X-ray production by cascading stages of a High-Gain Harmonic Generation Free-electron Laser II: special topics*

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In this paper, we study the tolerance of a new approach to produce coherent x-ray by cascading several stages of a High-Gain Harmonic Generation (HGHG) Free-Electron Laser (FEL). Being a harmonic generation process, a small noise in the initial fundamental signal will lead to a significant noise-to-signal (NTS) ratio in the final harmonic, so the noise issue is studied in this paper. We study two sources of noise: the incoherent undulator radiation, which is a noise with respect to the seed laser; and the noise of the seed laser itself. In reality, the electron beam longitudinal current profile is not uniform. Since the electron beam is the amplification medium for the FEL, this non-uniformity will induce phase error in the FEL. Therefore, this effect is studied. Phase error due to the wakefield and electron beam self-field is also studied. Synchrotronization of the electron beam and the seed laser is an important issue determining the success of the HGHG. We study the timing jitter induced frequency jitter in this paper. We also show that an HGHG FEL poses a less stringent requirement on the emittance than a SASE FEL does, due to a Natural Emittance Effect Reduction (NEER) mechanism. This NEER mechanism suggests a new operation mode, i.e., the HGHG FEL could adopt a high current, though unavoidable, a high emittance electron beam. Study in this paper shows that, production of hard x-rays with good longitudinal coherence by cascading stages of a HGHG FEL is promising. However, technical improvement is demanded.

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I. INTRODUCTION

Short wavelength Free-Electron Lasers (FELs) are perceived as the next generation of synchrotron light sources. In the past decade, significant advances have been made in the theory and technology of high-brightness electron beams and single-pass FELs. These developments facilitate the construction of practical vacuum ultraviolet (VUV) FELs and make x-ray FELs possible. Self-Amplified Spontaneous Emission (SASE) [1–15] and High-Gain Harmonic Generation (HGHG) [16–24] are the two leading candidates for VUV and x-ray FELs. The first HGHG proof-of-principle experiment [19, 20] succeeded in August, 1999 in Brookhaven National Laboratory. The experimental results agree with the theory prediction. The following advantages of the HGHG FEL over the SASE FEL were confirmed: 1. much better longitudinal coherence, 2. much narrower bandwidth, 3. more stable central wavelength. These HGHG FEL advantages were further confirmed recently in ultraviolet wavelength regime [23, 24]. These stimulated our interest in investigating whether it is now feasible to produce an x-ray FEL by the HGHG-based scheme. This was the purpose of the preceding paper [25] (hereafter referred as paper I) and this paper.

The basic theory of cascading stages of HGHG has been laid down in paper I. In this paper, we will study some special topics.

In Sec. II, the tolerance to noise in this new approach is studied. It is well known that in the harmonic generation process, the noise in the initial signal will lead to significant noise-to-signal (NTS) ratio in the final harmonic [26, 27]. Hence, the tolerance to noise in the initial signal is studied in this paper. We consider two sources of noises. The first is the incoherent undulator radiation, which is noise with respect to the seed laser; because the undulator radiation is chaotic in nature [28], it will affect the phase coherence of the final output. The second is the noise of the seed laser itself. The seed laser is modelled as a coherent state, and noise is then discussed.

In reality, the longitudinal electron beam current profile is not uniform. Since the electron beam is the amplification medium for the FEL, this nonuniformity in the electron beam will induce phase error in the FEL. We study this effect in Sec. III. If this phase error is large, after stages of harmonic generation, the longitudinal coherence will be degraded. Hence, in real design this should be taken into consideration.

In real machine, due to the interaction between the beam and its environment, there will be wakefields [31]. Further more, high peak current electron beam also generates high self-fields. The wakefields and self-fields in-

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duce energy modulation on the electron beam. Again, since the electron beam is the amplification medium for the FEL, this energy modulation will induce phase error via the dispersion strength in the dispersion section and also the undulators. We study these issues in Sec. IV. Synchrotronization of the electron beam and the seed laser is an important factor determining the success of the HGHG. The timing jitter induced frequency jitter is studied also in Sec. IV.

We study the emittance effect in the HGHG FEL in Sec. V. Study on the middle dispersion section which is absent in the SASE FEL unveils the possibility of reduce the emittance effect. For an HGHG FEL, the bunching is mainly produced in the dispersion section; while for a SASE FEL, bunching is produced in the undulator. The emittance effect is much smaller in the dispersion section than that in a undulator; therefore, the emittance requirement in the HGHG FEL turns out to be less stringent than that in the SASE FEL. This is called the Natural Emittance Effect Reduction (NEER) mechanism. This NEER mechanism suggests a new operation mode, i.e., adopting a high current, though unavoidably high emittance electron beam in the harmonic generation stages.

II.   TOLERANCE TO NOISE

Although SASE FEL have excellent transverse modes, poor temporal coherence is a limitation imposed by the start-up shot noise; because the coherence length $c\,\sigma_t = c/(2\sigma_\omega)$ is usually much shorter than the bunch length. Here, σ_ω is the spectral bandwidth of the SASE radiation. Without a proper coherent seed at x-ray wavelength, an HGHG FEL resorts to a coherent seed at longer wavelengths. Hence, it is crucial that the coherence of the input seed laser can be preserved; so that the final x-ray HGHG FEL will have good temporal coherence.

Let us first briefly review the basic theory [26] of noise problem in harmonic generation. According to the study described in Sec. IV of paper I, the HGHG process has an intrinsic advantage of good power stability, i.e., the intensity fluctuation in the harmonic radiation is reduced during a multistage HGHG process. Here we will focus on the phase fluctuation.

We model the phase noise as a phase modulation in the signal, i.e., the signal is

$$
\begin{aligned}
E_1(\alpha,\omega,\Omega) &= E_{10}\sin(\omega t + \alpha\sin\Omega t)\\
&= E_{10}\sum_{m=-\infty}^{+\infty}(-1)^m J_m(\alpha)\sin(\omega + m\Omega)t. \quad (1)
\end{aligned}
$$

The signal has a frequency of ω, and the noise is modelled as the phase modulation with a frequency of Ω. For the nth harmonic, the initial noise manifested itself as

$$
\begin{aligned}
E_n(\alpha,\omega,\Omega) &= E_{n0}\sin(n\omega t + n\alpha\sin\Omega t)\\
&= E_{n0}\sum_{m=-\infty}^{+\infty}(-1)^m J_m(n\alpha)\sin(n\omega + m\Omega)t.
\end{aligned}
$$
$$(2)$$

For the first sideband, the initial NTS ratio is $[J_1(\alpha)/J_0(\alpha)]^2$, and the final NTS ratio is $[J_1(n\alpha)/J_0(n\alpha)]^2$. In the case of a small value of α and $n\,\alpha$, the NTS ratio increases by a factor of n^2. Thus, a negligible value of the NTS ratio in the initial signal will lead to a large NTS ratio in the final harmonic. The first sideband in the final harmonic is located at $n\omega \pm \Omega$, hence the relative location is $\Omega/(n\,\omega)$. This suggests that, in reality, even if the first sideband is not negligible, it can be outside of the FEL gain bandwidth. If this is the case, a monochrometer can filter out the noise component.

A.   Tolerance to the incoherent undulator radiation

Incoherent undulator radiation is produced in every undulator. Undulator radiation is chaotic [28] in nature, hence its phase is random. The incoherent undulator radiation is superimposed on the seed laser. After the harmonic generation process, this amount of undulator radiation will degrade the coherence of the final harmonic.

The above formulae (1) and (2) provide a good description of our model. We have a seed laser at frequency ω. The undulator radiation also peaks at ω, with a full bandwidth of $1/N_u$, where N_u is the number of periods in the undulator. Thus $\Omega \approx \omega/(2\,N_u)$, according to Eq. (1). Let us now compute the noise power in the following.

1.   Start-up noise power

First, we need compute the start-up shot noise in the first two power e-folding lengths [10, 29].

The spontaneous radiation power spectrum in a unit solid angle in the forward direction and at the resonant frequency is given by [30]

$$
\begin{aligned}
B_0 &\equiv \left(\frac{d^2 P_{\text{spont}}}{\frac{d\omega}{\omega}d\Omega}\right)_{\theta=0;\,\omega=\omega_r}\\
&= \frac{eZ_0 I_0}{4\pi}N_u^2\gamma^2\frac{K^2}{\left(1+\frac{K^2}{2}\right)^2}[JJ]^2\omega_r, \quad (3)
\end{aligned}
$$

where e is the absolute value of the electron charge; $Z_0 \approx 120\pi\,\Omega$ is the vacuum impedance; I_0 is the electron beam current; N_u is the number of period in the undulator; γ is the Lorentz factor; $K = e\lambda_u B_u/(2\pi mc) \approx 93.4\lambda_u B_u$ is the dimensionless undulator vector potential with λ_u the period of the undulator and B_u the peak field in the

undulator; $\omega_r = c2\pi/\lambda_r$ is the radiation frequency with λ_r the radiation wavelength; and the Bessel factor $[JJ]$ is

$$[JJ] = J_0\left[\frac{a_w^2}{2\left(1 + a_w^2\right)}\right] - J_1\left[\frac{a_w^2}{2\left(1 + a_w^2\right)}\right], \quad (4)$$

with $a_w = K/\sqrt{2}$; J_0 and J_1 are the zeroth order and first order Bessel function. The radiation opening angle is

$$\theta_w \equiv \sqrt{\frac{2\lambda_r}{L_u}}, \quad (5)$$

where, L_u is the length of the undulator. Hence, the power spectrum at the exit of the undulator is,

$$S_0 = \pi\theta_w^2 B_0. \quad (6)$$

The bandwidth of the FEL is [4],

$$b_w \equiv \sqrt{2\pi}\frac{\sigma_\omega}{\omega} = \sqrt{\frac{3\sqrt{3}\rho}{N_u}}, \quad (7)$$

where ρ is the Pierce parameter [2]. Hence the total spontaneous radiation in the bandwidth of FEL is

$$P_{\text{spont}} = S_0 b_w. \quad (8)$$

As we know, the start-up region is the first two power e-folding length (one field e-folding length); hence what we really need is the effective spontaneous radiation power at that location, i.e.,

$$P_{\text{spont}}^{2L_G} = \frac{2L_G}{L_u}P_{\text{spont}} = \frac{2L_G}{L_u}S_0 b_w. \quad (9)$$

But only the exponentially growth mode will survive finally; hence, we need find the effectively coupling factor for such mode. The coupling factor is found to be [10],

$$C_n(a) \approx \frac{\sqrt{3}}{\pi a^2}\exp\left[-\frac{1}{a\sqrt{1 + a^2}}\left(\beta_0 + \beta_1\frac{1}{a^2}\right)\right], \quad (10)$$

where, n is the index referring to a certain mode being excited; $a = \sqrt{4\rho k_u k_r}R_0$ is the scaled beam size with $k_u = 2\pi/\lambda_u$, $k_r = 2\pi/\lambda_r$, and R_0 the real edge of an electron beam. The above formula is a good approximation, when $a > 0.25$. Within such range, $\beta_0 = 1.093$, and $\beta_1 = -0.02$ for the fundamental guiding mode. So, in the undulator, the FEL power reads

$$P_{\text{SASE}}(z) = C_1\frac{1}{9}e^{\frac{z}{L_G}}\frac{2L_G}{L_u}P_{\text{spont}} = C_1\frac{1}{9}e^{\frac{z}{L_G}}P_{\text{spont}}^{2L_G} \quad (11)$$

The meaning of the formula is very clear: the factor $\frac{2L_G}{L_u}$ multiplied by the spontaneous power P_{spont} is the power of the first two power e-folding lengths (one field e-folding length). This is then multiplied by a coupling factor $C_1(a)$, representing the equivalent start-up noise as an equivalent input seed. This equivalent seed is to be

amplified by a factor of $\frac{1}{9}e^{z/L_G}$, where the factor $\frac{1}{9}$ represents the well known lethargy distance for an input signal to be amplified before reaching an exponential growth in an 1-D theory. Hence finally we get, for the fundamental guiding mode, the effective start-up noise power as

$$\begin{aligned}P_{\text{SASE}}^{\text{Start-up}} &= C_1 P_{\text{spont}}^{2L_G}\\ &= C_1\frac{2L_G}{L_u}\pi\left(\frac{2\lambda_r}{L_u}\right)\sqrt{\frac{3\sqrt{3}\rho}{N_u}}\\ &\times \frac{eZ_0 I_0}{4\pi}N_u^2\gamma^2\frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2}[JJ]^2\omega_r. \quad (12)\end{aligned}$$

2. Calculation on the new approach

Let us now study the performance of the coherent x-ray production scheme of Fig. 3 in paper I.

The main contribution of the incoherent undulator radiation comes from the first stage since, according to the above analysis, this amount of noise will present itself in the final NTS ratio with an amplification factor of N^2, where N is the harmonic number. In the modulator, according to Eqs. (11) and (12), the undulator noise power is about 60 W, and the input signal is 1 GW. The noise power is essentially given by the power in the first upper sideband and the first lower sideband. Thus there is approximately 30 W in the first upper sideband. Therefore $\alpha \approx 2\sqrt{30/(1\times 10^9)}$, where we have used the fact that $J_1(\alpha) \approx \alpha/2$ and $J_0(\alpha) \approx 1$ for small α. Shown in Fig. 3, the modulator of the first stage is designed to be resonant at $2,250$ Å, and the final radiation wavelength is 1.5 Å, so that $N = 2250/1.5 = 1500$. The final NTS ratio is then about, $2N^2(\alpha/2)^2 \approx 14\%$, where the 2 indicates the contribution from both the first upper sideband and the first lower sideband. More generally,

$$P_{\text{NTS}}^{\text{final}} \approx 2\left[\frac{J_1(n\alpha)}{J_0(n\alpha)}\right]^2 \approx 14\%, \quad (13)$$

where, α is obtained by solving $[J_1(\alpha)/J_0(\alpha)]^2 = 30/(1\times 10^9)$. Hence, 1 GW input power will ensure that the final NTS ratio is small. The first sideband is located at $1/(N\,2\,N_u)$, where $N_u = 1.8/0.069$, hence $1/(N\,2\,N_u) \approx 1.3\times 10^{-5}$. As discussed in Sec. VI.C of paper I, the FWHM bandwidth of the final radiation is 1.8×10^{-5}, hence a monochrometer may filter out this sideband. In short, to overcome the noise degradation, we would increase the power of the seed laser, and essentially shorten the modulator length and therefore also reduce the incoherent undulator radiation.

In the radiator, together with the HGHG FEL, there will be a SASE FEL. In order to reduce the contribution of the SASE FEL, we increase the energy modulation $\Delta\gamma$ produced in the modulator, and reduce the dispersion strength $d\psi/(d\gamma)$ in the dispersion section accordingly to keep $\Delta\gamma\times d\psi/(d\gamma)$ constant. Therefore, according to

Eq. (1) of paper I, i.e.,

$$b_n(r, z) = \exp\left[-\frac{1}{2}\sigma_\gamma^2\left(\frac{d\psi}{d\gamma}\right)^2\right]\left|J_n\left[\Delta\gamma\left(\frac{d\psi}{d\gamma}\right)\right]\right|,$$
(14)

the bunching factor b_n will increase. Now, according to Eq. (18) of paper I, i.e.,

$$P_1^{\mathrm{Coh}}(z) = \frac{Z_0 I_{\mathrm{peak}}^2}{8}\frac{1}{4\pi\sigma_x^2}\left(\frac{K[JJ]}{\gamma}\right)^2\left(\int_0^z \bar{b}_m(z)\,dz\right)^2,$$
(15)

the start-up coherent emission power $P^{\mathrm{Coh}} \propto |b_n|^2$. Hence, the start-up coherent emission power will increase. However, since the energy modulation is an effective energy spread, the power e-folding length increases to $L_{Gr} = 0.9$ m, which should be compared to the number for the case without energy modulation, i.e., $L_{Gr} = 0.6$ m, given in Fig. 3 of paper I. Because of the larger energy spread in the electron beam, the saturation power is also reduced. At about $L_{\mathrm{Rad.}} = 4$ m, the system has reached saturation. The SASE FEL is extremely small, since there are only about 4 power e-folding lengths. According to Eqs. (11) and (12), in such a radiator the SASE FEL is only about 2 kW. Recall that the HGHG FEL has a power of 10 GW, and now $N = 450/1.5 = 300$. Similar calculation gives us a final NTS ratio of about 2%, which is even smaller. In the radiator $N_u = 4.0/0.046$, hence the first sideband is located at $1/(N\,2\,N_u) \approx 1.9 \times 10^{-5}$. As discussed in Sec. VI.C of paper I, the FWHM bandwidth of the final radiation is 1.8×10^{-5}, hence a monochrometer may filter out this sideband. In short, by increasing the energy modulation from the modulator, and accordingly reducing the dispersion strength in the dispersion section, the noise produced in the radiator will be greatly reduced.

The contribution from the other undulators is even smaller becasue the harmonic number N is reduced along the device. Hence, the above analysis tells us that the undulator radiation effect should not be a serious problem.

B. Tolerance to the noise in the seed laser itself

Now let us study the noise in the seed laser itself. First, let us compute the noise power in the seed laser. Suppose that the 1-GW seed laser is amplified from an oscillator of average power of 500 mW with a repetition rate of 100 MHz. The pulse energy is $500 \times 10^{-3}/(100 \times 10^6) = 5 \times 10^{-9}$ J. Suppose the pulse is 100 fs long, then the peak power is $P = 5 \times 10^{-9}/(100 \times 10^{-15}) = 50$ kW. Since we need a 1-GW input seed laser, the gain is then $G = 1 \times 10^9/(50 \times 10^3) = 2 \times 10^4$. The seed laser is at $2,250$ Å, and the pulse length is $\tau = 100$ fs, therefore the

bandwidth is $\Delta\nu = 1/\tau$. The noise power is then [32]

$$\begin{aligned}
P_n &= G\,h\,\nu\,\Delta\nu \\
&= 2 \times 10^4 \times 6.626 \times 10^{-34} \\
&\quad \times \frac{3.0 \times 10^8}{2,250 \times 10^{-10}} \times \frac{1}{100 \times 10^{-15}} \\
&\approx 1.77\,W,
\end{aligned}$$
(16)

where h is the Plank constant. Hence, compared with the 1-GW design power, this noise power is negligible, even after the harmonic generation.

Now let us look at the noise in the phase. We model the seed laser as a coherent state. Then the phase variance σ_ϕ is related to the mean photon number $\bar{n}$ by [33]

$$\sigma_\phi = \frac{1}{2\sqrt{\bar{n}}}.$$
(17)

We will still use the above numerical example. The pulse energy is $E = 5 \times 10^{-9}$ J. Each photon energy is $h\,\nu$. Hence the number of photons is

$$\bar{n} = \frac{E}{h\,\nu} = \frac{5 \times 10^{-9}}{6.626 \times 10^{-34} \times \frac{3.0 \times 10^8}{2,250 \times 10^{-10}}} = 5.66 \times 10^9,$$
(18)

and according to Eq. (17), we have $\sigma_\phi \approx 6.65 \times 10^{-6}$. Therefore, in the final radiation, the corresponding variance of the phase is $N\sigma_\phi \approx 0.01$, which is very small.

The above analysis tells us that the noises in the seed laser power or in the phase are negligible.

III. NON-UNIFORMITY OF THE ELECTRON BEAM

In most published papers, researchers have dealt with a coasting electron beam. Since the electron beam is the amplification medium, non-uniformity of the electron beam will result in non-uniformity of the radiation. Any variation in the output of a HGHG stage will affect the final radiation at 1.5 Å. In Sec. IV of paper I, we have discussed how this non-uniformity of the electron beam current affect the radiation power. Due to the intrinsic stability mechanism, the radiation power fluctuation will be reduced in the multi stage HGHG. Here we study how this electron beam current non-uniformity affect the phase of the final radiation. We focus on the phase variation in the exponential growth region.

As we emphasized in paper I, there is exponential growth only in the radiator of the first stage. Also, since the harmonic number is reduced in the following stages, the effect on the final radiation is reduced. Hence, we focus on the first stage. If we use a multi-bunch scheme, and if the laser pulse is located at the center of the electron beam, then the variation of the electron beam density is at its minimum. But if we use one bunch for the entire device then, for the first stage, the seed laser pulse has to stay at the back end of the electron beam, where

the density variation is not minimum. We therefore study the case of using one electron bunch for the entire 5 stages and the final amplifier. If the electron bunch is flat-top longitudinally, then there will not be a phase variation, but if the bunch is longitudinally a Gaussian, then because of the variation of the density, the phase of the FEL also varies.

Here we use an 1-D model, following the notation of Refs. [4] and [16]. We focus only on the high-gain guided mode [4], $E(z,t) \propto e^{-i\lambda_1 2\rho k_w z}$, where $\lambda_1 = e^{i\frac{2\pi}{3}}$, the unstable solution of the well-known cubic equation [2] for the case of zero detune. The slowly varying part of the phase for this mode is then

$$\phi = \rho\, k_w\, z, \tag{19}$$

therefore,

$$\Delta\phi = \frac{\Delta\rho}{\rho}\frac{1}{2\sqrt{3}}\frac{z}{L_G^{1-D}}, \tag{20}$$

where $L_G^{1-D} = 1/(2\sqrt{3}\,\rho\, k_w)$, and ρ is the Pierce parameter [2], which is proportional to $n^{\frac{1}{3}}$, i.e.,

$$\rho = \rho_0\, e^{-\frac{1}{3}\frac{t^2}{2\sigma_t^2}}. \tag{21}$$

Suppose the seed laser is located at $t = \sigma_t$, then the variation of ρ is

$$\frac{\Delta\rho}{\rho} = -\frac{1}{3}\frac{t-\sigma_t}{\sigma_t} - \frac{1}{9}\frac{(t-\sigma_t)^2}{\sigma_t^2} + \frac{4}{81}\frac{(t-\sigma_t)^3}{\sigma_t^3}. \tag{22}$$

The linear term introduces a frequency shift and the quadratic term introduces a frequency chirp. Both of these can be corrected; the only term that will produce phase distortion is the cubic term. In fact this naturally induced frequency chirp can be helpful, if one wants to compress the FEL pulse.

A. Phase distortion

The first radiator length is about $4\,L_G$. Suppose the electron bunch rms length is $\sigma_t = 100$ fs, and the seed laser pulse length is $\tau = 10$ fs, then the phase distortion is

$$\begin{aligned}
\Delta\phi &= \frac{4}{81}\left(\frac{\tau}{\sigma_t}\right)^3\frac{1}{2\sqrt{3}}\frac{z}{L_G}\\
&= \frac{4}{81}\left(\frac{10}{100}\right)^3\frac{1}{2\sqrt{3}}\frac{4\,L_G}{L_G} \approx 5.7\times10^{-5}. \tag{23}
\end{aligned}$$

The wavelength of this radiation is 450 Å, hence the phase distortion in the final 1.5 Å radiation is $(450/1.5)\times 5.7\times10^{-5} \approx 0.017$, which is very small.

B. Frequency shift

The frequency shift due to the linear term is then

$$\Delta\omega = \frac{1}{3}\frac{1}{\sigma_t}\frac{1}{2\sqrt{3}}\frac{z}{L_G} \approx 3.85\times10^{12}. \tag{24}$$

Therefore,

$$\frac{\Delta\omega}{\omega} = \frac{3.85\times10^{12}}{\frac{2\pi c}{\lambda}} = \frac{3.85\times10^{12}}{\frac{2\pi 3.0\times10^8}{450\times10^{-10}}} \approx 9.2\times10^{-5}. \tag{25}$$

This frequency shift leads to a relative frequency shift of 3.1×10^{-7} in the final 1.5-Å radiation. This is negligible compared with the 1.8×10^{-5} bandwidth.

IV. EFFECT OF ENERGY CHIRP

The wakefields [31] in the beam line and the self-fields of the electron beam cause the electron to experience energy modulation along the beam. We here model this effect as a sinusoidal energy modulation:

$$\begin{aligned}
\gamma(t) &= \gamma_0 + \sqrt{2}\sigma_\gamma \sin\left(\frac{2\pi}{l}t\right)\\
&\approx \gamma_0 + \sqrt{2}\sigma_\gamma\left[\frac{2\pi}{l}t - \frac{1}{6}\left(\frac{2\pi}{l}t\right)^3\right], \tag{26}
\end{aligned}$$

where l is the electron bunch duration and $\sqrt{2}\sigma_\gamma$ is the amplitude of the energy modulation.

A. Phase distortion

The t^3 term in the above energy modulation will cause phase distortion in the radiation pulse. Again the most significant contribution comes from the first stage. The energy distortion leads to a phase distortion due to the dispersion strength in the modulator, the dispersion section and also the start-up region of the radiator. Let us compute the distortion of the phase in the 450-Å radiation. According to Eqs. (9), (10), and (11) of paper I, the phase distortion is

$$\begin{aligned}
\Delta\psi &= \sqrt{2}\sigma_\gamma\frac{1}{6}\left(\frac{2\pi}{l}t\right)^3\\
&\times \left(n\frac{2k_{u\,m}}{\gamma_0}\frac{1}{2}L_{mod} + \frac{d\psi}{d\gamma}\bigg|_{disp} + \frac{2k_{u\,r}}{\gamma_0}2L_{G\,r}\right) \tag{27}
\end{aligned}$$

Suppose that the electron pulse duration is 200 fs, and the seed laser is 10 fs long. We further assume that the energy modulation is the same as the local energy spread. We use the numbers in Fig. 3 of paper I to compute the

phase distortion:

$$
\begin{aligned}
\Delta\psi \;=\; & \sqrt{2} \times 5.0 \times 10^{-4} \frac{2 \times 10^{9}}{0.511 \times 10^{6}} \frac{1}{6} \left(\frac{2\pi \times 5}{200} \right)^{3} \\
& \times \left(5 \frac{2\frac{2\pi}{0.069}}{\frac{2 \times 10^{9}}{0.511 \times 10^{6}}} \frac{1}{2} 1.8 + 0.23 + \frac{2\frac{2\pi}{0.046}}{\frac{2 \times 10^{9}}{0.511 \times 10^{6}}} 2 \times 0.9 \right) \\
& \approx \; 0.001. \qquad (28)
\end{aligned}
$$

Hence, in the final 1.5-Å radiation, the phase distortion will be around $0.001 \times (450/1.5) = 0.3$. This requires some correction. If the energy modulation is produced in the beam line before the e-beam enters the final undulator, a carefully designed e-beam by-pass [27] beam line will wash out this energy modulation. Then the energy spread will be uniform along the electron beam.

B. Frequency shift

Now let us study the frequency shift due to the linear term. We still focus on the first stage:

$$
\begin{aligned}
\Delta\omega \;=\; & \sqrt{2}\sigma_\gamma \left(\frac{2\pi}{l} \right) \\
& \times \left(n \frac{2k_{u\,m}}{\gamma_0} \frac{1}{2} L_{mod} + \frac{d\psi}{d\gamma}\bigg|_{disp} + \frac{2k_{u\,r}}{\gamma_0} 2L_{G\,r} \right) (29)
\end{aligned}
$$

We will still use the numbers in Fig. 3 of paper I, thus

$$
\begin{aligned}
\Delta\omega \;=\; & \sqrt{2} \times 5.0 \times 10^{-4} \frac{2 \times 10^{9}}{0.511 \times 10^{6}} \frac{2\pi}{200 \times 10^{-15}} \\
& \times \left(5 \frac{2\frac{2\pi}{0.069}}{\frac{2 \times 10^{9}}{0.511 \times 10^{6}}} \frac{1}{2} 1.8 + 0.23 + \frac{2\frac{2\pi}{0.046}}{\frac{2 \times 10^{9}}{0.511 \times 10^{6}}} 2 \times 0.9 \right) \\
& \approx \; 4.9 \times 10^{13}. \qquad (30)
\end{aligned}
$$

This frequency shift leads to a relative frequency shift in the 450-Å radiation of

$$
\frac{\Delta\omega}{\omega} = \frac{\Delta\omega}{\frac{2\pi c}{\lambda_r}} = \frac{\Delta\omega}{\frac{2\pi 3.0 \times 10^{8}}{450 \times 10^{-10}}} \approx 1.2 \times 10^{-3}. \qquad (31)
$$

Thus, in the final 1.5-Å radiation, the relative frequency shift is about 3.9×10^{-6}. This is small compared with the 1.8×10^{-5} bandwidth.

C. Frequency jitter due to the timing jitter

Let us now study the frequency jitter due to the timing jitter. We write the energy modulation as

$$
\begin{aligned}
\gamma(t) \;=\; & \gamma_0 + \sqrt{2}\sigma_\gamma \sin\left[\frac{2\pi}{l}(t - t_0) \right] \\
\approx\; & \gamma_0 + \sqrt{2}\sigma_\gamma \left[-\sin\left(\frac{2\pi}{l} t_0 \right) + \frac{2\pi t}{l} \cos\left(\frac{2\pi}{l} t_0 \right) \right],
\end{aligned}
$$
$$
(32)
$$

where we explicitly use t_0 to stand for the location of the center of the laser pulse. This should be compared with Eq. (26). Now, because of the time jitter, t_0 is varying from time to time. Therefore the energy modulation has a jitter, which is also a function of t_0:

$$
\delta\gamma(t, t_0) = \sqrt{2}\sigma_\gamma \frac{2\pi t}{l} \cos\left(\frac{2\pi}{l} t_0 \right). \qquad (33)
$$

Because of the dispersion strength in the undulators and in the dispersion section, this energy variation will lead to phase variation along the electron bunch. But since the amount of variation is a function of t_0, it has a jitter. Let us still focus on the first stage, and look at this variation in the 450 Å radiation. Now the frequency shift is a function of the location t_0,

$$
\begin{aligned}
\Delta\omega \;=\; & \sqrt{2}\sigma_\gamma \frac{2\pi}{l} \cos\left(\frac{2\pi}{l} t_0 \right) \\
& \times \left(n \frac{2k_{u\,m}}{\gamma_0} \frac{1}{2} L_{mod} + \frac{d\psi}{d\gamma}\bigg|_{disp} + \frac{2k_{u\,r}}{\gamma_0} 2L_{G\,r} \right) \\
\equiv\; & (\Delta\omega)_{max} \cos\left(\frac{2\pi}{l} t_0 \right) \\
=\; & (\Delta\omega)_{max} \cos\left[\frac{2\pi}{l}(t_{00} + \Delta t_0) \right] \\
\approx\; & (\Delta\omega)_{max} \left[\cos\left(\frac{2\pi}{l} t_{00} \right) - \sin\left(\frac{2\pi}{l} t_{00} \right) \frac{2\pi}{l} \Delta t_0 \right],
\end{aligned}
$$
$$
(34)
$$

where we set $t_0 = t_{00} + \Delta t_0$ to show the jitter explicitly. This should be compared with Eq. (29). In fact, the frequency shift given in Eq. (29) is the maximum, i.e., for the worst case. Now, from Eq. (34), we find that the maximum frequency shift jitter is

$$
[\delta(\Delta\omega)]_{max} = (\Delta\omega)_{max} \frac{2\pi\Delta t_0}{l}. \qquad (35)
$$

Still taking the numbers in Fig. 3 of paper I, and making use of the result in Eq. (30), we have

$$
[\delta(\Delta\omega)]_{max} = 4.9 \times 10^{13} \frac{2\pi \times 10}{200} \approx 1.54 \times 10^{13}, \quad (36)
$$

where we have assumed the timing jitter to be 10 fs. This leads to a relative frequency jitter of 3.68×10^{-4} in the 450-Å radiation. Therefore, in the final 1.5-Å radiation, the relative frequency jitter is 1.2×10^{-6}. This is negligible compared with the 1.8×10^{-5} bandwidth of the final 1.5-Å radiation. In a realistic case, the timing jitter is around 100 fs, then we should not use the final expansion in Eq. (34). A simple analysis on $(\Delta\omega)_{max} \cos[(2\pi/l)(t_{00} + \Delta t_0)]$ directly tells us $[\delta(\Delta\omega)]_{max} = 9.8 \times 10^{13}$. This leads to a frequency shift jitter of 7.8×10^{-6}, which is still smaller than the bandwidth of the final radiation at 1.5-Å.

It is interesting to explore Eq. (34) a little more. As we pointed out above, Eq. (29) is in fact the maximum frequency shift due to a sinusoidal energy modulation given

by Eq. (26), or by Eq. (32) in a more explicit way. Now Eq. (34) tells us that, at $t_{00} = 0$, there is a maximum frequency shift $(\Delta\omega)_{max}$, but the frequency shift jitter is the minimum, i.e., $\delta(\Delta\omega) = 0$, since $\sin[(2\pi/l)t_{00}]\,|_{t_{00}=0} = 0$. At $(2\pi/l)t_{00} = \pi/2$, $\Delta\omega = 0$, but $\delta(\Delta\omega)$ reaches maximum. In short, Eq. (29) and Eq. (35) both give the value for the worst case. To conclude, for the electron beam of the DESY TTF project, where $\sigma_\gamma/\gamma = 5.0 \times 10^{-4}$, a sinusoidal energy modulation with an amplitude of $\sqrt{2}\sigma_\gamma$ will not affect the final 1.5 Å radiation seriously.

V. EMITTANCE EFFECTS

In this section, we will study the emittance effect in the undulator and also in the dispersion section. As we will show, the phase mixing due to the emittance in the dispersion section is much smaller than that in the undulator. This is what we call Natural Emittance Effect Reduction (NEER) mechanism.

A. Wigglers

In a planar wiggler with a parabolic pole face, the magnetic field reads [34]

$$
\begin{aligned}
\vec{B_w} = &-\frac{B_{w0}}{k_y}\{\hat{x}k_x\sinh(k_x x)\sinh(k_y y)\sin(k_w z)\\
&+ \hat{y}k_y\cosh(k_x x)\cosh(k_y y)\sin(k_w z)\\
&+ \hat{z}k_w\cosh(k_x x)\sinh(k_y y)\cos(k_w z)\},
\end{aligned}
\tag{37}
$$

with

$$
k_x^2 + k_y^2 = k_w^2.
\tag{38}
$$

As usual, z is the electron propagating direction, x is the horizontal direction, and y is the vertical direction. We are interested in equal focusing in the x and y planes. Hence,

$$
k_x = k_y = \frac{k_w}{\sqrt{2}}.
\tag{39}
$$

Solving the single-electron kinetic equations, we find for the transverse slow betatron motion,

$$
x_\beta = x_{\beta 0}\cos\left(k_{\beta n}z + \phi_x\right),
\tag{40}
$$

and

$$
y_\beta = y_{\beta 0}\cos\left(k_{\beta n}z + \phi_y\right),
\tag{41}
$$

where $k_{\beta n} = [K/(2\gamma)]k_w$, with $k_w = 2\pi/\lambda_w$, and λ_w is the wiggler period; and $K = e\,B_{w0}/(m\,c\,k_w)$ is the wiggler parameter. Hence the wiggler provides natural focusing [34].

Superimposed on the slow betatron motion is the fast wiggling motion,

$$
x'_w \approx -\frac{K}{\gamma}\left(1 + \frac{1}{2}k_x^2 x_\beta^2 + \frac{1}{2}k_y^2 y_\beta^2\right)\cos(k_w z).
\tag{42}
$$

Averaging out the fast wiggling motion, we find the longitudinal velocity to be

$$
\beta_\parallel \approx 1 - \frac{1 + \frac{K^2}{2}}{2\gamma^2} - \frac{1}{2}k_{\beta n}^2\left(x_{\beta 0}^2 + y_{\beta 0}^2\right).
\tag{43}
$$

If we assume a round beam, then, in the 'Courant-Snyder' notation [35],

$$
\beta_\parallel \approx 1 - \frac{1 + \frac{K^2}{2}}{2\gamma^2} - k_{\beta n}\epsilon,
\tag{44}
$$

where, $\epsilon = k_{\beta n}x_{\beta 0}^2 = k_{\beta n}y_{\beta 0}^2$ is the geometric emittance of the electron beam.

The variation of the ponderomotive phase, $\psi = (k_s + k_w)z - \omega_s t$, along the propagation is

$$
\frac{d\psi}{dz} = k_w + k_s - k_s\,\beta_\parallel^{-1} \approx k_w\frac{2\Delta\gamma}{\gamma_0} - k_s\,k_{\beta n}\,\epsilon.
\tag{45}
$$

Hence the emittance acts like an effective energy spread, which is

$$
\frac{\sigma_\gamma}{\gamma}{}^{\text{undul}}_{\text{eff},\epsilon} = \frac{k_s\,k_{\beta n}}{2\,k_w}\,\epsilon.
\tag{46}
$$

This can be generalized for a general focusing case to get

$$
\frac{\sigma_\gamma}{\gamma}{}^{\text{undul}}_{\text{eff},\epsilon} = \frac{k_s\,k_\beta}{2\,k_w}\,\epsilon.
\tag{47}
$$

Note that k_β is determined by both the natural focusing and a possible external focusing, as long as the 'smooth approximation' [36] is adopted. This means that we neglect the variation of the betatron function.

B. Dispersion Section

In the dispersion section, the emittance-induced path-length difference is far smaller than that due to the local energy spread. This is the key point why in the HGHG scheme the emittance will be a less important factor, and it suggests a new operation mode, i.e., to adopt an electron beam with higher current, though unavoidably with higher emittance. To illustrate this, let us first compute an ideal case, i.e., we assume that the idealized dispersion section is divided into three sections with total length L_s. The field is

$$
B(z) = \begin{cases}
B & 0 \le z < \frac{L_s}{4}\\
-B & \frac{L_s}{4} < z < \frac{3L_s}{4}\\
B & \frac{3L_s}{4} < z \le L_s,
\end{cases}
\tag{48}
$$

The x-z plane is the bending plane, and y is only a drift space.

It is easy to find that:

- If the electron has an initial radial off-set but no angular off-set, then the path it takes will have the same length as the one without radial off-set.

Stage 1			Stage 2		
Mod.	Disp.	Rad.	Mod.	Disp.	Rad.
3.1×10^{-5}	1.0×10^{-5}	1.0×10^{-4}	1.0×10^{-4}	5.5×10^{-5}	3.6×10^{-4}
Stage 3			Stage 4		
1.4×10^{-5}	3.6×10^{-7}	4.5×10^{-5}	4.5×10^{-5}	2.6×10^{-6}	1.3×10^{-4}
Stage 5					
1.3×10^{-4}	5.9×10^{-6}	3.0×10^{-4}			

TABLE I: The effective relative local energy spread σ_γ/γ due to the emittance ϵ of the electron beam.

- In the x-z plane, the one with an initial angular off-set, x', will lead to a path difference of

$$\Delta s_{x'} = 2R\{\arcsin[\sin\theta + \sin x'] + \arcsin[\sin\theta - \sin x']\} - 4R\theta$$
$$\approx \frac{L_s}{2} x'^2, \tag{49}$$

where R is the bending radius, and θ is the bending angle of each dipole. In the y-z plane, it is just a drift, hence,

$$\Delta s_{y'} = \frac{L_s}{2} y'^2. \tag{50}$$

- The energy off-set of an electron will lead to a path-length difference of

$$\Delta s_{\Delta\gamma} = -\frac{1}{48} \frac{L_s^3}{R^2} \frac{\Delta\gamma}{\gamma}. \tag{51}$$

Hence, in an idealized dispersion section, the emittance acts like an effective energy spread, which is

$$\frac{\sigma_\gamma}{\gamma}{}^{\text{disp}}_{\text{eff},\epsilon} = \frac{48 R^2 \epsilon}{L_s^2 \beta} \approx \frac{k_s L_s \epsilon}{\beta \gamma \frac{d\psi}{d\gamma}{}_{\text{disp}}}, \tag{52}$$

where k_s is the wavenumber in the radiator. In getting the final expression, we made an approximation and kept the first non-zero order in an expansion with respect to the bending angle. This is justified because the bending angle is always small.

Let us now analyze the device in Fig. 3 of paper I. Shown in Table I, are the effective relative local energy spread due to the emittance in the dispersion sections and the undulators in each stage. In our calculation, we use $L_d = 0.32$ m. There are 5 stages as in Fig. 3 of paper I: 'Mod.' stands for the modulator; 'Disp.', the dispersion section; and 'Rad.', the radiator. We found that the effective relative local energy spread due to the emittance in the dispersion section is far smaller than that in the undulators. Now recall that, in an HGHG FEL bunching is produced mainly in the dispersion section, while in a SASE FEL, bunching is produced in the undulators. Hence, the emittance effect in an HGHG FEL is much smaller than that in a SASE FEL. This is called the Natural Emittance Effect Reduction (NEER) mechanism. The NEER mechanism suggests a new operation mode, i.e., we may adopt an electron beam with a higher current, even though unavoidably higher emittance, in the Harmonic Generation stages, i.e., in the converter, though in the amplifier we would still use a low-emittance electron beam. Detailed investigation is underway [37].

VI. CONCLUSION AND DISCUSSION

In this paper, we study the noise tolerance for the new scheme of x-ray production by cascading stages of an HGHG FEL. Study shows that by increasing the seed laser power, this noise effect in the first modulator could be under control. In the first radiator, the noise is greatly suppressed, if we increase the energy modulation from the first modulator, and accordingly reduce the dispersion strength in the dispersion section. We also study the phase error induced by a possible non-uniformity in the electron beam current profile. Further more, phase error due to wakefields and self-fields is studied. Timing jitter induced frequency jitter is also studied. We then show that the emittance requirement in the HGHG scheme is less stringent than that in the SASE scheme. This is due to the NEER mechanism. A concrete example is the device shown in Fig. 3 of paper I, with the results summarized in Table I. This then suggests a new design possibility, i.e., one may adopt high current, though unavoidably high emittance electron beam in the harmonic generation stages.

In conclusion, based on the detailed study in this paper, we find that production of x-ray with good longitudinal coherence by cascading stages of a High-Gain Harmonic Generation Free-electron Laser is very promising, though challenging.

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