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AN ALTERNATIVE SET OF OPERATORS FOR THREE-PARTICLE SCATTERING*

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ABSTRACT

An alternative set of operators for three-particle scattering is obtained in a natural way from a wavefunction formulation of the three-particle problem. These operators, intermediate between the more conventional Faddeev and Lovelace-type operators, are found to combine attractive features of both.

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I. INTRODUCTION

It has by now become conventional to discuss three-particle scattering theory either in terms of the Faddeev operators T^{β} or $M_{\beta\alpha}$, or the Lovelace-type operators $U_{\beta\alpha}$, $\alpha, \beta = 1, 2, 3$. In this paper we present a different set of operators, which we label $K_{\beta\alpha}$. Although already introduced more or less explicitly by a number of authors,[1] this set — intermediate between the above mentioned operators — is shown here to combine attractive features of both $M_{\beta\alpha}$ and $U_{\beta\alpha}$, and to be the most natural one to consider when starting from a wave-function approach to the three-particle scattering problem.

In the next section we discuss the significance of the operators $K_{\beta\alpha}$ and their relation to three-particle wavefunctions. In Section III we present their connection with the more familiar three-body operators, and with the interesting Faddeev kernels. We also show that all physical transition amplitudes can be simply expressed in terms of the set $K_{\beta\alpha}$, and mention some straightforward applications of the formalism. Finally, in the last section we explicitly perform an angular momentum expansion of matrix elements of $K_{\beta\alpha}$, and compare this with the corresponding expansion for $U_{\beta\alpha}$. The expansion for $K_{\beta\alpha}$ is shown to converge more rapidly than the corresponding expansion for $U_{\beta\alpha}$, and this enables us to point out that while physical transition amplitudes are as simply expressed in terms of $K_{\beta\alpha}$ as in terms of $U_{\beta\alpha}$, the former operators may in some cases be more suited for use than the latter.

II. THE WAVEFUNCTION APPROACH

Consider a three-particle scattering process in which the initial state is a two-particle bound state and a free particle, described by the channel eigenstate $|\Phi_{\alpha n}\rangle = |\psi_{\alpha n}\rangle\ |\vec{q}\rangle$. In the wavefunction approach to three-particle scattering theory,[2] the outgoing wave scattering solution is obtained from the resolvent $G(s) = (H-s)^{-1}$ of the total Hamiltonian $H = H_0 + V = H_0 + V_\alpha + V_\beta + V_\gamma$ as

$$|\Psi_\alpha^+\rangle = -i\epsilon G(E+i\epsilon)|\Phi_{\alpha n}\rangle \tag{2.1}$$

where E is the energy. The splitting of the wavefunction into Faddeev components $|\Psi_\alpha^+\rangle = \Sigma_\beta |\Psi_\alpha^\beta\rangle$ corresponds to a splitting according to which pair of particles interacts last, which in terms of the resolvent means

$$G = G_0 - G_0 V G = \Sigma_\beta \left[\delta_{\beta\alpha} G_0 - G_0 V_\beta G\right] \tag{2.2}$$

Thus,

$$|\Psi_\alpha^\beta\rangle = -i\epsilon \left[\delta_{\beta\alpha} G_0(E+i\epsilon) - G_0(E+i\epsilon) V_\beta G(E+i\epsilon)\right] |\Phi_{\alpha n}\rangle$$

Upon taking the limit $\epsilon \to 0$ — performed by using the resolvent identity $G = G_\alpha - G\bar{V}_\alpha G_\alpha$, where G_α is the resolvent of $H_0 + V_\alpha$ — this expression can be used to define a set of operators such that

$$|\Psi_\alpha^\beta\rangle = \left[\delta_{\beta\alpha} - G_0(E+i0) K_{\beta\alpha}(E+i0)\right] |\Phi_{\alpha n}\rangle \tag{2.3}$$

where

$$K_{\beta\alpha} = \bar{\delta}_{\beta\alpha} V_\beta - V_\beta G \bar{V}_\alpha , \qquad \bar{\delta}_{\beta\alpha} = 1 - \delta_{\beta\alpha} , \qquad \bar{V}_\alpha = V - V_\alpha \tag{2.4}$$

If the initial state is not a bound pair plus a free particle but rather three free particles — described by $|\Phi_0\rangle = |\vec{p}\rangle\,|\vec{q}\rangle$ — the necessary modifications are minor. For the Faddeev components of the outgoing wave scattering state we again arrive at

$$|\Psi_0^\beta\rangle = \left[\delta_{\beta\alpha} - G_0(E+i0) K_{\beta\alpha}(E+i0)\right] |\Phi_\alpha\rangle \tag{2.5}$$

where here $|\Phi_\alpha\rangle = \left[1 - G_\alpha(E+i0) V_\alpha\right] |\Phi_0\rangle = |\Psi_\alpha\rangle\,|\vec{q}\rangle$, and $|\Psi_\alpha\rangle$ is a two-particle outgoing wave scattering state. The choice of channel α is of course in this case arbitrary.

From the two relations (2.3) and (2.5) it is evident that the wavefunction formulation of three-particle scattering theory will be very closely related to a formulation in terms of the set of operators $K_{\beta\alpha}$[3]. In the next section we outline such a formulation.

III. THE OPERATORS $K_{\beta\alpha}$

Let us write down the Faddeev-type equations for the operators introduced above:

$$K_{\beta\alpha} = \bar{\delta}_{\beta\alpha} t_\beta - t_\beta G_0 \Sigma_{\gamma\neq\beta} K_{\gamma\alpha} \tag{3.1}$$

$$K_{\beta\alpha} = \bar{\delta}_{\beta\alpha} t_\beta - \Sigma_{\gamma\neq\alpha} K_{\beta\gamma} G_0 t_\gamma \tag{3.2}$$

From (3.1) we see explicitly that the property of $|\Psi_\alpha^\beta\rangle$ of describing a situation where the last interacting pair is the β-pair has been carried over to $K_{\beta\alpha}$. This is of course a property the $K_{\beta\alpha}$ operator shares with the Faddeev operators $M_{\beta\alpha} = \delta_{\beta\alpha} V_\alpha - V_\beta G V_\alpha$ and ${}^\beta T = V_\beta - V_\beta G V$; however, the more commonly used operators of the Lovelace type, e.g., the AGS[4] operators $U_{\beta\alpha} = -\bar{\delta}_{\beta\alpha} G_\alpha^{-1} + \bar{V}_\beta - \bar{V}_\beta G \bar{V}_\alpha$ do not have this property. In fact, the $K_{\beta\alpha}$ set can be considered as the result of splitting the $U_{\beta\alpha}$ set into components according to which pair of particles interacts last. This is evident from the expression

$$U_{\beta\alpha} = -\bar{\delta}_{\beta\alpha} G_0^{-1} + \Sigma_{\gamma\neq\beta} K_{\gamma\alpha} \tag{3.3}$$

which follows from the definitions of the operators involved.

It is also evident from (3.3) that the main advantage of the Lovelace-type operators, namely their close relationship to the physical transition amplitudes, will be shared by $K_{\beta\alpha}$. To see this in detail, consider the transition amplitudes for three-particle scattering processes starting with one free particle and a bound pair in the α-channel. The amplitudes for elastic, rearrangement and breakup scattering are,

$$\langle\Phi_\beta|\bar{V}_\beta|\Psi_\alpha^+\rangle = \langle\Phi_\beta|\left[-\bar{\delta}_{\beta\alpha}G_0^{-1} + \Sigma_{\gamma\neq\beta}K_{\gamma\alpha}\right]|\Phi_{\alpha n}\rangle \tag{3.4}$$

$$\langle\Phi_0|V|\Psi_\alpha^+\rangle = \langle\Phi_0|\Sigma_\gamma K_{\gamma\alpha}|\Phi_\alpha\rangle = \langle\Phi_\beta|\left[-\bar{\delta}_{\beta\alpha}G_0^{-1}+\Sigma_{\gamma\neq\beta}K_{\gamma\alpha}\right]|\Phi_{\alpha n}\rangle \tag{3.5}$$

where it is understood that the matrix elements are on-shell, $S=E+i0$, $E_\alpha=E_\beta=E_0=E$. In order to obtain these expressions, we have used the relation $V_\alpha|\Phi_{\alpha n}\rangle = -G_0^{-1}(E_\alpha+i0)|\Phi_{\alpha n}\rangle$ and that consequently on the energy shell $\langle\Phi_0|V_\alpha|\Phi_{\alpha n}\rangle = \langle\Phi_\beta|V_\beta-V_\alpha|\Phi_{\alpha n}\rangle = 0$. Furthermore, $|\Phi_\beta\rangle = \left[1-G_\beta(E+i0)V_\beta\right]|\Phi_0\rangle$ has the same interpretation as $|\Phi_\alpha\rangle$ in (2.5), and again the choice of channel index is arbitrary.

Formulae similar to (3.4) and (3.5) can be obtained also for the transition amplitudes describing processes that start with three free particles, and end up either with three free particles or one free particle and a two-particle bound state. In fact, with the appropriate interpretation of $|\Phi_\alpha\rangle$ and $|\Phi_\beta\rangle$, the transition amplitude for any three-particle process can be written as

$$\langle\Phi_\beta|\left[-\bar{\delta}_{\beta\alpha}G_0^{-1} + \Sigma_{\gamma\neq\beta}K_{\gamma\alpha}\right]|\Phi_\alpha\rangle \tag{3.6}$$

Thus, apart from a possible "particle exchange" term $-\bar{\delta}_{\beta\alpha}G_0^{-1}$, the three-particle transition amplitudes are simply matrix elements of $K_{\beta\alpha}$.

Having seen how the $K_{\beta\alpha}$ operators combine features of both the Faddeev and the Lovelace-type operators, it is interesting to note that the optimal three-particle amplitudes of Osborn and Kowalski[5] can be understood as matrix elements of the $K_{\beta\alpha}$ operators introduced here. In fact, (for choice a of

Ref. 5, so that $\bar{A}_{\beta\alpha} = \mathscr{K}_{\beta\alpha}$)

$$\mathscr{K}_{\beta\alpha}(\vec{p},\vec{q};\vec{p}'_{\alpha};s) = \langle \vec{p},\vec{q}\,|K_{\beta\alpha}(s)G_0(s)\,V_{\alpha}\,|\Phi_{\alpha n}\rangle$$

$$= -\langle\vec{p},\vec{q}\,|K_{\beta\alpha}(E+i0)\,|\Phi_{\alpha n}\rangle \quad \text{if } S=E+i0 \qquad (3.7)$$

and their Faddeev equations (IV.8) for the $\mathscr{K}_{\beta\alpha}$ amplitudes can be obtained from (3.1) by post-multiplying with G_0V_{α} and taking matrix elements. The functions $\mathscr{K}_{\beta\alpha}$ and the related functions $\mathscr{L}_{\beta\alpha}$,

$$\mathscr{L}_{\beta\alpha}(\vec{p},\vec{q};\vec{p}'_{\alpha};s) = \mathscr{K}_{\beta\alpha}(\vec{p},\vec{q};\vec{p}'_{\alpha};s) - \delta_{\beta\alpha}\langle\vec{p}\,\vec{q}\,|V_{\alpha}\,|\Phi_{\alpha n}\rangle$$

are central in Faddeev's treatment of the three-particle scattering problem.[6]

For some purposes it is convenient to redefine $K_{\beta\alpha}$ to include the "particle exchange" term of (3.6), obtaining

$$L_{\beta\alpha} = -\delta_{\beta\alpha}G_0^{-1} + K_{\beta\alpha} \qquad (3.8)$$

The operators $L_{\beta\alpha}$ satisfy the Faddeev equations

$$L_{\beta\alpha} = -\delta_{\beta\alpha}G_0^{-1} - t_{\beta}G_0\Sigma_{\gamma\neq\beta}\,L_{\gamma\alpha}$$
$$L_{\beta\alpha} = -\delta_{\beta\alpha}G_0^{-1} - \Sigma_{\gamma\neq\alpha}\,L_{\beta\gamma}G_0t_{\gamma} \qquad (3.9)$$

and are related to the Faddeev functions $\mathscr{L}_{\beta\alpha}$ in the same way as $K_{\beta\alpha}$ is related to $\mathscr{K}_{\beta\alpha}$. Combining (3.8) with (2.3), it is clear that the operators $L_{\beta\alpha}$ generate matrix elements of the wave operators for the Faddeev components of the three-particle wavefunction.

It is worth noticing that most approximation techniques which have been used to solve the Faddeev equations for the Lovelace-type operators are also applicable to the $K_{\beta\alpha}$ or $L_{\beta\alpha}$ formalism. For instance, let $t=t'+t''$ be any splitting of the two-particle transition operators, and let $L'_{\beta\alpha}$ be the solution to the Faddeev equations (3.9) with t' as the two-particle operator. Then it can be shown that

$$L_{\beta\alpha} = L'_{\beta\alpha} + \Sigma_\gamma L'_{\beta\gamma} G_0 t''_\gamma G_0 \Sigma_{\sigma\neq\gamma} L_{\sigma\alpha}$$

$$L_{\beta\alpha} = L'_{\beta\alpha} + \Sigma_\gamma L_{\beta\gamma} G_0 t''_\gamma G_0 \Sigma_{\sigma\neq\gamma} L'_{\sigma\alpha} \qquad (3.10)$$

These relations evidently constitute the appropriate starting point for an application of the AGS quasi-particle approach[4] and related approximation techniques to the formalism discussed here.

Furthermore, a formalism in terms of the $K_{\beta\alpha}$ provides, for obvious reasons, a straightforward operator framework to Noyes' wavefunction approach to three-particle scattering theory.[3]

It might be of interest to note the remarkably simple but apparently unrecognized way in which $K_{\beta\alpha}$ and $M_{\beta\alpha}$ are related,

$$K_{\beta\alpha} = \Sigma_{\gamma\neq\alpha} M_{\beta\gamma}\,, \qquad M_{\beta\alpha} = \delta_{\beta\alpha} t_\alpha - K_{\beta\alpha} G_0 t_\alpha \qquad (3.11)$$

Also,

$$U_{\beta\alpha} = -\bar{\delta}_{\beta\alpha} G_0^{-1} + \Sigma_{\gamma\neq\beta} K_{\gamma\alpha}\,, \qquad K_{\beta\alpha} = -t_\beta G_0 U_{\beta\alpha} \qquad (3.12)$$

Equations (3.11) and (3.12) clearly display the intermediate character of the set $K_{\beta\alpha}$ with respect to $M_{\beta\alpha}$ and $U_{\beta\alpha}$.

To conclude this section we point out that instead of the set $K_{\beta\alpha}$, an alternative set of operators $\widetilde{K}_{\beta\alpha} = \Sigma_{\gamma\neq\beta} M_{\gamma\alpha}$ would have been obtained, had we started from the dual state vectors

$$<\Psi_\alpha^+| = <\Phi_{\alpha n}|\left[1 - \widetilde{K}_{\alpha\beta}(E+i0)\, G_0(E+i0)\right] \qquad (3.13)$$

The reason for this is that the adjoint of $K_{\beta\alpha}$ is not related to $K_{\alpha\beta}$ but to $\widetilde{K}_{\alpha\beta}$,

$$\left[<\Phi_\beta|K_{\beta\alpha}(s^*)|\Phi_\alpha>\right]^* = <\Phi_\alpha|\widetilde{K}_{\alpha\beta}(s)|\Phi_\beta> \neq <\Phi_\alpha|K_{\alpha\beta}(s)|\Phi_\beta> \quad . \qquad (3.14)$$

IV. ANGULAR MOMENTUM

From the preceding discussions it can be expected that appropriate matrix elements of $K_{\beta\alpha}$ will carry some information specific to the β-channel interaction. In effect, we show here that the number of important matrix elements of $K_{\beta\alpha}$ – in an angular momentum basis where one of the labels is the angular momentum ℓ_β of the β-pair and for a given total angular momentum – will be determined by the number of important partial waves in the β-channel two-particle interaction. This is to be contrasted with the analogous situation with the operators $U_{\beta\alpha}$, for which such a connection cannot be established. The reason for this is of course that a relative angular momentum label ℓ that simultaneously fits both the $\alpha\neq\beta$ and $\gamma\neq\beta$ two-particle channels present in $U_{\beta\alpha}$ cannot be defined.

In order to see more explicitly how this difference between the $K_{\beta\alpha}$ and the $U_{\beta\alpha}$ sets comes about, consider the Faddeev-type equations for the angular momentum matrix elements of these operators (we assume distinguishable, spinless particles)

$$
\begin{aligned}
U_{\beta\alpha}^{\ell_\beta\lambda_\beta}(p_\beta, q_\beta; \vec{p}'_\alpha \vec{q}'_\alpha; E) = &-(p_\beta^2+q_\beta^2-E)\,\delta(p_\beta^2-\bar{\bar{p}}_\beta^2)\,\bar{\delta}_{\beta\alpha} \times \\
&\times \sum_{\ell'_\alpha\lambda'_\alpha} \mathscr{R}_{\beta\alpha}^{\ell_\beta\lambda_\beta;\ell'_\alpha\lambda'_\alpha}(\bar{\bar{p}}_\beta, q_\beta; p'_\alpha, q'_\alpha)\, \mathscr{Y}^{M}_{J\ell'_\alpha\lambda'_\alpha}(\hat{p}'_\alpha, \hat{q}'_\alpha) \\
&- \sum_{\gamma\neq\beta} \sum_{\ell'_\gamma\lambda'_\gamma} \iint p''^2_\gamma dp''_\gamma q''^2_\gamma dq''_\gamma\, \frac{\bar{p}_\gamma}{2}\, \mathscr{R}_{\beta\gamma}^{\ell_\beta\lambda_\beta;\ell'_\gamma\lambda'_\gamma}(p_\beta, q_\beta; \bar{p}_\gamma q''_\gamma)\, t_\gamma^{\ell'_\gamma}(\bar{p}_\gamma, p''_\gamma; E-q''^2_\gamma) \times \\
&\times \frac{1}{p''^2_\gamma+q''^2_\gamma-E-i0}\, U_{\gamma\alpha}^{\ell'_\gamma\lambda'_\gamma}(p''_\gamma, q''_\gamma; \vec{p}'_\alpha, \vec{q}'_\alpha; E) \qquad (4.1)
\end{aligned}
$$

$$K^{\ell_\beta \lambda_\beta}_{\beta\alpha}(p_\beta, q_\beta; \vec{p}'_\alpha \vec{q}'_\alpha; E) = t^{\ell_\beta}_\beta(p_\beta, \bar{\bar{p}}_\beta; E-q^2_\beta) \frac{\bar{\bar{p}}_\beta}{2} \bar{\delta}_{\beta\alpha} \times$$

$$\times \sum_{\ell'_\alpha \lambda'_\alpha} \mathscr{R}^{\ell_\beta \lambda_\beta; \ell'_\alpha \lambda'_\alpha}_{\beta\alpha}(\bar{\bar{p}}_\beta, q_\beta; p'_\alpha, q'_\alpha)\, \mathscr{Y}^M_{J\ell'_\alpha \lambda'_\alpha}(\hat{p}'_\alpha, \hat{q}'_\alpha)$$

$$- \sum_{\gamma \neq \beta} \iint p''^2_\gamma dp''_\gamma q''_\gamma dq''_\gamma\, t^{\ell_\beta}_\beta(p_\beta, \bar{p}_\beta; E-q^2_\beta) \frac{\bar{p}_\beta}{2} \sum_{\ell'_\gamma \lambda'_\gamma} \mathscr{R}^{\ell_\beta \lambda_\beta; \ell'_\gamma \lambda'_\gamma}_{\beta\gamma}(\bar{p}_\beta, q_\beta; p''_\gamma, q''_\gamma) \times$$

$$\times \frac{1}{p''^2_\gamma + q''^2_\gamma - E - i0} K^{\ell'_\gamma \lambda'_\gamma}_{\gamma\alpha}(p''_\gamma, q''_\gamma; \vec{p}'_\alpha, \vec{q}'_\alpha; E) \qquad (4.2)$$

$$\bar{\bar{p}}^2_\beta = p'^2_\alpha + q'^2_\alpha - q^2_\beta, \qquad \bar{p}^2_\beta = p''^2_\gamma + q''^2_\gamma - q^2_\beta, \qquad \bar{p}^2_\gamma = p^2_\beta + q^2_\beta - q''^2_\gamma$$

Here we have suppressed the labels referring to the total angular momentum J and its axis-projection M. ℓ_β is the relative angular momentum of the β-pair subsystem, λ_β is the angular momentum of the β-particle relative to the β-pair subsystem, and $\mathscr{R}^{\ell_\beta \lambda_\beta; \ell'_\alpha \lambda'_\alpha}_{\beta\alpha}$ is a geometrical recoupling coefficient defined as

$$\mathscr{R}^{\ell_\beta \lambda_\beta; \ell'_\gamma \lambda'_\gamma}_{\beta\gamma}(p_\beta, q_\beta; p'_\gamma, q'_\gamma) = \frac{4}{p_\beta q_\beta} \iint d\hat{p}'_\gamma d\hat{q}'_\gamma\, \mathscr{Y}^{M*}_{J\ell_\beta \lambda_\beta}(\hat{p}'_\beta, \hat{q}'_\beta) \delta(q^2_\beta - q'^2_\beta)\, \mathscr{Y}^M_{J\ell'_\gamma \lambda'_\gamma}(\hat{p}'_\gamma, \hat{q}'_\gamma) \qquad (4.3)$$

with $p^2_\beta + q^2_\beta = p'^2_\gamma + q'^2_\gamma$. The $\mathscr{Y}$-functions are the usual angular momentum functions for two particles defined by Blatt and Weiskopf.[7]

Equation (4.2) carries a factor $t^{\ell_\beta}_\beta$ in both driving term and kernel; thus, if $t^{\ell_\beta}_\beta$ is significant only for $\ell_\beta \lesssim L$, say, exactly the same property will be carried over to the elements $K^{\ell_\beta \lambda_\beta}_{\beta\alpha}$. In Eq. (4.1), on the other hand, matrix elements of $U_{\beta\alpha}$ with arbitrarily large ℓ_β will be connected to terms with $t^{\ell'_\gamma}_\gamma$ with $\ell'_\gamma \lesssim L$ through recoupling coefficients. Thus, such matrix elements will decrease with ℓ_β only to the extent that those recoupling coefficients themselves decrease.

It is clear from Eqs. (4.1) and (4.2) that this property of the $K_{\beta\alpha}$ matrix elements does not imply — as might at first be assumed — a reduction in the number of coupled equations one needs to consider (up to a given accuracy) for the set $K_{\beta\alpha}$ as compared with the set $U_{\beta\alpha}$; indeed, in (4.1) the sum over ℓ'_γ can be truncated at $\ell'_\gamma \approx L$ thanks to the factor $t_\gamma^{\ell'_\gamma}$, even if the matrix elements $U_{\gamma\alpha}^{\ell'_\gamma\lambda'_\gamma}$ stay large for large ℓ'_γ. In Eq. (4.2), on the other hand, the $t_\beta^{\ell_\beta}$ factor is outside the sum over ℓ'_γ, and we have to use the just established fact that the matrix elements $K_{\gamma\alpha}^{\ell'_\gamma\lambda'_\gamma}$ themselves are small for $\ell'_\gamma > L$ in order to truncate the sum over ℓ'_γ.

Hence, in both cases the sum over ℓ'_γ can be cut off at approximately the same ℓ'_γ value, and the complexity of the two sets (4.1) and (4.2) remains the same from this point of view. The difference lies of course in that once a particular set of equations for the $K_{\beta\alpha}^{\ell_\beta\lambda_\beta}$ has been solved, all large matrix elements of $K_{\beta\alpha}$ are known; in contrast many matrix elements $U_{\beta\alpha}^{\ell_\beta\lambda_\beta}$ exist, apart from the ones obtained by solving the equivalent set of equations for the $U_{\beta\alpha}$, that are still large. If needed, these remaining important elements must be computed by quadratures of Eqs. (4.1) by taking $\ell_\beta > L$ to the left and using the solved-for $U_{\gamma\alpha}^{\ell'_\gamma\lambda'_\gamma}$ to the right.

Thus we see that the choice of $K_{\beta\alpha}$ rather than $U_{\beta\alpha}$ provides a much simpler angular momentum expansion, even though it does not decrease the complexity of the Faddeev equations for the angular momentum matrix elements. As an example, consider three alternative formulae for the breakup transition amplitude,

$$<\Phi_0 | \Sigma_\gamma K_{\gamma\alpha} | \Phi_{\alpha n}> \qquad (4.4)$$

$$<\Phi_0 | \Sigma_\gamma t_\gamma G_0 U_{\gamma\alpha} | \Phi_{\alpha n}> \qquad (4.5)$$

$$<\Phi_0 | \frac{1}{2} \Sigma_\gamma U_{\gamma\alpha} | \Phi_{\alpha n}> \qquad (4.6)$$

We now see that, using (4.6) rather than (4.4) involves the evaluation of many more angular momentum matrix elements, due to the slow convergence in ℓ of $U_{\gamma\alpha}$ as compared to $K_{\gamma\alpha}$. This situation can be avoided by using (4.5) instead of (4.6), since rapid convergence in ℓ is assured by the factor $t_{\gamma}^{\ell_{\gamma}}$; but an additional integration is then needed in order to obtain the amplitude.[5]

The preceding discussion was inspired by a comment by D. D. Brayshaw,[8] to the effect that $M_{\beta\alpha}$ is at an advantage over ${}^{\beta}T$ from the point of view of an angular momentum expansion.

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