SLAC-PUB-10360
UH-511-1045-04
hep-ph/0402280
February 2004

Real radiation at NNLO: $e^+e^- \to 2$ jets through $\mathcal{O}(\alpha_s^2)$

Charalampos Anastasiou

Stanford Linear Accelerator Center,
Stanford University, Stanford, CA 94309

Kirill Melnikov

Department of Physics and Astronomy, University of Hawaii,
2505 Correa Rd., Honolulu, HI 96822

Frank Petriello

Department of Physics, Johns Hopkins University,
3400 North Charles St., Baltimore, MD 21218

We present a calculation of the differential two jet cross section in e^+e^- annihilation through next-to-next-to-leading order in the strong coupling constant α_s. The calculation is performed using a new method for dealing with real radiation suggested recently by us in [1]. For the first time, the two jet event rate is computed directly, without any reference to the inclusive cross-section $e^+e^- \to$ hadrons. We also calculate the energy distribution of the leading jet in $e^+e^- \to 2$ jets and find significant modifications of the shape of this distribution at NNLO.

PACS numbers:

High-energy physics will begin to explore a new energy frontier when the Large Hadron Collider at CERN turns on in 2007. Our understanding of physics at very small distances will dramatically improve. However, a detailed investigation of the new physics we discover will require a careful study of Standard Model backgrounds, detector responses, and other similar issues. Since many layers separate interesting physics from raw experimental data, a dedicated effort is required to fully utilize LHC results. There have been significant advances towards this goal in the past few years; we now have an increased understanding of parton distribution functions and jet algorithms, improved Monte Carlo event generators, methods for automating next-to-leading order calculations with large number of external legs and, finally, new technology for next-to-next-to-leading order (NNLO) computations.

NNLO calculations are certainly not required for all processes at the LHC or existing colliders; however, there are a few situations in which NNLO calculations are highly desirable. These include processes for which the one-loop corrections are abnormally large (e.g. the production of the SM Higgs boson at hadron colliders [2]) or for measurements in which high experimental precision is either achieved (e.g. the α_s determination from the three-jet event rate in e^+e^- annihilation [3] or the W mass measurement at the Tevatron) or expected (e.g. W and Z boson production at the LHC [4]). These calculations should also inform us how accurate NLO calculations really are, beyond the standard checks of stability with respect to renormalization and factorization scales variations. We should learn to estimate the significance of NNLO corrections without performing the calculations, given the required precision for an observable and the kinematic regions in which it is measured. For this purpose, exclusive NNLO calculations are needed, since experimental cuts on the final state can have a strong impact on the convergence of perturbative expansion. Unfortunately, not a single calculation of a fully differential QCD observable at NNLO has been performed, either for lepton or hadron colliders.

In this Letter we remedy this situation and present the calculation of the two jet cross-section in e^+e^- annihilation at NNLO in perturbative QCD. Although jets and their properties have been studied very extensively at lepton colliders [5], we believe that such calculation is important for the following reasons: 1) it is the first-ever calculation of a *fully differential* observable at NNLO; 2) although the total rate for two jet events in e^+e^- annihilation is known through NNLO from indirect calculations, our results for distributions in two jet events are new; 3) this calculation is possible because of a new method we recently suggested for handling real radiation in hard processes [1]; it is important to demonstrate its efficiency by applying it to a non-trivial example.

There is a strong correlation between the complexity of higher order calculations and the level of exclusiveness desired. Traditionally, it was thought that calculations at higher orders are difficult because of multi-loop integrals. A significant effort therefore went into developing flexible, easily automated methods for performing higher

Work supported in part by the Department of Energy contract DE-AC03-76SF00515.

loop computations [6]. As a result, calculations up to two loops are no longer prohibitively difficult; for example, the two-loop virtual corrections for $1 \to 3$ and all partonic $2 \to 2$ processes at hadron colliders have been computed [7]. Surprisingly, the major obstacle in obtaining differential results at higher orders are tree-level processes with additional final-state partons. While it is very easy to write down the corresponding matrix element, it is not possible to integrate it numerically over the restricted (exclusive) phase-space without first extracting the singular structure of the integrand in the soft and collinear limits. Analytic integrations also become extremely difficult because of the arbitrariness of final-state cuts. This problem has been successfully solved at NLO using both the slicing and dipole subtraction methods [8, 9]. Attempts have been made recently to generalize the dipole formalism to NNLO [10]; so far, they have not completely succeeded. It is therefore productive to look for an alternative method of dealing with multi-particle final states in the presence of arbitrary constraints on their phase space.

What are the ideal features of such a method? Given the complexity of higher order calculations, it should satisfy the following requirements: 1) the singularities should be extracted in an algorithmic fashion; 2) the method should be easy to automate; 3) it should be easily generalizable, at least in principle, to arbitrary numbers of partons in the final state; 4) the method should work efficiently in the presence of arbitrary constraints on the final state; 5) it should lead to a fast and accurate numerical evaluation of physical quantities.

We have proposed such a method recently in [1]. We now briefly describe its salient features. Consider a perturbative tree-amplitude $\mathcal{M}$ with n particles in the final state. Its contribution to the differential cross-section can be written as

$$\mathrm{d}\sigma^{(n)} = \int \mathrm{d}\Gamma_n |\mathcal{M}|^2 J(\{p_i\}), \tag{1}$$

where $\mathrm{d}\Gamma_n$ denotes the n-particle phase-space and $J(\{p_i\})$ imposes restrictions on the final state (e.g. the jet algorithm) which define the experimentally observed process. Throughout this Letter we use dimensional regularization (with $d = 4-2\epsilon$ dimensions) for both infrared and collinear divergences. If all particles are well separated (resolved), $|\mathcal{M}|^2$ is finite; however, when the integration in Eq.(1) is attempted, there are divergences associated with soft and collinear kinematic configurations. A direct numerical integration of Eq.(1) is therefore not possible.

In Ref. [1] we demonstrated that by mapping the invariant masses onto algorithmically chosen new integration variables, it is possible to extract the ϵ poles explicitly. We then obtain an expansion in ϵ,

$$\mathrm{d}\sigma^{(n)} = \sum_{k=0}^{2(n-2)} \frac{\mathrm{d}F_k}{\epsilon^{2(n-2)-k}} + \mathcal{O}(\epsilon), \tag{2}$$

where the coefficients $\mathrm{d}F_k$ are well-defined, ϵ-independent multi-dimensional integrals for a generic function $\mathcal{J}$. It needs to be specified only at the stage of numerical evaluation. This allows us to derive results for arbitrary jet algorithms and experimental observables.

It is relatively easy to derive such an expansion at NLO, where at most one parton can become unresolved. In this case, trivial mappings [1] of the phase-space variables onto variables with range from 0 to 1 produce integrals with the following singular structure:

$$I_1 = \int_0^1 \mathrm{d}x\mathrm{d}y \; x^{\epsilon-1} y^{\epsilon-1} \; J(x,y). \tag{3}$$

All singularities in the above integral can be extracted by writing $x^{\epsilon-1} = \delta(x)/\epsilon + [1/x]_+ + \epsilon[\ln(x)/x]_+ + ...$, for both x and y in the integrand.

Beyond NLO, two or more partons may become unresolved, which gives rise to a more complicated structure of overlapping singularities. Typically, we find integrals similar to

$$I_2 = \int_0^1 \mathrm{d}x\mathrm{d}y \; \frac{x^{\epsilon} y^{\epsilon}}{(x+y)^2} \; J(x,y). \tag{4}$$

The procedure described above does not work because the singularities are not factorized. To solve this problem, we apply the technique of sector decomposition [11]; we divide the integration region in Eq.(4) into patches with a definite ordering of the integration variables ($x < y$ and $y < x$) and reweight all variables in each patch so that the integrations again range from 0 to 1. This leads to factorization of the singular limits. This procedure can be completely automated. The same method should, in principle, work for any number of particles in the final state, both massless and massive, and for any restrictions on the final-state phase space.

For $e^+e^- \to 2$ jets through NNLO, the largest multiplicity of particles in the final state is four (e.g. $e^+e^- \to q\bar{q}gg$). A parameterization of the $1 \to 4$ particle phase-space in terms of five independent variables which is suitable for sector decomposition and extracting infrared divergences was given in [1]. In the same reference, we gave a more detailed description of the method, and considered a number of relatively simple examples. In this Letter we apply the method to a fully realistic and non-trivial problem – the calculation of the $e^+e^- \to 2$ jets cross section at NNLO. Traditionally, the inclusive 2-jet rate is calculated at NNLO indirectly, by first computing the total inclusive cross-section for $e^+e^- \to$ hadrons and then subtracting from it the $e^+e^- \to 4$ jets and $e^+e^- \to 3$ jets cross-sections at LO and NLO, respectively. This paper presents the first direct calculation of the 2-jet rate at NNLO. Using our method, we can also obtain differential results at NNLO, which can not be derived indirectly. We illustrate this by computing the energy distribution of the leading jet in $e^+e^- \to 2$ jets through NNLO.

The cross-section for e^+e^- annihilation into hadrons through order $\mathcal{O}(\alpha_s^2)$ can be written as

$$\sigma = \sigma_0 \left(\delta_{j,2} + \left(\frac{\alpha_s}{\pi}\right)\left(C_1^{(2)}\delta_{j,2} + C_1^{(3)}\delta_{j,3}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(C_2^{(2)}\delta_{j,2} + C_2^{(3)}\delta_{j,3} + C_2^{(4)}\delta_{j,4}\right)\right), \quad (5)$$

where $\sigma_0 = 4\pi\alpha_{\rm QED}\sum_q Q_q^2/s$ is the tree level cross-section for $e^+e^- \to q\bar{q}$, $\sqrt{s}$ is the center of mass energy, $\alpha_s = \alpha_s(s)$ is the $\overline{\rm MS}$ QCD coupling constant and the coefficients $C_i^{(j)}$ describe jet cross-sections in various orders in perturbation theory, as indicated by the Kronecker symbols. From inclusive calculations of the cross-section [12], we find

$$C_1^{(2)} + C_1^{(3)} = 1, \qquad C_2^{(2)} + C_2^{(3)} + C_2^{(4)} = \frac{365}{24} - 11\zeta(3) - \left(\frac{11}{12} - \frac{2\zeta(3)}{3}\right)N_f \approx 1.99 - 0.115 N_f, \quad (6)$$

where N_f is the number of massless fermion flavors.

An important goal of this Letter is the calculation of the coefficient $C_2^{(2)}$, the NNLO correction to the two-jet production rate. For this, we need the two-loop virtual correction to $e^+e^- \to q\bar{q}$, the one-loop correction to the $e^+e^- \to q\bar{q}g$ process, and the tree level processes $e^+e^- \to q\bar{q}gg$ and $e^+e^- \to q\bar{q}q_1\bar{q}_1$. We also require the coupling constant renormalization of the NLO result. At order $\mathcal{O}(\alpha_s^2)$, all of these processes contain divergent contributions to the two jet cross-sections; the highest singularity is $1/\epsilon^4$. The singularities cancel when individual contributions are combined to form a physical observable.

The two-loop virtual corrections to $e^+e^- \to q\bar{q}$ are well-known [13]. We have outlined above how the tree-level four parton final state is handled in our approach. We note that a global parameterization of the four particle phase space, which we used in [1] for the N_f terms, leads to large analytic expressions which are difficult to evaluate numerically. We found it much more convenient to select a different parameterization for the invariant masses in each individual term, thereby reducing the number of sector decompositions required to extract the singularities. This choice of parameterization can be done automatically once the basis topologies appearing in the matrix element are identified. With this clever choice of parameterization, the size of the computer code for the fully differential NNLO $e^+e^- \to 2$ jets process is not much larger than what we have found in simpler examples in [1]. The required CPU time is also not very large; to achieve the precision on the jet rates presented in this paper, about four hours are needed on a PC with a 3 GHz Pentium 4 processor.

We now briefly comment on the calculation of the one-loop corrections to the $q\bar{q}g$ final state. It might seem that a different technique is needed to handle this contribution, since a virtual loop integration is involved. However, this is not the case [1]. Once the virtual loop integrals are expressed through Feynman parameters, they can be treated identically to phase-space integrals. We found it convenient to express them through standard hypergeometric functions, and use the one-dimensional integral representation for the hypergeometric functions, together with the three-parton phase space parameterization. Although this procedure is not necessary, it is useful because it provides an economical input for sector decomposition.

Since our approach to the problem is numerical, including the cancellation of $1/\epsilon$ poles, we must consider issues of numerical accuracy. The simplest check is the comparison of the direct and indirect evaluations of the total two-jet event rate. The indirect result is obtained by taking the difference between the $\mathcal{O}(\alpha_s^s)$ contribution to the inclusive cross-section, given in Eq. (6), and subtracting from it the four-jet cross-section at LO and the three-jet cross-section at NLO. Both of these quantities are computed in our code. We use the JADE algorithm [14] to identify jets in the final state. However, the jet definition is an independent subroutine in our code that can be trivially changed if desired. Choosing the jet separation parameter for the JADE algorithm $y_{\rm cut} = 0.1$, we obtain

$$C_2^{(2),{\rm indirect}} = (-49.2 \pm 0.4) + (1.7974 \pm 0.0011)\,N_f, \quad (7)$$

where the errors denote our integration uncertainties for the 3 and 4 jet cross sections. A direct computation of the same quantity yields

$$\begin{aligned} C_2^{(2)} \;=\; & \frac{10^{-6}}{\epsilon^4} + \frac{10^{-4}}{\epsilon^3} + \frac{10^{-3}}{\epsilon^2} + \frac{(-4 \pm 4) \times 10^{-2}}{\epsilon} \\ & + \frac{(-0.3 \pm 4) \times 10^{-4}}{\epsilon} N_f + (-49.8 \pm 0.4) \\ & + (1.798 \pm 0.002)\,N_f \;. \end{aligned} \quad (8)$$

We have included the integration errors found during an actual run for the $1/\epsilon$ poles to demonstrate the level of cancellation; the magnitudes indicated for the higher poles are typical of results found using our code. Comparing Eq.(7) and Eq.(8), we conclude that our program provides a precision on the finite part of the NNLO correction to the two-jet rate better than 1%. We also conclude that our numerical cancellation of $1/\epsilon$ poles works very efficiently. These features do not change significantly when the jet separation parameter $y_{\rm cut}$ is varied.

Our approach permits us to also compute differential distributions in addition to the total rate. As an example, we present below the energy distribution of the leading jet in two jet events at NNLO. At leading order, this distribution is simple; since two massless quarks are produced, each jet contains half of the total energy. The distribution becomes more interesting at NLO, when it becomes possible for one of the jets to have an invariant mass different from zero. At NNLO, configurations when the invariant masses of both jets are different from zero appear for the first time. We compute this distribution by a simple modification of the jet function; after an event is identified as a two jet event, the energies of the

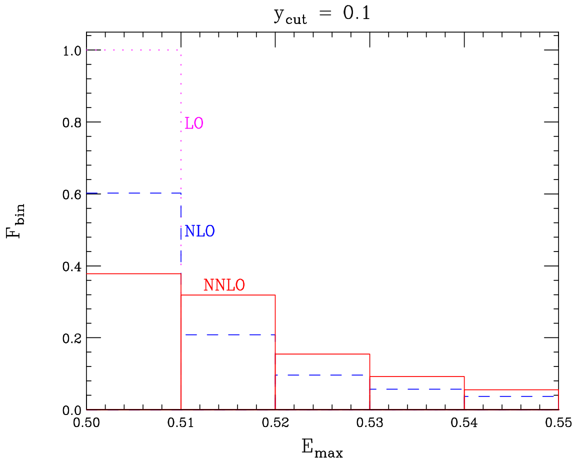

FIG. 1: Bin-integrated energy distribution for $y_{\rm cut} = 0.1$. The fractions of events in each energy bin are shown. The dotted histogram denotes the LO result, the dashed histogram the NLO result, and the solid histogram the NNLO result.

two jets are computed and the jet with the largest energy is identified. This number is then stored in the appropriate bin of a histogram. The corresponding bin-integrated energy distribution is shown in Fig. 1 for $y_{\rm cut} = 0.1$, $N_f = 5$, and $\alpha_s = 0.121$. The distribution is significantly distorted by NNLO QCD corrections; the corrections are large for this $y_{\rm cut}$, and many situations that look like three and four jet events are identified as two jet events. Smaller $y_{\rm cut}$ choices lead to large logarithms in the perturbative expansion that invalidate the fixed order result.

In conclusion, we have presented the first calculation of the NNLO corrections to a fully differential observable in QCD. We have demonstrated our approach using the non-trivial example of $e^+e^- \to 2$ jets. We have computed the NNLO corrections to the energy distribution of the two jets in e^+e^- annihilation, and have shown that the shape of the distribution changes when the NNLO corrections are included. Our method allows other phenomenologically interesting distributions in 2-jet events to be easily computed; these will be discussed elsewhere. Since our approach is fully numerical, we have presented convincing evidence that reasonable precision and control of numerical stability can be achieved. The method we developed for this calculation is quite flexible; it generalizes straightforwardly to an arbitrary number of partons in the final state, both massive and massless. Given unlimited computing resources, it provides a complete solution to the problem of real radiation at higher orders in perturbative QCD. In practice, significant effort and some ingenuity will be required to apply it to more complicated processes of direct phenomenological relevance. We look forward to this challenge.

K. M. thanks the KITP, UC Santa Barbara for its hospitality. C. A. thanks the Johns Hopkins University for its hospitality. This research was supported by the US Department of Energy under contracts DE-AC03-76SF0515, DE-FG03-94ER-40833 and the Outstanding Junior Investigator Award DE-FG03-94ER-40833, and‘ by the National Science Foundation under contracts P420D3620414350, P420D3620434350 and partially under Grant No. PHY99-07949.

[1] C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0311311.
[2] R. V. Harlander and W. B. Kilgore, Phys. Rev. Lett. **88**, 201801 (2002); C. Anastasiou and K. Melnikov, Nucl. Phys. **B646**, 220 (2002); V. Ravindran, J. Smith and W. L. van Neerven, Nucl. Phys. **B665**, 325 (2003).
[3] For a review, see S. Bethke, J. Phys. **G26**, R27 (2000).
[4] M. Dittmar, F. Pauss and D. Zürcher, Phys. Rev. **D56**, 7284 (1997); V. A. Khoze, A. D. Martin, R. Orava and M. G. Ryskin, Eur. Phys. J. **C19**, 313 (2001); W.T. Giele and S.A. Keller, hep-ph/0104053; R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. **B359**, 343 (1991) [Erratum-ibid. **B644**, 403 (2002)]; C. Anastasiou, L. J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. Lett. **91**, 182002 (2003).
[5] For a review and references to earlier work, see S. Catani, *"Jet physics at LEP and SLC,"*, hep-ph/9411361.
[6] F. V. Tkachov, Phys.Lett. **B100**, 65 (1981);. K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. **B192**, 345 (1981); S. Laporta, Mod. Phys. **A15**, 5087 (2000).
[7] See e.g. C. Anastasiou, et.al, Nucl. Phys. **B605**, 486 (2001); Z. Bern et. al, JHEP 03, 018 (2002); L. W. Garland et. al, Nucl. Phys. **B642**, 227 (2002).
[8] W. T. Giele and E. W. N. Glover, Phys. Rev. **D46**, 1980 (1992).
[9] S. Catani and M. H. Seymour, Nucl. Phys. **B485**, 291 (1997) [Erratum-ibid **B510**, 503 (1997)].
[10] D.A. Kosower, Phys. Rev. **D67**, 116003 (2003) S. Weinzierl, JHEP 0303, 062 (2003); JHEP 0307, 052 (2003).
[11] T. Binoth and G. Heinrich, Nucl. Phys. **B585**, 741 (2000).
[12] M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979); K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Phys. Lett. **B85**, 277 (1979); W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **44**, 560 (1980).
[13] W. L. van Neerven, Nucl. Phys. **B268**, 453 (1986).
[14] W. Bartel *et al.* [JADE Collaboration], Z. Phys. **C33**, 23 (1986).