [14], they agree perfectly [12] with the previously determined $\mathcal{N} = 4$ amplitudes [11]. The analytical results presented here can therefore serve as benchmarks for testing algorithms [15] for direct numerical evaluation of high-multiplicity one-loop amplitudes in QCD.

The experimentally-driven computations are technically complicated, so they require more powerful methods than textbook ones. In the past decades, a number of new approaches have been developed to cope with this complexity, including helicity methods [16], color decompositions [17–20], recursion relations [8], supersymmetry Ward identities [21], ideas based on string theory [22, 23], and the unitarity-based method [9, 10, 24–26]. The latter technique has been applied to numerous calculations, most recently the two-loop calculation of all helicity amplitudes for gluon–gluon scattering [27] and the two-loop splitting amplitudes $g \to gg$ [26]. (The latter agree with the explicit collinear limits of certain two-loop amplitudes [26, 28].)

Forefront calculations still tax available technologies to their utmost, which motivates the continuing development of new techniques. The twistor-space representation introduced along with the topological string theories mentioned above, shows great promise as a source of new methods for performing cutting-edge calculations. Indeed, Cachazo, Svrček, and Witten [23] have used observations about the twistor-space structure of amplitudes to formulate a new and simple set of rules based on *MHV vertices* (off-shell continuations of the Parke-Taylor MHV amplitudes) which can be used to compute all tree-level amplitudes in unbroken gauge theory. Their rules extend to processes with external fermions and scalars as well as gluons [29], and lead to explicit results for next-to-MHV amplitudes [23, 29, 30] and checks of 'googly' amplitudes (with two positive-helicity, and the rest negative-helicity gluons) [31–33]. The rules can be implemented either analytically or numerically. The computational efficiency of these rules may be further enhanced with a recursive rearrangement [34].

A critical question is how to extend these ideas to loop calculations. The challenge is to reduce loop calculations to a purely algebraic problem of polynomial complexity, avoiding the exponential explosion in the complexity of intermediate stages that would be encountered with brute force tensor reduction and integration methods. A direct topological-string approach appears to lead to complications, mixing in non-unitary states from conformal supergravity [35]. However, in an important step, Brandhuber, Spence, and Travaglini [36] have shown that one can compute one-loop amplitudes using exactly the same MHV vertices that work at tree level. They showed explicitly how to compute the infinite sequence of one-loop MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. Many technical steps in their computation parallel the original cut-based method [9]; however, the new computation is quite different conceptually, and probes the off-shell continuation of the MHV vertices. The BST calculation also reaffirms the basic simplicity of the twistor-space structure of one-loop amplitudes. Taking into account [37] a 'holomorphic anomaly' brings an earlier investigation of this structure [38] into agreement with the picture emerging from the BST calculation. The holomorphic anomaly has already been applied by Cachazo [39], in conjunction with unitarity, to derive algebraic equations relating coefficients of scalar box integrals to the rational functions making up a cut integrand.

In this paper we present new results for amplitudes in the $\mathcal{N} = 4$ gauge theory. We will give compact formulæ for all four helicity configurations required for the seven-point next-to-MHV (NMHV) amplitude. These amplitudes were computed by evaluating unitarity cuts in various channels, the same method used previously to obtain the all-multiplicity MHV helicity amplitudes [9] and the six-point NMHV helicity amplitudes [10]. In this case, however, the unitarity cuts are more complicated. We have reduced them to a standard set of (cut) scalar box integrals using integral reduction methods implemented on the computer. The coefficients of the scalar boxes in the full amplitude are equal to the coefficients of the cut scalar boxes that have a cut in the channel considered. The resulting expressions for the coefficients are analytical, but quite large. However, remarkably simple expressions exist for these quantities. The simple expressions can be found by considering the analytic behavior of the coefficients in various limiting regions of the seven-point phase space, and using that information to build ansätze for the coefficients. The ansätze can then be checked numerically with high precision against the large expressions at random kinematic points.

There are a large number of box coefficients to determine (92, after invoking some reflection symmetries). However, their structure can be fit into general patterns, which are simple to describe in twistor-space language. The twistor-space structure of the full amplitude, including the box integrals multiplying these coefficients, is less transparent because of the holomorphic anomaly which affects the integrals [37]. On the other hand, the twistor-space properties of the coefficients square well with the recent application of the holomorphic anomaly to computation of $\mathcal{N} = 4$ box coefficients [39].

We also give an all-n formula for the coefficient of a particular class of (three-mass) box integrals in the adjacent-minus NHMV amplitude, the helicity configuration $(---++\cdots++)$. We study the twistor-space structure of this part of the amplitude and find that it is consistent with expectations of simplicity emerging from the BST calculation and the holomorphic anomaly. These results should provide a useful guide to the analytic structure likely to emerge in other scalar-box coefficients in this amplitude and in other amplitudes.

As this paper was being completed, ref. [40] appeared, in which one of the four seven-point helicity amplitudes presented here is computed via the holomorphic anomaly and unitarity.

This paper is organized as follows. In section II we review the color and helicity decompositions of tree-level and one-loop gauge amplitudes, as well as the structure of the

MHV amplitudes at these orders. In section III we describe the application of the unitarity-based technique to the seven-point NMHV computation, including an outline of the integral reduction approach and its implications for the denominators of different classes of box coefficients. We also sketch how the coefficients were simplified. Section IV details the results for the seven-point amplitudes by listing the independent box integral coefficients. Section V describes consistency checks that were applied to the results. These checks include the behavior as two gluons i and j become collinear, *i.e.* as the kinematic invariant $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j \to 0$. They also include the behavior as multi-particle invariants vanish; that is, $s_{ijk} = (k_i + k_j + k_k)^2 \to 0$. In section VI we give one of the box integral coefficients in the all-n NMHV helicity amplitude $(---++\cdots++)$. Section VII analyzes the twistor-space properties of the seven-point box coefficients, and of the term in the all-n NMHV helicity amplitude from section VI. In section VIII we present our conclusions. There are two appendices. Appendix A collects the dimensionally-regulated scalar box integral functions appearing in the $\mathcal{N} = 4$ amplitudes. Appendix B describes an example of how one of the seven-point helicity amplitudes can be factorized onto a multi-particle pole.

II. REVIEW OF PREVIOUS AMPLITUDE RESULTS

We now briefly summarize results for previously computed amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. At tree level it is convenient to use the color decomposition [17, 18] of amplitudes

$$\mathcal{A}_n^{\rm tree}(\{k_i, \lambda_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n^{\rm tree}(\sigma(1^{\lambda_1}, \ldots, n^{\lambda_n}))\,, \tag{1}$$

where S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{λ_j} denotes the j-th momentum and helicity λ_j. The T^a are fundamental representation ${\rm SU}(N_c)$ color matrices normalized so that ${\rm Tr}(T^a T^b) = \delta^{ab}$. The color-ordered amplitude $A_n^{\rm tree}$ is invariant under a cyclic permutation of its arguments.

We describe the amplitudes using the spinor helicity formalism. In this formalism amplitudes are expressed in terms of spinor inner-products,

$$\langle j\, l\rangle = \langle j^- | l^+ \rangle = \bar{u}_-(k_j) u_+(k_l)\,, \qquad [j\, l] = \langle j^+ | l^- \rangle = \bar{u}_+(k_j) u_-(k_l)\,, \tag{2}$$

where $u_\pm(k)$ is a massless Weyl spinor with momentum k and plus or minus chirality [16, 18]. Our convention is that all legs are outgoing. The notation used here follows the standard

QCD literature, with

$$\langle i\,j\rangle\,[j\,i] = 2k_i\cdot k_j = s_{ij}\,. \tag{3}$$

(Note that the square bracket $[i\,j]$ differs by an overall sign compared to the notation commonly used in twistor-space studies [1].) For non-MHV amplitudes we also use the abbreviated notation,

$$\begin{aligned}\left\langle i^+\right|(a+b)\left|j^+\right\rangle &= \left\langle i^+\right|(\not k_a+\not k_b)\left|j^+\right\rangle\,,\\ \left\langle i^-\right|(a+b)(c+d)\left|j^+\right\rangle &= \left\langle i^-\right|(\not k_a+\not k_b)(\not k_c+\not k_d)\left|j^+\right\rangle\,.\end{aligned} \tag{4}$$

We denote the sums of cyclicly-consecutive external momenta by

$$K^\mu_{i\cdots j} \equiv k^\mu_i + k_{i+1} + \cdots + k_{j-1} + k_j, \tag{5}$$

where all indices are mod n for an n-gluon amplitude. The invariant mass of this vector is $s_{i\ldots j} = K^2_{i\ldots j}$. In the seven-point case, using momentum conservation we just need to consider two- and three-particle invariant masses, which are denoted by

$$s_{ij} \equiv (k_i+k_j)^2 = 2k_i\cdot k_j, \qquad\qquad s_{ijk} \equiv (k_i+k_j+k_k)^2. \tag{6}$$

In color-ordered amplitudes only invariants with cyclicly-consecutive arguments appear, $s_{i,i+1}$ and $s_{i,i+1,i+2}$.

The simplest of the partial amplitudes are the maximally helicity-violating (MHV) Parke-Taylor tree amplitudes [7] with two negative-helicity gluons and the rest positive,

$$A^{\text{tree MHV}}_{jk}(1,2,\ldots,n) = i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,, \tag{7}$$

where j and k label the negative helicity legs.

For one-loop amplitudes, the color decomposition is similar [19]. When all internal particles transform in the adjoint representation of $\mathrm{SU}(N_c)$, as is the case for $\mathcal{N}=4$ supersymmetric Yang-Mills theory, we have

$$\mathcal{A}^{\text{1-loop}}_n(\{k_i,\lambda_i,a_i\}) = \sum_{c=1}^{\lfloor n/2\rfloor+1}\ \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\,A_{n;c}(\sigma)\,, \tag{8}$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x. The leading color-structure factor

$$\mathrm{Gr}_{n;1}(1) = N_c\ \mathrm{Tr}(T^{a_1}\cdots T^{a_n})\,, \tag{9}$$

is N_c times the tree color factor. The subleading color structures are given by

$$\text{Gr}_{n;c}(1) = \text{Tr}(T^{a_1} \cdots T^{a_{c-1}})\ \text{Tr}(T^{a_c} \cdots T^{a_n}). \tag{10}$$

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant.

The one-loop subleading-color partial amplitudes are given by a sum over permutations of the leading-color ones [9],

$$A_{n;c}(1,2,\ldots,c-1;c,c+1,\ldots,n) \;=\; (-1)^{c-1} \sum_{\sigma\in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma)\,, \tag{11}$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1, c-2, \ldots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \ldots, n-1, n\}$, and $COP\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,2,\ldots,n\}$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i. To obtain the full amplitude, therefore, we need only compute the leading-color single-trace partial amplitudes. The simple relation (11) between leading- and subleading-color contributions is special to one loop; at higher loops, new non-planar structures enter the subleading-color partial amplitudes.

We also collect here the results for the MHV partial amplitudes in the $\mathcal{N}=4$ theory [9]. These amplitudes appear in various kinematic limits of the NMHV amplitudes, so it is useful to understand their structure. The MHV amplitudes are a simple linear combination of certain box integral functions,

$$A_{n;1}^{\mathcal{N}=4\ \text{MHV}} = c_\Gamma\, A_n^{\text{tree}} \times V_n^g\,. \tag{12}$$

The factor V_n^g $(n \geq 5)$ depends on whether n is odd $(n = 2m+1)$ or even $(n = 2m)$,

$$\begin{aligned}
(\mu^2)^{-\epsilon} V_{2m+1}^g \;&=\; \sum_{r=2}^{m-1}\sum_{i=1}^{n} F^{2\text{m}\,e}\big(s_{i\ldots(i+r)}, s_{(i-1)\ldots(i+r-1)}, s_{i\ldots(i+r-1)}, s_{(i+r+1)\ldots(i-2)}\big) \\
&\quad + \sum_{i=1}^{n} F^{1\text{m}}\big(s_{i-3,i-2}, s_{i-2,i-1}, s_{i\ldots(i-4)}\big)\,, \\
(\mu^2)^{-\epsilon} V_{2m}^g \;&=\; \sum_{r=2}^{m-2}\sum_{i=1}^{n} F^{2\text{m}\,e}\big(s_{i\ldots(i+r)}, s_{(i-1)\ldots(i+r-1)}, s_{i\ldots(i+r-1)}, s_{(i+r+1)\ldots(i-2)}\big) \\
&\quad + \sum_{i=1}^{n} F^{1\text{m}}\big(s_{i-3,i-2}, s_{i-2,i-1}, s_{i\ldots(i-4)}\big) \\
&\quad + \sum_{i=1}^{n/2} F^{2\text{m}\,e}\big(s_{i\ldots(i+m-1)}, s_{(i-1)\ldots(i+m-2)}, s_{i\ldots(i+m-2)}, s_{(i+m)\ldots(i-2)}\big)\,.
\end{aligned} \tag{13}$$

The box integral functions F are defined in appendix A. They are essentially scalar box integrals, but multiplied by convenient normalization factors in order to remove all power-law behavior in the kinematic invariants, leaving only logarithmic behavior. By dimensional analysis, the scale μ enters all dimensionally-regulated (unrenormalized) one-loop amplitudes as an overall factor of $(\mu^2)^\epsilon$.

The four-point and five-point amplitudes appear in the multi-particle factorization limits of the seven-point amplitudes, as discussed in section V and appendix B. The $n = 4$ case is given by

$$A_{4;1}^{\mathcal{N}=4} = 2\, c_\Gamma\, (\mu^2)^\epsilon\, A_4^{\rm tree} \times F^{\rm 0m}(s_{12}, s_{23})\,, \tag{14}$$

for all non-vanishing helicity choices. For $n = 5$, the expression (13) reduces to

$$\begin{aligned} A_{5;1}^{\mathcal{N}=4} \ =\ & c_\Gamma\, (\mu^2)^\epsilon\, A_5^{\rm tree} \Big[F^{\rm 1m}(s_{12}, s_{23}, s_{45}) + F^{\rm 1m}(s_{23}, s_{34}, s_{51}) + F^{\rm 1m}(s_{34}, s_{45}, s_{12}) \\ & + F^{\rm 1m}(s_{45}, s_{51}, s_{23}) + F^{\rm 1m}(s_{51}, s_{12}, s_{34}) \Big]\,. \end{aligned} \tag{15}$$

Besides the $\mathcal{N} = 4$ MHV amplitudes, a number of other infinite series of one-loop amplitude have been computed. The n-point MHV gluon amplitudes with an $\mathcal{N} = 1$ chiral multiplet in the loop were also computed using the unitarity method. In QCD one-loop n-point amplitudes with identical helicities are also known [41]. At two loops less is known. A conjecture for the planar two-loop MHV amplitudes in terms of one-loop amplitudes was presented in ref. [12], suggesting that the simplicity uncovered at one loop persists to higher loops in the planar, leading-color limit $N_c \to \infty$.

The one-loop non-MHV six-point amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory were computed in ref. [10]. Here we will compute all non-MHV seven-point amplitudes. (For $n = 6$ and $n = 7$, 'non-MHV' coincides with 'next-to-MHV'.) We expect that these new amplitudes will be helpful for unraveling their full n-point twistor-space structure. Indeed, their structure provided important guidance for obtaining the coefficients of a class of three-mass box integrals appearing in the all-n next-to-MHV amplitudes, which we present in section VI.

III. CALCULATIONAL APPROACH

It seems clear that calculating the seven-point amplitude by computing the 227,585 contributing Feynman diagrams one by one is not the best way to proceed. And an all-multiplicity result is simply not accessible to this standard approach.

Fortunately, the unitarity-based technique is better for this application. (For a recent review, see ref. [26].) The basic idea is to reconstruct a one-loop amplitude from its cuts or absorptive pieces, which are products of tree amplitudes. In general, one must calculate these cuts in D dimensions. In this case there are no ambiguities in reconstructing amplitudes in any massless theory. In the special case of supersymmetric theories, one can even evaluate the cuts 'in four dimensions' — that is, by assigning four-dimensional helicities to the states crossing the cut — without encountering any ambiguities, for all terms in the amplitudes which vanish as $\epsilon \to 0$ [10]. This property reflects the better ultraviolet behavior of these theories. Then the one-loop cuts reduce to products of tree-level helicity amplitudes. Starting from tree amplitudes rather than diagrams means that the extensive cancellations that occur in gauge theories are taken into account before any loop integrations are done, which greatly reduces the complexity of the calculations. Because infinite series of tree-level amplitudes are known, it also opens the door to calculating infinite series of one-loop amplitudes.

The reconstruction is done by identifying the corresponding Feynman integrals whose cuts yield the original integrands. This is easier than doing the dispersion integrals explicitly, because we take advantage of an extensive machinery for simplifying and computing such integrals.

In calculating an n-point amplitude in gauge theory, one expects to encounter n-point tensor integrals with tensor rank up to n. In supersymmetric theories, the maximal tensor rank is lower, a reflection of the better ultraviolet behavior of the theory [10]. In $\mathcal{N} = 4$ theories, the maximal tensor rank is reduced to $n - 4$ [9, 42]. In addition, thorough use of the spinor-helicity representation of external gluon polarization vectors can reduce both the tensor rank and multiplicity of integrals we encounter upon reconstructing them from the cuts.

Nonetheless, we do have to treat some higher-point and tensor integrals. At one loop, on general grounds, one can reduce any loop integral in D dimensions to a combination of box integrals, triangles, and bubbles. In $\mathcal{N} = 4$ supersymmetric theories, only scalar box integrals are needed, and no triangles or bubbles. The tensor higher-point integrals encountered, where loop momenta appear in the numerator of the integrand, could be reduced to scalar ones via brute-force Brown–Feynman/Passarino–Veltman techniques [43], or alternatively via Feynman-parameter methods. However, we have found it useful to follow a method adapted from that originally proposed by van Neerven and Vermaseren [44], relying on the fact that any vector in four dimensions can be written in terms of a basis of four vectors. For example, we may expand the vector $\ell^\mu_{[4]}$ as

$$\epsilon^{p_1p_2p_3p_4} \times \ell^\mu = \ell_{[4]} \cdot p_1 \; \epsilon^{\mu p_2p_3p_4} + \ell_{[4]} \cdot p_2 \; \epsilon^{p_1\mu p_3p_4} + \ell_{[4]} \cdot p_3 \; \epsilon^{p_1p_2\mu p_4} + \ell_{[4]} \cdot p_4 \; \epsilon^{p_1p_2p_3\mu} \,, \quad (16)$$

where $\epsilon^{\mu p_2p_3p_4}$ is shorthand for $\epsilon^\mu_{\nu\sigma\rho} p_2^\nu p_3^\sigma p_4^\rho$, and $\epsilon_{\mu\nu\sigma\rho}$ is the Levi-Civita tensor. Of course, we are working with dimensionally-regulated integrals, but we can split the loop momentum vector ℓ^μ into two parts, the four-dimensional components and the (-2ϵ)-dimensional ones. The four-dimensional components can be expanded according to eq. (16), where $\{p_1, p_2, p_3, p_4\}$ where the p_i will be chosen to be either external momenta or sums of external momenta (see below). In this case, we can take advantage of the fact that

$$\ell_{[4]} \cdot p_i = \frac{1}{2}(\ell_{[4]}^2 - (\ell_{[4]} - p_i)^2 + p_i^2) \,, \quad (17)$$

in order to cancel propagators, or reduce the tensor rank of the integral. In the unitarity-based method, ℓ^2 is taken to be zero, and so when we are using the spinor-helicity representation, what appears in the numerator are only dot products of the loop momentum, and spinor 'sandwiches' of the loop momentum between spinors representing external momenta. These objects depend only on the four-dimensional components of the loop momentum. (Because we are evaluating the cuts in four dimensions, we have dropped some terms in the cut integrand which are proportional to the square of the (-2ϵ)-dimensional components. These pieces lead only to $\mathcal{O}(\epsilon)$-suppressed terms in the amplitudes, for supersymmetric theories.)

We do not choose a fixed basis of external vectors for all expansions. For terms containing multiple factors of the loop momentum in the numerator, we proceed in stages, expanding the successive occurrences of the loop momentum one by one. At each stage, we choose the basis set for the expansion to consist of external momenta, not of the amplitude, but of the subject integral. In particular, after the first stage of this procedure, one of the external legs will consist of a *sum* of the original external momenta, because of the cancellation of one of the propagators using eq. (17). We choose such a sum to be an element of the basis set for the second stage. Using the new basis set in the second stage ensures that the loop-momentum-containing quantities $\ell_{[4]}^2$ and $(\ell_{[4]} - p_i)^2$ appearing on the right-hand-side of eq. (17) will continue to cancel some propagator in the integral at this stage. We proceed similarly in the further stages. We call this approach a 'pivoting' technique. Since dot products of the loop momentum with external legs of any one-loop integral are always expressible in terms of inverse propagators and external invariants, as in eq. (17), our choice always allows a given integral to be reduced into a sum of lower-rank and lower-point integrals. (Were we to have chosen a fixed external basis set, expansions beyond the first would *not* necessarily be expressible in terms of inverse propagators of the daughter integrals, complicating the analysis.)

In principle the pivoting technique can be applied to any cut (or even to full uncut diagrams if desired). For the $\mathcal{N} = 4$ application, however, it is only necessary to consider cuts in three-particle channels, those with invariants $s_{i,i+1,i+2}$ crossing the cut. That is because box integrals contain cuts in several channels, and for the seven-point process at least one channel is always a three-particle channel. Such a cut is the product of a five-point tree amplitude and a six-point tree amplitude, or it may be a sum of such terms, the sum corresponding to multiple members of the $\mathcal{N} = 4$ supermultiplet crossing the cut. It is useful to divide the cuts into 'singlet' and 'nonsinglet' pieces. The singlet pieces have a total helicity of ± 2 crossing the cut. They can only come from gluonic intermediate states. The nonsinglet pieces have a total helicity of 0 crossing the cut, and receive contributions from the entire $\mathcal{N} = 4$ supermultiplet, gluons, fermions and scalars. One can further divide the singlet and nonsinglet pieces according to whether the six-point tree amplitude(s) are MHV or non-MHV. So there are four types of cut building blocks:

- *singlet-MHV*. This cut is the simplest type to evaluate, and can be done directly in a compact form for all n; see ref. [39] and section VI).

- *nonsinglet-MHV*. This cut is also simple; the sum over $\mathcal{N} = 4$ states can be performed as in ref. [10], resulting in an expression which is identical to the singlet-MHV case, except for overall, loop-momentum-independent, prefactors.

- *singlet-non-MHV*. This cut is more complicated, but still only the product of two helicity amplitudes.

- *nonsinglet-non-MHV*. This cut is the most complicated, but only needed to be evaluated as a cross-check.

It is also possible to consider *generalized* cuts which provide very useful additional information about loop amplitudes [26, 45]. The example of a 'triple cut' is shown in fig. 1. It is

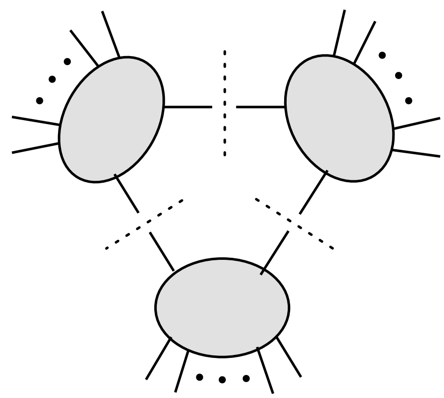

FIG. 1: A generalized 'triple cut'. The three propagators cut by the dashed lines are required to be 'open'.

not a cut in the traditional sense of having a phase-space integral which generates the imaginary part in a single channel. It corresponds to a collection of underlying Feynman diagrams which have three propagators 'open', or uncancelled, in the same way that an ordinary cut requires two propagators to be open. Terms which vanish as the 'cut' propagators go on shell may be ignored. The contributing terms can be represented as the product of three tree amplitudes. We will see in section VI that analysis of such cuts provides additional information, often quite easily.

Application of the pivoting technique to each type of cut yields an expression in terms of external spinor invariants, and scalar integrals up to n points. To reduce the six- and seven-point scalar integrals we used the formulæ from appendix VI of ref. [46]. To reduce the five-point scalar integrals we made use of two different reduction formulæ. The first formula is also from ref. [46] in the equivalent form from ref. [47],

$$I_5 = \frac{1}{2}\sum_{j=1}^{n} c_j I_4^{(j)} + \epsilon\, c_0\, I_5^{D=6-2\epsilon}, \tag{18}$$

where $I_4^{(j)}$ is the daughter box integral derived from the pentagon by omitting (the negative of) the propagator between legs $(j-1)$ and j; and where

$$c_j = \sum_{l=1}^{n} S^{-1}_{jl}\,,$$
$$c_0 = \sum_{l=1}^{n} c_l\,,$$

in terms of $S_{ij} = S_{ji} = -K^2_{i\ldots(j-1)}/2$ (and $S_{ii} = 0$). The pentagon integral in $6-2\epsilon$

dimensions, $I_5^{D=6-2\epsilon}$, is finite as $\epsilon \to 0$, and it has a manifest prefactor of ϵ in eq. (18), so its contribution may be neglected to the order we are working.

These reductions are valid in dimensional regularization. The six- and seven-point reduction formulæ are equivalent to those of Melrose [48], and van Neerven and Vermaseren [44]. The five-point reduction formula is equivalent after dropping the $\mathcal{O}(\epsilon)$ terms. The five-point reduction formula (18) is valid for integrals, but does not hold point-by-point for the integrands. We found it convenient for certain parts of the calculation to use a reduction formula which is valid point-by-point. We can obtain such a formula by adding appropriate terms containing the Levi-Civita tensor and linear in the loop momentum, $\epsilon^{\mu\cdots}\ell_\mu\ldots$, which vanish upon performing the loop integration. The advantage of including these terms in the reduction is that one can check all algebra numerically, directly on the integrand.

Once we have reduced the integrand to a sum of four-point integrands, we are done, because amplitudes in the maximally supersymmetric theory can be expressed in terms of scalar boxes, with no triangle or bubble integrals. (Due to $\mathcal{N}=4$ supersymmetry, each term in the cut integrand with $r-2$ non-cut propagators starts out with no more than degree $r-4$ tensor integrals. Given this starting point, the pivoting reduction ensures that tensor boxes never appear.)

This procedure gives us the coefficients of the standard box integral functions. The forms emerging directly from the reductions are generally quite complicated. However, we can simplify them further, as discussed below. Somewhat surprisingly, many of them have a remarkably simple form. Indeed, the most 'complicated' integrals that appear, the three-mass boxes, have single-term coefficients, and the most complicated coefficients have only four terms.

The coefficients have various momentum-space singularities. Some of the coefficient singularities are also singularities of the full one-loop amplitude, such as the collinear singularities or poles in multi-particle invariants s_{ijk}. Other singularities are spurious. These are introduced by the reduction procedure. Some of these singularities disappear coefficient by coefficient; others survive. They will cancel in the amplitude as a whole, as we will discuss in section V; but this cancellation involves the box integrals and their behavior in an essential way. That is, each coefficient can (and does in general) possess singularities not present in the amplitude as a whole. The pentagon reduction (18), in particular, gives rise to denominators which induce such singularities. They can be read off from eq. (19) by inspecting the determinant of the appropriate pentagon S_{ij} matrix. In the seven-point calculation, we encounter one-mass and two-mass pentagons. The former can be treated as a special case of the latter. The non-adjacent two-mass pentagon (with external legs (k_1, k_2, P, k_3, Q), capital letters denoting massive legs) gives rise to the following denominator,

$$2k_1 \cdot k_2((P+k_2)^2(P+k_3)^2 - P^2(P+k_2+k_3)^2)((Q+k_3)^2(Q+k_1)^2 - Q^2(Q+k_1+k_3)^2). \quad (19)$$

Each of the last two factors can be further factorized using the spinor identity [45],

$$(P+k_1)^2(P+k_2)^2 - P^2(P+k_1+k_2)^2 = \langle 1^-|\not{P}|2^-\rangle\langle 2^-|\not{P}|1^-\rangle. \quad (20)$$

This type of denominator thus develops singularities when P lies in the (space-time) plane spanned by k_1 and k_2,

$$P^\mu = a_1 k_1^\mu + a_2 k_2^\mu, \quad (21)$$

for arbitrary values of a_1 and a_2. For this reason, we call the factor in eq. (20) and the corresponding singularity a ‘planar’ or ‘back-to-back’ one.

The adjacent-mass two-mass pentagon (with external legs (k_1, k_2, k_3, P, Q)) gives rise to the denominator,

$$4k_1 \cdot k_2\, k_2 \cdot k_3 \left[(k_1+k_2+k_3)^2(Q+k_1)^2(P+k_3)^2 - 2k_1 \cdot k_2\, P^2(Q+k_1)^2 - 2k_2 \cdot k_3\, Q^2(P+k_3)^2\right]. \quad (22)$$

The last factor, in brackets, also has a spinor factorization (for $k_1 + k_2 + k_3 + P + Q = 0$),

$$\langle 1^-|\not{Q}\not{P}|3^+\rangle\langle 3^+|\not{P}\not{Q}|1^-\rangle. \quad (23)$$

It therefore vanishes whenever

$$P = a_1 k_1 + a_2 k_2 + a_3 k_3, \quad (24)$$

with

$$a_2 = -a_1 \frac{k_1 \cdot k_2 + (1+a_3) k_1 \cdot k_3}{a_1 k_1 \cdot k_2 + (1+a_3) k_2 \cdot k_3}\,. \quad (25)$$

To distinguish it from the other kind of denominator and singularity, we will call this type ‘cubic’.

In computing the all-n coefficient discussed in section VI, one also encounters denominators arising from the reduction of certain three-mass pentagon integrals. It turns out that they also can be expressed in terms of these sorts of ‘cubic’ factors.

Knowledge of all of the singularities of the box coefficients allows one to simplify them 'numerically'. The basic idea is to generate kinematic points numerically which are close to the various singular regions: nearly collinear kinematics for each of the seven color-adjacent pairs $(i,\ i+1)$; kinematic points where each of the seven multi-particle invariants $s_{i,i+1,i+2}$ almost vanish; and points near the relevant 'planar' and 'cubic' singularities. By studying the behavior of the large analytical expressions for the box coefficients in these regions, one can write down all the allowed factors in the denominator of the expression. In the collinear limit where k_i is nearly parallel to k_{i+1}, it is helpful to be able to rotate k_i and k_{i+1} about their common sum. The complex phase behavior under this rotation reveals whether the denominator contains $\langle i\,(i+1)\rangle$ or $[i\,(i+1)]$ (or both).

The numerator of the box coefficient can often be completely determined by the spinor homogeneity properties, which dictate how many net powers of $\langle i^-|$ *vs.* $\langle i^+|$ occur, in terms of the helicity of leg i. In a few cases, we solved for a coefficient by writing down a sum of all possible expressions of the right dimension and spinor homogeneity, and comparing it numerically to the actual expression at a number of randomly-generated phase-space points.

The easy-two-mass box coefficients are the simplest to determine in this way, because they cannot have any 'cubic' singularities, and because they turn out to be composed of just one term. (The easy-two-mass box integral has a massless leg interposed between each massive one, which prevents it from being obtained from an adjacent-mass pentagon by cancelling a propagator. The adjacent-mass pentagons are the only source of cubic singularities.) For example, in examining the coefficient c_{147} for $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$, one quickly finds that it vanishes strongly as legs 2 and 3 become collinear, and as legs 6 and 7 become collinear. The factors of $\langle 2\,3\rangle^3[6\,7]^3$ in the numerator of eq. (39) can be established in this way. The denominator factors in eq. (39) are all easily obtained numerically as well, except for the question of whether $\langle 1^+|\,(6+7)\,|5^+\rangle$ or its complex conjugate $\langle 1^-|\,(6+7)\,|5^-\rangle$ correctly describes the 'planar' singularity, and similarly for $\langle 6^+|\,(7+1)\,|2^+\rangle$ *vs.* $\langle 6^-|\,(7+1)\,|2^-\rangle$. However, only one of these four possibilities can give the right spinor homogeneity for legs 1, 2, 5 and 6. (The possibility of additional factors in the numerator is excluded in this case by dimensional analysis.)

In addition to simplifying the coefficients obtained from a direct calculation, we can also employ another tool. The structure of infrared singularities provides equations which can be used as consistency checks, as we shall discuss in section V, or alternatively to solve for some of the coefficients. The infrared singularities in the amplitude are known on general grounds [49] to be,

$$A_{n;1}^{\mathcal{N}=4}\Big|_{\epsilon\ \text{pole}} = -\frac{c_\Gamma}{\epsilon^2}\sum_{i=1}^{n}\Big(\frac{\mu^2}{-s_{i,i+1}}\Big)^{\epsilon}\times A_n^{\text{tree}}. \tag{26}$$

That is, the $1/\epsilon$ poles are proportional to the tree amplitude and contain only logarithms of nearest-neighbor two-particle invariants. This implies that the coefficient of any given $\ln(-s_{i,i+1})/\epsilon$ must be equal to the tree; and the coefficient of any $\ln(-s_{i,i+1,i+2})/\epsilon$ must vanish. On the other hand, the scalar box functions whose coefficients we are computing contain both of these sorts of terms, and so both types of equations are non-trivial. Each box function in eqs. (A2)–(A5) contains various $\ln(-s_{i,i+1})/\epsilon$ and $\ln(-s_{i,i+1,i+2})/\epsilon$ terms with coefficients 0, ± 1 or $\pm\frac{1}{2}$. Thus the constraints arising from eq. (26) become simple linear relations among the coefficients, some of which involve the tree amplitude. (Other linear relations may be determined from numerical evaluation of the box coefficients.) We use the $1/\epsilon$ pole information as part of the simplification process, but the simplest forms of the coefficients do not satisfy it manifestly. That is, we shall obtain alternate, compact representations for the tree amplitudes, by re-imposing the constraints from eq. (26) after completing the simplifications.

It is also possible to relate box coefficients from one of the four seven-point NMHV helicity amplitudes to those from the other three, using properties of the tree amplitudes which enter the cuts. Such relations are possible any time a cut that is sensitive to a given box integral does *not* contain the 'nonsinglet-non-MHV' type of contribution. Under such circumstances, there are supersymmetry Ward identities (SW) [21] which can be used to permute around the helicities on one side of the cut.

Consider for example the s_{123} cut of $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^+,4^-,5^+,6^+,7^+)$. It has a singlet-non-MHV term, which is given by

$$\text{S-NMHV}_{123} = i\, A_5^{\text{tree}}((-\ell_1)^+,1^-,2^-,3^+,\ell_4^+)\, A_6^{\text{tree}}((-\ell_4)^-,4^-,5^+,6^+,7^+,\ell_1^-), \tag{27}$$

integrated over phase space. Using a SWI on the left-side of the cut, the cut becomes

$$\text{S-NMHV}_{123} = i\frac{[1\,2]^4}{[2\,3]^4}\, A_5^{\text{tree}}((-\ell_1)^+,1^+,2^-,3^-,\ell_4^+)\, A_6^{\text{tree}}((-\ell_4)^-,4^-,5^+,6^+,7^+,\ell_1^-). \tag{28}$$

Except for the spinor-product prefactor, this expression is equivalent to the singlet-non-MHV piece of the s_{234} cut of $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$, after the relabelling $1 \leftrightarrow 4$,

$2 \leftrightarrow 3$, $5 \leftrightarrow 7$ (and an overall minus sign). On the other hand, the nonsinglet-MHV term in the above s_{123} cut is given by

$$
\begin{aligned}
\text{NS-MHV}_{123} &= i \sum_{\lambda} A_5^{\text{tree}}((-\ell_1)^{-\lambda}, 1^-, 2^-, 3^+, \ell_4^{\lambda})\, A_6^{\text{tree}}((-\ell_4)^{-\lambda}, 4^-, 5^+, 6^+, 7^+, \ell_1^{\lambda}) \\
&= i \frac{\langle 3^- |\, (1+2)\, | 4^- \rangle^4}{\langle 1^- |\, (2+3)\, | 4^- \rangle^4} \sum_{\lambda} A_5^{\text{tree}}((-\ell_1)^{-\lambda}, 1^+, 2^-, 3^-, \ell_4^{\lambda}) \\
&\qquad \times A_6^{\text{tree}}((-\ell_4)^{-\lambda}, 4^-, 5^+, 6^+, 7^+, \ell_1^{\lambda}). \qquad (29)
\end{aligned}
$$

Again, except for the spinor-product prefactor, this expression is equivalent to the nonsinglet-MHV piece of the s_{234} cut of $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$, after applying the same relabelling and minus sign.

IV. RESULTS

In this section we present the results for the four independent one-loop 7-point NMHV amplitudes in $\mathcal{N} = 4$ SYM: $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$, $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$, $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+)$, and $A_{7;1}^{\mathcal{N}=4}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+)$. All other $A_{7;1}^{\mathcal{N}=4}$ NMHV helicity amplitudes can be obtained from these by the three operations:

1. *parity*, which exchanges + and − helicities and is implemented on the basic spinor products by the action $\langle i\, j \rangle \leftrightarrow [j\, i]$,

2. *reflection symmetry*, essentially charge conjugation invariance, which states that $A_{n;1}^{\mathcal{N}=4}(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}) = (-1)^n A_{n;1}^{\mathcal{N}=4}(n^{\lambda_n}, \ldots, 2^{\lambda_2}, 1^{\lambda_1})$, and

3. *cyclic symmetry*, since each $A_{n;1}^{\mathcal{N}=4}$ is the coefficient of a cyclicly-invariant color trace.

The subleading-color partial amplitudes appearing in eq. (8), $A_{7;c}^{\mathcal{N}=4}$ for $c > 1$, are obtained by sums over permutations of the leading-color $A_{7;1}^{\mathcal{N}=4}$ [9], using eq. (11).

The $\mathcal{N} = 4$ SYM amplitude is expressible as a sum of scalar box integrals $\mathcal{I}_4$, or equivalently the box functions F given in appendix A. For the color-order amplitudes $A_{7;1}$, the kinematics of each box integral can be obtained from the heptagon diagram with external legs in the order 1,2,3,4,5,6,7, by deleting three different propagators, say i, j, k. We label each box function F, and each kinematic coefficient c multiplying it, by this triplet of integers. However, to avoid confusion with a twistor-space 'line operator' F_{ijk} to appear in

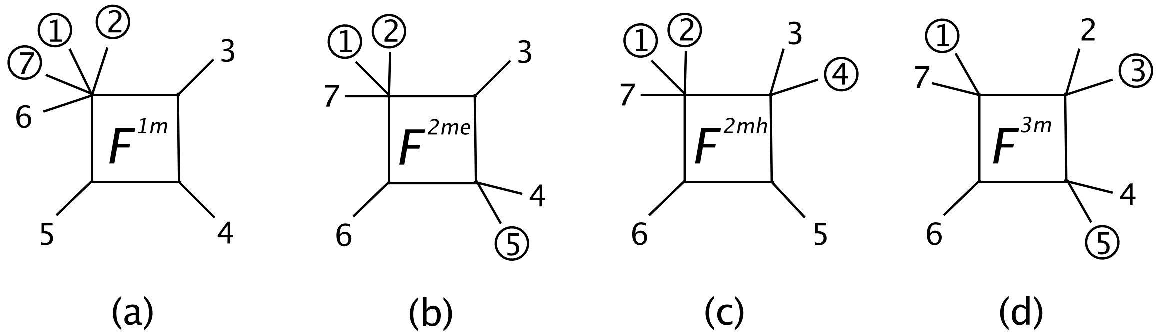

FIG. 2: Examples of box integral functions $B(i, j, k)$ appearing in 7-point amplitudes; the arguments i, j, k are circled: (a) the one-mass box $B(1, 2, 7) = F^{1m}(s_{34}, s_{45}, s_{345})$, (b) the 'easy' two-mass box $B(1, 2, 5) = F^{2me}(s_{345}, s_{456}, s_{45}, s_{712})$, (c) the 'hard' two-mass box $B(1, 2, 4) = F^{2mh}(s_{56}, s_{345}, s_{712}, s_{34})$, and (d) the three-mass box $B(1, 3, 5) = F^{3m}(s_{671}, s_{456}, s_{71}, s_{23}, s_{45})$.

section VII, we call these box functions $B(i, j, k)$ instead of $F(i, j, k)$. Thus we write the leading-color partial amplitude as

$$A_{7;1}^{\mathcal{N}=4} = ic_\Gamma\,(\mu^2)^\epsilon \sum_{i,j,k} c_{ijk} B(i, j, k), \tag{30}$$

where

$$c_\Gamma \;=\; \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{31}$$

is a ubiquitous prefactor. We label the propagators so that in the heptagon diagram the external leg labeled p lies between propagators labeled p and $p+1$ (mod 7). Thus in the box $B(i, j, k)$, leg i is grouped into a single massive leg with leg $i-1$, and similarly for legs j and k. If two members of the set $\{i, j, k\}$ are cyclicly adjacent, the corresponding massive leg of the box integral contains three massless legs. If all three of $\{i, j, k\}$ are cyclicly adjacent, that massive leg contains four massless legs. Some examples of this labeling are shown in fig. 2.

The kinematic arguments of the boxes are s and t, the squares of the sums of momenta emerging from two adjacent vertices of the box, followed by the 'masses', the squares of the sums of momenta emerging from individual vertices of the box (when these are nonzero). The easy two-mass box has two reflection symmetries, so it doesn't matter which invariant is s and which t. For the hard two-mass box, s is the invariant formed by the two adjacent massless legs. For the three-mass box, s is the invariant formed by the massless leg and the first massive leg following it cyclicly.

For a given helicity amplitude, the number of box functions, and box coefficients, is the number of un-ordered integer triplets (i,j,k), where each integer runs from 1 to 7, and all three are unequal. This number is just $\binom{7}{3}$, or 35. Another way to count this number is to observe that there are 5 different kinds of box in the 7-point case:

- the one-mass box shown in fig. 2a, plus cyclic permutations (7 boxes),
- the easy-two-mass box shown in fig. 2b, plus cyclic permutations (7 boxes),
- the hard-two-mass box shown in fig. 2c, plus its 'reflection' in which the 2-leg cluster precedes the 3-leg cluster, plus cyclic permutations (14 boxes),
- the three-mass box shown in fig. 2d, plus cyclic permutations (7 boxes).

In the remainder of this section we present the coefficients c_{ijk} of all the box functions, using symmetries whenever possible. In section VII we shall describe the twistor-space structure and other general properties of these coefficients, which fit nicely into a uniform pattern.

A. $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$

This partial amplitude possesses a reflection, or "flip" symmetry. We define the flip operation (for this partial amplitude) by

$$X|_{\text{flip}} = -X(1 \leftrightarrow 3, 4 \leftrightarrow 7, 5 \leftrightarrow 6), \tag{32}$$

i.e. a two-fold exchange of labels accompanied by an overall minus sign. The tree amplitude obeys

$$A_7^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)\Big|_{\text{flip}} = A_7^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+,7^+). \tag{33}$$

as does the one-loop amplitude,

$$A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)\Big|_{\text{flip}} = A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+). \tag{34}$$

We can write all of the box coefficients in terms of the following quantities:

$$c_A = \frac{-\langle 2\,3\rangle^3 \langle 5^+|\,(6+7)\,|1^+\rangle^3}{s_{234}s_{671}\,\langle 3\,4\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle}\ , \tag{35}$$

$$c_B = \frac{-\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle^3}{s_{345}s_{671}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 2^+|\,(7+1)\,|6^+\rangle}\,, \tag{36}$$

$$c_{136} = \frac{\langle 2\,3\rangle^3\langle 7^+|\,(5+6)\,|4^+\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7\,1]\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle\,\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle}, \tag{37}$$

$$c_{146} = [5\,6]^3 \times \frac{\langle 1\,2\rangle^3}{\langle 3\,4\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 2^-|\,(3+4)(5+6)\,|7^+\rangle} \times \frac{\langle 2\,3\rangle^3}{\langle 7\,1\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle\,\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle}\,, \tag{38}$$

$$c_{147} = \frac{\langle 2\,3\rangle^3[6\,7]^3}{s_{671}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[7\,1]\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle}\,, \tag{39}$$

$$c_{247} = -\frac{\langle 5^-|\,(6+7)(1+2)\,|3^+\rangle^3}{[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 7^-|\,(1+2)(3+4)\,|5^+\rangle}\,, \tag{40}$$

$$c_{346} = -\frac{\langle 2\,3\rangle^3 s_{567}^3}{s_{234}\,\langle 3\,4\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 2^-|\,(3+4)(5+6)\,|7^+\rangle}\,, \tag{41}$$

$$c_{347} = \frac{\langle 4^+|\,(2+3)\,|1^+\rangle^3}{s_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle}\,, \tag{42}$$

$$c_{567} = \frac{s_{123}^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle}\,. \tag{43}$$

$$\tag{44}$$

Note that c_B and c_{567} are flip symmetric.

The quantities (35)–(43), plus their images under the flip operation, are not all independent. They satisfy

$$[c_{136}]|_{\rm flip} + c_{346} + c_{567} = c_{247} + c_{347}\,, \tag{45}$$

$$c_{146} + c_{147} + c_{346} = c_A + c_{136}\,, \tag{46}$$

and

$$A_7^{\rm tree} = [c_A + c_{136}]|_{\rm flip} + c_{147} + c_{346} + c_{567} \tag{47}$$

$$= c_B + c_{347} + [c_{347}]|_{\rm flip}\,. \tag{48}$$

Equation (48) in particular provides a compact representation of the tree amplitude which has the proper collinear behavior manifest in all channels, resembling some "twistor-space" constructions [23]. Equation (48) is manifestly flip symmetric, whereas eq. (47) is not.

In terms of these quantities, the remaining box coefficients are given by

$$c_{235} = c_{237} = c_{256} = c_{257} = c_{357} = c_{367} = 0, \tag{49}$$

$$c_{236} = c_{267} = c_{356} = c_{567}, \tag{50}$$
$$c_{134} = c_A + c_{147}, \tag{51}$$
$$c_{137} = c_A + c_{347}, \tag{52}$$
$$c_{167} = c_A + c_{136} + c_{347}, \tag{53}$$
$$c_{234} = c_{247} + c_{347}, \tag{54}$$
$$c_{345} = c_B + c_{147} + c_{347}, \tag{55}$$
$$c_{457} = c_B + c_{147}, \tag{56}$$
$$c_{467} = c_{346} + c_{347}, \tag{57}$$

and

$$c_{123} = c_{234}|_{\text{flip}}\,, \tag{58}$$
$$c_{124} = c_{134}|_{\text{flip}}\,, \tag{59}$$
$$c_{125} = c_{347}|_{\text{flip}}\,, \tag{60}$$
$$c_{126} = c_{346}|_{\text{flip}}\,, \tag{61}$$
$$c_{127} = c_{345}|_{\text{flip}}\,, \tag{62}$$
$$c_{135} = c_{247}|_{\text{flip}}\,, \tag{63}$$
$$c_{145} = c_{147}|_{\text{flip}}\,, \tag{64}$$
$$c_{156} = c_{467}|_{\text{flip}}\,, \tag{65}$$
$$c_{157} = c_{457}|_{\text{flip}}\,, \tag{66}$$
$$c_{245} = c_{137}|_{\text{flip}}\,, \tag{67}$$
$$c_{246} = c_{136}|_{\text{flip}}\,, \tag{68}$$
$$c_{456} = c_{167}|_{\text{flip}}\,. \tag{69}$$

The flip relations can also be summarized as

$$c_{ijk} = c_{\tilde{i}\tilde{j}\tilde{k}}|_{\text{flip}}\,, \tag{70}$$

where

$$\tilde{i} = (5 - i) \bmod 7, \tag{71}$$

and $\tilde{i}\tilde{j}\tilde{k}$ should be written in ascending order.

B. $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^+,4^-,5^+,6^+,7^+)$

This partial amplitude is alone among the NMHV seven-point amplitudes in having no flip symmetry. The box coefficients c_{ijk} are given in terms of the following quantities:

$$c_A = -\frac{\langle 2\,4\rangle^4 \langle 5^+|\,(6+7)\,|1^+\rangle^3}{s_{671}s_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle}\ , \tag{72}$$

$$c_B = \frac{\langle 1\,2\rangle^3 \langle 7^+|\,(5+6)\,|4^+\rangle^3}{s_{123}s_{456}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\ , \tag{73}$$

$$c_C = \frac{-\langle 4^-|\,(3+5)(6+7)\,|1^+\rangle^4}{s_{345}s_{671}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 2^+|\,(7+1)\,|6^+\rangle\,\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle}, \tag{74}$$

$$c_D = \frac{\langle 1\,2\rangle^3 \langle 6^+|\,(3+5)\,|4^+\rangle^4}{s_{345}s_{712}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 7\,1\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle\,\langle 5^-|\,(3+4)(1+2)\,|7^+\rangle}\ , \tag{75}$$

$$c_E = \frac{\langle 1\,2\rangle^3 \langle 3^+|\,(5+6)\,|4^+\rangle^4}{s_{456}s_{712}\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 7\,1\rangle\,\langle 3^+|\,(4+5)\,|6^+\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle\,\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle}\ , \tag{76}$$

$$c_{125} = \frac{\langle 7^+|\,(1+2)\,|4^+\rangle^4}{s_{712}\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7\,1]\,\langle 2^+|\,(7+1)\,|6^+\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle}\ , \tag{77}$$

$$c_{135} = \frac{\left(\langle 3^+|\,(6+7)\,|1^+\rangle\,\langle 4\,6\rangle + \langle 3^+|\,5\,|4^+\rangle\,\langle 1\,6\rangle\right)^4}{[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 3^+|\,(4+5)\,|6^+\rangle\,\langle 2^+|\,(7+1)\,|6^+\rangle} \times \frac{1}{\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\ , \tag{78}$$

$$c_{136} = \frac{\langle 2\,4\rangle^4 \langle 7^+|\,(5+6)\,|4^+\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7\,1]\,\langle 1^+|\,(2+3)\,|4^+\rangle} \times \frac{1}{\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle\,\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle}\ , \tag{79}$$

$$c_{145} = \frac{\langle 1\,2\rangle^3 [3\,5]^4}{s_{345}\,[3\,4]\,[4\,5]\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 3^+|\,(4+5)\,|6^+\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle}\ , \tag{80}$$

$$c_{146} = \frac{\langle 1\,2\rangle^3 \langle 2\,4\rangle^4 [5\,6]^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 7\,1\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle} \times \frac{1}{\langle 2^-|\,(3+4)(5+6)\,|7^+\rangle\,\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle}\ , \tag{81}$$

$$c_{147} = \frac{\langle 2\,4\rangle^4 [6\,7]^3}{s_{671}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[7\,1]\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle}\ , \tag{82}$$

$$c_{236} = \frac{\langle 3^+|\,(1+2)\,|4^+\rangle^4}{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle}\ , \tag{83}$$

$$c_{237} = \frac{\langle 1\,2\rangle^3 s_{567}^3}{s_{123}\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}\ , \tag{84}$$

$$c_{246} = \frac{\langle 1\,2\rangle^3 \langle 3^+|\,(5+6)\,|7^+\rangle^4}{[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle} \times \frac{1}{\langle 2^-|\,(3+4)(5+6)\,|7^+\rangle\,\langle 5^-|\,(3+4)(1+2)\,|7^+\rangle}\,, \tag{85}$$

$$c_{247} = \frac{\langle 4^-|\,(1+2)(6+7)\,|5^+\rangle^4}{[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle} \times \frac{1}{\langle 7^-|\,(1+2)(3+4)\,|5^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}\,, \tag{86}$$

$$c_{256} = \frac{\langle 1\,2\rangle^3 [5\,6]^3}{s_{456}\,\langle 2\,3\rangle\,[4\,5]\,\langle 7\,1\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle}\,, \tag{87}$$

$$c_{257} = \frac{\langle 1\,2\rangle^3 \langle 3\,4\rangle^3 [6\,7]^3}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle} \times \frac{1}{\langle 5^-|\,(6+7)(1+2)\,|3^+\rangle\,\langle 1^-|\,(6+7)(4+5)\,|3^+\rangle}\,, \tag{88}$$

$$c_{346} = -\frac{\langle 2\,4\rangle^4 s_{567}^3}{s_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 2^-|\,(3+4)(5+6)\,|7^+\rangle}\,, \tag{89}$$

$$c_{347} = \frac{\langle 3^+|\,(2+4)\,|1^+\rangle^4}{s_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle}\,, \tag{90}$$

$$c_{357} = \frac{\langle 1\,2\rangle^3 \langle 5^+|\,(6+7)\,|1^+\rangle^3}{\langle 2\,3\rangle\,[4\,5]\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle} \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\,. \tag{91}$$

Again the quantities (72)–(91) are not independent. They satisfy

$$c_A + c_{136} = c_{146} + c_{147} + c_{346}, \tag{92}$$

$$c_C + c_{125} + c_{237} + c_{257} + c_{347} = c_D + c_{147} + c_{236} + c_{246} + c_{346}, \tag{93}$$

$$c_D + c_{246} + c_{256} = c_E + c_{145} + c_{146}, \tag{94}$$

$$c_{236} + c_{246} + c_{346} = c_{237} + c_{247} + c_{347}, \tag{95}$$

$$c_B + c_{357} = c_{237} + c_{256} + c_{257}, \tag{96}$$

$$c_A + c_{135} + c_{145} = c_C + c_{347} + c_{357}. \tag{97}$$

The tree amplitude may be written in terms of the box coefficients in several different ways. The shortest expression we have found is:

$$A_7^{\rm tree} = c_B + c_C + c_{125} + c_{347} + c_{357}. \tag{98}$$

The remaining box coefficients are given by

$$c_{123} = c_E + c_{136} + c_{236} + c_{256}, \tag{99}$$

$$c_{124} = c_D + c_{145}, \tag{100}$$

$$c_{126} = c_E + c_{256}, \tag{101}$$

$$c_{127} = c_C + c_{125} + c_{145} + c_{257}, \tag{102}$$

$$c_{134} = c_A + c_{147}, \tag{103}$$

$$c_{137} = c_A + c_{347}, \tag{104}$$

$$c_{156} = c_E + c_{125}, \tag{105}$$

$$c_{157} = c_C + c_{145}, \tag{106}$$

$$c_{167} = c_A + c_{136} + c_{347}, \tag{107}$$

$$c_{234} = c_{236} + c_{246} + c_{346}, \tag{108}$$

$$c_{235} = c_B + c_{256}, \tag{109}$$

$$c_{245} = c_D + c_{125}, \tag{110}$$

$$c_{267} = c_{236} + c_{237}, \tag{111}$$

$$c_{345} = c_C + c_{147} + c_{347} + c_{357}, \tag{112}$$

$$c_{356} = c_B + c_{236}, \tag{113}$$

$$c_{367} = 0, \tag{114}$$

$$c_{456} = c_D + c_{125} + c_{246} + c_{256}, \tag{115}$$

$$c_{457} = c_C + c_{147}, \tag{116}$$

$$c_{467} = c_{346} + c_{347}, \tag{117}$$

$$c_{567} = c_B + c_{236} + c_{357}. \tag{118}$$

C. $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+)$

For this partial amplitude we define the "flip" operation,

$$X|_{\text{flip}} = -X(1 \leftrightarrow 2, 3 \leftrightarrow 7, 4 \leftrightarrow 6), \tag{119}$$

which is a symmetry of the tree amplitude,

$$A_7^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+)\Big|_{\text{flip}} = A_7^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+). \tag{120}$$

and of the one-loop amplitude. The box coefficients c_{ijk} are given in terms of the following quantities:

$$c_A = \frac{\langle 1\,2\rangle^3 \langle 7^+|\,(4+6)\,|5^+\rangle^4}{s_{123}s_{456}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\,, \tag{121}$$

$$c_B = \frac{\langle 1\,2\rangle^3 \langle 4^+|\,(6+7)\,|5^+\rangle^4}{s_{123}s_{567}\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}\,, \tag{122}$$

$$\begin{aligned} c_C = & -\frac{\langle 5^-|\,(3+4)(6+7)\,|1^+\rangle^4}{s_{345}s_{671}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 2^+|\,(7+1)\,|6^+\rangle} \\ & \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle}\,, \end{aligned} \tag{123}$$

$$\begin{aligned} c_D = & \frac{\langle 1^-|\,(6+7)(3+4)\,|2^+\rangle^4}{s_{234}s_{671}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 5^+|\,(6+7)\,|1^+\rangle} \\ & \times \frac{1}{\langle 4^-|\,(2+3)(7+1)\,|6^+\rangle}\,, \end{aligned} \tag{124}$$

$$\begin{aligned} c_{136} = & \frac{\left(\langle 7^+|\,(3+4)\,|2^+\rangle\,\langle 5\,4\rangle + \langle 7^+|\,6\,|5^+\rangle\,\langle 2\,4\rangle\right)^4}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7\,1]\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle} \\ & \times \frac{1}{\langle 6^-|\,(7+1)(2+3)\,|4^+\rangle\,\langle 2^-|\,(7+1)(5+6)\,|4^+\rangle}\,, \end{aligned} \tag{125}$$

$$c_{147} = \frac{\langle 2\,5\rangle^4 [6\,7]^3}{s_{671}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[7\,1]\,\langle 1^+|\,(6+7)\,|5^+\rangle\,\langle 6^+|\,(7+1)\,|2^+\rangle}\,, \tag{126}$$

$$c_{236} = \frac{\langle 3^+|\,(1+2)\,|5^+\rangle^4}{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle}\,, \tag{127}$$

$$\begin{aligned} c_{247} = & \frac{\langle 5^-|\,(3+4)(6+7)\,|5^+\rangle^4}{[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 1^+|\,(6+7)\,|5^+\rangle} \\ & \times \frac{1}{\langle 5^-|\,(6+7)(1+2)\,|3^+\rangle\,\langle 5^-|\,(3+4)(1+2)\,|7^+\rangle}\,, \end{aligned} \tag{128}$$

$$c_{256} = \frac{\langle 1\,2\rangle^3 [4\,6]^4}{s_{456}\,\langle 2\,3\rangle\,[4\,5]\,[5\,6]\,\langle 7\,1\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle}\,, \tag{129}$$

$$\begin{aligned} c_{257} = & \frac{\langle 1\,2\rangle^3 \langle 3\,5\rangle^4 [6\,7]^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle} \\ & \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}\,, \end{aligned} \tag{130}$$

$$\begin{aligned} c_{357} = & \frac{\langle 1\,2\rangle^3 \langle 4^+|\,(6+7)\,|1^+\rangle^4}{\langle 2\,3\rangle\,[4\,5]\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle\,\langle 5^+|\,(6+7)\,|1^+\rangle} \\ & \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\,, \end{aligned} \tag{131}$$

$$c_{367} \;=\; \frac{\langle 1\,2\rangle^3 [6\,7]^3}{s_{567}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,\langle 5^+|\,(6+7)\,|1^+\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle}\;. \tag{132}$$

Note that c_D and c_{256} are flip-symmetric.

The quantities (121)–(132) obey the linear relations (plus their flips),

$$\begin{aligned}
c_A + c_{357} + c_{367} \;&=\; c_B + c_{256} + c_{257}, && (133)\\
c_A + c_C - c_{147} - c_{236} + c_{357} + c_{367} \;&=\; [c_A + c_C - c_{147} - c_{236} + c_{357} + c_{367}]\big|_{\rm flip}\,, && (134)\\
c_C + c_{257} \;&=\; [c_B - c_{236}]\big|_{\rm flip} + c_{147} + c_{247}, && (135)\\
c_D + c_{136} + c_{367} \;&=\; [c_C + c_{357}]\big|_{\rm flip} + c_{147}, && (136)\\
c_A - c_{136} - c_{236} \;&=\; [c_A - c_{136} - c_{236}]\big|_{\rm flip}\,. && (137)
\end{aligned}$$

The tree amplitude can be written in a manifestly flip-symmetric form as

$$A_7^{\rm tree} = c_{247} + c_{256} + c_B + c_{147} + [c_B + c_{147}]\big|_{\rm flip}\,. \tag{138}$$

The remaining independent box coefficients are given by

$$\begin{aligned}
c_{123} \;&=\; [c_A]\big|_{\rm flip} + c_{136} + c_{236} + c_{256}, && (139)\\
c_{127} \;&=\; c_C + c_{257} + [c_{236} + c_{367}]\big|_{\rm flip}\,, && (140)\\
c_{134} \;&=\; c_D + c_{147}, && (141)\\
c_{157} \;&=\; c_C + [c_{367}]\big|_{\rm flip}\,, && (142)\\
c_{235} \;&=\; c_A + c_{256}, && (143)\\
c_{237} \;&=\; c_B + c_{367}, && (144)\\
c_{267} \;&=\; c_B + c_{236}, && (145)\\
c_{345} \;&=\; c_C + c_{147} + [c_{147}]\big|_{\rm flip} + c_{357}, && (146)\\
c_{356} \;&=\; c_A + c_{236}, && (147)\\
c_{457} \;&=\; c_C + c_{147}, && (148)\\
c_{567} \;&=\; c_A + c_{236} + c_{357} + c_{367}. && (149)
\end{aligned}$$

The final 16 box coefficients are obtained by the flip symmetry via eq. (70), where

$$\tilde\imath = (4 - i) \bmod 7, \tag{150}$$

and again $\tilde\imath\tilde\jmath\tilde k$ should be written in ascending order.

D. $A_{7;1}^{\mathcal{N}=4}(1^-,2^+,3^-,4^+,5^-,6^+,7^+)$

For this partial amplitude we define the "flip" operation,

$$X|_{\text{flip}} = -X(1\leftrightarrow 5, 2\leftrightarrow 4, 6\leftrightarrow 7), \tag{151}$$

which is a symmetry of the tree amplitude,

$$A_7^{\text{tree}}(1^-,2^+,3^-,4^+,5^-,6^+,7^+)\Big|_{\text{flip}} = A_7^{\text{tree}}(1^-,2^+,3^-,4^+,5^-,6^+,7^+). \tag{152}$$

and of the one-loop amplitude. The box coefficients c_{ijk} are given in terms of the following quantities:

$$c_A = \frac{\langle 1\,3\rangle^4 \langle 7^+|\,(4+6)\,|5^+\rangle^4}{s_{123}s_{456}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}, \tag{153}$$

$$c_B = \frac{\langle 1\,3\rangle^4 \langle 4^+|\,(6+7)\,|5^+\rangle^4}{s_{123}s_{567}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}, \tag{154}$$

$$c_C = \frac{\langle 1^-|\,(6+7)(2+4)\,|3^+\rangle^4}{s_{234}s_{671}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 5^+|\,(3+4)\,|2^+\rangle\,\langle 5^+|\,(6+7)\,|1^+\rangle\,\langle 4^-|\,(2+3)(7+1)\,|6^+\rangle}, \tag{155}$$

$$c_D = \frac{-\langle 5^-|\,(4+6)(7+2)\,|1^+\rangle^4}{s_{456}s_{712}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 7\,1\rangle\,\langle 3^+|\,(4+5)\,|6^+\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle\,\langle 4^-|\,(5+6)(7+1)\,|2^+\rangle}, \tag{156}$$

$$\begin{aligned} c_{136} &= \frac{(\langle 7^+|\,(2+4)\,|3^+\rangle\,\langle 5\,4\rangle + \langle 7^+|\,6\,|5^+\rangle\,\langle 3\,4\rangle)^4}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7\,1]\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle} \\ &\quad \times \frac{1}{\langle 4^-|\,(5+6)(7+1)\,|2^+\rangle\,\langle 4^-|\,(2+3)(7+1)\,|6^+\rangle}\,, \end{aligned} \tag{157}$$

$$c_{236} = \frac{\langle 2^+|\,(1+3)\,|5^+\rangle^4}{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle}\,, \tag{158}$$

$$\begin{aligned} c_{246} &= \frac{(\langle 4^+|\,(2+7)\,|1^+\rangle\,\langle 5\,7\rangle + \langle 4^+|\,6\,|5^+\rangle\,\langle 1\,7\rangle)^4}{\langle 1\,2\rangle\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 3^+|\,(1+2)\,|7^+\rangle} \\ &\quad \times \frac{1}{\langle 7^-|\,(1+2)(3+4)\,|5^+\rangle\,\langle 7^-|\,(5+6)(3+4)\,|2^+\rangle}\,, \end{aligned} \tag{159}$$

$$c_{256} = \frac{\langle 1\,3\rangle^4 [4\,6]^4}{s_{456}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[4\,5]\,[5\,6]\,\langle 7\,1\rangle\,\langle 4^+|\,(5+6)\,|7^+\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle}\,, \tag{160}$$

$$\begin{aligned} c_{257} &= \frac{\langle 1\,3\rangle^4 \langle 3\,5\rangle^4 [6\,7]^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6^+|\,(4+5)\,|3^+\rangle\,\langle 7^+|\,(1+2)\,|3^+\rangle} \\ &\quad \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle\,\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle}\,, \end{aligned} \tag{161}$$

$$c_{347} = \frac{[2\,4]^4 \langle 1\,5\rangle^4}{s_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2^+|\,(3+4)\,|5^+\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle}\,, \tag{162}$$

$$c_{357} = \frac{\langle 1\,3\rangle^4 \langle 4^+|\,(6+7)\,|1^+\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[4\,5]\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 4^+|\,(2+3)\,|1^+\rangle\,\langle 5^+|\,(6+7)\,|1^+\rangle} \times \frac{1}{\langle 3^-|\,(4+5)(6+7)\,|1^+\rangle\,\langle 6^-|\,(4+5)(2+3)\,|1^+\rangle}\,, \tag{163}$$

$$c_{367} = \frac{\langle 1\,3\rangle^4 [6\,7]^3}{s_{567}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,\langle 5^+|\,(6+7)\,|1^+\rangle\,\langle 7^+|\,(5+6)\,|4^+\rangle}\,. \tag{164}$$

Note that c_D, c_{257} and c_{347} are flip symmetric.

The linear relations obeyed by the quantities (153)–(164) are

$$c_A + c_{357} + c_{367} = c_B + c_{256} + c_{257}, \tag{165}$$

$$c_C + c_D + c_{136} + c_{367} = c_B - c_{236} + c_{256} + [c_B - c_{236} + c_{256}]|_{\text{flip}} + c_{257} + c_{347}, \tag{166}$$

$$c_C + c_{136} + c_{367} = [c_C + c_{136} + c_{367}]|_{\text{flip}}\,, \tag{167}$$

$$c_B - c_{236} - c_{246} + c_{347} = [c_C - c_{357}]|_{\text{flip}}\,. \tag{168}$$

plus the equations related by the flip symmetry. The simplest, manifestly flip-symmetric representation we have been able to find for the tree amplitude is

$$A_7^{\text{tree}} = c_{257} + c_{347} + c_B + c_{256} + [c_B + c_{256}]|_{\text{flip}}\,. \tag{169}$$

The remaining independent box coefficients are given by

$$c_{123} = c_D + c_{136} + c_{236} + c_{256}, \tag{170}$$

$$c_{126} = c_D + c_{256}, \tag{171}$$

$$c_{134} = c_C + [c_{367}]|_{\text{flip}}\,, \tag{172}$$

$$c_{137} = c_C + c_{347}, \tag{173}$$

$$c_{167} = c_C + c_{136} + c_{347} + c_{367}, \tag{174}$$

$$c_{234} = c_B + c_{347} + c_{367} + [c_{357}]|_{\text{flip}}\,, \tag{175}$$

$$c_{235} = c_A + c_{256}, \tag{176}$$

$$c_{237} = c_B + c_{367}, \tag{177}$$

$$c_{267} = c_B + c_{236}, \tag{178}$$

$$c_{356} = c_A + c_{236}, \tag{179}$$

$$c_{567} = c_A + c_{236} + c_{357} + c_{367}. \tag{180}$$

The final 16 box coefficients are obtained by the flip symmetry (70), where

$$\tilde{\imath} = (7 - i) \bmod 7, \tag{181}$$

and again $\tilde{\imath}\tilde{\jmath}\tilde{k}$ should be written in ascending order.

V. CONSISTENCY OF THE RESULTS

In this section we describe various consistency checks that we performed on the amplitudes. Amplitudes, in general, must satisfy a stringent set of constraints on their analytic properties. In particular, all kinematic poles in the amplitudes must correspond to the physical propagation of particles. Such constraints are so restrictive that they can be used, for example, to construct ansätze for infinite sequences of all-plus-helicity amplitudes in gauge and gravity theories [41, 50]. The poles may be divided into three categories: collinear, multi-particle and spurious. We consider each of these categories in turn.

A. Collinear Behavior

An important constraint on the amplitudes arises from the region in phase space where the momenta of two legs a and b become collinear. In the collinear region, $k_a \to zk_P$, $k_b \to (1-z)k_P$, where k_P is the momentum of the quasi-on-shell intermediate state P, with helicity λ. In this limit, massless color-ordered tree amplitudes behave as

$$A_n^{\rm tree} \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm} \Big(\mathrm{Split}^{\rm tree}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})\, A_{n-1}^{\rm tree}(\ldots (a+b)^\lambda \ldots) \,, \tag{182}$$

where $\mathrm{Split}^{\rm tree}_{-\lambda}$ are tree-level splitting amplitudes [18]. At one-loop, the generalization is,

$$\begin{aligned} A_n^{\text{1-loop}} \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm} \Big(& \mathrm{Split}^{\rm tree}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})\, A_{n-1}^{\text{1-loop}}(\ldots (a+b)^\lambda \ldots) \\ & + \mathrm{Split}^{\text{1-loop}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})\, A_{n-1}^{\rm tree}(\ldots (a+b)^\lambda \ldots) \Big) \,, \end{aligned} \tag{183}$$

where the $\mathrm{Split}^{\text{1-loop}}_{-\lambda}$ are one-loop splitting amplitudes, which are tabulated in the second appendix of ref. [9]. This reference also contains a discussion of the behavior of the collinear limits of one-loop amplitudes and integral functions.

We have verified numerically that all helicity amplitudes presented in section IV have the proper collinear behavior in all channels, as dictated by eq. (183). This, by itself, provides a rather powerful check on their form.

B. Multi-Particle Factorization

A related constraint is that of multi-particle factorization. This factorization plays a central role in the tree-level CSW construction. Presumably, it will also play an important role in extending the CSW construction to loop level. These properties are not as well-discussed in the literature as the collinear properties, so we present the key points here, and include an example in appendix B.

At tree level, multi-particle factorization is reasonably straightforward. For $(k_i + k_{i+1} + \cdots + k_{i+r-1})^2 \equiv K^2 \to 0$ (with $r > 2$), the amplitude behaves as

$$A_n^{\rm tree} \stackrel{K^2\to 0}{\longrightarrow} \sum_{\lambda=\pm} A_{r+1}^{\rm tree}(k_i, \ldots, k_{i+r-1}, K^\lambda)\, \frac{i}{K^2}\, A_{n-r+1}^{\rm tree}((-K)^{-\lambda}, k_{i+r}, \ldots, k_{i-1})\,, \qquad (184)$$

where λ is the helicity of the intermediate state with momentum K.

At loop level, in infrared-divergent gauge theories, multi-particle factorization is more subtle. For such theories loop amplitudes do not factorize in any naive sense, due to the emission of soft gluons which induce non-trivial logarithmic corrections to factorization formulæ. These logarithmic corrections do however obey universal formulæ [51] because of their close relation to the universal infrared divergences [49]. More explicitly, for $K^2 \to 0$ the factorization properties for one-loop amplitudes are described by [51],

$$\begin{aligned} A_{n;1}^{\text{1-loop}} \stackrel{K^2\to 0}{\longrightarrow} \sum_{\lambda=\pm} \Bigg[& A_{r+1;1}^{\text{1-loop}}(k_i, \ldots, k_{i+r-1}, K^\lambda)\, \frac{i}{K^2}\, A_{n-r+1}^{\rm tree}((-K)^{-\lambda}, k_{i+r}, \ldots, k_{i-1}) \\ & + A_{r+1}^{\rm tree}(k_i, \ldots, k_{i+r-1}, K^\lambda)\, \frac{i}{K^2}\, A_{n-r+1;1}^{\text{1-loop}}((-K)^{-\lambda}, k_{i+r}, \ldots, k_{i-1}) \\ & + A_{r+1}^{\rm tree}(k_i, \ldots, k_{i+r-1}, K^\lambda)\, \frac{i}{K^2}\, A_{n-r+1}^{\rm tree}((-K)^{-\lambda}, k_{i+r}, \ldots, k_{i-1})\, c_\Gamma\, \mathcal{F}_n(K^2; k_1, \ldots, k_n) \Bigg]\,, \end{aligned} \qquad (185)$$

where the one-loop *factorization function* $\mathcal{F}_n$ is independent of helicities. This formula is similar to the one for an amplitude which factorizes naively, depicted in fig. 3, except that $\mathcal{F}_n$ contains kinematic invariants with momenta from both sides of the propagator carrying momentum K. For example $\ln(-s_{i-1,i})$ is one such logarithm: k_{i-1} is a momentum belonging to one of the factorized amplitudes on the right-hand-side of eq. (185) and k_i to to the other amplitude.

In general, the factorization function is composed of 'factorizing' and 'non-factorizing' components,

$$\mathcal{F}_n = \mathcal{F}_n^{\rm fact} + \mathcal{F}_n^{\text{non-fact}}\,. \qquad (186)$$

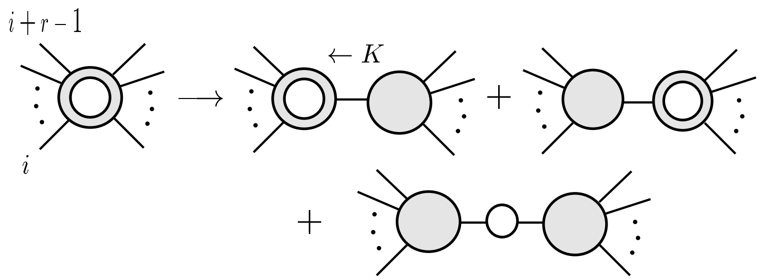

FIG. 3: A schematic depiction of multi-particle factorization at one loop.

The factorizing contributions are easily obtained by computing bubble Feynman diagrams. For the case of $\mathcal{N}=4$ supersymmetric gauge theory the factorizing contributions vanish, leaving only the 'non-factorizing' contributions.

These 'non-factorizing' contributions are linked to the infrared divergences, as shown in ref. [51]. A constructive proof of the universality of these non-factorizing contributions was also given in that reference, as well as explicit formulæ for determining their values in any theory. All non-factorizing contributions are determined from the infrared divergences present in a given process. The infrared divergences in the $\mathcal{N}=4$ theory at one loop are

$$A_{n;1}^{\mathcal{N}=4}(1,2,\ldots,n)\Big|_{\epsilon\ \text{pole}} = -\frac{c_\Gamma}{\epsilon^2}\sum_{j=1}^{n}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon} A_n^{\text{tree}}(1,2,\ldots,n)\,. \tag{187}$$

From Table 2 or ref. [51], the appearance of the singularities

$$\left[-\frac{c_\Gamma}{\epsilon^2}\left(\frac{\mu^2}{-s_{i-1,i}}\right)^{\epsilon} - \frac{c_\Gamma}{\epsilon^2}\left(\frac{\mu^2}{-s_{i+r-1,i+r}}\right)^{\epsilon}\right] A_n^{\text{tree}} \tag{188}$$

implies that in the factorization channel $K^2 = s_{i\ldots(i+r-1)} \to 0$, the factorization function is

$$\begin{aligned}\mathcal{F}_n^{\mathcal{N}=4}(K^2;k_1,\ldots,k_n) &= 2(\mu^2)^{\epsilon}\Big(F^{2\text{m}\,h}_{n:r-1;i+1} + F^{2\text{m}\,h}_{n:n-r-1;i+r+1} + \frac{2}{\epsilon^2}(-s_{i\ldots(i+r-1)})^{-\epsilon}\Big)\\ &= 2(\mu^2)^{\epsilon}\Big(F^{2\text{m}\,h}(s_{i-1,i}, s_{i\ldots(i+r-1)}, s_{(i+1)\ldots(i+r-1)}, s_{(i+r)\ldots(i-2)})\\ &\qquad + F^{2\text{m}\,h}(s_{i+r-1,i+r}, s_{i\ldots(i+r-1)}, s_{(i+r+1)\ldots(i-1)}, s_{i\ldots(i+r-2)})\\ &\qquad + \frac{2}{\epsilon^2}(-s_{i\ldots(i+r-1)})^{-\epsilon}\Big)\,,\end{aligned} \tag{189}$$

where we use the notation from ref. [51] on the first line. On the subsequent lines we use the notation in appendix A for the integral functions.

We have checked that all amplitudes in section IV satisfy the correct multi-particle factorizations dictated by eq. (185), with the factorization function (189). In appendix B, we present a sample evaluation of a multi-particle factorization limit. All the remaining multi-particle factorization limits, for any of the seven-point helicity amplitudes given in section IV, are similar.

C. Cancellation of spurious singularities

In addition to the collinear and multi-particle poles discussed in the previous two subsections, the coefficients also contain ‘planar’ and ‘cubic’ singularities as defined in section III. Unlike the collinear and multi-particle singularities, these cannot be singularities of the whole amplitude. Factorization (even in massless gauge theories) does not allow them. We therefore expect them to cancel between different integral functions. Their origin in the integral reductions also suggests this.

This is indeed what one finds. For example, in $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$, there are eight coefficients that contain a denominator factor of $\langle 1^+|\,(2+3)\,|4^+\rangle$. As noted in section III, these coefficients — c_{136}, c_{167}, c_{236}, c_{267}, c_{346}, c_{356}, c_{467}, and c_{567} — will therefore be separately singular when $k_2+k_3=a_1k_1+a_4k_4$. One can nonetheless verify that these singularities cancel in the amplitude as a whole. Indeed, the cancellation occurs separately within two linear combinations of terms in the amplitude,

$$\begin{aligned} & c_{136}B(1,3,6)+c_{236}B(2,3,6)+c_{346}B(3,4,6)+c_{356}B(3,5,6), \\ \text{and}\qquad & c_{167}B(1,6,7)+c_{267}B(2,6,7)+c_{467}B(4,6,7)+c_{567}B(5,6,7). \end{aligned} \tag{190}$$

While a subset of terms in the whole amplitude will allow the cancellation of any given singularity — typically four terms for the planar singularities, and five for the cubic ones — there does not appear to be any subset of terms in which *all* spurious singularities cancel.

We have checked numerically that all spurious singularities do indeed cancel in the entire amplitude, for each of the four NMHV helicity configurations. This cancellation is a strong check on the amplitude, precisely because it involves the analytic behavior of the integral functions in a non-trivial way.

VI. AN ALL-MULTIPLICITY COEFFICIENT

The computation of amplitudes beyond a fixed number of external legs can provide additional information and clues to general structure. In particular, as we shall discuss in section VII, a discussion of the twistor-space structure of the seven-point amplitude requires an accounting for the 'holomorphic anomaly' pointed out by Cachazo, Svrček, and Witten [37]. Some aspects of the structure of an all-n amplitude also require such an accounting; but we will be able to describe most aspects without reference to the 'anomaly'.

We limit ourselves here to confirming the coefficient presented in ref. [39]; presenting one new set of coefficients, of a class of three-mass box functions; and discussing a number of vanishing coefficients.

Consider the adjacent-minus amplitude $A_{n;1}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$. The simplest of the cuts is the cut in the s_{123} channel, represented by the product of tree amplitudes,

$$C_{123} \equiv i\, A_5^{\rm tree}((-\ell_1)^+, 1^-, 2^-, 3^-, \ell_4^+)\, A_5^{\rm tree}((-\ell_4)^-, 4^+, 5^+, \ldots n^+, \ell_1^-), \qquad (191)$$

integrated over phase space. This cut is particularly simple because only MHV amplitudes participate, and only gluons can cross the cut (so the result is the same in $\mathcal{N}=4$ or pure $\mathcal{N}=4$ super-Yang-Mills theory, or QCD). Inserting eq. (7) twice, the cut becomes

$$C_{123} = \frac{i(s_{123})^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle \cdots \langle (n-1)\,n\rangle}\, \frac{1}{[\ell_1\,1]\,[3\,\ell_4]\,\langle \ell_4\,4\rangle\,\langle n\,\ell_1\rangle}\,. \qquad (192)$$

We denote the running loop momenta by ℓ_i; that is, $\ell_n = \ell_1 + k_n$, ℓ_1, $\ell_2 = \ell_1 - k_1$, $\ell_3 = \ell_1 - k_1 - k_2$, $\ell_4 = \ell_1 - k_1 - k_2 - k_3$. To obtain eq. (191) we have used $\langle \ell_1\,\ell_4\rangle\,[\ell_1\,\ell_4] = (\ell_1 - \ell_4)^2 = s_{123}$. We can also clear out the loop-momentum-dependent spinor-products from the denominator, in favor of standard propagators,

$$C_{123} = \frac{i(s_{123})^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle \cdots \langle (n-1)\,n\rangle}\, \frac{\langle 3^-|\, \not{\ell}_4 |4^-\rangle \langle 1^-|\, \not{\ell}_1 |n^-\rangle}{\ell_n^2 \ell_2^2 \ell_3^2 \ell_5^2}\,. \qquad (193)$$

It turns out that the algebraic steps for integrating the cut are essentially identical to those needed in the six-point case [10]. One can actually do a bit better, and avoid any integration, by multiplying and dividing by the spinor strings,

$$\left\langle n^- \right| (1+2) \left| 3^- \right\rangle \left\langle 4^- \right| (2+3) \left| 1^- \right\rangle\,, \qquad (194)$$

whose appearance is motivated by the form (20) of the denominators encountered in reducing non-adjacent two-mass pentagon integrals. One obtains,

$$\begin{aligned} C_{123} \;=\; & \frac{i(s_{123})^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle \cdots \langle (n-1)\,n\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 3^+|\,(1+2)\,|n^+\rangle} \\ & \times \frac{{\rm tr}\left[\frac{1}{2}(1+\gamma_5)(1+2)3\,\not{\ell}_4 4(2+3)1\,\not{\ell}_1 n\right]}{\ell_n^2 \ell_2^2 \ell_3^2 \ell_5^2}\,. \end{aligned} \qquad (195)$$

The trace of Dirac γ matrices, with a chiral projection inserted, is easily evaluated. The γ_5 terms are proportional to the Levi-Civita tensor and integrate to zero. One can algebraically reduce the non-Levi-Civita terms to a combination of four cut box integrals,

$$\begin{aligned} A^{\text{1-loop}}(1^-,2^-,3^-,4^+,\ldots,n^+)\Big|_{123\text{ cut}} \\ = ic_\Gamma \frac{s_{123}^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\ldots\langle (n-1)\,n\rangle\,\langle 1^+|\,(2+3)\,|4^+\rangle\,\langle 3^+|\,(1+2)\,|n^+\rangle} \\ \times\Big(F^{1\text{m}}(s_{12},s_{23},s_{123}) + F^{2\text{m}\,h}(s_{n1},s_{123},s_{23},s_{4\ldots(n-1)}) \\ + F^{2\text{m}\,h}(s_{34},s_{123},s_{5\ldots n},s_{12}) \\ + F^{2\text{m}\,e}(s_{n\ldots 3},s_{1\ldots 4},s_{123},s_{5\ldots(n-1)})\Big)\,, \end{aligned} \tag{196}$$

where the integral functions F are given in appendix A. This result reduces to eqs. (43) and (50) for $n=7$. It also confirms Cachazo's [39] very elegant evaluation of this coefficient, obtained by exploiting the holomorphic anomaly [37].

Now consider the coefficient of a class of three-mass box functions in the same amplitude. In this case NMHV tree amplitudes appear in the cuts. We used the simplified form the NMHV amplitudes appropriate for use in the cuts obtained in ref. [30]. The cut starts out with the structure of a tensor hexagon integral (independent of n), and can be cleanly reduced to a sum of scalar box integrals. Here we present the coefficient of the set of three-mass box integrals,

$$F^{3\text{m}}(s_{2\ldots(c_1-1)},s_{(c_2+1)\ldots 2},s_{3\ldots(c_1-1)},s_{c_1\ldots c_2},s_{c_2+1\ldots 1}) \tag{197}$$

shown in fig. 4. This coefficient can be written as a single term,

$$\begin{aligned} ic_\Gamma \frac{(\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{c_1\ldots c_2})^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle} \times \frac{\langle (c_1-1)\,c_1\rangle\,\langle c_2\,(c_2+1)\rangle}{s_{c_1\ldots c_2}\,\langle (c_1-1)^-|\,\not{K}_{c_1\ldots c_2}\not{K}_{(c_2+1)\ldots 1}\,|2^+\rangle\,\langle c_1{}^-|\,\not{K}_{c_1\ldots c_2}\not{K}_{(c_2+1)\ldots 1}\,|2^+\rangle} \\ \times \frac{1}{\langle c_2{}^-|\,\not{K}_{c_1\ldots c_2}\not{K}_{3\ldots(c_1-1)}\,|2^+\rangle\,\langle (c_2+1)^-|\,\not{K}_{c_1\ldots c_2}\not{K}_{3\ldots(c_1-1)}\,|2^+\rangle}\,. \end{aligned} \tag{198}$$

We have checked that this set of coefficients correctly reproduces the appropriate box coefficients for $n=6$ and 7.

We defer a more detailed discussion of this computation, as well as results for other box coefficients, to a future publication. While the all-n NMHV loop amplitude contains everything from one-mass to three-mass boxes (four-mass boxes are absent), a generic term

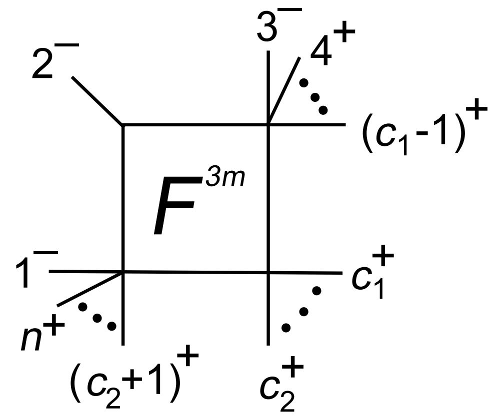

FIG. 4: The class of three-mass box functions whose coefficient is given in eq. (198).

contains either a three-mass box, or an easy two-mass box. The latter coefficients are more complicated in structure, while the former are simpler.

Using the triple cut shown in fig. 1, it is easy to see that *the coefficient of any box integral which contains two adjacent clusters of legs, each having only positive-helicity gluons, must vanish.* This statement is true for any amplitude in any cut-constructible theory. The reasoning is as follows: Consider a triple cut where each of the two positive-helicity clusters is identified with the set of external legs emerging from a blob in fig. 1. Then the vanishing of the tree amplitudes $A_m^{\rm tree}(\pm{+}{+}\cdots{+})$ implies that the particle crossing the cut line between these two blobs must have $-$ helicity on both sides of the cut, which is not a valid helicity assignment. A corollary is that *any box integral with one all-plus cluster adjacent to two consecutive massless plus-helicity legs must have a vanishing coefficient.* A simple consequence of this result for the seven-point amplitude $({-}{-}{-}{+}{+}{+}{+})$ is that $c_{235} = c_{237} = c_{257} = c_{357} = 0$, as given in eq. (49). Clearly a large number of box coefficients for the all-n amplitude $({-}{-}{-}{+}{+}\cdots{+}{+})$ vanish by these rules.

VII. TWISTOR-SPACE PROPERTIES

The target space for Witten's candidate topological string theory is $\mathbb{CP}^{3|4}$, otherwise called projective (super-)twistor space. Points in twistor space correspond to null momenta or equivalently to light cones in space-time. The correspondence is specified by a 'half-Fourier' transform. More precisely, if we represent a null momentum by the tensor product of a spinor λ^a and a conjugate spinor $\tilde\lambda^{\dot a}$, then twistor quantities are obtained by Fouriertransforming with respect to all the $\tilde{\lambda}^{\dot{a}}$.

Amplitudes in twistor space, as it turns out, have rather simple properties. At tree level, they are non-vanishing only on certain curves. This implies that they contain factors of delta functions (or derivatives thereof) whose arguments are the characteristic equations for the curves. The coefficients of the delta functions, however, have been quite difficult to calculate directly.

As Witten pointed out in his original paper [1], however, we do not need the twistor-space amplitudes in order to establish the structure of the delta functions they contain. In momentum space, the Fourier transform turns the polynomials into differential operators (polynomial in the λ_i, and derivatives with respect to the $\tilde{\lambda}_i$), which will annihilate the amplitude. One particularly useful building block for these differential operators is the line annihilation operator, expressing the condition that three points in twistor space lie on a common 'line' or $\mathbb{CP}^1$. If the coordinates of the three points, labeled i, j, k, are $Z_i^I = (\lambda_i^a, \mu_i^{\dot{a}})$, *etc.*, then the appropriate condition is

$$\epsilon_{IJKL} Z_i^I Z_j^J Z_k^K = 0 \tag{199}$$

for all choices of L. Choosing $L = \dot{a}$, and translating this equation back to momentum space using the identification $\mu^{\dot{a}} \leftrightarrow -i\partial/\partial\tilde{\lambda}_{\dot{a}}$, we obtain the operator,

$$F_{ijk} = \langle i\, j\rangle \frac{\partial}{\partial\tilde{\lambda}_k} + \langle j\, k\rangle \frac{\partial}{\partial\tilde{\lambda}_i} + \langle k\, i\rangle \frac{\partial}{\partial\tilde{\lambda}_j}. \tag{200}$$

Two important sufficient conditions for F_{ijk} to annihilate an expression, *i.e.* for it to have support only when i, j, k lie on a line in twistor space, are [1]

1. The expression is completely independent of $\tilde{\lambda}_i$, $\tilde{\lambda}_j$, and $\tilde{\lambda}_k$, or

2. $\tilde{\lambda}_i$, $\tilde{\lambda}_j$, $\tilde{\lambda}_k$ appear only via a sum of momenta containing them, of the form

$$P^{a\dot{a}} = (\cdots + k_i + k_j + k_k + \cdots)^{a\dot{a}} = \cdots + \lambda_i^a\tilde{\lambda}_i^{\dot{a}} + \lambda_j^a\tilde{\lambda}_j^{\dot{a}} + \lambda_k^a\tilde{\lambda}_k^{\dot{a}} + \cdots \tag{201}$$

The first condition is obvious from the definition (200); the second holds because of the Schouten identity,

$$\langle i\, j\rangle\, \lambda_k + \langle j\, k\rangle\, \lambda_i + \langle k\, i\rangle\, \lambda_j = 0. \tag{202}$$

The tree-level MHV amplitude, for example, is annihilated by F_{ijk}, because it is independent of the $\tilde{\lambda}_i$. Any possible delta functions vanish for generic momenta, because they take the form $\delta(\langle i\,j\rangle)$. At one loop, Cachazo, Svrček, and Witten [37] pointed out that such delta functions, arising from the spinor analog of the fact that $\partial_{\bar{z}}(1/z) \neq 0$, do arise. They must be taken into account for a proper analysis of the twistor-space structure of amplitudes.

We will not compute the relevant 'holomorphic anomaly' terms for the amplitudes in this paper, and so we will not be able to fully exhibit their twistor-space structure. While the 'anomaly' terms enter into the action of the differential operators on the box integrals, their action on the coefficients is unaffected by it. The properties of the coefficient are also important, and we discuss them.

In addition to the line operator F_{ijk}, we will employ the planar operator [1],

$$K_{ijkl} \equiv \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L = \langle i\,j\rangle\, \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}} \frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} \pm [5 \text{ permutations}], \tag{203}$$

whose vanishing implies that four points lie in a plane in twistor space.

In the case of the seven-point amplitude, all line operators F and planar operators K are affected by the 'anomaly'. For higher-point amplitudes, however, there are classes of unaffected operators. We will study their action on the term in the amplitude discussed in section VI. We shall show it is consistent with the simple twistor-space structure expected from a generalization of the BST calculation.

A. Properties of seven-point box coefficients c_{ijk}

Before describing the twistor-space structure of the coefficients c_{ijk} from section IV, we list some of their other (related) properties here.

- Each box coefficient can be written as the sum of a small number of 'terms', or 'one-term coefficients': namely, the auxiliary quantities c_A, c_B, c_C, c_D, c_E (as needed), plus the c_{ijk} that are given directly in section IV in terms of spinor strings.
- The sum contains at most four terms. In the case of $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$, it has at most three terms.
- Only the coefficients of the one-mass box integrals, $c_{i,i+1,i+2}$ (with indices mod 7), require the use of three or four terms.
- The coefficients of the hard two-mass box integrals, $c_{i,i+1,i+3}$ or $c_{i-2,i,i+1}$, require at most two terms.

- The coefficients of the easy two-mass box integrals, $c_{i,i+1,i+4}$, and of the three-mass box integrals, $c_{i,i+2,i+4}$, are all one-term expressions.

- Each term resembles a product of off-shell MHV vertices [23], in terms of the types of denominators that occur: multi-particle invariants $s_{i,i+1,i+2}$, but not full two-particle poles $s_{i,i+1}$; spinor products continued 'off-shell', like $\langle a\, B^* \rangle = \langle a^- |\, (b+c)\, | d^- \rangle$, where $k_B = k_b + k_c$, and d temporarily plays the role of the arbitrary reference spinor in the CSW formalism (see also ref. [30]). The choice of d varies from term to term, however. The longer strings of the form $\langle a^- |\, (b+c)(d+e)\, | f^+ \rangle$ could correspond to continuing both spinors off-shell.

- The resemblance extends to the numerator factors, which always appear raised to the third or fourth power, like the factor of $\langle i\, j \rangle^4$ in the numerator of the Parke-Taylor amplitudes [7]. Probably they should always be considered raised to the fourth power, because whenever such a factor appears raised to the third power, it always seems to be of the type that 'could have appeared' in the denominator, such as $\langle i,\, i+1 \rangle$, $[i,\, i+1]$, $s_{i,i+1,i+2}$, or even the more complicated spinor strings which are 'square roots' of the integral reduction factors discussed in section III. Other numerator factors which cannot occur in denominators, such as $\langle i,\, i+2 \rangle$, or $\langle a^- |\, (b+c)\, | d^- \rangle$ where a, b, c, d are *not* cyclicly consecutive, always appear raised to the fourth power, if they appear in the numerator at all.

- Perhaps the most interesting of the one-term coefficients are those exemplified by c_{135} in $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$. Here a new structure appears in the numerator, unlike any of the denominator factors. The unexpected form made this coefficient more difficult to guess. In the end, we deduced it from its collinear behavior in particular channels. Using the Schouten identity and momentum conservation, one can show that it has the following collinear limits,

$$\left\langle 3^+ \right| (6+7) \left| 1^+ \right\rangle \langle 4\, 6 \rangle + \left\langle 3^+ \right| 5 \left| 4^+ \right\rangle \langle 1\, 6 \rangle \;\to\; + \left\langle 3^+ \right| (6+7) \left| 1^+ \right\rangle \langle 4\, 6 \rangle \quad \text{for } 4||5, \tag{204}$$

$$\to\; - \left\langle 3^+ \right| (2+4) \left| 1^+ \right\rangle \langle 4\, 6 \rangle \quad \text{for } 5||6, \tag{205}$$

$$\to\; - \left\langle 3^+ \right| (1+2) \left| 4^+ \right\rangle \langle 1\, 6 \rangle \quad \text{for } 6||7, \tag{206}$$

$$\to\; + \left\langle 3^+ \right| (5+6) \left| 4^+ \right\rangle \langle 1\, 6 \rangle \quad \text{for } 7||1. \tag{207}$$

The most general twistor-space property of the box coefficients is that *all points lie in a plane*. That is, K_{mnpq} for every choice of m, n, p, q annihilates every one-term coefficient, and hence, by linearity, it annihilates every box coefficient c_{ijk}. This property is not so easy to see analytically, but it is straightforward to verify numerically by evaluating the expressions $K_{mnpq}\, c_{ijk}$ at random points in 7-particle phase space.

On the other hand, the way the 7 coplanar points get distributed into lines varies from term to term, yet it does so in a very systematic way. Also, the correct behavior can be determined by inspection. From the amplitude $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$, consider, for example:

- the easy two-mass box coefficient c_{125} given in eq. (77). Here legs 3,4,5,6 only appear as λ_i spinors, not $\tilde{\lambda}_i$ spinors. So legs 3,4,5,6 must lie on a line for c_{125}. On the other hand, legs 7,1,2 appear sometimes as $\tilde{\lambda}_i$ spinors, as in $[1\,2]$ and $[7\,1]$, so they should be placed at generic points in the plane, as shown in fig. 5a.

- the hard two-mass box coefficient c_{237} given in eq. (84). Here legs 1,2,3 either appear as λ_i spinors, or through the momentum sum $k_1 + k_2 + k_3$. The latter appearance is easy to see in $s_{123} = (k_1 + k_2 + k_3)^2$, but it is also true for $\langle 4^+|\,(2+3)\,|1^+\rangle = \langle 4^+|\,\gamma^\mu\,|1^+\rangle\,(k_1 + k_2 + k_3)^\mu$, since $\not{k}_1|1^+\rangle = 0$; and similarly for $\langle 3^-|\,(1+2)(6+7)\,|5^+\rangle$. Thus legs 1,2,3 lie on a line. Likewise, legs 5,6,7 lie on a different line. Leg 4 appears as $\tilde{\lambda}_4$, so it sits in the plane, away from the 2 lines, as shown in fig. 5b.

- the three-mass box coefficient c_{135} given in eq. (78). From the denominator factors alone, one would conclude, as in the previous case that legs 4,5,6 lie in a line, and legs 6,7,1 lie in a line. Thus the two lines intersect at a common point, leg 6. The form of the numerator in eq. (78) makes manifest the proper 6,7,1 behavior. However, it is not apparent that it allows 4,5,6 to lie in a line until one rewrites

$$\left\langle 3^+\right|(6+7)\left|1^+\right\rangle \langle 4\,6\rangle + \left\langle 3^+\right| 5 \left|4^+\right\rangle \langle 1\,6\rangle = \left\langle 3^+\right|(5+6)\left|4^+\right\rangle \langle 1\,6\rangle + \left\langle 3^+\right| 7 \left|1^+\right\rangle \langle 4\,6\rangle\ . \tag{208}$$

Numerical investigation reveals that the pictures shown in fig. 5 are accurate, in that there are no 'hidden' twistor-space collinearities. Note also that each configuration in fig. 5 can be described as follows: *all seven points lie in a plane, distributed among three lines, and two of the lines can be chosen to intersect at one of the seven points.*

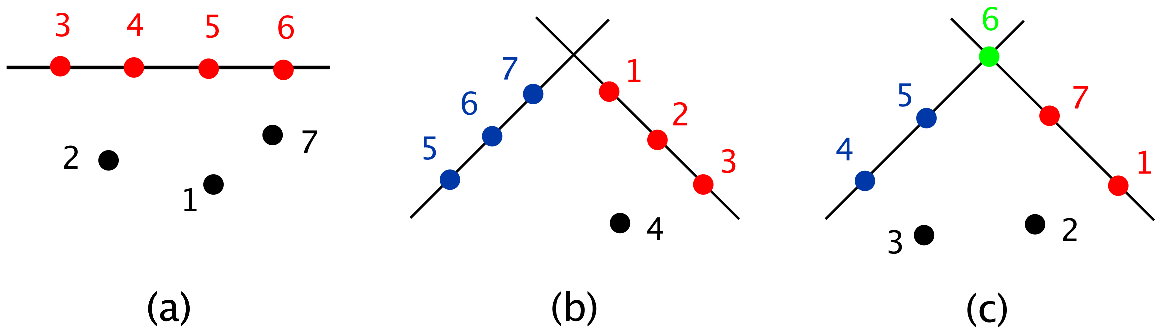

FIG. 5: Examples of twistor-space configurations for single-term box coefficients in the helicity amplitude $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^+,4^-,5^+,6^+,7^+)$. In every case, all the points lie in a plane. (a) the easy two-mass box coefficient c_{125}, (b) the hard two-mass box coefficient c_{237}, (c) the three-mass box coefficient c_{135}.

In fact, the configurations shown in fig. 5, after permuting the external leg labels around, are the complete set of twistor-space configurations for *all* the one-term coefficients for all four helicity amplitudes. Furthermore,

- The three-mass box coefficients (always one term) always have the configuration of fig. 5c. The massless leg in the corresponding integral always lies at the intersection of the two lines.

- The easy two-mass box coefficients (always one term) always have the configuration of fig. 5a. The three points lying off the line in fig. 5a are those belonging to the three-leg cluster forming one of the external masses in the integral.

- The hard two-mass box coefficients are the sums of (at most two) terms with the configurations of fig. 5a and fig. 5b. In the fig. 5a term, the three points lying off the line are those belonging to the two-leg cluster and the massless leg adjacent to it. In the fig. 5b term, the point lying off of both lines is the massless leg adjacent to the three-leg cluster.

- The one-mass box coefficients generally involve all three types of configurations in fig. 5. However, for all helicity amplitudes the full coefficients do obey $F_{712}F_{456}c_{123} = 0$ and $F_{123}F_{456}c_{123} = 0$, plus all cyclic permutations thereof.

The coefficients of the box integral functions in the six-point NMHV amplitude [10] have an analogous structure, only simpler because there are fewer points: All six points lie in

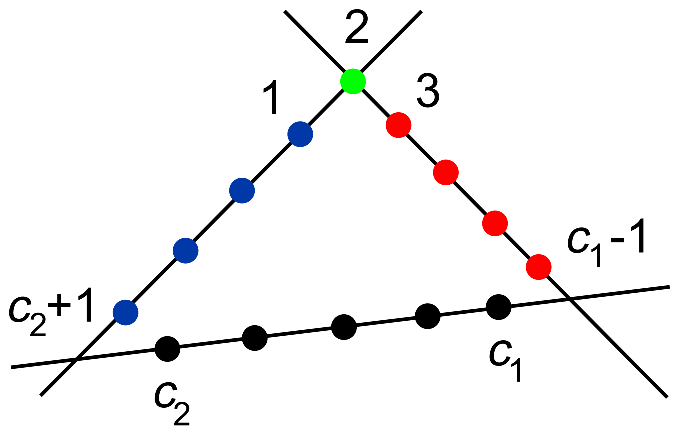

FIG. 6: The twistor-space configuration for the class of three-mass-box coefficients in the helicity amplitude $(---++\cdots++)$ discussed in the text. All points lie in a plane.

a plane in twistor space, and for each single term in a box coefficient, three of the points always lie on a line.

B. Properties of an all-n box coefficient

The result for the class of three-mass-box coefficients for the all-n NMHV amplitude $(---++\cdots++)$ is given in eq. (198). We now analyze its twistor-space structure. First observe that all points in the set $\{c_1, c_1+1, \ldots, c_2\}$ (where the indices are understood cyclicly mod n) lie on a line, because they only appear through the momentum sum $K_{c_1\ldots c_2}$ or through λ_{c_1} or λ_{c_2}. The points $\{c_2+1, c_2+2, \ldots, 1, 2\}$ also lie on a line, for a similar reason. (Note that $\not{K}_{(c_2+1)\ldots 1}|2^+\rangle = \not{K}_{(c_2+1)\ldots 2}|2^+\rangle$, and that leg 2 otherwise only appears as λ_2.) Finally, the points $\{2, 3, \ldots, c_1-1\}$ lie on a line. Numerical investigation for a number of randomly-chosen box coefficients for $n = 8, 9, 10$ confirms that all n points lie in a plane. This twistor-space structure is summarized in fig. 6.

This structure generalizes the behavior of the seven-point 'one-term' coefficients, which was illustrated in fig. 5. Again a 'one-term' expression can be described in twistor space as having all points lying on three lines within a single plane; and one of the intersections of the lines always contains one of the n points.

In the full amplitude, the coefficient (198) appears multiplied by the three-mass-box integral function $F^{3\mathrm{m}}(s_{2\ldots(c_1-1)}, s_{(c_2+1)\ldots 2}, s_{3\ldots(c_1-1)}, s_{c_1\ldots c_2}, s_{c_2+1\ldots 1})$ in eq. (197). The action of the F and K twistor-space operators on F^{3m} is affected by the holomorphic anomaly. The behavior of leg 2 in particular is affected, because of singularities in the integral when the loop momentum becomes collinear with k_2. At the level of the F operator, none of the other legs are affected by this issue, since they are safely buried into massive clusters. So, the *lines* in fig. 6, excluding the point 2 and ignoring their intersections, also describe the twistor-space collinear behavior of the box function multiplying the coefficient. (This property is also manifest from the form of the arguments of F^{3m}.) On the other hand, the coplanarity of (the finite parts of) this integral is not apparent, and perhaps doubtful, even taking into account the holomorphic anomaly.

VIII. CONCLUSIONS

In this paper we computed all the next-to-MHV one-loop seven-gluon amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory, as well as a class of contributions to an n-gluon next-to-MHV amplitude. These results provide a useful guide to the spinor-product structure and (related) twistor-space properties of higher-point $\mathcal{N} = 4$ amplitudes. The coefficients of the box integral functions appearing in the amplitudes can be written as simple terms built out of spinor strings, or sums of just a few such terms. Each term exhibits a simple twistor-space structure: all points lie on three lines confined to a single plane, and two of the lines always intersect at one of the n points.

The co-planarity of the NMHV box coefficients is a very intriguing result. This twistor-space localization is stronger than would seem to be implied by generic MHV loop diagrams. For NMHV one-loop amplitudes these diagrams have three vertices. The one-particle-irreducible MHV-diagrams (triangle diagrams) heuristically appear to be planar. (An MHV vertex is dual to a line in twistor space; a propagator in the loop is dual to an intersection between two lines [1, 23, 38].) On the other hand, the one-particle-reducible MHV-diagrams (bubbles with an additional MHV vertex attached to an external leg) do not appear to have to be planar. Perhaps the sum is better behaved than individual MHV diagrams. It would be very interesting to know whether or not the NMHV box coefficients for other $\mathcal{N} = 4$ amplitudes are planar.

The distribution of the seven points into lines in the plane, for the one-term coefficients in $A_{7;1}^{\mathcal{N}=4}$, as depicted in fig. 5, is also intriguing. This division is independent of the particular

NMHV helicity configuration, and only depends on the type of integral being considered. This helicity-independence seems obscure from the point of view of MHV diagrams.

The simplicity of the structure we have uncovered suggests that the computation of further multi-leg one-loop $\mathcal{N} = 4$ amplitudes will be quite tractable, and provides strong motivation for additional investigations in this direction.

The seven-point results presented here may also have some practical relevance. They can be used to benchmark purely numerical approaches to complicated multi-parton loop amplitudes in QCD, which are likely to become available in the near future. Such amplitudes will contribute to a better understanding of cross sections for multi-jet production at hadron colliders. Multi-jet rates are important to understand for their own sake, but also because such processes form both irreducible and reducible backgrounds (the latter when jets fake leptons or photons, for example) to processes involving the electroweak interactions. We are optimistic that combining insights from unitarity and twistor space will provide new practical tools for computing experimentally-relevant loop amplitudes in massless gauge theories.

Acknowledgments

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APPENDIX A: BOX INTEGRALS

In this appendix we collect the dimensionally-regulated integral functions appearing in the $\mathcal{N} = 4$ amplitudes; these integral functions were obtained from refs. [46, 53]. The reader is referred to these papers for further details of their derivation. Through $\mathcal{O}(\epsilon^0)$, we have

$$\begin{aligned} F^{4\mathrm{m}}(s,t,K_1^2,K_2^2,K_3^2,K_4^2) = \frac{1}{2}\Bigg\{ & -\mathrm{Li}_2\left(\frac{1}{2}(1-\lambda_1+\lambda_2+\rho)\right) + \mathrm{Li}_2\left(\frac{1}{2}(1-\lambda_1+\lambda_2-\rho)\right) \\ & - \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1-\lambda_1-\lambda_2-\rho)\right) + \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1-\lambda_1-\lambda_2+\rho)\right) \\ & - \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2^2}\right)\ln\left(\frac{1+\lambda_1-\lambda_2+\rho}{1+\lambda_1-\lambda_2-\rho}\right)\Bigg\}, \end{aligned} \tag{A1}$$

$$
\begin{aligned}
F^{3\mathrm{m}}(s,t,K_2^2,K_3^2,K_4^2) = &-\frac{1}{2\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}-(-K_2^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}\Big] \\
&-\frac{1}{2}\ln\left(\frac{-K_2^2}{-t}\right)\ln\left(\frac{-K_3^2}{-t}\right)-\frac{1}{2}\ln\left(\frac{-K_3^2}{-s}\right)\ln\left(\frac{-K_4^2}{-s}\right) \\
&+\mathrm{Li}_2\left(1-\frac{K_2^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{K_4^2}{t}\right)-\mathrm{Li}_2\left(1-\frac{K_2^2K_4^2}{st}\right) \\
&+\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right),
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
F^{2\mathrm{m}\,h}(s,t,K_3^2,K_4^2) = &-\frac{1}{2\epsilon^2}\Big[(-s)^{-\epsilon}+2(-t)^{-\epsilon}-(-K_3^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}\Big] \\
&-\frac{1}{2}\ln\left(\frac{-K_3^2}{-s}\right)\ln\left(\frac{-K_4^2}{-s}\right)+\mathrm{Li}_2\left(1-\frac{K_3^2}{t}\right)+\mathrm{Li}_2\left(1-\frac{K_4^2}{t}\right) \\
&+\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right),
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
F^{2\mathrm{m}\,e}(s,t,K_2^2,K_4^2) = &-\frac{1}{\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}-(-K_2^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}\Big] \\
&+\mathrm{Li}_2\left(1-\frac{K_2^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{K_2^2}{t}\right)+\mathrm{Li}_2\left(1-\frac{K_4^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{K_4^2}{t}\right) \\
&-\mathrm{Li}_2\left(1-\frac{K_2^2K_4^2}{st}\right)+\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right),
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
F^{1\mathrm{m}}(s,t,K_4^2) = &-\frac{1}{\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}-(-K_4^2)^{-\epsilon}\Big] \\
&+\mathrm{Li}_2\left(1-\frac{K_4^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{K_4^2}{t}\right)+\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)+\frac{\pi^2}{6},
\end{aligned} \tag{A5}
$$

$$
F^{0\mathrm{m}}(s,t) = -\frac{1}{\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}\Big]+\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)+\frac{\pi^2}{2}, \tag{A6}
$$

where the k_i denote massless momenta and the K_i massive momenta. The external momentum arguments $K_1,\ldots,K_4$ are sums of external momenta k_i from the n-point amplitude. The kinematic variables appearing in the integrals are

$$
s=(k_1+k_2)^2\,, \qquad\qquad t=(k_2+k_3)^2\,, \tag{A7}
$$

or with k relabeled as K for off-shell (massive) legs. The functions appearing in $F_4^{4\mathrm{m}}$ are

$$
\rho \;\equiv\; \sqrt{1-2\lambda_1-2\lambda_2+\lambda_1^2-2\lambda_1\lambda_2+\lambda_2^2}\,, \tag{A8}
$$

and

$$
\lambda_1=\frac{K_1^2\,K_3^2}{(K_1+K_2)^2\,(K_2+K_3)^2}\,, \qquad\qquad \lambda_2=\frac{K_2^2\,K_4^2}{(K_1+K_2)^2\,(K_2+K_3)^2}\,. \tag{A9}
$$

We have rearranged the expressions for $F^{3\mathrm{m}}$ and $F^{2\mathrm{m}\,h}$ to make the poles in ϵ more transparent. We have also corrected some signs in $F^{4\mathrm{m}}$ in ref. [9] and in the published version of ref. [46].

APPENDIX B: MULTI-PARTICLE FACTORIZATION EXAMPLE

In this appendix we explicitly evaluate the multi-particle factorization of an NMHV seven-point amplitude in $\mathcal{N}=4$ super-Yang-Mills theory. As discussed in the text, one-loop multi-particle factorization is described by a universal formula (185), even though naive factorization does not hold for infrared-divergent gauge theories.

As an example, consider the amplitude $A_{7;1}^{\mathcal{N}=4}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$. According to the general factorization formula, in the limit $K^2 = s_{234} \to 0$, where $K = k_2+k_3+k_4$, we have

$$A_{7;1}^{\mathcal{N}=4} \overset{s_{234}\to 0}{\longrightarrow} \Bigg[A_{4;1}^{\mathcal{N}=4}(2^-,3^-,4^+,(-K)^+)\,\frac{i}{s_{234}}\,A_5^{\rm tree}(K^-,5^+,6^+,7^+,1^-) + A_4^{\rm tree}(2^-,3^-,4^+,(-K)^+)\,\frac{i}{s_{234}}\,A_{5;1}^{\mathcal{N}=4}(K^-,5^+,6^+,7^+,1^-) + A_4^{\rm tree}(2^-,3^-,4^+,(-K)^+)\,\frac{i}{s_{234}}\,A_5^{\rm tree}(K^-,5^+,6^+,7^+,1^-) \times c_\Gamma\,\mathcal{F}_7^{\mathcal{N}=4}(s_{234};k_1,\ldots,k_7)\Bigg] . \tag{B1}$$

The subtlety in loop-level multi-particle factorization can be exposed by inspecting the infrared divergences on both sides of eq. (B1). On the left-hand-side, the divergences are [49],

$$A_{7;1}^{\mathcal{N}=4}(1,2,3,4,5,6,7)\Big|_{\epsilon\ \rm pole} = -\frac{c_\Gamma}{\epsilon^2}\sum_{j=1}^{7}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon} A_7^{\rm tree}(1,2,3,4,5,6,7) . \tag{B2}$$

On the other hand, in the products of amplitudes appearing in the first two terms on the right-hand-side of eq. (B1), we have the singular terms

$$-\frac{c_\Gamma}{\epsilon^2}A_7^{\rm tree}(1,2,3,4,5,6,7)\Bigg[\left(\frac{\mu^2}{s_{K2}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{23}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{34}}\right)^{\epsilon}+\left(\frac{\mu^2}{s_{4K}}\right)^{\epsilon} +\left(\frac{\mu^2}{-s_{K5}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{67}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{71}}\right)^{\epsilon}+\left(\frac{\mu^2}{-s_{1K}}\right)^{\epsilon}\Bigg]\Bigg|_{s_{234}\to 0} . \tag{B3}$$

The mismatch between eq. (B2) and eq. (B3) must be absorbed into the factorization function $\mathcal{F}_7$ appearing in the third term in eq. (B1). This mismatch implies, after expanding in ϵ, that the factorization function must contain the logarithmic terms

$$\frac{\ln(-s_{12})}{\epsilon}+\frac{\ln(-s_{45})}{\epsilon} . \tag{B4}$$

Neither of these terms allows for a separation of momenta $\{k_2,k_3,k_4\}$ from $\{k_5,k_6,k_7,k_1\}$, which one might have expected from a naive interpretation of eq. (B1), or from fig. 3 with $i=2$ and $r=3$.

The mismatched infrared singularities in eq. (B4) encode the factorization functions. From eq. (189), the infrared singularities (B4) imply that the factorization function in this channel is (recall $\mathcal{F}_n^{\mathcal{N}=4} = \mathcal{F}_n^{\text{non-fact}}$),

$$\begin{aligned}
&\mathcal{F}_7^{\text{non-fact}}(s_{234}; k_1, \ldots, k_7) \\
&\quad = 2\,(\mu^2)^\epsilon \Big(F^{2\text{m}\,h}(s_{12}, s_{234}, s_{34}, s_{567}) + F^{2\text{m}\,h}(s_{45}, s_{234}, s_{671}, s_{23}) + \frac{2}{\epsilon^2}(-s_{234})^{-\epsilon} \Big) \\
&\quad = 2(\mu^2)^\epsilon \Big\{ -\frac{1}{\epsilon^2}\Big[(-s_{12})^{-\epsilon} - (-s_{567})^{-\epsilon} - (-s_{34})^{-\epsilon} \Big] \\
&\qquad - \frac{1}{2\epsilon^2}\frac{(-s_{567})^{-\epsilon}(-s_{34})^{-\epsilon}}{(-s_{12})^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{-s_{12}}{-s_{234}}\right) \\
&\qquad - \frac{1}{2}\ln^2\left(\frac{-s_{34}}{-s_{234}}\right) - \frac{1}{2}\ln^2\left(\frac{-s_{567}}{-s_{234}}\right) - \frac{\pi^2}{3} \Big\} \\
&\qquad + \{k_1 \leftrightarrow k_5, k_2 \leftrightarrow k_4, k_6 \leftrightarrow k_7\}\,.
\end{aligned} \tag{B5}$$

Now compare this prediction to the results for $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ given in section IV. From that section, eight of the box functions have an s_{234} kinematic pole in their coefficients. Thus,

$$\begin{aligned}
&A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)\Big|_{s_{234}\to 0} \\
&\qquad \longrightarrow\; c_{134}\,B(1,3,4) + c_{137}\,B(1,3,7) + c_{167}\,B(1,6,7) + c_{234}\,B(2,3,4) \\
&\qquad\quad + c_{345}\,B(3,4,5) + c_{346}\,B(3,4,6) + c_{347}\,B(3,4,7) + c_{467}\,B(4,6,7)\,.
\end{aligned} \tag{B6}$$

In this limit the coefficients appearing with an s_{234} pole behave simply. The first five coefficients give,

$$\{c_{134}, c_{234}, c_{345}, c_{346}, c_{347}\} \to A_4^{\text{tree}}(2^-, 3^-, 4^+, (-K)^+)\frac{1}{s_{234}}A_5^{\text{tree}}(K^-, 5^+, 6^+, 7^+, 1^-)\,, \tag{B7}$$

while the remaining three give,

$$\{c_{137}, c_{167}, c_{467}\} \to 2A_4^{\text{tree}}(2^-, 3^-, 4^+, (-K)^+)\,\frac{1}{s_{234}}\,A_4^{\text{tree}}(K^-, 5^+, 6^+, 7^+, 1^-)\,. \tag{B8}$$

In the $s_{234} \to 0$ factorization limit, an inspection of the explicit form of the integrals in appendix A reveals, after expanding in ϵ, that the ones with an s_{234} pole in their coefficients behave as (see also table 5 of ref. [51]),

$$B(2,3,4) = F^{1\text{m}}(s_{56}, s_{67}, s_{567}) \longrightarrow F^{1\text{m}}(s_{56}, s_{67}, s_{567})\,, \tag{B9}$$

$$B(3,4,5) = F^{1\text{m}}(s_{67}, s_{71}, s_{671}) \longrightarrow F^{1\text{m}}(s_{67}, s_{71}, s_{671})\,, \tag{B10}$$

$$B(1,3,4) = F^{2\mathrm{m}\,h}(s_{56}, s_{671}, s_{71}, s_{234}) \longrightarrow F^{1\mathrm{m}}(s_{671}, s_{56}, s_{71}) + \frac{1}{\epsilon^2}(-s_{234})^{-\epsilon} - \frac{1}{2\epsilon^2}\frac{(-s_{234})^{-\epsilon}(-s_{71})^{-\epsilon}}{(-s_{56})^{-\epsilon}} - \mathrm{Li}_2\Big(1 - \frac{s_{71}}{s_{56}}\Big)\,, \quad \text{(B11)}$$

$$B(3,4,6) = F^{2\mathrm{m}\,h}(s_{71}, s_{567}, s_{234}, s_{56}) \longrightarrow F^{1\mathrm{m}}(s_{71}, s_{567}, s_{56}) + \frac{1}{\epsilon^2}(-s_{234})^{-\epsilon} - \frac{1}{2\epsilon^2}\frac{(-s_{234})^{-\epsilon}(-s_{56})^{-\epsilon}}{(-s_{71})^{-\epsilon}} - \mathrm{Li}_2\Big(1 - \frac{s_{56}}{s_{71}}\Big)\,, \quad \text{(B12)}$$

$$B(3,4,7) = F^{2\mathrm{m}\,e}(s_{567}, s_{671}, s_{234}, s_{67}) \longrightarrow F^{1\mathrm{m}}(s_{567}, s_{671}, s_{67}) + \frac{1}{\epsilon^2}(-s_{234})^{-\epsilon}\,, \quad \text{(B13)}$$

$$B(1,6,7) = F^{1\mathrm{m}}(s_{23}, s_{34}, s_{234}) \longrightarrow F^{0\mathrm{m}}(s_{23}, s_{34}) + \frac{1}{\epsilon^2}(-s_{234})^{-\epsilon}\,. \quad \text{(B14)}$$

The two remaining integrals,

$$F^{2\mathrm{m}\,h}(s_{12}, s_{234}, s_{34}, s_{567}), \qquad F^{2\mathrm{m}\,h}(s_{45}, s_{234}, s_{671}, s_{23}), \quad \text{(B15)}$$

do not have particularly simple properties as $s_{234} \to 0$, although the polylogarithms do simplify to logarithms.

Using eqs. (B7)–(B14), we may rewrite eq. (B6) as

$$\begin{aligned} &A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) \\ &\quad \longrightarrow c_\Gamma\,(\mu^2)^\epsilon A_4^{\mathrm{tree}}(2^-, 3^-, 4^+, (-K)^+)\,\frac{i}{s_{234}}\,A_5^{\mathrm{tree}}(K^-, 5^+, 6^+, 7^+, 1^-) \\ &\qquad \times\Big[F^{1\mathrm{m}}(s_{56}, s_{67}, s_{567}) + F^{1\mathrm{m}}(s_{67}, s_{71}, s_{671}) + F^{1\mathrm{m}}(s_{671}, s_{56}, s_{71}) \\ &\qquad\quad + F^{1\mathrm{m}}(s_{71}, s_{567}, s_{56}) + F^{1\mathrm{m}}(s_{567}, s_{671}, s_{67}) + 2F^{0\mathrm{m}}(s_{23}, s_{34}) + \mathcal{R}_{234}\Big]\,, \end{aligned} \quad \text{(B16)}$$

where the remainder is

$$\begin{aligned} \mathcal{R}_{234} \;=\; & \frac{5}{\epsilon^2}(-s_{234})^{-\epsilon} - \frac{1}{2\epsilon^2}\frac{(-s_{234})^{-\epsilon}(-s_{71})^{-\epsilon}}{(-s_{56})^{-\epsilon}} - \mathrm{Li}_2\Big(1 - \frac{s_{71}}{s_{56}}\Big) \\ & - \frac{1}{2\epsilon^2}\frac{(-s_{234})^{-\epsilon}(-s_{56})^{-\epsilon}}{(-s_{71})^{-\epsilon}} - \mathrm{Li}_2\Big(1 - \frac{s_{56}}{s_{71}}\Big) \\ & + 2F^{2\mathrm{m}\,h}(s_{12}, s_{234}, s_{34}, s_{567}) + 2F^{2\mathrm{m}\,h}(s_{45}, s_{234}, s_{671}, s_{23})\,. \end{aligned} \quad \text{(B17)}$$

Comparing eq. (B16) to eq. (B1), we may identify

$$A_{4;1}^{\mathcal{N}=4}(2^-, 3^-, 4^+, (-K)^+) = 2\,c_\Gamma\,(\mu^2)^\epsilon\,A_4^{\mathrm{tree}}\,F^{0\mathrm{m}}(s_{23}, s_{34})\,, \quad \text{(B18)}$$

in agreement with eq. (14), and

$$A_{5;1}^{\mathcal{N}=4}(K^-, 5^+, 6^+, 7^+, 1^-) \;=\; c_\Gamma\,(\mu^2)^\epsilon\,A_5^{\mathrm{tree}}\Big[F^{1\mathrm{m}}(s_{56}, s_{67}, s_{567}) + F^{1\mathrm{m}}(s_{67}, s_{71}, s_{671})$$

$$+ F^{1m}(s_{71}, s_{567}, s_{56}) + F^{1m}(s_{567}, s_{671}, s_{67})$$
$$+ F^{1m}(s_{671}, s_{56}, s_{71})\Bigg] , \tag{B19}$$

in agreement with with eq. (15). The remainder may also be identified with the factorization function,

$$\mathcal{F}_7(s_{234}; k_1, \ldots, k_7) = (\mu^2)^{\epsilon}\, \mathcal{R}_{234}\,. \tag{B20}$$

The last equation requires use of a dilogarithm identity and expansion through $\mathcal{O}(\epsilon^0)$. Combining this result with eqs. (B18) and (B19), we thus find that the amplitude $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ has precisely the expected factorization properties in the limit $s_{234} \to 0$.

The other channels work similarly. For all the amplitudes in section IV, for each multi-particle limit the coefficients of precisely eight integrals have poles in the channel of interest. Five of the integrals belong with $A_{5;1}^{\mathcal{N}=4}$, one belongs with $A_{4;1}^{\mathcal{N}=4}$, and the remaining two combine with the discontinuities from the other integrals to form the factorization function.

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