Cosmology Quantized in Cosmic Time[*]

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This paper discusses the problem of inflation in the context of Friedmann-Robertson-Walker Cosmology. We show how, after a simple change of variables, to quantize the problem in a way which parallels the classical discussion. The result is that two of the Einstein equations arise as exact equations of motion and one of the usual Einstein equations (suitably quantized) survives as a constraint equation to be imposed on the space of physical states. However, the Friedmann equation, which is also a constraint equation and which is the basis of the Wheeler-deWitt equation, acquires a welcome quantum correction that becomes significant for small scale factors. We discuss the extension of this result to a full quantum mechanical derivation of the anisotropy $(\delta\rho/\rho)$ in the cosmic microwave background radiation, and the possibility that the extra term in the Friedmann equation could have observable consequences. To clarify the general formalism and explicitly show why we choose to weaken the statement of the Wheeler-deWitt equation, we apply the general formalism to de Sitter space. After exactly solving the relevant Heisenberg equations of motion we give a detailed discussion of the subtleties associated with defining physical states and the emergence of the classical theory. This computation provides the striking result that quantum corrections to this long wavelength limit of gravity eliminate the problem of the *big crunch*. We also show that the same corrections lead to possibly measurable effects on the CMB radiation. For the sake of completeness, we discuss the special case, $\Lambda = 0$, and its relation to Minkowski space. Finally, we suggest interesting ways in which these techniques can be generalized to cast light on the question of chaotic or eternal inflation. In particular, we suggest one can put an experimental lower bound on the distance to a universe with a scale factor very different from our own, by looking at its effects on our CMB radiation.

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I. INTRODUCTION

The COBE[1] and WMAP[2][3][4] measurements of the anisotropy in the cosmic microwave background(CMB) radiation agree remarkably well the predictions of slow-roll inflation[5]. This agreement provides a strong reason to believe that the paradigm for computing the fluctuations[6] in $\delta\rho/\rho$ is correct. Perhaps the most striking feature of this result is that they represent an imprinting of the structure of the quantum state of the field theory, at the time inflation begins, onto the electromagnetic radiation that comes to us from the surface of last scattering. Unfortunately, derivations of this effect usually mix classical and quantum ideas and so, it is difficult to determine how they would change given a fully quantum mechanical treatment. This paper fills this gap. We begin by showing how, working in fixed, co-moving coordinates, one can canonically quantize the theory of the Friedmann-Robertson-Walker(FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x} \cdot d\vec{x},$$

(1)

and the spatially constant part of the inflaton field, $\Phi(t)$, in a straightforward manner. We then show that the quantized system has states for which the expectation values of the scale factor and inflaton field satisfy the equations associated with the inflationary scenario. This, of course means that starting in one of these states one can construct the usual perturbative analysis but, with the added benefit, that the formalism will automatically generate terms responsible for back reaction.

It is important to emphasize that our approach assumes that getting quantum mechanics to describe the evolution of the system in *cosmic time* is paramount. For this reason we find that we cannot impose a strong form of the Wheeler-deWitt equation. In our formalism, geometry, which is defined by the condition that the Einstein equations

be true, is an emergent phenomenon. It exists only for some quantum states and then, only when the scale factor becomes large. To clarify the subtle way in which this works we begin by setting out the general formalism and then, we exactly solve our equations for the case of de Sitter space. Next, we identify a class of states which correspond to systems which, at large times, behave both classically and in complete accord with the full set of Einstein equations. As we will show, a bonus of this approach is that the quantum corrections to the Einstein equations, which become important when the scale factor is small, completely eliminate the problem of the big crunch. In the latter sections of this paper we discuss ways to extend this derivation to compute possible experimental consequences of this extra term.

We should note that some of the results presented in this paper were discussed in two earlier preprints[7].

II. THE TWO MEANINGS OF "CLASSICAL"

There are, in fact, two ways in which current derivations of $\delta\rho/\rho$ invoke classical arguments. First, these derivations begin by treating both the scale factor $a(t)$ and the spatially constant part of the inflaton field, $\Phi(t)$, as classical time-dependent background fields. One then studies the physics of the classical action

$$\mathcal{S} = \mathbf{V} \int dt \sqrt{-g} \left[\frac{R(t)}{2\kappa^2} + \frac{1}{2} \frac{d\Phi(t)}{dt}^2 - V(\Phi(t)) \right]. \tag{2}$$

The second appearance of classical ideas occurs when one adds back spatially varying fluctuations in the Newtonian potential and the inflaton field as quantum operators. Many familiar derivations then discuss the perturbative evolution of these fields in the time-dependent background of the classical solution and, at an appropriate point in the discussion, say "and then the field goes classical". This much less important introduction of classical ideas is used to convert the quantum computation of the two-point correlation function for the density operator to an ensemble average of gaussian fluctuations. In reality, this statement is just a way of avoiding any discussion of the physics of *squeezed states* and *quantum non-demolition variables*[8]. While we do not discuss this issue in this paper, we will return to it in a longer, more pedagogical paper, which is in preparation. This longer paper will show how to extend the results presented here to a full quantum treatment of $\delta\rho/\rho$. The point we wish to emphasize at this juncture is that a full quantum treatment of the spatially constant part of the problem, appropriately extended to include the spatially varying modes of the fields to second order, provides a complete quantum picture of all of the physics which can be experimentally tested in the foreseeable future.

III. THE CLASSICAL PROBLEM

Before discussing our approach to the quantum treatment of FRW cosmology it is important to demonstrate that the classical version of our formalism does no violence to the usual Einstein theory. We demonstrate this in this section. In the next section we show how to canonically quantize the same theory.

Simplifying the usual derivations of the Einstein equations for FRW cosmology is easily accomplished if one observes that experimentally we are dealing with a spatially flat universe and so it is perfectly adequate to formulate the problem in a definite coordinate system. In the discussion which follows, we take this to be co-moving coordinates in which the metric takes the general form shown in Eq.1.

We already noted that, restricting attention to the classical problem of a scalar field in an FRW cosmology, the action reduces to the form shown in Eq.2, where $\mathbf{V}$ is the volume of the region in which the theory is being defined, $\sqrt{-g} = a(t)^3$ and the scalar curvature times $\sqrt{-g}$ is given by

$$\sqrt{-g}\, R(t) = \frac{3}{\kappa^2}\, a(t) \frac{da(t)}{dt}^2 + \frac{3}{\kappa^2}\, a(t)^2 \frac{d^2 a(t)}{dt^2}. \tag{3}$$

(Clearly, when we generalize to the computation of $\delta\rho/\rho$, the volume, $\mathbf{V}$, must be taken to be larger than the horizon volume at the time of inflation in order to avoid edge effects.)

Substituting these expressions into Eq.2 and integrating by parts, to eliminate the term with $d^2 a(t)/dt^2$, we obtain

$$\mathcal{S} = \mathbf{V} \int dt \left[-\frac{3}{\kappa^2} a(t) \left(\frac{da(t)}{dt} \right)^2 + \frac{1}{2} a(t)^3 \left(\frac{d\Phi(t)}{dt} \right)^2 - a(t)^3 V(\Phi(t)) \right]. \tag{4}$$

Next, to simplify the analysis of the quantum version of this problem, we make the change of variables $u(t)^2 = a(t)^3$, which leads to the action

$$S = \mathbf{V} \int dt \left[-\frac{4}{3\kappa^2} \left(\frac{du(t)}{dt} \right)^2 + \frac{1}{2} u(t)^2 \left(\frac{d\Phi(t)}{dt} \right)^2 - u(t)^2 V(\Phi(t)) \right]. \tag{5}$$

This change of variables merely simplifies the classical discussion, however it has a greater significance for the quantized theory. This is because we can choose $-\infty \leq u \leq \infty$, whereas the only physically allowable range for a is $0 \leq a \leq \infty$. It is only for the space of square-integrable functions on the interval $-\infty \leq u \leq \infty$ that the Heisenberg equations of motion can be obtained by canonical manipulations.

There are only two Euler-Lagrange equations for this system:

$$\frac{8}{3\kappa^2} \frac{d^2 u(t)}{dt^2} + 2u(t) \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 - V(\Phi(t)) \right) = 0 \quad \text{and} \quad -u(t)^2 \left(\frac{d^2\Phi(t)}{dt^2} + 3\mathcal{H}(t) \frac{d\Phi(t)}{dt} + \frac{dV(\Phi)}{d\Phi(t)} \right) = 0; \tag{6}$$

where the Hubble parameter, $\mathcal{H}$, is defined as

$$\mathcal{H} = \frac{1}{a(t)} \frac{da(t)}{dt} = \frac{2}{3u(t)} \frac{du(t)}{dt}. \tag{7}$$

Thus, by quantizing in this fixed gauge, we fail to obtain the full set of Einstein equations. The missing equations are the Friedmann equation and its time derivative

$$\mathcal{H}(t)^2 = \frac{\kappa^2}{3} \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 + V(\Phi(t)) \right) \quad \text{and} \quad \frac{d\mathcal{H}(t)}{dt} = -\frac{\kappa^2}{2} \left(\frac{d\Phi(t)}{dt} \right)^2. \tag{8}$$

A sophisticated way of explaining why we fail to obtain these equations is to note that by fixing the form of the metric to be that given in Eq.1, we have lost the freedom to vary the lapse and shift functions. But this is what we must do to obtain the missing equations from a Lagrangian formulation. This predicament is not unique to gravity; it occurs in ordinary electrodynamics if one chooses $A_0 = 0$ gauge. As is well known, in this gauge we obtain all of the Maxwell equations except Coulomb's law, $\vec{\nabla} \cdot \vec{E} - \rho = 0$, as exact equations of motion. However, it follows from the equations we do have, that Coulomb's law commutes with the evolution, *i.e.* if we set it equal to zero it remains zero. Hence, in this gauge, while there are many solutions to the equations of motion, we can select the ones we choose to call physical by imposing an extra time-independent constraint.

The situation with the Friedmann equation and its time derivative is analogous to the situation in electrodynamics. We will now show that while Eqs.8, are not equations of motion, if they are imposed at any one time, then they will continue to be true at all later times. (In other words they are constraint equations.)

To prove these constraints are preserved by the equation of motion we begin by differentiating $\mathcal{H}$ with respect to t to obtain

$$\frac{d^2 u(t)}{dt^2} = \frac{3u(t)}{2} \left(\frac{d\mathcal{H}(t)}{dt} + \frac{3}{2} \mathcal{H}(t)^2 \right). \tag{9}$$

Substituting this into Eq.6 and rearranging terms we obtain

$$\frac{2u(t)}{\kappa^2} \left(2\frac{d\mathcal{H}(t)}{dt} + 3\mathcal{H}(t)^2 + \kappa^2 \left(\frac{d\Phi(t)}{dt} \right)^2 - \kappa^2 \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 + V(\Phi(t)) \right) \right) = 0, \tag{10}$$

which can be immediately rewritten in the form

$$\frac{2u(t)}{\kappa^2} \left[\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2 \left(\frac{d\Phi(t)}{dt} \right)^2 \right) + 3 \left(\mathcal{H}(t)^2 - \frac{\kappa^2}{3} \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 + V(\Phi(t)) \right) \right) \right] = 0. \tag{11}$$

If we, for convenience, define

$$\mathbf{G} = \mathcal{H}(t)^2 - \frac{\kappa^2}{3} \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 + V(\Phi(t)) \right), \tag{12}$$

the equation of motion for $\Phi(t)$ implies

$$\frac{d\mathbf{G}}{dt} = 2\mathcal{H}(t)\frac{d\mathcal{H}(t)}{dt} + \kappa^2\mathcal{H}(t)\left(\frac{d\Phi(t)}{dt}\right)^2 = 2\mathcal{H}(t)\left(\frac{d\mathcal{H}(t)}{dt} + \frac{\kappa^2}{2}\left(\frac{d\Phi(t)}{dt}\right)^2\right). \tag{13}$$

The missing Einstein equations are equivalent to requiring that both $\mathbf{G}$ and $d\mathbf{G}/dt$ vanish for all time. Substituting these definitions into Eq.11, we obtain the exact equation of motion

$$\frac{2u(t)}{\kappa^2}\left(\frac{1}{\mathcal{H}(t)}\left(\frac{d\mathbf{G}}{dt}\right) + 3\mathbf{G}\right) = 0. \tag{14}$$

This equation shows that if, at time $t = t_0$, $\mathbf{G} = 0$, then the exact equation of motion implies $d\mathbf{G}/dt$ will also vanish. Given that this equation is a first order differential equation for $\mathbf{G}(t)$, it follows that $\mathbf{G}(t) = 0$ exactly. In other words, we arrive at the desired result. The Friedmann equation is, in analogy to Coulomb's law in $A_0 = 0$ gauge, a constraint which can be imposed at a single time and which will continue to be true at all later times. We will show that a similar theorem can be proven for the quantum theory; however, we will argue that the analogy with QED is not perfect.

Before moving on to the quantum theory, let us spend a few moments discussing the Hamiltonian version of the classical theory. We do this to show why it is possible to confuse the Friedmann equation with the Hamiltonian at the classical level .

Following the usual prescription, we vary Eq.4 with respect to du/dt and $d\Phi/dt$ to obtain

$$p_u = -\mathbf{V}\frac{8}{3\kappa^2}\frac{du(t)}{dt} \quad ; \quad p_\Phi = \mathbf{V}u^2\frac{d\Phi(t)}{dt}. \tag{15}$$

We then construct the Hamiltonian

$$\mathbf{H} = p_u\frac{du(t)}{dt} + p_\Phi\frac{d\Phi(t)}{dt} - \mathcal{L} = -\frac{3\kappa^2}{16\mathbf{V}}p_u^2 + \frac{1}{2\mathbf{V}u^2}p_\phi^2 + \mathbf{V}u^2V(\Phi) \tag{16}$$

An important feature of this Hamiltonian is that due to the minus sign in front of the p_μ^2 term, it has no minimum. Fortunately, this doesn't matter. To see this, we simply rewrite the Hamiltonian in terms of du/dt and $d\Phi/dt$. This leads to the expression

$$\mathbf{H} = \mathbf{V}\left[-\frac{4}{3\kappa^2}\left(\frac{du(t)}{dt}\right)^2 + u^2\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right)\right]. \tag{17}$$

Substituting the definition of $\mathcal{H}$, this becomes

$$\mathbf{H} = -\mathbf{V}u^2\left[\frac{4}{3\kappa^2}\frac{1}{u^2}\left(\frac{du(t)}{dt}\right)^2 - \left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right)\right] = -\mathbf{V}u^2\left[\frac{3\mathcal{H}^2}{\kappa^2} - \left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right)\right]$$

$$\mathbf{H} = -\mathbf{V}\frac{3u^2}{\kappa^2}\mathbf{G}. \tag{18}$$

This shows that the Hamiltonian, $\mathbf{H}$, is proportional to the constraint, $\mathbf{G}$. It follows that setting $\mathbf{G} = 0$ means $\mathbf{H} = 0$, which tells us that the Hamiltonian vanishes for physical solutions. In other words, if we start a system out at $t = t_0$ in a configuration which has zero energy, it will stay at zero energy and never explore the region of arbitrarily negative energy. The identification of the Hamiltonian with the constraint equation is the content of the Wheeler-DeWitt equation.

IV. CANONICAL QUANTIZATION OF THE THEORY

Now that we have seen that our formalism, including the change of variables from $a(t)$ to $u(t)$, does no violence to the classical theory, we will proceed to a discussion of the quantum mechanics.

Starting from the classical Lagrangian, we define the quantum Hamiltonian

$$\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}}p_u^2 + \frac{1}{2\mathbf{V}u^2}p_\Phi^2 + \mathbf{V}u^2V(\Phi) \tag{19}$$

where the operators u, Φ and their conjugate momenta have the commutation relations

$$[p_u, u] = -i \quad ; \quad [p_\Phi, \Phi] = -i. \tag{20}$$

All other commutators vanish. To derive the Heisenberg equations of motion, note that for any operator $\mathbf{O}$, the Heisenberg operator is $O(t) = e^{i\mathbf{H}t} \mathbf{O} e^{-i\mathbf{H}t}$. Commuting $\mathbf{H}$ with the operators u and Φ, we obtain

$$
\begin{aligned}
\frac{du(t)}{dt} &= i\,[\mathbf{H}, u] = -\frac{3\kappa^2}{8\mathbf{V}} p_u \\
\frac{d\Phi(t)}{dt} &= i\,[\mathbf{H}, \Phi] = \frac{1}{u^2\mathbf{V}} p_\Phi \\
\frac{d^2u(t)}{dt^2} &= i\left[\mathbf{H}, \frac{du(t)}{dt}\right] = -\frac{3\kappa^2}{4} u \left[\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 - V(\Phi)\right] \\
\frac{d^2\Phi(t)}{dt^2} &= \frac{3\kappa^2}{16\mathbf{V}} \left(\frac{1}{u^2} p_u \frac{1}{u} + \frac{1}{u} p_u \frac{1}{u^2} + p_u \frac{1}{u^3} + \frac{1}{u^3} p_u\right) p_\Phi - \frac{dV(\Phi)}{d\Phi}.
\end{aligned}
\tag{21}
$$

This shows that the two dynamical equations of the classical theory are also exact operator equations of motion in the quantum theory. What is missing, as in the classical theory, are the constraint equations. In order to find the constraint equations that commute with the Hamiltonian, we begin by rewriting the equation for Φ in the suggestive form

$$\frac{d^2\Phi(t)}{dt^2} + 3\mathcal{H}\frac{d\Phi(t)}{dt} + \frac{dV(\Phi)}{d\Phi} = 0, \tag{22}$$

where the quantum version of the *Hubble* operator $\mathcal{H}$ is perforce

$$\mathcal{H} = -\frac{\kappa^2}{8\mathbf{V}}\left(p_u\frac{1}{u} + \frac{1}{u^3} p_u u^2\right). \tag{23}$$

Next we compute its time derivative from the equation

$$\frac{d\mathcal{H}}{dt} = i\,[\mathbf{H}, \mathcal{H}]. \tag{24}$$

Finally, to find the quantum version of the conserved constraint operator, $\mathbf{G}$, we follow the classical procedure and write

$$\frac{d^2u(t)}{dt^2} = \frac{3u}{2}\left(\frac{d\mathcal{H}}{dt} + \frac{3}{2}\mathcal{H}^2 - \frac{9\kappa^4}{128\mathbf{V}^2 u^4}\right). \tag{25}$$

The extra term is the quantum correction to the classical formula. It is obtained by explicitly taking the difference between the expression for d^2u/dt^2 and the combination $(3u/2)\left(d\mathcal{H}/dt + 3\mathcal{H}^2/2\right)$. (This step involves commutator gymnastics better left to Maple.) Once again, paralleling the classical discussion, we substitute the expression for d^2u/dt^2 into the Heisenberg equation of motion for u, obtaining

$$\frac{3u}{2}\left(\frac{d\mathcal{H}}{dt} + \frac{3}{2}\mathcal{H}^2 - \frac{9\kappa^4}{128\mathbf{V}^2 u^4}\right) + \frac{3\kappa^2 u}{4}\left(\left(\frac{d\Phi(t)}{dt}\right)^2 - \left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right)\right) = 0. \tag{26}$$

At this point it is tempting to parallel the classical discussion and define the operator

$$\mathbf{G} = \mathcal{H}^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right) + \mathbf{Q}, \tag{27}$$

where

$$\mathbf{Q} = -\frac{3\kappa^4}{64\mathbf{V}^2 u^4}, \tag{28}$$

and show that if it annihilates a state at any one time, then it annihilates it for all times. We will now show that this can be done, however we will then argue that identifying the kernel of this operator with the space of physical states is incorrect. To proceed with the proof substitute this definion into Eq.26 to obtain the operator equation of motion

$$\frac{3u}{4}\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2 + 3\mathbf{G}\right) = 0. \tag{29}$$

This is almost what we need to show that the space of physical states, defined to be those which obey the condition $\mathbf{G}(t)|\psi\rangle = 0$, is invariant under Hamiltonian evolution. Clearly, we will be able to use Eq.29 to complete the proof if we can show that there exists an operator $\mathbf{A}$ such that

$$\frac{d\mathbf{G}}{dt} = \mathbf{A}\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2\right). \tag{30}$$

To find $\mathbf{A}$, explicitly compute

$$\begin{aligned}
\frac{d\mathbf{G}}{dt} &= i\left[\mathbf{H}, \mathbf{G}\right] \\
&= \mathcal{H}\frac{d\mathcal{H}(t)}{dt} + \frac{d\mathcal{H}(t)}{dt}\mathcal{H} + \kappa^2\mathcal{H}\left(\frac{d\Phi(t)}{dt}\right)^2 + [\mathbf{H}, \mathbf{Q}] \\
&= \mathcal{H}\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2\right) + \left[\frac{d\mathcal{H}(t)}{dt}, \mathcal{H}\right] + [\mathbf{H}, \mathbf{Q}],
\end{aligned} \tag{31}$$

and substitute this result into Eq.30. The resulting equation can then be rearranged into the form

$$(\mathbf{A} - \mathcal{H})\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2\right) = [\mathbf{H}, \mathbf{Q}] + \left[\frac{d\mathcal{H}(t)}{dt}, \mathcal{H}\right]. \tag{32}$$

Solving this equation for $\mathbf{A}$, we obtain

$$\mathbf{A} = \mathcal{H} + \left([\mathbf{H}, \mathbf{Q}] + \left[\frac{d\mathcal{H}(t)}{dt}, \mathcal{H}\right]\right)\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2\right)^{-1}, \tag{33}$$

which allows us to rewrite the Heisenberg equation of motion for u as

$$\frac{3u}{4}\left(\frac{1}{\mathbf{A}}\frac{d\mathbf{G}}{dt} + 3\mathbf{G}\right) = 0. \tag{34}$$

Given that Eq.34 is an exact operator equation of motion, we see that if we could define the space of states by the condition $\mathbf{G}(\mathbf{t_0})|\psi\rangle = 0$, then Eq.34 proves that this condition will hold for all time. Note, however, that given this definition of $\mathbf{G}$, it follows immediately that

$$\mathbf{G}(t) = -\frac{\kappa^2}{3\mathbf{V}u(t)^2}\mathbf{H}. \tag{35}$$

Thus, while we can define the space of physical states, to be those which are annihilated by the Hamiltonian, obviously this immediately leads to a contradiction between the Schroedinger and Heisenberg picture. This is because $H|\psi\rangle = 0$ implies that the state does not evolve in the Schroedinger picture, whereas we have already shown that the operators $u(t)$ and $\Phi(t)$ do evolve in time.

V.   A BETTER STATE CONDITION

One way to avoid the inconsistency between the Schroedinger picture and Heisenberg equations of motion is to observe that we can define a one-parameter family of possible gauge-conditions as follows:

$$\mathbf{G}_\alpha = \mathcal{H}^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right) + \alpha\mathbf{Q} \tag{36}$$

Then, following the previous arguments, we can show that any of these $\mathbf{G}_\alpha$ also satisfies an equation of the form

$$\frac{3u}{4}\left(\frac{1}{\mathbf{A}_\alpha}\frac{d\mathbf{G}_\alpha}{dt} + 3\mathbf{G}_\alpha\right) = 0. \tag{37}$$

This means that a state is annihilated by $\mathbf{G}_\alpha(t)$ at any one time, t_0, will be annihilated by $\mathbf{G}_\alpha(t)$ for all times. Since, independent of the value of α, the extra pieces in all of these modified Friedmann equations vanish for large $u(t)$ it follows that, as before, in the limit of large $u(t)$ the expectation values of the dynamical fields will satisfy all of the Einstein equations.

To see that these alternative forms of the gauge-condition avoid direct conflict between the Schroedinger and Heisenberg picture simply substitute the explicit form of the Hubble operator, Eq.23, and the definition of $\mathbf{Q}$, Eq.28, into the definition of $\mathbf{G}_\alpha$ and rewrite it as

$$
\begin{aligned}
\mathbf{G}_\alpha &= \mathcal{H}^2 + \alpha\mathbf{Q} - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right) \\
&= \frac{\kappa^4}{16\mathbf{V}^2 u^2}p_u^2 + (1-\alpha)\frac{3\kappa^4}{64\mathbf{V}^2 u^4} - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right) \\
&= -\frac{\kappa^2}{3\mathbf{V}u^2}\mathbf{H} + (1-\alpha)\frac{3\kappa^4}{64\mathbf{V}^2 u^4}.
\end{aligned}
\tag{38}
$$

Now, since the Hamiltonian, $\mathbf{H}$, is time independent, we see that

$$
\mathbf{G}_\alpha(t) = -\frac{\kappa^2}{3\mathbf{V}u(t)^2}\left[\mathbf{H} - (1-\alpha)\frac{9\kappa^2}{64\mathbf{V}u(t)^2}\right].
\tag{39}
$$

Thus, it is only for $\alpha = 1$ that $\mathbf{G}_\alpha|\Psi\rangle = 0$ implies that the Hamiltonian annihilates the state.

Unfortunately, as we will see in the next section, in exactly solvable models we can explicitly show that the solutions to the equation $\mathbf{G}_\alpha|\psi\rangle = 0$ are not normalizable, and attempting to impose this strong condition for any α leads to problems interpreting the quantum mechanical theory. For this reason we propose a weak form of the condition, namely: a state is physical if

$$
\lim_{t\to\pm\infty}\mathbf{G}(t)|\Psi\rangle = 0.
\tag{40}
$$

It should be clear from the fact that $\mathbf{Q}$ vanishes for large t that Eq.40 guarantees that geometry, in the sense that the familiar Einstein equations become arbitrarily accurate, emerges dynamically at large times. Another way of characterizing the difference between this approach and the Wheeler-DeWitt equation is that, for reasons which will be apparent in the next section, the underlying physics is more closely related to a scattering problem, rather than an eigenvalue problem.

In the next sections, where we discuss the exact solution of de Sitter space, we show that this asymptotic condition is satisfied for a large class of states. Furthermore, the exact solution demonstrates why imposing a stronger condition on physical states is neither necessary nor desirable.

VI. DE SITTER SPACE: AN EXACTLY SOLVABLE PROBLEM

Since our assertion that it is unnecessary to adopt a strong version of the gauge condition flies in the face of conventional wisdom, it is important to show how things work in an exactly solvable example. For this reason we devote the next few sections of this paper to a discussion of de Sitter space.

Begin by considering the general action of the FRW problem, but with $V(\Phi)$ replaced by a cosmological constant Λ, so that the Hamiltonian takes the form

$$
\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}}p_u^2 + \frac{1}{2\mathbf{V}u^2}p_\Phi^2 + \mathbf{V}u^2\Lambda
\tag{41}
$$

We then note that, since the conjugate variable to p_Φ doesn't appear in the Hamiltonian, we are free to work in sectors of the Hilbert space in which p_Φ takes a definite value. For the particular sector defined by the condition $p_\Phi|\psi\rangle = 0$ the Hamiltonian takes the simpler form

$$
\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}}p_u^2 + \mathbf{V}u^2\Lambda,
\tag{42}
$$

which we immediately recognize as a theory with a cosmological constant, whose solution at the classical level is just de Sitter space.

Direct commutation of the Hamiltonian, Eq.42, with the operators $u(t)$ and $p_u(t)$ yields the following Heisenberg equations of motion for $u(t)$ and $p_u(t)$:

$$\frac{du(t)}{dt} = -\frac{3\kappa^2}{8\mathbf{V}}p_u \;;\; \frac{d^2u(t)}{dt^2} = \frac{3\kappa^2\Lambda}{4}u. \tag{43}$$

The exact solution to these equations, written in terms of the operators $u(t=0) = u$ and $p_u(t=0) = p_u$ are

$$
\begin{aligned}
u(t) &= \cosh(\omega t)u - \frac{3\kappa^2}{8\mathbf{V}\omega}\sinh(\omega t)p_u \\
p_u(t) &= \cosh(\omega t)p_u - \frac{8\mathbf{V}\omega}{3\kappa^2}\sinh(\omega t)u,
\end{aligned}
\tag{44}
$$

where we have defined

$$\omega = \sqrt{\frac{3\kappa^2\Lambda}{4}}. \tag{45}$$

It is convenient to rewrite Eq.44 in terms of exponentials; i.e.,

$$u(t) = \frac{e^{\omega t}}{2}\left(u - \frac{3\kappa^2}{8\mathbf{V}\omega}p_u\right) + \frac{3\kappa^2 e^{-\omega t}}{16\mathbf{V}\omega}\left(p_u + \frac{8\mathbf{V}\omega}{3\kappa^2}u\right) \tag{46}$$

and to introduce the canonically conjugate asymptotic operators

$$u_\infty = \frac{1}{\sqrt{2}}\left(u - \frac{3\kappa^2}{8\mathbf{V}\omega}p_u\right) \;;\; p_\infty = \frac{1}{\sqrt{2}}\left(p_u + \frac{8\mathbf{V}\omega}{3\kappa^2}u\right). \tag{47}$$

In terms of these operators the solution for the operator $u(t)$ and the Hamiltonian take the simple forms

$$u(t) = \frac{1}{\sqrt{2}}e^{\omega t}u_\infty + \frac{1}{\sqrt{2}}\frac{3\kappa^2}{8\mathbf{V}\omega}e^{-\omega t}p_\infty, \tag{48}$$

and

$$\mathbf{H} = \frac{\sqrt{3\Lambda}\kappa}{4}\left(u_\infty p_\infty + p_\infty u_\infty\right). \tag{49}$$

Eq.48 shows why, in the preceding section, we stated that the underlying physics is more closely related to the physics of a scattering problem than an eigenvalue problem. To establish the parallel all we need do is identify the *in*-states with the eigenstates of p_∞ and the *out*-states with the eigenstates of u_∞. Note, since the Hamiltonian is time-independent (Eq.49), the expectation value of the energy in any state is perforce time independent too.

From this point on all of the technical work is finished, the only chore which remains is to extract the physical significance of these results.

VII.   MORE ABOUT PHYSICAL STATES

Before discussing the physical states of the quantum theory, it is worth spending a few moments considering what the preceding results mean in the context of the classical theory. Obviously, Eqs.48 and 49 are equally true for both the classical and quantum versions of the theory; the only difference between these cases being is that in the classical theory u_∞ and p_∞ are simply numbers, whereas in the quantum theory they are non-commuting operators. Thus, for the classical theory, imposing the condition that the energy vanishes is the same as requiring either u_∞ or p_∞ to vanish. This is, of course, just the usual result: i.e., for the case of a cosmological constant, the full, non-linear, set of Einstein equations, admit only an expanding, or contracting, solution for $a(t)$ or $u(t)$. This is why running the expanding solution back in time (or the contracting solution forward in time) always leads to a *big crunch*.

Clearly, the situation is different for the quantum theory since it is not possible to simply set an operator to zero. If one chooses the gauge-condition which corresponds to $\alpha = 1$, i.e. the Wheeler-deWitt equation, then one is looking for states annihilated by the Hamiltonian. Given that we can write $p_\infty = -i\frac{d}{du_\infty}$, for a function of the form $|\psi\rangle = e^{S(u_\infty)}$, this equation takes the simple form

$$2\,u_\infty \frac{dS(u_\infty)}{du_\infty} = -1, \tag{50}$$

which has the solution

$$S(u_\infty) = -\ln(\sqrt{u_\infty}). \tag{51}$$

This of course means that $|\psi\rangle$ is of the form

$$|\psi\rangle \approx \frac{1}{\sqrt{u_\infty}} \tag{52}$$

which is not normalizable. The situation is no better if one chooses one of the gauge-conditions for which $\alpha \neq 1$. It is because working with these non-normalizable states makes interpreting the quantum theory so problematic that we adopt the weaker asymptotic condition defined in Eq.40

Intuitively, given the exact solution for $u(t)$, we see that any state for which $\mathbf{H}|\Psi\rangle$ has a finite norm will, for sufficiently large $|t|$, satisfy Eq.40 to arbitrary accuracy. This means that essentially any Gaussian wave packet in u_∞ will be a physical state. It also means that for large times all the physics measured in such a state will be compatible with the full set of Einstein equations. (Actually there is the additional requirement that the wave-function, $\mathbf{H}|\psi\rangle$, vanishes sufficiently rapidly at zero when written as a function u_∞ or p_∞. This, however, can be accomplished by multiplying any shifted Gaussian in u_∞ by an appropriate polynomial in u_∞. This subtlety will not seriously affect the considerations of the sections to follow and so we will ignore it. It is, however, important when we consider more complicated situations.)

VIII.   QUANTUM HISTORIES

Now that we have settled upon shifted Gaussian wavepackets as good candidates for physical states, we turn to a discussion of the only two physical observables in this theory; the expansion rate and the volume of the universe. Since we are working in the Heisenberg picture, where the choice of state determines the entire subsequent evolution of the system, we will henceforth refer to the choice of an allowed quantum state as a choice of *quantum history*. What we wish to ascertain is to what degree the value of each of the observables depends upon the specific choice of *quantum history*.

Obviously, the exact solution given in Eq.48 shows that, at large times, the expansion rate is attached to the scale factor and is totally independent of the state. This, however, is not true of the volume. Thus, in the remainder of this section we will discuss the degree to which the measured properties of the volume operator differ from quantum history to quantum history.

Since we started off quantizing in a volume with coordinate size $\mathbf{V}$, the volume of the universe at any time is given by

$$\begin{aligned} V(t) &= \mathbf{V}u(t)^2 \\ &= \frac{\mathbf{V}}{2}\left[e^{2\omega t}u_\infty^2 + \left(\frac{3\kappa^2}{8\mathbf{V}\omega}\right)^2 e^{-2\omega t}p_\infty^2 + \frac{3\kappa^2}{8\mathbf{V}\omega}\left(u_\infty p_\infty + p_\infty u_\infty\right)\right]. \end{aligned} \tag{53}$$

A surprising feature of this formula is that for large times in the past and future the volume operator $V(t)$ behaves classically. By this we mean that, if one measures $V(t)$ at some late time, t_1, and obtain a definite value, then we will be able to predict the value we will obtain if we measure $V(t)$ at some later time t_2. To see that this is the case we note that Eq.53 tells us that, for very large positive times, $V(t)$ is, to arbitrarily high accuracy, proportional to the single operator u_∞^2 (at large negative times it is proportional to p_∞^2). Thus we see that a measurement of $V(t_1)$, for sufficiently large t_1, corresponds to a measurement of u_∞^2, which means that we know $V(t)$ for all times $t_2 > t_1$.

From the fact that u_∞ and p_∞ are canonically conjugate variables we see that if we were to try and identify a quantum history with an eigenstate of p_∞, then the volume operator would be well-determined in the past, but completely undetermined in the future. Conversely, eigenstates of u_∞ correspond to states for which the volume operator is completely well determined in the future, but completely undetermined in the past. Fortunately, the condition that physical states must be normalizable states for which $\langle\psi|\mathbf{H}^2|\psi\rangle < \infty$ is true, tells us that we cannot identify such states with quantum histories. States which can be identified with admissible quantum histories are Gaussian packets of the form,

$$|\Psi\rangle = e^{-\frac{\gamma}{2}u_\infty^2} \tag{54}$$

and the coherent states, $|u_0, p_0, \gamma\rangle$, obtained from them. These coherent states are defined by

$$|u_0, p_0, \gamma\rangle = e^{ip_0 u_\infty}\, e^{-iu_0 p_\infty}|\Psi\rangle, \tag{55}$$

and the expectation values of u_∞ and p_∞ in these states are given by

$$\langle u_0, p_0, \gamma | u_\infty | u_0, p_0, \gamma \rangle = u_0, \quad \langle u_0, p_0, \gamma | p_\infty | u_0, p_0, \gamma \rangle = p_0. \tag{56}$$

Moreover, the relevant products of these operators have the values

$$\langle u_0, p_0, \gamma | u_\infty^2 | u_0, p_0, \gamma \rangle = u_0^2 + \frac{1}{2\gamma},$$

$$\langle u_0, p_0, \gamma | p_\infty^2 | u_0, p_0, \gamma \rangle = p_0^2 + \frac{\gamma}{2},$$

$$\langle u_0, p_0, \gamma | u_\infty p_\infty + p_\infty u_\infty | u_0, p_0, \gamma \rangle = 2\Re(\langle u_\infty \, p_\infty \rangle) = 2 \, u_0 p_0.$$

The nice thing about such coherent states is that they are the kind of states we would expect to obtain if, in the past, we make a measurement which determines $V(-t)$ to have a central value $\frac{\mathbf{V}}{2} e^{\omega|t|} p_0^2$, with a width parameterized by γ. For this same packet, measurements of $V(t)$ in the distant future will produce results centered about the value $\frac{\mathbf{V}}{2} e^{\omega|t|} u_0^2$, with a width parameterized by $1/\gamma$.

IX. EQUIVALENCE CLASSES OF HISTORIES

From this point on we will restrict the term quantum history to mean a coherent state of the form defined above. What we wish to show next is that many of these histories are equivalent to one another in a way which we will make precise. Begin by considering

$$\langle V(t) \rangle = \langle u_0, p_0, \gamma | V(t) | u_0, p_0, \gamma \rangle = \frac{\mathbf{V}}{2} \left[e^{2\omega t} \langle u_\infty^2 \rangle + \left(\frac{3\kappa^2}{8\mathbf{V}\omega} \right)^2 e^{-2\omega t} \langle p_\infty^2 \rangle + \frac{3\kappa^2}{8\mathbf{V}\omega} (2\Re(\langle u_\infty p_\infty \rangle)) \right]. \tag{57}$$

It is obvious from Eq.57 that at large times the volume behaves as a single exponential, as expected from the solution of the classical Einstein equations. More interesting, however, is the fact that letting $t \to t + t_0$, where t_0 is defined by the condition

$$e^{2\omega t_0} = \frac{3\kappa^2}{8\mathbf{V}\omega} \sqrt{\frac{\langle p_\infty^2 \rangle}{\langle u_\infty^2 \rangle}}, \tag{58}$$

allows us to rewrite Eq.57 as

$$\langle V(t) \rangle = \frac{3\kappa^2 \sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}}{8\omega} \left[\cosh(\omega t) + \frac{\Re(\langle u_\infty p_\infty \rangle)}{\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}} \right]$$

$$= \frac{\kappa^2 \sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}}{4\mathcal{H}} \left[\cosh(\omega t) + \frac{\Re(\langle u_\infty p_\infty \rangle)}{\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}} \right] \tag{59}$$

Thus, we see $\langle V(t) \rangle$ corresponds to a system which is contracting at large times in the past and which then bounces and begins to re-expand in the future. During most of this history the system satisfies the Friedmann equation to high accuracy and expands (or contracts) with a Hubble constant equal to

$$\mathcal{H} = \frac{2}{3}\omega = \sqrt{\frac{\kappa^2 \Lambda}{3}}. \tag{60}$$

However, there is a period in time where the quantum corrections to the Friedmann equation dominate the behavior; namely, at times $t \approx 1/\omega$. Assuming, for the sake of argument, that were to set $1/\kappa\mathcal{H} \approx 10^3$, as it is in many models of slow roll inflation, and assuming $\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}$ to be of order unity, then the minimum volume of the universe at the time of the bounce is on the order of 10^3 Planck volumes; i.e., on the order to 10 Planck-lengths in each dimension. This sets the order of magnitude of the scale at which the quantum corrections become important. It is gratifying that these quantum corrections keep the system from contracting forever and ending in a *big crunch*.

Another very interesting feature of Eq.59 is that it is characterized by only two numbers, $\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}$ and $\Re\langle u_\infty p_\infty \rangle / \sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}$. The first number is unrestricted in magnitude and roughly determines the physical volume of the universe at the time of the bounce. The second number, is constrained by the Schwarz inequality to lie between plus and minus one, and parameterizes the degree to which the behavior of the system during the time of

the bounce deviates from a pure hyperbolic cosine. If the time over which the deviation takes place is characterized by $1/\omega \approx 1/\mathcal{H}$, then the minimum size to which the system contracts is characterized by the ratio of the energy density in the state to the cosmological constant. This statement follows from taking the expectation value of the Hamiltonian as written in Eq.49, which implies

$$\Re(\langle u_\infty p_\infty \rangle) = \frac{2}{\kappa\sqrt{3\Lambda}} \langle \mathbf{H} \rangle. \tag{61}$$

Note, it would appear from the Schwarz inequality that in principle one could have a history for which the universe actually shrinks to zero size before it bounces. Fortunately it is easy to see that this can only occur if u_0 or p_0 diverges, which violates the condition on allowable physical states, since such states would have infinite values for $\langle \mathbf{H}^2 \rangle$.

Finally, Eq.59 shows that any two quantum histories which give the same values for $\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}$ and $\Re(\langle u_\infty p_\infty \rangle)\sqrt{\langle u_\infty^2 \rangle \langle p_\infty^2 \rangle}$, see the same physics. They only differ by the time at which they see the bounce occur. For Gaussian packets we see that this will be true for states which are related by the transformation

$$u_0 \rightarrow \lambda u_0, \ p_0 \rightarrow \frac{p_0}{\lambda}, \text{ and } \gamma \rightarrow \frac{\gamma}{\lambda^2}. \tag{62}$$

It is easy to check that this can be implemented by a unitary transformation. The values of u_0 and p_0 can be changed by means of the shift operators used to define the coherent states in the first place. The width of the Gaussian can be changed by application of a unitary *squeezing operator* of the form

$$e^{\left(\alpha(\gamma)\, a^{\dagger 2} - \alpha(\gamma)^*\, a^2\right)}, \tag{63}$$

where the creation and annihilation operators are defined such that

$$u_\infty = \frac{1}{\sqrt{2\gamma}}\left(a^\dagger + a\right) \quad \text{and} \quad p_\infty = -i\sqrt{\frac{\gamma}{2}}\left(a^\dagger - a\right). \tag{64}$$

X.   MINKOWSKI SPACE $\Lambda = 0$

Finally, we would like to discuss what happens when we take $\Lambda = 0$, because, in this case, things work quite a bit differently. The $\Lambda = 0$ Hamiltonian is

$$\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}} p_u^2 \tag{65}$$

and the Heisenberg equations of motion take the form

$$\frac{du}{dt}(t) = -\frac{3\kappa^2}{16\mathbf{V}} p_u \quad ; \quad \frac{dp_u}{dt}(t) = 0. \tag{66}$$

The exact solution to these equations is

$$u(t) = u - \frac{3\kappa^2}{16\mathbf{V}} p_u\, t \quad ; \quad p_u(t) = p_u \tag{67}$$

Taking the square of $u(t)$ we obtain the volume operator

$$V(t) = \mathbf{V}u^2(t) = \mathbf{V}\left[u^2 - \frac{3\kappa^2}{16\mathbf{V}}\left(u\, p_u + p_u\, u\right) t + \left(\frac{3\kappa^2}{16\mathbf{V}}\right)^2 p_u^2\, t^2\right]. \tag{68}$$

It follows once again that, as in the de Sitter case, the volume operator becomes classical at large times in the past and the future. In this case however there is a state which, while non-normalizable, satisfies the condition $G(t)|\Psi\rangle = 0$ for all times; namely, the eigenstate of p_u with eigenvalue 0. Now, however, this condition is consistent with the Heisenberg equations of motion, because in this eigenstate $u(t) = u$ and is independent of time. Moreover, this state satisfies the requirement that $\langle \mathbf{H}^2 \rangle$ is finite. Obviously, this state is the limit of sequence Gaussian packets in p_u of smaller and smaller width. If we choose this quantum history then, after we absorb the scale factor into $\vec{x}$, we find that this history corresponds to a time-independent Minkowski space.

It is interesting to ask what other, less special, histories correspond to. Let us assume we are working with an arbitrary coherent state of the form discussed in the previous section. Then, the expectation value of the volume operator is

$$\langle V(t) \rangle = \mathbf{V} \left[\langle u^2 \rangle - 2 \frac{3\kappa^2}{16\mathbf{V}} \, \Re(\langle u \, p_u \rangle) \, t + \left(\frac{3\kappa^2}{16\mathbf{V}} \right)^2 \langle p_u^2 \rangle \, t^2 \right], \tag{69}$$

which can be rewritten in the form

$$\langle V(t) \rangle = \mathbf{V} \left[\frac{\langle u^2 \rangle \langle p_u^2 \rangle - \Re(\langle u \, p_u \rangle)^2}{\langle p_u^2 \rangle} + \left(\frac{3\kappa^2}{16\mathbf{V}} \right)^2 \langle p_u^2 \rangle \left(t - \frac{16\mathbf{V}\Re(\langle u \, p_u \rangle)}{3\kappa^2 \, \langle p_u^2 \rangle} \right)^2 \right]. \tag{70}$$

Thus, we see that for the generic history, the case of zero cosmological constant actually corresponds to a universe for which the volume factor is expanding like t^2, or for which the scale factor $a(t)$ is growing like $t^{2/3}$. Surprisingly this corresponds to a universe dominated by non-relativistic matter. In other words, a non-vanishing energy density present in the quantum excitations of the scale factor produce the same effect as cold matter.

A final point worth mentioning is that, as in the case of de Sitter space, the Schwarz inequality guarantees that the volume never shrinks to zero for any allowable physical state; i.e., we never are in the situation of a *big crunch*. It is interesting to note that in this formalism the big crunch is averted due to the quantum physics of the long wavelength modes of the gravitational field and not short distance physics.

XI. RECOVERING THE CLASSICAL THEORY

To this point we have set up our general formalism and we have shown how things work for the exactly solvable case of de Sitter space, defined by the two conditions $V(\Phi) = \Lambda$ and $p_\phi = 0$. The question which comes up at this juncture is how things work in the more general setting when $V(\Phi)$ is not a constant and we wish to deal with the usual slow-roll theory of inflation. Before jumping into this discussion we must say a few words about the related question of what happens if we relax the second condition and set p_ϕ to some arbitrary constant.

When $p_\phi \neq 0$ the Heisenberg equations for the system defined by Eq.41 are no longer exactly solvable; nevertheless, we can always write the full time development operator of the theory as

$$U(t) = e^{iH_0 t} S(t) \tag{71}$$

where H_0 is the Hamiltonian of the theory obtained by setting $p_\phi = 0$ and $S(t)$ satisfies the differential equation

$$-i \frac{dS(t)}{dt} = V_I(t) = e^{iH_0 t} \left[\frac{p_\phi^2}{2 \, u^2} \right] e^{iH_0 t} = \frac{p_\phi^2}{2 \left(\cosh(\omega t) u + \sinh(\omega t) p \right)^2}. \tag{72}$$

(i.e., $S(t)$ is given, as usual, by path ordered exponential of $V_I(t)$). The form of $V(t)$ suggests that for an appropriate subclass of the space of physical states (those which vanish for a region around $u_\infty = 0$) the evolution of the system is asymptotically controlled by H_0. Thus, for these states, we expect our general description of how things evolve in phase space to apply. In any event, we see that in each sector of fixed p_ϕ, the evolution of the system is well defined and unitary. Clearly, going beyond these considerations and relaxing the condition that $V(\Phi)$ is a constant, will force us to work with packets in p_ϕ. This is of course even more difficult to analyze in detail. Thus, it is important to ask to what degree we can be sure that we will arrive at the usual inflationary scenario, be it fast or slow-roll inflation, using this approach. This is the question we address in the rest of this section.

Recovering the classical picture of inflation, in the limit in which our asymptotic condition holds to high accuracy, is straightforward. Since we are working with the Heisenberg equations of motion, all we have to do is assume that we start from a coherent state $|\psi\rangle$, such that $\mathbf{G}|\psi\rangle = 0$. Furthermore, we assume that $\langle\psi|u|\psi\rangle$, $\langle\psi|p_u|\psi\rangle$, $\langle\psi|\Phi|\psi\rangle$ and $\langle\psi|p_\Phi|\psi\rangle$ satisfy the initial conditions required for a classical theory of slow-roll inflation. In this case, it makes sense to rewrite the Heisenberg operators as

$$u(t) = \widehat{u}(t) + \delta u(t) \quad ; \quad \Phi(t) = \widehat{\Phi}(t) + \delta\Phi(t), \tag{73}$$

where $\hat{u}(t)$ and $\hat{\Phi}(t)$ are c-number functions such that $\langle\psi|u(t)|\psi\rangle = \hat{u}(t)$ and $\langle\psi|\Phi(t)|\psi\rangle = \hat{\Phi}(t)$. Given these assumptions, we wish to show that if these c-number functions satisfy the classical slow-roll equations for inflation, then as a consequence of inflation, the quantum corrections to the Heisenberg equations of motion will be strongly suppressed.

Substituting these definitions into the Heisenberg equation of motion for the operator $u(t)$ in Eq.21 and the form of the equation of motion for the operator $\Phi(t)$ in Eq.22, we see that if the classical functions $\hat{u}(t)$ and $\hat{\Phi}(t)$ satisfy the classical equations for slow-roll inflation, the c-number terms all cancel, and one is left with equations for the operators $\delta u(t)$ and $\delta\Phi(t)$. At first glance, solving these equations seems difficult; however, the situation improves significantly if we look at the constraint equations

$$\left[\mathcal{H}^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(d\Phi(t)/dt\right)^2 + V(\Phi(t))\right) + \mathbf{Q}(t)\right]|\psi\rangle = 0, \quad ; \quad \left[d\mathcal{H}/dt + \frac{\kappa^2}{2}\left(d\Phi(t)/dt\right)^2\right]|\psi\rangle = 0. \tag{74}$$

In the rest of this section we will ignore the operator $\mathbf{Q}(t)$, since it is proportional to $\hat{u}(t)^{-4}$, which we expect to be small in the inflationary and FRW eras.

The second equation says that in the sector of physical states we can replace the operator $\left(d\Phi(t)/dt\right)^2$ by $-2(d\mathcal{H}/dt)/\kappa^2$. Substituting this in the first constraint yields the equation

$$\left[3\mathcal{H}(t)^2 + \frac{d\mathcal{H}(t)}{dt} - \kappa^2 V(\Phi(t))\right]|\psi\rangle = 0. \tag{75}$$

Substituting Eq.73 into the definition of $\mathcal{H}$, we get

$$\mathcal{H}(t) = \widehat{\mathcal{H}}(t) + \delta\mathcal{H}(t), \tag{76}$$

where the form one obtains for $\delta\mathcal{H}(t)$ would seem to imply that the operator shrinks rapidly during the inflationary era because of the inverse power of $\hat{u}(t)$ appearing in its definition. Nevertheless, it behooves us to check that the operators $\delta u(t)$, *etc.* do not behave badly. Substituting Eq.76 into the constraint equation and using $\Phi(t) = \widehat{\Phi}(t) + \delta\Phi(t)$, cancelling out the contributions of the c-number functions and keeping terms of first order in $\delta\mathcal{H}$ and $\delta\Phi(t)$, we obtain

$$\left[6\,\widehat{\mathcal{H}}\,\delta\mathcal{H} - \kappa^2\left(\frac{dV(\widehat{\Phi})}{d\Phi}\right)\delta\Phi(t)\right]|\psi\rangle = 0. \tag{77}$$

Taking the expectation value of this equation in the state $|\psi\rangle$, noting that by assumption $\langle\psi|\delta\Phi(t)|\psi\rangle = 0$, it follows that

$$6\,\widehat{\mathcal{H}}(t)\,\langle\psi|\delta\mathcal{H}(t)|\psi\rangle + \frac{d\langle\psi|\delta\mathcal{H}(t)|\psi\rangle}{dt} = 0. \tag{78}$$

The solution to this equation is

$$\langle\psi|\delta\mathcal{H}(t_f)|\psi\rangle = e^{-6\int_{t_i}^{t_f} dt\widehat{\mathcal{H}}(t)}\,\langle\psi|\delta\mathcal{H}(t_i)|\psi\rangle. \tag{79}$$

If we recall that the classical function $\widehat{\mathcal{H}}(t)$ is just $d\left(\ln(a(t))\right)/dt$, we see that the integral in Eq.79 is just the number of e-foldings during inflation. Thus, the contribution of the operator $\delta\mathcal{H}$ is strongly suppressed.

XII. HOW BIG ARE THE CORRECTIONS?

Obviously, if we limit ourselves to the exactly solved case of de Sitter space, estimating the size of the corrections to Einstein's equations in a given quantum history reduces to calculating the ratio of the exponentially increasing term in the expression for the volume as a function of time, to the exponentially decreasing term. The question now is how big are the corrections to the Einstein equations in a more general setting, and when does make sense to ignore their effects on the CMB calculation. Another way to ask the same question is to ask whether it makes sense to ignore the operator $\mathbf{Q} = -3\kappa^4/64\mathbf{V}^2 u^4$ at the onset of inflation.

Obviously, the issue boils down to how large this term is relative to the operator $\kappa^2 V(\Phi)/3$. To establish this ratio, we must first specify the value of the quantization volume $\mathbf{V}$. Clearly, there is no upper limit for the value one can choose for $\mathbf{V}$. There is, however, a lower limit, since $\mathbf{V}$ must be chosen larger than the horizon volume at the time of inflation to avoid boundary effects which are not seen in the WMAP data. Thus, $\mathbf{V} > 1/\mathcal{H}_I^3$, where $\mathcal{H}_I$ is the value of the Hubble parameter at the onset of inflation.

To estimate the size of $\mathcal{H}_I$, if the classical approximation dominates, we use the classical version of the Friedmann equation. This equation tells us that

$$\mathcal{H}_I^2 \approx \frac{\kappa^2}{3}V(\Phi). \tag{80}$$

Substituting this into the expression for $\mathbf{Q}$ we obtain that

$$
\begin{aligned}
\mathbf{Q} &= -\frac{3\kappa^4}{64}\mathcal{H}_I^6\frac{1}{u^4} \\
&= -\frac{\kappa^{10}}{576}V(\Phi)^3\frac{1}{u^4},
\end{aligned}
\tag{81}
$$

which is to be compared to $\kappa^2 V(\Phi)/3$. Thus the statement that $\mathbf{Q}$ can be ignored at the onset of inflation is equivalent to

$$
\frac{\kappa^2}{3}V(\Phi) \gg \frac{\kappa^{10}}{576}V(\Phi)^3\frac{1}{u^4}.
\tag{82}
$$

It is convenient to multiply this equation by a factor of κ^2 to obtain

$$
\kappa^4 V(\Phi) \gg \frac{1}{192}\left(\kappa^4 V(\Phi)\right)^3\frac{1}{u^4},
\tag{83}
$$

or equivalently

$$
\frac{1}{192}\left(\kappa^4 V(\Phi)\right)^2\frac{1}{u^4} \ll 1.
\tag{84}
$$

At this point we note that the product $\kappa^4 V(\Phi)$ is usually constrained to be less than or on the order of 10^{-6}. Thus, if $u(t)$ is chosen to be the order of unity at the time inflation starts, the effects of $\mathbf{Q}$ will be negligible. Note however that $1/u(t)^4 = 1/a(t)^6$, so one cannot extrapolate very many e-foldings back from the starting point before quantum corrections become important.

XIII.   REMARKS CONCERNING THE COMPUTATION OF CMB ANISOTROPY

While we have not yet done any detailed computations, it is clear that the fact that the quantum system deviates from pure exponential growth at a finite time in the past could have implications for the usual derivation of CMB fluctuations. It is entirely possible that the delay in the time at which the long wavelength modes of the scalar field exit the horizon relative to the shorter wavelength modes might produce visible effects in the predicted measurement of $\delta\rho/\rho$. If this is so then one should be able to put an experimental limit on how far back in time one can push the start of the usual computation.

XIV.   POSSIBLE EXTENSIONS

In order to extend this model to a complete treatment of the CMB anisotropy, one has both to add an extra field $\chi(t,x)$ to the metric, in order to model Newtonian fluctuations, and to put the the spatially varying part to the Φ field back into the action. In other words, the metric is taken to have the form

$$
ds^2 = -(1 + 2\epsilon\,\chi(t,\vec{x}))\,dt^2 + a(t)^2\,(1 - 2\epsilon\,\chi(t,\vec{x}))\,d\vec{x}^2,
\tag{85}
$$

and the action is taken to be

$$
\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{R(g)}{2\kappa^2} + \frac{1}{1+2\epsilon\,\chi(t,\vec{x})}\left(\frac{d\Phi(t)}{dt} + \frac{\epsilon d\phi(t,\vec{x})}{dt}\right)^2 - \frac{\epsilon^2\vec{\nabla}\phi(t,\vec{x})\cdot\vec{\nabla}\phi(t,\vec{x})}{a(t)^2\,(1-2\epsilon\,\chi(t,\vec{x}))} - V(\Phi(t)+\epsilon\phi(t,\vec{x}))\right].
\tag{86}
$$

If we now expand this formula up to order ϵ^2 we see that: the ϵ^0 term is the problem we have been considering; the ϵ^1 term vanishes due to the equations of motion; the remaining terms are quadratic in the fields $\chi(t,x)$ and $\phi(t,x)$. Thus, if we start in the coherent state discussed in the previous sections, $\chi(t,x)$ and $\phi(t,x)$ are simply free fields evolving in a time-dependent background and their Heisenberg equations of motion can be solved exactly.

This analysis allows one to completely reproduce the usual computations for $\delta\rho/\rho$. (A complete treatment of this will appear in a forthcoming pedagogical paper.) In other words, this effective theory is capable of reproducing the theory of all CMB measurements within a quantum framework in which the long wavelength part of the gravitational field satisfies the exact Einstein equations, while the shorter wavelengths are treated perturbatively. The small size of the CMB fluctuations tells us this is a reasonable approach.

This extension of the model gives us a canonical Hamiltonian picture of the evolution of the theory. This means that the back-reaction caused by the changes in $\chi(t,x)$ and $\phi(t,x)$ are completely specified. Clearly, it is important to ask if these back-reaction effects, or the effects $\mathbf{Q}$ causes on the evolution of the system, leave an observable imprint on the CMB fluctuation spectrum.

There are two less obvious but very interesting directions in which one can extend this work.

The first is to reintroduce some very long wavelength modes of $\chi(t,x)$ and $\phi(t,x)$ into the part of the Lagrangian that we treat exactly. To be specific, we can expand the field $\chi(t,x)$ and $\phi(t,x)$ in some sort of wavelet basis, for which the low-lying wavelets represent changes over fractions of the horizon scale; *i.e.*

$$\chi(t,x) = \sum_{j=1}^{N} b_j(t) w_j(x) + \epsilon\chi(t,x)^{'} \quad ; \quad \phi(t,x) = \sum_{j=1}^{N} c_j(t) w_j(x) + \epsilon\phi(t,x)^{'}. \tag{87}$$

Next, we can plug this expansion into the action and expand it to second order in ϵ.

In contrast to the previous case, the order ϵ^0 part of the action will now be the theory of $2N$ non-linearly coupled variables. Clearly, we should be able to parallel the discussion given in this paper and proceed to: first, derive the canonical Hamiltonian, then derive the Heisenberg equations of motion for the system, which will not be the full set of Einstein equations, and, finally, construct the proper constraint operators to fill out the full set of equations of motion. The resulting theory should be equivalent to a theory in which we have finite size boxes with independent scale factors that are weakly coupled to one another. Since the usual CMB results imply that fluctuations on all wavelengths are small, it must be true that in this version of the theory the scale factors in neighboring boxes (or pixels) can't get very far from one another without affecting the CMB fluctuations. We suggest that one way to see how a fully quantized theory of gravity behaves at shorter wavelengths is to pixelize the theory in this way. Then we can study what happens to the quantum problem as we add operators corresponding to higher frequency fluctuations in the quantum fields back into the fully non-linear problem.

Another interesting direction is to pixelize a problem initially quantized in a region extending over several horizon volumes. Since the pixelization to volumes smaller than the scale set by the horizon during inflation certainly leads to coupling terms between the pixels, there will be analogous (presumably weaker) couplings between what will now be neighboring horizon-size pixels. The interesting question here is whether or not the scale factors in neighboring volumes can get very far from one another without causing observable changes in the CMB fluctuation spectrum seen by an observer in any individual volume. In other words, can we—within the context of chaotic or eternal inflation—put experimental limits on how near to us a very different universe from ours can be, without leaving a visible imprint on the CMB radiation in our universe? If such limits can be found, we will have found a way to see the unseeable.

XV. SUMMARY

In this paper we showed how to fully quantize the theory of inflation and $\delta\rho/\rho$, at least if one takes the point of view that getting a sensible evolution of a quantum system as a function of cosmic time takes precedence over forcing a purely geometrical interpretation.

Our focus throughout was on the formulation of the part of the problem that involved the spatially constant fields. As we demonstrated, in both the classical and quantum theory, working in a fixed gauge yields only two of the four relevant Einstein equations as equations of motion. In the classical theory we showed that the Friedmann equation and its time derivative must be treated as constraints whose constancy in time requires a proof. Our proof followed from the two equations of motion we did have. Next, we showed that, in the quantum version of the theory, the same two Einstein equations appear as operator equations of motion; however, surprisingly, due to quantum corrections there were a one-parameter family of possible choices for constraint equations. We argued that the simplest of these constraint equations, that which corresponds to the Wheeler-deWitt equation, cannot be used to define the space of physical states, since it leads to a direct conflict between the Schroedinger and Heisenberg pictures. However, we showed that that problem does not exist for the other possible choices for constraint operators. Nevertheless, we insisted that we would still run into trouble if we imposed a strong form of any of these constraints and suggested a weaker form of the constraint which avoids these problems. In order to clarify how our weaker conditions work in detail, we applied the general formalism to the case of de Sitter (and Minkowski) space. Our goal was to show, in an explicit, exactly solvable, case how the formalism works. The most important result of this discussion is that, in the case of de Sitter space, the system deviates from the expected pure exponential expansion at a finite time in the past. We also went on to discuss variants of the de Sitter problem and then discussed the recovery of the usual inflationary picture in the more complicated problem. One possible consequence of the fact that the behavior of the universe at early times differs from pure exponential expansion, is that it implies one might measure the effects of

the quantum corrections to the pure Einstein equations as deviations from the conventionally predicted form of $\delta\rho/\rho$. Failing that, one might bound the earliest time at which one is free to set initial conditions on the state of the inflaton and other fields in the system. To put it another way, there may either be measurable consequences following from the quantum nature of the problem at early times, or one will have to face up to the problem of how and when to set initial conditions.

While, as it stands, the formalism we have presented is by no means a candidate for a theory of everything, we feel that the interesting results obtained by proceeding along these lines suggests it is a good candidate for a theory of something. Namely, a fully quantum theory of the measured fluctuations in the CMB radiation.

XVI. ACKNOWLEDGEMENTS

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