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Coupled-Bunch Beam Breakup due to Resistive-Wall Wake*

Jiunn-Ming Wang†
National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973

Juhao Wu‡
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

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The coupled-bunch beam breakup problem excited by the resistive wall wake is formulated. An approximate analytic method of finding the asymptotic behavior of the transverse bunch displacement is developed and solved.

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I. INTRODUCTION

The coupled-bunch beam breakup (bbu) problem in a periodic linac excited by the resonance wake is well understood [1–3]. However, there are no systematic studies for the corresponding problem excited by the resistive wall impedance. This study of the resistive wall bbu problem is necessitated by the recently proposed PERL project [4]. For PERL, the light source consists of twelve undulators, each twelve meters long, totally 144 meters. The beam is shielded from the environment by circular copper pipes of a very small radius b=2.5 mm. The proposed injection cycle is twelve hours. It is crucial to know if the PERL beam can survive the bbu. We present our theoretical results for the resistive-wall coupled-bunch beam breakup problem in this paper. Some of the results obtained here has been briefly reported in Ref. [5].

The paper is organized as follows: In Sec. II, we set up the equations of motion and then solve the related eigenvalue problem. Physically, the eigenfunction so obtained describe the beam coherent-oscillation of an "extended problem". In Sec. III, we give a formal solution for the initial value problem. The solution consists of an integral representation for the transverse position of the M-th bunch at a longitudinal position z in terms of the eigenfunctions obtained in the previous section [6]. The asymptotic limit, $M \to \infty$, of the transient solution is then obtained in Sec. IV for two extreme cases: the No Focusing (NF) case and the Strong Focusing (SF) case. In Sections II $\sim$ V, we treat the case where only one bunch is offset initially. While in Sections II $\sim$ IV, we treat the case where every bucket of the linac is filled by the same amount of charge, we treat in Section V the case where the filling pattern is such that the beam has periodically unfilled gaps. The results of the section V is compared to the results of the preceding sections. The conclusion we draw from the comparison is that the asymptotic resistive-wall coupled-bunch bbu is a locally averaged current problem. In Section VI we go back to the problem where each bucket is symmetrically filled. The difference between this section and the section IV is that here we treat the case where initially the transverse position of every bunch is offset by the same amount – injection error. By comparing the results of Section VI with those of Section IV, we observe "Screen Effects" for the injection error case.

II. EQUATION OF MOTION AND THE EIGENVALUE PROBLEM

An electron bunch train consisting of a series of identical point like bunches passes through a circularly cylindrical pipe of radius b and conductivity σ. The entrance to the pipe is located at $z = 0$, and the M-th bunch, $M = 0, 1, 2...$, moves in the z direction according to $z = c\,t - M\,c\,\tau_{\rm B}$, where $\tau_{\rm B}$ is the bunch separation in units of seconds. Inside the pipe, the equation of motion for a particle in the bunch M is

$$\hat{O}\, y_M \equiv y''_{\rm M}(z) + k_y^2 y_{\rm M}(z) = \sum_{N=0}^{M-1} S(M-N) y_{\rm N}(z)\ , \quad (1)$$

where the prime "$\prime$" stands for d/dz. The right hand side of Eq. (1) represents the effects of the wake force , and for the resistive wall wake [7]

$$S(M) = a/\sqrt{M}\ , \quad (2)$$

with

$$a = 4\frac{I_{\rm B}}{I_{\rm A}}\frac{1}{b^3}\delta_{\rm skin}\ , \quad (3)$$

where $I_{\rm B} = e\,N_{\rm B}/\tau_{\rm B}$, $e\,N_{\rm B}$ = bunch charge, $I_{\rm A} \equiv 4\,\pi\,\epsilon_o\,mc^3\gamma/e = \gamma\,I_{\rm Alfvèn}$, $I_{\rm Alfvèn} \approx 17,000$ Amp, γ = the relativistic energy factor, and $\delta_{\rm skin} = \sqrt{2/\mu_o\sigma\omega_{\rm B}}$ = the

*Work supported by Department of Energy contract DE–AC02–98CH10886 (Jiunn-Ming Wang) and contract DE–AC03–76SF00515 (Juhao Wu)
†Electronic address: jmwang@bnl.gov
‡Electronic address: jhwu@SLAC.Stanford.EDU

skin depth corresponding to the bunch frequency $f_{\rm B} = \omega_{\rm B}/2\pi = 1/\tau_{\rm B}$. We ignore the effects of the wake force of a bunch on itself; as a consequence, the upper limit of the sum in Eq. (1) is $M-1$ instead of M. The thickness of the beam pipe is assumed to be ∞ for convenience. Also notice that the bunch N is in front of the bunch M if $M > N$.

In writing the above equations, we assumed the linac to be uniformly filled. For such a case, the locally averaged current $I_{\rm average} = I_{\rm B}$. For the case of non-uniform filling, an example of that will be discussed in Section V, the equation (1) has to be modified.

The right hand side of Eq. (1) is a convolution sum, therefore, it can be diagonalized by a Fourier transform. Define

$$F(\theta) = \sum_{M=1}^{\infty} \frac{1}{\sqrt{M}}\, e^{iM\,\theta} \;, \tag{4}$$

and

$$\xi(\theta, z) = \sum_{M=0}^{\infty} y_{\rm M}(z) e^{iM\,\theta} \;, \tag{5}$$

then

$$y_{\rm M}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-iM\theta} \xi(\theta, z) \;, \tag{6}$$

and

$$\xi''(\theta, z) + k_y^2 \xi(\theta, z) = aF(\theta)\xi(\theta, z) \;. \tag{7}$$

The last equation is an eigenvalue equation, with the parameter θ playing the role of distinguishing different eigenvalues. For the coherent mode θ, we see from Eq. (5) that the parameter θ is the phase difference of the adjacent bunches in this mode. Recall that in a storage ring, a symmetric coupling bunch mode n is characterized by the Courant-Sessler phase factor $\exp(i\,2\,\pi\,n/h)$ [8], where h is the number of the bunches in the ring. We can think of the phase $\exp(i\theta)$ as the limit of the Courant-Sessler factor as both n and $h \to \infty$ while $n/h = \theta$ remains finite. The eigenvalue for the mode θ is, from Eq. (7),

$$k_c(\theta) = \sqrt{k_y^2 - aF(\theta)} \;, \tag{8}$$

and the corresponding eigenvetors are

$$\cos[k_c(\theta)z], \qquad \text{or} \qquad \sin[k_c(\theta)z] \tag{9}$$

The function $F(\theta)$ can be written as [9]

$$\begin{aligned} F(\theta) &= \sqrt{\frac{i\pi}{\theta}} + \sum_{n=0}^{\infty} \zeta_{\rm Riemann}\left(\frac{1}{2} - n\right) \frac{(i\theta)^n}{n!} \\ &\approx \sqrt{\frac{i\pi}{\theta}} - 1.460 - 0.208\, i\,\theta + O(\theta^2) \;, \end{aligned} \tag{10}$$

where $\zeta_{\rm Riemann}(x)$ is the Riemann's Zeta function. The function $F(\theta)$ has a branch point at $\theta = 0$, therefore, through Eq. (7), $\xi(\theta, z)$ also has a singular point at the same position. Since Eq. (6) is the inverse of Eq. (5) and we look for $y_{\rm M}$ with $M > 0$, causality requires this singularity to lie below the contour of Eq. (6) on the θ plane. In order to explain this point more clearly, let us introduce

$$\zeta \equiv e^{i\theta} \;. \tag{11}$$

In term of this variable, Eqs. (4) $\sim$ (7) become

$$F(\zeta) = \sum_{M=1}^{\infty} \frac{1}{\sqrt{M}}\, \zeta^M \;, \tag{12}$$

$$\xi(\zeta, z) = \sum_{M=0}^{\infty} y_{\rm M}(z) \zeta^M \;, \tag{13}$$

$$y_{\rm M}(z) = \frac{1}{2\pi i} \oint d\zeta\, \zeta^{-(M+1)} \xi(\zeta, z) \;, \tag{14}$$

and

$$\xi''(\zeta, z) + k_y^2 \xi(\zeta, z) = aF(\zeta)\xi(\zeta, z) \;. \tag{15}$$

When expressing a function of θ, for example the function $F(\theta)$, in terms of ζ, we write $F(\zeta) = F(\theta)$ above instead of creating a new symbol; this should not introduce any unnecessary confusion. We adopt this convention throughout this paper. The singularity of $F(\theta)$ at $\theta = 0$, corresponds to a singularity of $F(\zeta)$ at $\zeta = 1$. The singular part of $F(\zeta)$ is

$$F(\zeta) \cong \sqrt{\frac{\pi}{1-\zeta}} \quad \text{for } \zeta \to 1 \;. \tag{16}$$

Equation (13) is a power series expansion of the function ξ in the variable ζ. The radius of the convergence circle of this series is 1, since the closest singularity of ξ is at $\zeta = 1$, i.e., at $\theta = 0$. From the residue theorem, Eq. (14) is clearly the inverse of Eq. (13) provided that the integration contour lies inside of the convergence circle, and the contour encircle the origin $\zeta = 0$ counterclockwise. It is convenient to take the contour to be the unit circle and the singularity to be located at $\zeta = 1 + \epsilon$ with a small and positive ϵ. On the ζ-plane, we make a cut on the real axis from $\zeta = 1 + \epsilon$ to $\zeta = \infty$, and make all the following calculation on the first sheet of the Riemann surface. Expressed in the θ variable in Eq. (6), the singularity is at $\theta = -\,i\,\log(1+\epsilon)$, i.e., below the contour of Eq. (6). The cut on the θ-plane is at the lower-half of the imaginary axis, i.e., θ from $-i\epsilon$ to $-i\infty$.

We solve in the next section the transient bbu problem by relating it to the coherent solutions given by Eqs. (8) and (9).

III. INITIAL VALUE PROBLEM

One can carry out the bbu calculations in terms of either the ζ or the θ variable. We choose to use the variable ζ here. (The paper [5] is carried out in the variable θ.)

We show in this section that the transient solution to the equation of motion (1) is

$$\begin{aligned} y_{\rm M}(z) &= y_{\rm M0}\cos(k_y z) + y'_{\rm M0}\sin(k_y z)\,/k_y \\ &+ \frac{1}{2\pi i}\sum_{N=0}^{M-1} y_{\rm N0}\oint d\zeta\,\zeta^{-(M-N+1)}\cos[k_c(\zeta)z] \\ &+ \frac{1}{2\pi i}\sum_{N=0}^{M-1} y'_{\rm N0}\oint d\zeta\,\zeta^{-(M-N+1)}\frac{\sin[k_c(\zeta)z]}{k_c(\zeta)}\ , \end{aligned} \tag{17}$$

where y_{M0} and y'_{M0} are, respectively, the initial values (values at $z=0$) of $y_{\rm M}(z)$ and $y'_{\rm M}(z)$.

First, we find the transient solution of Eq. (7). This equation yields

$$\tilde{\xi}(\zeta,s) = \frac{s\,\xi(\zeta,0)+\xi'(\zeta,0)}{s^2+k_y^2-a\,F(\zeta)}\ , \tag{18}$$

where

$$\tilde{\xi}(\zeta,s) \equiv \int_0^\infty dz\,\xi(\zeta,z)\,e^{-sz}. \tag{19}$$

After carrying out the inverse Laplace transform of (19), using (18), we obtain

$$\xi(\zeta,z) = \xi(\zeta,0)\cos[k_c(\zeta)z] + \xi'(\zeta,0)\frac{\sin[k_c(\zeta)z]}{k_c(\zeta)}\ . \tag{20}$$

In order to obtain (17), we substitute the above result (20) into (14) and then use $\xi(\zeta,0)=\sum_{M=0}^{\infty} y_{\rm M,0}\zeta^M$ and $\xi'(\zeta,0)=\sum_{M=0}^{\infty} y'_{\rm M,0}\zeta^M$. The result is (17). We shall apply the solution (17) to some specific cases in the next section.

IV. INITIAL SINGLE BUNCH OFFSET

We study in this section the equation (17) for the case where only the first bunch, i.e. $M=0$, is initially offset transversely from the center of the chamber, $y_{\rm M0}=y_{00}\delta_{\rm M,0}$, and $y'_{\rm M0}=0, \forall\, M$. In this case, Eq. (17) becomes, for $M\neq 0$,

$$\begin{aligned} y_{\rm M}(z) &= \frac{1}{2\pi i}y_{00}\oint d\zeta\zeta^{-(M+1)}\cos[k_c(\zeta)z] \\ &\equiv \frac{y_{00}}{4\pi}\left[\eta_{\rm M}^{(+)}(z)+\eta_{\rm M}^{(-)}(z)\right]\ , \end{aligned} \tag{21}$$

where

$$\eta_{\rm M}^{(\pm)}(z) \equiv \frac{1}{i}\oint d\zeta\exp\left\{\Psi_{\rm M}^{(\pm)}(\zeta)\right\}\ , \tag{22}$$

with

$$\Psi_{\rm M}^{(\pm)}(\zeta) = \pm i k_c(\zeta)z - (M+1)\log(\zeta)\ . \tag{23}$$

We wish to find the asymptotic behavior of $y_{\rm M}$ as given by Eq. (21) as $M\to\infty$; we shall use the well known saddle point method for this purpose.

The asymptotic behavior of the integral (21) is determined by the behavior of $\cos[k_c(\zeta)z]$ near $\zeta=1$, or $\theta=0$, where the phase difference between adjacent bunches approaches zero. In other words, the saddle point $\zeta_{saddle}\to 1$, or equivalently, $\theta_{saddle}\to 0$ in the limit of $M\to\infty$. The behavior of $\cos[k_c(\zeta)z]$ near $\zeta=1$ is, from Eq. (8), controlled by the behavior of $F(\zeta)$ in the same neighborhood, where $F(\zeta)$ is given in Eq. (16). We shall use the approximation for $F(\zeta)$ in Eq. (16) throughout the rest of this paper. Combining the last expression with Eqs. (8), (15) and (16), we have

$$k_c(\zeta) = \sqrt{k_y^2 - a\sqrt{\pi/(1-\zeta)}}\ , \tag{24}$$

and

$$\xi''(\zeta,z) + k_y^2\xi(\zeta,z) = a\sqrt{\frac{\pi}{1-\zeta}}\xi(\zeta,z)\ . \tag{25}$$

The last equation together with the equation(14) make up the basis for the remainder of this section.

We shall carry out below the asymptotic analysis of the following two cases:

First Case: This is the case where either $k_y=0$, or M is so large that the $a\sqrt{\pi/(1-\zeta)}$ term dominate over k_y^2 in Eq. (24). As a consequence, we can use the approximate expression

$$k_c(\zeta) = a_1 i(1-\zeta)^{-1/4} \tag{26}$$

where $a_1=\sqrt{a\sqrt{\pi}}$. This case will be referred to as the no focusing case. Clearly, in order for this approximation to be valid, the condition $|a_1(1-\zeta_{\rm NF})^{-1/4}|\gg k_y$, has to be satisfied, where $\zeta_{\rm NF}$ is the saddle point.

Second Case: This is the case where M is so large that Eq. (16) is valid, and yet k_y^2 in Eq. (24) dominates over the $a\sqrt{\pi/(1-\zeta)}$ term. As a consequence,

$$k_c(\zeta) \cong k_y - 2a_2(1-\zeta)^{-1/2}\ , \tag{27}$$

where $a_2=a\sqrt{\pi}/(4k_y)$. We shall refer to this case as the strong focusing case. The condition for the validity of this approximation is $k_y\gg|a_1(1-\zeta_{\rm SF})^{-1/4}|$, where $\zeta_{\rm SF}$ is the saddle point.

The remainder of this section is devoted to detailed treatment of these two cases.

A. No Focusing (NF) case

We wish to carry out the saddle point analysis to the integrals (21) and (22) with

$$\Psi_{\rm M}^{(\pm)}(\zeta) = (M+1)[\mp 4\alpha_1(1-\zeta)^{-1/4} - \log(\zeta)]\ , \tag{28}$$

$$\dot{\Psi}_{\rm M}^{(\pm)}(\zeta) = (M+1)[\pm\alpha_1(1-\zeta)^{-5/4} - 1/\zeta] \ , \qquad (29)$$

and

$$\ddot{\Psi}_{\rm M}^{(\pm)}(\zeta) = (M+1)[\mp(5/4)\alpha_1(1-\zeta)^{-9/4} + 1/\zeta^2] \ , \quad (30)$$

where "$\cdot$" stands for $d/d\zeta$, and $\alpha_1 \equiv a_1 z/[4(M+1)]$. The function $\Psi_{\rm M}^{(\pm)}(\zeta)$ has branch points at $\zeta = 0$ and $\zeta = 1$. Let us draw cuts in the ζ plane from $\zeta = -\infty$ to 0, and from $\zeta = 1$ to ∞. The integral (21) is performed on the first sheet of $\Psi_{\rm M}^{(\pm)}(\zeta)$ which is defined to be the sheet where $\Psi_{\rm M}^{(\pm)}(\zeta) =$ real for $0 < \zeta < 1$.

The saddle point $\zeta_{\rm NF}$ satisfies $\dot{\Psi}_{\rm M}^{(\pm)}(\zeta_{\rm NF}) = 0$, or

$$(1-\zeta_{\rm NF})^{5/4} = \pm\alpha_1\zeta_{\rm NF} \ . \qquad (31)$$

This equation can not be solved algebraically. However noting that $\alpha_1 = O(1/M)$ is small in the limit of $M \to \infty$, we solve the equation by perturbation. In terms of the variable $\tilde{\zeta} \equiv 1-\zeta$, Eq. (31) becomes, to the lowest order in α_1

$$\tilde{\zeta}_{\rm NF}^{5/4} = \mp\alpha_1 \ . \qquad (32)$$

Taking the fourth power of this equation, we have

$$\tilde{\zeta}_{\rm NF}^{5} = \alpha_1^4 \ , \qquad (33)$$

yielding the solutions

$$\tilde{\zeta}_{\rm NF} = \alpha_1^{4/5}(1, e^{\pm i2\pi/5}, e^{\pm i4\pi/5}) \ . \qquad (34)$$

The condition (33) is a necessary but not a sufficient condition for saddle points, (for example, we took the fourth power of Eq. (32) in order to obtain Eq. (33), we might in doing so have introduced spurious solutions.) Each of the solutions (34) has yet to be verified to be a relevant saddle point. It is straightforward to verify that $\zeta_{\rm NF}^{(-)} = 1-\alpha_1^{4/5}$ is the only saddle point of $\eta^{(-)}$, and $\zeta_{\rm NF}^{(+)} = 1-\alpha_1^{4/5}e^{\pm i4\pi/5}$ are the only saddle points of $\eta^{(+)}$ we have to consider.

The saddle point contribution to $\eta^{(\pm)}$ satisfies

$$\eta_M^{(\pm)} \propto \exp\left[\Psi^{(\pm)}\left(\zeta_{\rm NF}^{(\pm)}\right)\right] \ . \qquad (35)$$

Routine calculation gives the following results for the exponents:

$$\Psi_{\rm M}^{(-)}\left(\zeta_{\rm NF}^{(-)}\right) = 5(M+1)\alpha_1^{4/5} \ , \qquad (36)$$

and

$$\Psi_{\rm M}^{(+)}\left(\zeta_{\rm NF}^{(+)}\right) = 5(M+1)\alpha_1^{4/5}\exp(\pm i4\pi/5) \ . \qquad (37)$$

Notice that the real part of $\Psi^{(+)}$ above is negative; therefore, $\eta_M^{(+)} \to 0$ in the limit of $M \to \infty$. We shall ignore the $\eta_M^{(+)}$ term in Eq (21).

In order to perform the saddle point integral for $\eta_M^{(-)}$ we need, in addition to (36), the following

$$\ddot{\Psi}_{\rm M}^{(-)}\left(\zeta_{\rm NF}^{(-)}\right) = \frac{5(M+1)}{4\alpha_1^{4/5}} \ . \qquad (38)$$

We notice that $\ddot{\Psi}_{\rm M}^{(-)}\left(\zeta_{\rm NF}^{(-)}\right) \propto \alpha_1^{-9/5} \propto M^{9/5} \to \infty$ very fast, as $M \to \infty$. Such sharp dependence of the integrand of (22) in the neighborhood of the saddle saddle point validates the saddle point approximation.

From the above discussion, the equation

$$y_{\rm M}(z) = \frac{y_{00}}{4\pi}\eta_{\rm M}^{(-)}(z) \propto \exp\left[\Psi^{(-)}\left(\zeta_{\rm NF}^{(-)}\right)\right] \qquad (39)$$

together with Eqs. (36) and (38) are all we need for the saddle point estimate of the present bbu problem. However, before stating the results, let us have a discussion on the growth time $t_{\rm NF}$ of the mode under discussion.

The M-th bunch reaches the linac at time $t = M\tau_{\rm B}$. The quantity α_1 in the expression (36) can be written in terms of M and a. If we replace M or $M+1$ (recall that $M \gg 1$) in the resulting $\Psi^{(-)}$ by $t/\tau_{\rm B}$, we obtain

$$\Psi_{\rm M}^{(-)}\left(\zeta_{\rm NF}^{(-)}\right) = \left(\frac{t}{t_{\rm NF}}\right)^{1/5} \ , \qquad (40)$$

where the growth time

$$t_{\rm NF} = \frac{\tau_{\rm B}}{4\pi}\left(\frac{4}{5}\right)^5 \frac{1}{z^4}\frac{1}{a^2} \ , \qquad (41)$$

and the result of the saddle-point integral is

$$\begin{aligned} y_{\rm M}(z) &= \frac{y_{00}}{4\pi}\eta_{\rm M}^{(-)}(z) \\ &= \frac{y_{00}}{5\sqrt{2\pi}}\left(\frac{t_{\rm NF}}{t}\right)^{9/10}\frac{\tau_{\rm B}}{t_{\rm NF}}\exp\left\{\left(\frac{t}{t_{\rm NF}}\right)^{1/5}\right\} \end{aligned} \qquad (42)$$

So far we have been dealing with the case of a uniformly filled linac. If the filling is not uniform, (some buckets not filled,) the above results do not hold. In Section V, we shall treat an example of such non-uniform case. In order to facilitate later comparison, let us write Eq. (41) for $t_{\rm NF}$ in another form. Using Eq. (3), Eq. (41) becomes

$$t_{\rm NF} = \frac{\tau_{\rm B}}{\pi}\frac{16}{5^5}\frac{b^6}{z^4}\frac{1}{\delta_{\rm skin}^2}\frac{I_{\rm A}^2}{I_{\rm B}^2} \ . \qquad (43)$$

For the case of uniform filling, the $I_{\rm B} = e\,N_{\rm B}/\tau_{\rm B}$ above equals the locally averaged current $I_{\rm average}$. Therefore the above equation can be expressed as

$$t_{\rm NF} = \frac{\tau_{\rm B}}{\pi}\frac{16}{5^5}\frac{b^6}{z^4}\frac{1}{\delta_{\rm skin}^2}\frac{I_{\rm A}^2}{I_{\rm average}^2} \ . \qquad (44)$$

We shall compare later the above expressions (43) and (44) with the corresponding result for a non-uniformly filled beam.

B. Strong Focusing (SF) case

The treatment of this case is similar to the NF case. The exponent of the integrand of the integral (22) is, for this case,

$$\Psi_{\rm M}^{(\pm)}(\zeta) = \pm ik_y z \mp i2a_2 z(1-\zeta)^{-1/2} - (M+1)\log(\zeta) \,. \quad (45)$$

This function has branch points at $\zeta = 0$ and $\zeta = 1$. We cut the complex ζ plane from $\zeta = -\infty$ to 0, and from $\zeta = 1$ to ∞. The first two derivatives of $\Psi_{\rm M}^{(\pm)}(\zeta)$ are

$$\dot{\Psi}_{\rm M}^{(\pm)}(\zeta) = \mp ia_2 z(1-\zeta)^{-3/2} - \frac{M+1}{\zeta} \,, \quad (46)$$

and

$$\ddot{\Psi}_{\rm M}^{(\pm)}(\zeta) = \mp i\frac{3}{2}a_2 z(1-\zeta)^{-5/2} + \frac{M+1}{\zeta^2} \,. \quad (47)$$

The saddle point condition $\dot{\Psi}_{\rm M}^{(\pm)}(\zeta_{\rm SF}) = 0$ leads to

$$(1-\zeta_{\rm SF})^{3/2} = \mp i\alpha_2\zeta_{\rm SF} \,, \quad (48)$$

with $\alpha_2 \equiv a_2 z/(M+1)$. Since $\alpha_2 \to 0$, as $M \to \infty$, we could again find the saddle points by a perturbation method. The result is, to the leading order of α_2,

$$\zeta_{\rm SF} = \left(1-\alpha_2^{2/3}e^{i\pi/3}, 1+\alpha_2^{2/3}, 1-\alpha_2^{2/3}e^{-i\pi/3}\right) \,, \quad (49)$$

where we write the solutions of Eq. (48) as elements of a 1×3 row matrix.

The equation (49) is a necessary but not a sufficient condition for the saddle points. Simple algebraic calculations shows that the first element of the matrix (49) is a saddle point of $\eta_M^{(-)}$, and that the third element is a saddle point of $\eta_M^{(+)}$. The second element of (49) which is > 1 and lies on the branch cut is not accessible to the integration contour.

We need to evaluate $\Psi_{\rm M}^{(\pm)}$ and $\ddot{\Psi}_{\rm M}^{(\pm)}$ at the appropriate saddle points. They are

$$\Psi_{\rm M}^{(+)}(\zeta_{\rm SF,3}) = +ik_y z + 3(M+1)\alpha_2^{2/3}\exp\left\{-\frac{i\pi}{3}\right\} \,,$$

$$\ddot{\Psi}_{\rm M}^{(+)}(\zeta_{\rm SF,3}) = \frac{3(M+1)}{2\alpha_2^{2/3}}\exp\left\{\frac{i\pi}{3}\right\} \,,$$

$$\Psi_{\rm M}^{(-)}(\zeta_{\rm SF,1}) = -ik_y z + 3(M+1)\alpha_2^{2/3}\exp\left\{\frac{i\pi}{3}\right\} \,,$$

and

$$\ddot{\Psi}_{\rm M}^{(-)}(\zeta_{\rm SF,1}) = \frac{3(M+1)}{2\alpha_2^{2/3}}\exp\left\{-\frac{i\pi}{3}\right\} \,.$$

Using these results, we obtain the following asymptotic result for the displacement of the M-th bunch:

$$y_{\rm M}(z) = \frac{2y_{00}}{3\sqrt{2\pi}}\left(\frac{t_{\rm SF}}{t}\right)^{5/6}\frac{\tau_{\rm B}}{t_{\rm SF}}\exp\left\{\left(\frac{t}{t_{\rm SF}}\right)^{1/3}\right\} \times \cos\left[\sqrt{3}\left(\frac{t}{t_{\rm SF}}\right)^{1/3} - k_y z + \frac{\pi}{6}\right] \,, \quad (50)$$

where the growth time for this mode

$$t_{\rm SF} \equiv \tau_{\rm B}\left(\frac{2}{3}\right)^3\frac{1}{a_2^2 z^2} \,, \quad (51)$$

and again $t = (M+1)\tau_{\rm B}$, or $M\tau_{\rm B}$ since M is large

The results of this section have been applied to the parameters of PERL in Ref [5]. The conclusion of that study is that the PERL beam as designed can not survive the resistive wall bbu without feedback dampers.

V. BEAM WITH PERIODIC GAPS

The bunch filling pattern considered in this section is as follows: The beam is made of repetitive identical sequences where each sequence consists of p adjacent filled buckets followed by q empty buckets; there are in total $r = p + q$ buckets in a sequence.

A. Equations of motion

If all the buckets are filled, then

$$\hat{O}\, y_M \equiv \left(\frac{d^2}{dz^2} + k_y^2\right) y_M = \sum_{N=1}^{M-1} S(M-N)\, y_N \,, \quad (52)$$

where, $S(M)$ is the wake function given in Section II; $S(M) = 0$ for $M \le 0$, and $S(M-N) = a/\sqrt{M-N}$ for $M > 0$. The parameter a is given by Eq. (3). Note that we have made here a slight change of convention. We designated the bunches as $M = 1, 2, 3, ...$ above instead of $M = 0, 1, 2, ...$ as was done in Section II. We adopt this new convention throughout this section.

We have to generalize the above equation to include the case of a beam with periodic empty buckets. Let us use the notation $u = 1, 2, 3...$ for the sequence number, and $m = 1, 2....p$ for the bunch number in a sequence. It is convenient to define, corresponding to each u a $p \times p$ matrix $\mathbb{S}^{(u)}$ with its elements given by

$$\mathbb{S}_{m,n}^{(u)} = \mathbb{S}_{m-n}^{(u)} = S[ur + (m-n)] \,, \quad (53)$$

where the range of u for $\mathbb{S}^{(u)}$ is $u = 0, 1, 2, ...$ Corresponding to the above matrix, we define $1 \times p$ column vector

$$Y^{(u)} \equiv \begin{pmatrix} y_{u,1} \\ y_{u,2} \\ \vdots \\ y_{u,p} \end{pmatrix} \quad (54)$$

where $y_{u,m}$ is the transverse displacement of the m-th bunch in the u-th sequence.

The equation of motion for a beam with periodic gaps can now be written in a compact form similar to Eq. (1),

$$\hat{O}Y^{(u)} = \sum_{v=1}^{u} \mathbb{S}^{(u-v)} Y^{(v)} \ . \tag{55}$$

We solve this equation in the next subsection.

B. Solutions

The m-th component of the equation of motion (55) is

$$\hat{O}y_{u,m} = \sum_{v=1}^{u}\sum_{n=1}^{p} \mathbb{S}^{(u-v)}_{m-n} y_{v,n} \ . \tag{56}$$

The following generalization of Eqs. (4) and (5) is convenient:

$$\xi_m(\zeta) = \sum_{u=1}^{\infty} \zeta^u y_{u,m} \ , \tag{57}$$

$$\Delta_m(\zeta) = \sum_{u=0}^{\infty} \mathbb{S}^{(u)}_m \zeta^u \ . \tag{58}$$

Then the above three Eqs. (55) $\sim$ (58) lead to

$$\hat{O}\xi_m(\zeta) = \sum_{n=1}^{p} \Delta_{m-n}(\zeta)\xi_n(\zeta) \ . \tag{59}$$

Once the solution of the last equation is found, the displacement of the individual bunch is found by substituting the solution into the inverse of (57); namely,

$$y_{u,m} = \frac{1}{2\pi i}\oint d\zeta\, \zeta^{-(u+1)}\xi_m(\zeta) \ . \tag{60}$$

The method we use to solve Eq. (59) is a generalization of the method of Section IV. Note that for $u \to \infty$, the contribution to the integral (60) is dominated by the behavior of the integrand near $\zeta = 1$. Therefore we shall, in analogy to what we did in Section IV, approximate Δ_m by its singular part near $\zeta = 1$. The singular part is from [9]

$$\Delta_m(\zeta) \cong \frac{a}{\sqrt{r}}\sqrt{\frac{\pi}{1-\zeta}} \quad \forall \quad m \ , \tag{61}$$

and the corresponding approximation to Eq. (59) is

$$\hat{O}\,\xi_m(\zeta) \cong \frac{ap}{\sqrt{r}}\sqrt{\frac{\pi}{1-\zeta}}\,\xi_m(\zeta) \ , \quad \forall \quad m \ . \tag{62}$$

This equation together with Eq. (60) gives us the asymptotic behavior, $u \to \infty$, of $y_{u,m}$.

Observe the similarity of Eqs. (62) and (60) above to the following equations we obtained earlier for the uniform filling case, (Eqs. (25) and (14),)

$$\hat{O}\xi(\zeta,z) = a\sqrt{\frac{\pi}{1-\zeta}}\xi(\zeta,z) \ , \tag{63}$$

$$y_{\rm M}(z) = \frac{1}{2\pi i}\oint d\zeta\, \zeta^{-(M+1)}\xi(\zeta,z) \ . \tag{64}$$

The variable m appears as a passive parameter in Eqs. (62) and (60). Also, these equations can be obtained from Eqs. (63) and (64) by the following substitutions:

$$M, \text{ or } (M+1) \to \ u \ , \tag{65}$$

$$a \to ap/\sqrt{r} \ . \tag{66}$$

Therefore, we can obtain the results for Eqs. (62) and (60) from the corresponding results for the uniform filling case. We treat here the "No Focusing " case corresponding to the subsection IV-A. We specifically consider the growth time $t^{gap}_{\rm NF}$ for the beam with periodic gaps. The "Strong Focusing " case can be treated in a similar way.

We start from the exponent $\Psi^{(-)}_{\rm M}$ as given by (36). Expressing α_1 in terms of a, this equation is equivalent to

$$\Psi^{(-)}_{\rm M}\left(\zeta^{(-)}_{\rm NF}\right) = 5\pi^{1/5}(M+1)^{1/5}(z/4)^{4/5}a^{2/5} \ . \tag{67}$$

Now applying the substitution rules (65) and (66) to Eq. (67), we obtain

$$\Psi^{(-)}_{{\rm gap},u}\left(\zeta^{(-)}_{\rm NF}\right) = \left(\frac{u}{u_{\rm NF}}\right)^{1/5} \ , \tag{68}$$

where

$$u_{\rm NF} = \frac{1}{4\pi}\left(\frac{4}{5}\right)^5 \frac{1}{z^4}\frac{r}{a^2p^2} \tag{69}$$

is the growth time in units of sequences.

We have to translate u into time t. The bunch (u,m) reaches the linac at $t = (ur+m)\tau_{\rm B} \cong ur\tau_{\rm B}$. Therefore we should set $u \to t/r\tau_{\rm B}$ and

$$t^{\rm gap}_{\rm NF} = r\tau_{\rm B}u_{\rm NF} \tag{70}$$

$$= \frac{\tau_{\rm B}}{4\pi}\left(\frac{4}{5}\right)^5 \frac{1}{z^4}\frac{1}{a^2}\frac{r^2}{p^2} \tag{71}$$

$$= \frac{\tau_{\rm B}}{\pi}\frac{16}{5^5}\frac{b^6}{z^4}\frac{1}{\delta^2_{\rm skin}}\frac{I^2_{\rm A}}{I^2_{\rm B}}\frac{r^2}{p^2} \ . \tag{72}$$

These expressions differ from Eq. (41) or (43) by a factor of r^2/p^2. However, this difference is superficial. Let us calculate the average current of a sequence. It is clearly

$$I_{\rm average} = \frac{p}{r}I_{\rm B} \ . \tag{73}$$

In terms of $I_{\rm average}$, the growth time becomes

$$t^{\rm gap}_{\rm NF} = \frac{\tau_{\rm B}}{\pi}\frac{16}{5^5}\frac{b^6}{z^4}\frac{1}{\delta^2_{\rm skin}}\frac{I^2_{\rm A}}{I^2_{\rm average}} \ . \tag{74}$$

This is identical to (44). We therefore conclude that the coupled-bunch resistive-wall bbu is a locally-averaged current effect.

VI. INJECTION ERROR AND SCREEN EFFECT

In this section, we study Eq. (17) for the case where bbu is started up by an injection error, i.e., all the bunches are initially offset by the same amount, $y_{\rm M0} = y_{00}$, and $y'_{\rm M0} = 0, \forall\, M \geq 0$. Then, Eq. (17) becomes

$$\begin{aligned} y_{\rm M}(z) &= y_{00}\cos(k_y z) \\ &- \frac{1}{2\pi i} y_{00} \oint d\zeta \zeta^{-1}(1-\zeta)^{-1}\cos k_c(\zeta) z \\ &+ \frac{1}{2\pi i} y_{00} \oint d\zeta \zeta^{-(M+1)}(1-\zeta)^{-1}\cos k_c(\zeta) z \\ &= y_{00}\cos(k_y z) - y_{00}\cos(k_c(0) z) \\ &+ \frac{y_{00}}{4\pi}\left[\eta_{\rm M}^{(+)}(z) + \eta_{\rm M}^{(-)}(z)\right] , \end{aligned} \tag{75}$$

where, $\eta_{\rm M}^{(\pm)}(z)$ is given by Eq. (22) with

$$\Psi_{\rm M}^{(\pm)}(\zeta) = \pm i k_c(\zeta) z - (M+1)\log(\zeta) - \log(1-\zeta) . \tag{76}$$

Compared with Eq. (23), Eq. (76) has an additional term $-\log(1-\zeta)$ on the right hand side. We shall see presently that this term does not change the eigen solutions as given in Secs II $\sim$ V, but it will change the transient solutions. We shall also see that this term leads to an interesting "Screen Effect". From $k_c(0) = k_y$, Eq. (75) becomes simplified to

$$y_{\rm M}(z) = \frac{y_{00}}{4\pi}\left[\eta_{\rm M}^{(+)}(z) + \eta_{\rm M}^{(-)}(z)\right] . \tag{77}$$

Let us discuss as before two extreme cases: the No Focusing case and the Strong Focusing case.

A. No Focusing (NF) case

Similar to what was done in Sec. IV, we wish to carry out the saddle point analysis to the integral (22) with the exponent

$$\Psi_{\rm M}^{(\pm)}(\zeta) = \mp a_1 z(1-\zeta)^{-1/4} - (M+1)\log(\zeta) - \log(1-\zeta) . \tag{78}$$

The first two derivatives of the exponent are

$$\dot{\Psi}_{\rm M}^{(\pm)}(\zeta) = \mp\frac{1}{4}a_1 z(1-\zeta)^{-5/4} - \frac{M+1}{\zeta} + \frac{1}{1-\zeta} , \tag{79}$$

and

$$\ddot{\Psi}_{\rm M}^{(\pm)}(\zeta) = \mp\frac{5}{16}a_1 z(1-\zeta)^{-9/4} + \frac{M+1}{\zeta^2} + \frac{1}{(1-\zeta)^2} . \tag{80}$$

The saddle points are determined by $\dot{\Psi}_{\rm M}^{(\pm)}(\zeta_{\rm NF}) = 0$, i.e.,

$$0 = \mp\frac{1}{4}a_1 z(1-\zeta_{\rm NF})^{-5/4} - \frac{M+1}{\zeta_{\rm NF}} + \frac{1}{1-\zeta_{\rm NF}} , \tag{81}$$

which can not be solved algebraically. However, since the saddle points $\zeta_{\rm saddle} \to 1$ in the limit of $M \to \infty$, we could solve Eq. (81) by a perturbation method. In terms of $\tilde{\zeta} \equiv 1-\zeta$, Eq. (81) becomes

$$\mp\frac{1}{4}a_1 z\tilde{\zeta}_{\rm NF}^{-5/4} \pm \frac{1}{4}a_1 z\tilde{\zeta}_{\rm NF}^{-1/4} + \tilde{\zeta}_{\rm NF}^{-1} - 1 = M+1 . \tag{82}$$

Keeping the leading term in Eq. (82), we get

$$\mp\frac{1}{4}a_1 z\tilde{\zeta}_{\rm NF}^{-5/4} = M+1 . \tag{83}$$

The last equation is identical to Eq. (32), and therefore this yields the same first-order solution given in Eq. (34). We select now the relevant saddle points by repeating what we did before following Eq. (34), and then carry out the saddle point integral corresponding to the exponent (78). The result is

$$y_{\rm M}(z) = \mathcal{G}_{\rm NF}\frac{y_{00}}{5\sqrt{2\pi}}\left(\frac{t_{\rm NF}}{t}\right)^{9/10}\frac{\tau_{\rm B}}{t_{\rm NF}}\exp\left\{\left(\frac{t}{t_{\rm NF}}\right)^{1/5}\right\} , \tag{84}$$

where the growth time

$$t_{\rm NF} = \frac{\tau_{\rm B}}{4\pi}\left(\frac{4}{5}\right)^5\frac{1}{z^4}\frac{1}{a^2} , \tag{85}$$

and

$$\begin{aligned} \mathcal{G}_{\rm NF} &\equiv 5\left(\frac{t}{\tau_B}\right)^{4/5}\left(\frac{t_{\rm NF}}{\tau_B}\right)^{1/5} \\ &= 4\left(\frac{1}{4\pi a^2}\right)^{1/5}\left(\frac{M}{z}\right)^{4/5} . \end{aligned} \tag{86}$$

It is very interesting to compare with the above result (84) to the result (42) of the initial single-bunch offset case. (1) The growth time $t_{\rm NF}$ is the same for both cases, as it should be, since $t_{\rm NF}$ should depends only on the eigen solutions. (2) The only difference between the transient solutions is the factor $\mathcal{G}_{\rm NF}$ which is proportional to $M^{4/5}$ instead of to M, (recall that $t \propto M$.) This is surprising: Since $\theta_{saddle} \cong 0$, we would expect all the bunches preceding the bunch M to excite this bunch by the same amount leading to $\mathcal{G}_{\rm NF} \propto M$. Clearly, the preceding bunches are screening each other. (It can actually be shown that for large but not too large M, the function $\mathcal{G}_{\rm NF} \propto M$.)

B. Strong Focusing (SF) case

Let us not go into detailed discussion of this case, since the arguments are so similar to those of Section IV and Section VI-A. We just list the results:

$$y_{\rm M}(z) = \mathcal{G}_{\rm SF}\frac{2y_{00}}{3\sqrt{2\pi}}\left(\frac{t_{\rm SF}}{t}\right)^{5/6}\frac{\tau_{\rm B}}{t_{\rm SF}}\exp\left\{\left(\frac{t}{t_{\rm SF}}\right)^{1/3}\right\} \times \cos\left[\sqrt{3}\left(\frac{t}{t_{\rm SF}}\right)^{1/3} - k_y z - \frac{\pi}{6}\right], \quad (87)$$

where

$$t_{\rm SF} \equiv \tau_{\rm B}\left(\frac{2}{3}\right)^3\frac{1}{a_2^2 z^2}, \quad (88)$$

and

$$\mathcal{G}_{\rm SF} \equiv \frac{3}{2}\left(\frac{t}{\tau_B}\right)^{2/3}\left(\frac{t_{\rm SF}}{\tau_B}\right)^{1/3} = \left(\frac{16k_y^2}{\pi a^2}\right)^{1/3}\left(\frac{M}{z}\right)^{2/3}. \quad (89)$$

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