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Coherent radiation of electron cloud.*

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Abstract

The electron cloud in positron storage rings is pinched when a bunch passes by. For short bunches, the radiation due to acceleration of electrons of the cloud is coherent. Detection of such radiation can be used to measure the density of the cloud. The estimate of the power and the time structure of the radiated signal is given in this paper.

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1 Introduction

Effect of the electron cloud on the transverse beam stability in the storage ring has been discovered in the KEK Photon Factory[1] and studied extensively afterwards [2], [3]. The instability is one of the culprits limiting performance of the B-factories. The density of the cloud is the main parameter defining the instability. Diagnostics of the density could be important in limiting adverse effects of the cloud. Here we study the coherent radiation from the cloud as a possible tool for diagnostics of the e-cloud density. The radiation is the result of acceleration of electrons in transverse to the beam plane by the field of passing bunches. Radiation is coherent and propagates in the beam pipe. It can be detected with an rf antenna. The amplitude of the signal depends on the density of the cloud and can be used for the cloud diagnostics. We estimate here the power and the time structure of the signal.

2 Model

The beam is a train of the high energy positron bunches. Electron cloud is generated by synchrotron radiation and/or multipaction of secondary electrons. Electrons are accelerated radially by the field of a passing bunch and radiate during the passage of a bunch. Radiation field can be expanded over the eigen-modes $H^{(\pm)}_{m,n}$, $E^{(\pm)}_{m,n}$ in the beam pipe. Consider a round beam pipe with the radius b. The typical b are of few cm and the cut-off frequency of the beam pipe is several GHz. Let us assume the frequency dependence in the form $e^{-i\omega t}$, use the cylindrical coordinate system with the axis along the beam line, and denote by z_d the point of observation of the radiation. The Fourier harmonics of the magnetic field radiated at the frequency ω by the current induced in the electron cloud and propagating in $\pm z$ directions are

$$H^{\pm} = \sum_{m,n} a^{(\pm)}_{m,n} H^{(\pm)}_{m,n} e^{\pm i q_{m,n}(z-z_d)}, \tag{2.1}$$

and similar equation for $E^{\pm}_{m,n}$. Here m is the azimuthal number, and n the radial mode number, The propagating constant $q_{m,n} = +\sqrt{k^2 - \lambda^2_{m,n}}$ is defined with the cut in the complex plane of ω along the real axis from $k = -\lambda_{m,n}$ to $k = \lambda_{m,n}$. On the upper edge of the cut $q_{m,n} = +i\sqrt{\lambda^2_{m,n} - k^2}$. The small positive imaginary part is implied, $k = \omega/c + i\epsilon$, $\epsilon = 0+$, and $q_{m,n}(-k^*) = -q^*_{m,n}(k)$.

For TM mode, $\lambda_{m,n} = \nu_{m,n}/b$ where $\nu_{m,n}$ is the n-s root of the Bessel function $J_m(\nu_{m,n}) = 0$.

The explicit form of the eigen-modes can be found in the text books. In the following, we assume that the induced current in the cloud is axially symmetric and has only radial component. In this case, it is suffice to consider only $m = 0$ mode, and we drop this index below. The nonzero components of azimuthal $m = 0$ TM modes are

$$E_n^{s,r} = is\frac{q_n}{\lambda_n}J_0'(\lambda_n r)e^{isq_n(z-z_d)}, \quad E_n^{s,z} = J_0(\lambda_n r)e^{isq_n(z-z_d)}$$
$$H_n^{s,\phi} = \frac{ik}{\lambda_n}J_0'(\lambda_n r)e^{isq_n(z-z_d)}, \tag{2.2}$$

where prime means derivative over the argument, $s = \pm 1$, $\lambda_n = \nu_n/b$, and ν_n is the n-th root of $J_0(\nu_n) = 0$.

The modes are orthogonal with the norm $N_{n,m}^{s,s'}$ being proportional to the integral of the Pointing vector over cross section of the beam pipe,

$$N_{n,n'}^{s,s'} = \int dS(E_n^{s,r}H_{n'}^{s',\phi} - E_{n'}^{s',r}H_n^{s,\phi}) = 2\pi b^2 s'\frac{kq_n}{\lambda_n^2}J_1^2(\nu_n)\delta_{n,n'}\delta_{s,-s'}. \tag{2.3}$$

The amplitudes of the modes can be found from the identity following from Maxwell equations for the free field E_2, H_2 and the the fields E, H driven in the same volume V by the current j,

$$\int dS[E_1\times H_2 - E_2\times H_1) = -Z_0\int dV E_1 j_2^\omega. \tag{2.4}$$

Here $Z_0 = 4\pi/c_0 = 120\pi$ Ohms.

Substituting E, and H from Eq. 2.1 we get

$$a_n^\pm = \mp\frac{Z_0}{N_{nn}^{(-+)}}\int dV' j_\omega^r(r',z')E_n^{\mp,r}(r',z'). \tag{2.5}$$

The integral in $a^\pm$ over z is taken over the region where the modes were generated: $-\infty < z < z_d$ for a^+ and $z_d < z < \infty$ for a^-.

The relativistic bunch gives the radial kick to an electron of the cloud. Therefore, in the axially symmetric structures, the current has only radial component.

The current can be defined from the equation of motion for the trajectory $R(r,z,t)$

$$\frac{d^2R}{dt^2} = \frac{2N_b r_e c^2 R}{R^2+\sigma_\perp^2}\rho_B(z-ct), \tag{2.6}$$

where r_e is classical electron radius, N_b is bunch population, and $\rho_B(z)$ is the longitudinal density profile, $\int\rho_B(z)dz = 1$. We set the initial conditions at $t = -\infty$ $R(r_0,z,t\to-\infty) = r_0$, $\dot R(r_0,z,t\to-\infty) = 0$.

It follows from Eq. 2.6 that $R(r_0,z,t) = R(r_0,\zeta)$ depends only on the initial location r_0 and the parameter $\zeta = ct - z$. Let us assume the uniform density of the cloud, $n(r_0,z_0) = n_0$. Then, the current

$$j(r,z,t) = e\int 2\pi r_0 dr_0 dz_0 n(r_0,z_0)\dot R(r_0,z_0,t)\delta(z-z_0)\frac{\delta[r-R(r_0,z_0,t]}{2\pi r}$$
$$= \frac{en_0}{r}\int r_0 dr_0 \dot R(r_0,\zeta)\delta[r-R(r_0,\zeta)] \tag{2.7}$$

is also a function of r and $\zeta = ct - z$, $j(r, z, t) = j(r, \zeta)$.

The integral in Eq. 2.5 over dr' gives

$$\int r' dr' J_0'(\lambda_n r') j(r', z', t') = en_0 \int dt' e^{ikct'} \int r_0 dr_0 \frac{dR(r_0, \zeta')}{d\zeta'} J_0'(\lambda_n R(r_0, \zeta')). \tag{2.8}$$

Changing here integration over t' to integration over $d\zeta'$, $\zeta' = ct' - z'$, and substituting result in Eq. 2.5 we can carry out integration over dz'. That gives

$$a_n^{(\pm)} = \frac{4\pi e n_0}{k b^2 c_0 J_1^2(\nu_n)} \frac{e^{ikz_d}}{k \mp q_n} \int d\zeta e^{ik\zeta} \frac{\partial f(\zeta)}{\partial \zeta}, \tag{2.9}$$

where

$$f(\zeta) = \int_0^b r_0 dr_0 J_0[\lambda_n R(r_0, \zeta)]. \tag{2.10}$$

Let us assume that the amplitude of a signal in the detector situated at $z = z_d$ is determined by the component $E^r(r = b, z_d, t)$. The field at $z = z_d$ is

$$E_r(r, z_d, t) = \frac{1}{2} \sum_n [a_n^{(+)} E_n^{+,r} n(r, z_d) + a^{(-)} E_n^{-,r}(r, z_d)], \tag{2.11}$$

and takes the form

$$E_r(r, z_d, t) = \theta(ct - z_d) \sum_n \frac{4\pi e n_0 J_1(\nu_n r)}{\lambda_n^3 b^2 J_1^2(\nu_n)} [\frac{d^2 f(\zeta_d)}{d(\zeta_d)^2} + \lambda_n^2 f(\zeta_d) - \lambda_n^2 f(-\infty)]. \tag{2.12}$$

Here $\zeta_d = ct - z_d$, and the last term $f(-\infty) = (b^2/\nu_n) J_1(\nu_n)$ is obtained using initial condition $R(r_0, \zeta \to \infty) = r_0$.

In deriving Eq. 2.12 we used the integral

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dk}{k} [\frac{q_n(k)}{k + q_n} - \frac{q_n(k)}{k - q_n}] e^{ikz_d - ikc(t - \zeta'/c)} \\
&= -\frac{2}{\lambda_n^2} \int_{-\infty}^{\infty} \frac{dk}{k} [k - \frac{\lambda_n^2(k)}{k}] e^{ik(z_d - ct + \zeta')} \\
&= 2\pi i [-\frac{\partial}{\partial \zeta} \delta(z_d - ct + \zeta') + \lambda_n^2 \theta(z_d - ct + \zeta')],
\end{aligned} \tag{2.13}$$

where $\theta(x)$ is the step function.

The power radiated by bunch per turn is $P = \Delta U / T_0$ where T_0 is the revolution period and ΔU is radiated energy,

$$\Delta U = \frac{c}{4\pi} \int \frac{d\omega}{2\pi} |a_n|^2 |N_{n,n}|. \tag{2.14}$$

ΔU is given by the sum over modes propagating in the beam pipe. The sum converge very rapidly and the main contribution is given by the lowest mode.

To proceed further, we consider two extreme cases of very short and very long parabolic bunches.

In the first case, interaction with the bunch gives instantaneous kick. The trajectory given by Eq. 2.6

$$R(r_0, \zeta) = r_0 - \frac{2N_b r_e r_0}{\sigma_\perp^2 + r_0^2} \zeta. \tag{2.15}$$

defines the function Eq. 2.10, $f(\zeta) = b^2 F(\zeta/\zeta_0)$, where

$$F(\xi) = \int x dx J_0[\nu x(1 - \frac{\xi}{(\sigma_\perp/b)^2 + x^2})], \tag{2.16}$$

and

$$\frac{1}{\zeta_0} = \frac{2N_b r_e}{b^2} = \frac{eZ_0 I_{bunch}}{mc^2} \frac{R}{b^2}, \tag{2.17}$$

where mc^2 is electron mass.

The field at $r = b$ takes the form

$$E_r(b, z_d, t) = \Lambda \theta(\zeta)\, \Phi(\zeta/\zeta_0), \tag{2.18}$$

where

$$\Lambda = \frac{4\pi e n_0 R^2}{b\nu^3 J_1(\nu)} (\frac{eI_{bunch} Z_0}{mc^2})^2, \tag{2.19}$$

and Φ is given by Eq. 2.26,

$$\Phi(\xi) = \frac{d^2 F(\xi)}{d\xi^2} + (\frac{\nu L}{b})^2 F(\xi) - \nu (\frac{L}{b})^2 J_1(\nu), \tag{2.20}$$

where F is related to f, $f = b^2 F$.

Because the sum converge very rapidly, we retain only the first term in the sum $\nu = \nu_1 \simeq 2.4$.

With the PEP-II parameters $I_{bunch} = 2.5$ mA, $2\pi R = 2.2$ km, $b = 2.5$ cm, and assuming typical $n_0 = 10^7\ cm^{-3}$, we get $\Lambda = 1.9\, 10^5 (e/b^2)$. The function $\Phi(\xi)$ is shown in Fig. 1. It gives the shape of the signal measured at the wall for a short bunch provided the signal is proportional to E_r. For short bunches, the signal is due to radiation of the instantaneously accelerated electrons and variation in time is defined by the parameter ζ_0.

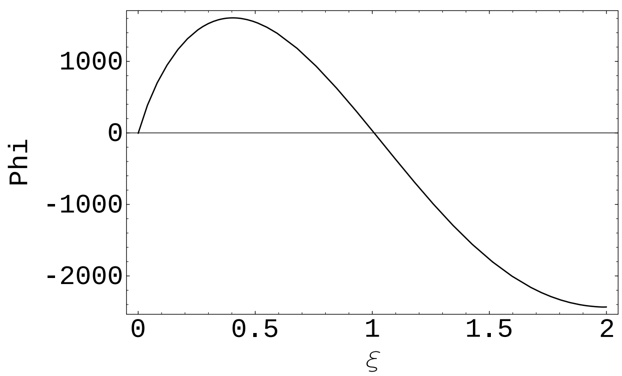

Figure 1: Time dependence of a signal proportional to E_r at the wall for short bunches.

For parabolic bunches with the length $2L$

$$\rho(\zeta) = \frac{3}{4L}\,(1 - \frac{\zeta^2}{L^2}), \tag{2.21}$$

and electrons within the beam, $r_0 < \sigma_\perp$, the equation of motion describes oscillations of the trapped electrons,

$$\begin{aligned} &\frac{d^2R}{d\xi^2} + \Omega^2(\xi)R = 0, \\ &\Omega^2(\xi) = \frac{2N_b r_e L^2}{\sigma_\perp^2}\rho(\xi) = \Omega_0^2(1-\xi^2), \end{aligned} \tag{2.22}$$

where $\xi = \zeta/L$, $\zeta = ct - z$ and

$$\Omega_0^2 = \frac{3N_b r_e L}{2\sigma_\perp^2}. \tag{2.23}$$

The field in this case is given by

$$E_r(b, z_d, t) = \Lambda\theta(\xi)\Phi(\xi), \tag{2.24}$$

where

$$\Lambda = \frac{4\pi e n_0 b^3}{\nu^3 L^2 J_1(\nu)}. \tag{2.25}$$

$\xi = \zeta/L$ and Φ is given by Eq. 2.26.

$$\Phi(\xi) = \frac{d^2F(\xi)}{d\xi^2} + (\frac{\nu\zeta_0}{b})^2 F(\xi) - \nu(\frac{\zeta_0}{b})^2 J_1(\nu). \tag{2.26}$$

With the same parameters for n_0 and b, and for $L = 10$ cm, $N_b = 10^{13}$, $\sigma_\perp = 5$ mm, we get $\Lambda = (e/b^2)\, 6.8\, 10^5$. The time dependence in this case is shown in Fig. 2 and is defined by the frequency Ω_0.

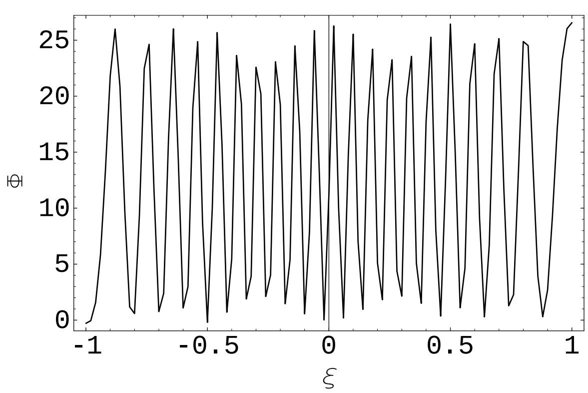

Figure 2: Time dependence of a signal proportional to E_r at the wall for long parabolic bunches.

Let us estimate the energy radiated by a long bunch. The radiated energy Eq. 2.14 is

$$\Delta U = \frac{4\pi}{\nu^2}(\frac{en_0}{J_1(\nu)})^2 \int dk \frac{q}{k} |\frac{1}{k-q}|^2 | \int d\zeta e^{ik\zeta} \frac{df}{d\zeta}|^2. \tag{2.27}$$

Here we neglected contribution of all trems except $n = 1$, and denote $\nu = \nu_1$, $q = \sqrt{k^2 - (\nu/b)^2}$.

The radiation is due to oscillations of electrons trapped in the bunch. Therefore, we can assume that for such electrons $R(r_0, \zeta) \simeq \sigma_\perp << b$. Because significant contribution comes from the first Bessel root $\nu_1 \simeq 1$, that allows us expand $J_0(\lambda_n R_0) \simeq 1 - (\lambda_n R_0/2)^2$.

The trajectory with the initial conditions $R(r_0, 0) = r_0$, and $(dR/d\zeta)_{\zeta=0} = 0$, is $R(r_0, \zeta) = r_0 \cos\psi(\zeta)$. In this approximation,

$$f(\zeta) = \frac{b^2}{2} - (\frac{\lambda_n b^2}{4})^2 \cos^2\psi(\zeta), \quad ,\zeta > 0;$$
$$\int d\zeta e^{ik\zeta} \frac{df}{d\zeta} = (\frac{\lambda_n b^2}{4})^2 \int d\zeta \Omega(\zeta)\, e^{ik\zeta} \sin(2\psi(\zeta)). \tag{2.28}$$

The last integral can be evaluated by the saddle-point method. The integral is exponentially small for $2\Omega_0/(kL) < 1$, and

$$|\int d\zeta \Omega(\zeta)\, e^{ik\zeta} \sin(2\psi(\zeta)|^2 \simeq (\frac{\nu b}{4})^4 (\frac{kL}{4})^2 \frac{2\pi}{\Omega_0\sqrt{(2\Omega_0/kL)^2 - 1}} \tag{2.29}$$

otherwise. The radiated energy is given by $k > \nu/b$ and the condition $2\Omega_0 b/(\nu L) > 1$

$$\Delta U = \frac{e^2}{b}\frac{1}{2\nu\Omega_0}(\frac{\pi n_0 b^3}{2J_1(\nu)})^2 (\frac{\nu}{4})^4 (\frac{L}{b})^2 S(\frac{2\Omega_0 b}{\nu L}), \tag{2.30}$$

where

$$S(p) = \int_1^p \frac{x^2 dx \sqrt{x^2 - 1}}{(x - \sqrt{x^2 - 1})^2 \sqrt{p^2 - x^2}}. \tag{2.31}$$

The function $S(p)$ grows fast with p as p^4, see Fig. 3.

Taking $N_{bunch} = 10^{13}$, the typical density of the cloud $n_0 = 10^7\ cm^{-3}$, $\sigma_\perp = 0.5$ cm, $L = 10$ cm, and $b = 2.5$ cm we get $\Omega_0 = 12.9$, $p = 2.7$, $S(p) = 181.7$, and $\Delta U = 3.17\, 10^{19}\, (e^2/b)$, about 0.7 μJ.

3 Summary and discussion

We give the estimate of the power radiated by the pinched electron cloud after passage of a positron bunch. The power seems to be detectable and the detection of the signal can provide information on the density of the cloud. The main difficulty, probably, is separation of the signal from the signal induced by the bunch and from the wake field induced by geometric discontinuities and the noise of the cloud. Placement of the detector in a straight pipe may help to suppress the wake fields. Dependence of the signal on current $P \propto n_0^2$ is quite different from the linear dependence of a signal on current due to wakefields because the electron cloud density in saturation n_0 itself proportional to beam current. The frequency content of the signal depends on the bunch length and on the transverse distribution of electrons in the cloud: electrons in the vicinity of the beam oscillate with plasma frequency $(\omega/c)^2 = 4\pi n r_e$ while electrons with initial radial position $r(0) >> \sigma_t$ shift slightly during the kick from the bunch. That also distinguish the signal from the signal due to the long-range wake fields which have a narrow bandwidth around

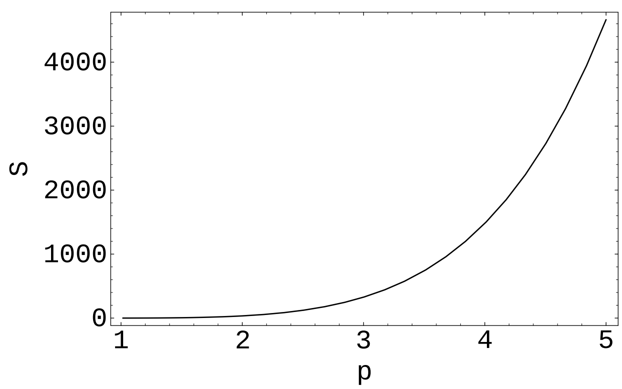

Figure 3: Function $S(p)$.

the HOM frequency. The time structure of the signal is different from the bunch profile. It would be interesting to study effect on the signal of the solenoidal magnetic field. Such a field generated in the beam pipe to suppress the cloud changes the dynamics of the electrons and would affect the signal.

It is worth mentioning the attempt to measure the density of the cloud by detecting the phase shift of the RF wave induced in the beam pipe [4]. Unexpectedly, it was discovered that the passing bunch strongly affects the detected signal. We suggest that the interference may be explained by the coherent radiation of the cloud described above.

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