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EVALUATION OF THE π-π SCATTERING LENGTHS USING ON-MASS-SHELL PIONS*

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ABSTRACT

We have evaluated the $\pi\pi$ scattering lengths, using current algebra but without the use of a power series expansion or extrapolation of the scattering amplitude. We have used the usual LSZ reduction and PCAC in terms of an axial current $J = A + c\partial\phi$ where A is the usual axial current, c is the Goldberger-Treiman constant, and ϕ is the pion field. The scattering amplitude is decomposed into four terms: Two of the three terms which are due to the equal-time commutators, are evaluated by assuming that the usual current algebra of the current A holds for the current J. Other terms are evaluated in terms of the single-particle intermediate states, among which we show that only s-waves contribute at threshold. Assuming that the ϵ-resonance is the only s-wave dominating the low-energy ($\pi\pi$) scattering, we find a relation between the form factors arising from the equal-time commutators of the current J, and the ϵ-pion coupling constant. Finally we obtain the scattering lengths, corresponding to isospins 0, and 2 and ϵ-resonance width 200 MeV, as $a_0 = 0.278\ m^{-1}$ and $a_2 = -0.044\ m^{-1}$.

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I. INTRODUCTION

Both experimental data and theoretical arguments have been extensively used in the discussion of the π-π scattering lengths.[1-7] Study of low energy π-π scattering has been made by many authors, using dispersion relations, phase shift analysis, and the current algebra. Due to the lack of adequate, accurate experimental data, the results of these calculations cannot be compared to well-established experimental numbers. Even so, the smallness of the scattering lengths as obtained by Weinberg[3] from a soft pion treatment seems valid.

However, some plausible arguments raise questions on this calculation. The main objections to the application of the soft pion treatment,[8] which have been pointed out by some of the previous authors, may be summarized as follows: First there is the well known PCAC assumption of the scattering amplitude with respect to k^2, for $0 \leq k^2 \leq m_\pi^2$, where k is a pion four-momentum.[9] Yet this assumption, as pointed out by Sucher et al.,[10] contradicts the result of the power series expansion of the amplitude which is involved in applying the soft pion limit in π-π scattering. Also, it is known that the results of the soft pion treatment of Weinberg do not satisfy the Adler sum rule.[11]

As a contribution to the clarification of these points we evaluate the π-π scattering lengths, using a technique described in a previous work[12] which allows all the pions to remain on the mass shell. This method avoids using the power series expansion of the amplitude, the Adler consistency condition,[13] and the extrapolation. We use, as usual, the Lehmann-Symanzik-Zimmermann (LSZ) formalism[14] to partially reduce the π-π scattering amplitude. The matrix element involving the pion fields is then reworked into one involving the current $J = A + c\partial\phi$, where A is the usual axial vector current, c is the Goldberger-Treiman coefficient, and ϕ is the pion interacting field. Note that $\partial_\mu J^\mu = c(\partial^2 + m^2)\phi$.

The current J contains no pion pole; this and the fact that the action of the Klein-Gordan operator on the pion field is included in this equation will allow us to find manageable expressions with the pions on the mass shell.

Our expression for the scattering amplitude consists of four terms: Three terms, W^{00}, W^{0}, and W^{1} due to the equal-time commutators, and one term, W, due to an unequal-time commutator of the current J. For the evaluation of W^{0} and W^{1} terms we assume, as done in Ref. 12, that the commutation rules of the current J are similar in form to those normally used for the current A.[15] The test and study of these commutation rules in the π-π case is another objective of the present work.

For the evaluation of the other two terms W^{00} and W , we assume that the main contributions can be gotten by inserting simple particle intermediate states. We show, however, that among these states only the spin zero particles contribute to the threshold values of these terms. We then further assume that there is only one spin zero particle significant for low-energy $(\pi\pi)$ scattering, namely the ϵ-resonance with $m_\epsilon = 720$ MeV; $\Gamma_\epsilon = 200$ MeV.[16, 17]

For the calculation of the scattering lengths we take advantage of the transformation properties of the above W-terms under the interchange of the pion states. In doing this, we also obtain a relation between the form factors of the scalar and vector terms W^{0} and W^{1}, and the ϵ-pion coupling constant.

Our results for the scattering lengths a_0 and a_2, (corresponding to the isospins 0 and 2), are $a_2 = -0.044\ m^{-1}$ and $a_0 = 0.278\ m^{-1}$ for $\Gamma_\epsilon = 200$ MeV. If the ϵ-resonance width is 400 MeV, then we find $a_2 = 0.03\ m^{-1}$ and $a_0 = 0.383\,m^{-1}$.

II. EVALUATION OF THE SCATTERING AMPLITUDE

A. General Formula

The two particle scattering amplitudes given by the LSZ formalism may be expressed as

$$T_{\beta b, \alpha a} = \frac{-i(2\pi)^4 \delta^4(p'+k'-p-k)}{(2\pi)^3 (4k_0 k_0')^{1/2} c^2} (W^{00} + W^0 + W' + W) \tag{1a}$$

where

$$W^0 = i \int d^4z \, e^{-ikz} \langle p'\beta | \delta(z_0) \left[J^o_b(0), \partial_\nu J^\nu_a(z) \right] | p\alpha \rangle \tag{1b}$$

$$W^1 = k'_\mu \int d^4z \, e^{-ikz} \langle p'\beta | \delta(z_0) \left[J^o_a(z), J^\mu_b(0) \right] | p\alpha \rangle \tag{1c}$$

$$W = -ik'_\mu k_\nu \int d^4z \, e^{-ikz} \langle p'\beta | T J^\mu_b(0) J^\nu_a(z) | p\alpha \rangle \tag{1d}$$

and

$$W^{00} = c \int d^4z \, e^{-ikz} \delta(z_0) \left\{ (p_0 - p_0') - i \frac{\partial}{\partial z_0} \right\} \times \langle p'\beta | \left[\partial_\mu J^\mu_b(z), \phi_a(0) \right] | p\alpha \rangle \tag{1e}$$

In these amplitudes, (k, a) and (p, α) are the momenta and isospin indices of the incident pions, while (k', b) and (p', β) are the corresponding items for the final pions. In writing Eq. (1), the axial vector current J is defined, as in Ref. 14, by

$$J^\mu_a = A^\mu_a + c\,\partial^\mu \phi_a \tag{2a}$$

$$\partial_\mu A^\mu_a = c\, m_\pi^2 \phi_a \tag{2b}$$

$$\partial_\mu J^\mu_a = c\left(\partial^2 + m_\pi^2\right) \phi_a \tag{2c}$$

Here A is the usual axial current of the hadrons, ϕ is the pion interacting field, and c is the Goldberger-Treiman coefficient. Equation (2b) is the usual PCAC relation,[9] and the Goldberger-Treiman constant is given by

$$\langle 0 | A^\mu_a(0) | k, b \rangle = c\, k^\mu \langle 0 | \phi_a(0) | k, b \rangle \, .$$

The amplitude (1e) is due to the last term in the expression of the time-ordered product[18]

$$\left(\partial_x^2 + m_\pi^2\right)\left(\partial_y^2 + m_\pi^2\right) T\left\{\phi_a(x)\ \phi_b(y)\right\}$$

$$= c^{-2} T\left\{\partial_\mu J_a^\mu(x),\ \partial_\nu J_b^\nu(y)\right\} + c^{-1}\delta(x_0 - y_0)\left[\partial_\mu J_a^\mu(x),\ \phi(y)\right]$$

which is used in writing (1a). Finally we note from definition (2c) that there are no pion poles in Eq. (1).

We shall see below that under the interchange of the pion states, while the sum of W's in Eq. (1a) is invariant, each of the W's is not. This fact allows us to derive two independent expressions for the ($\pi\pi$) scattering amplitudes in Section III. From these we shall obtain the scattering lengths for isospins 0 and 2, as well as a relation between the form factor of the scalar and vector terms W^0 and W^1, and the ϵ-pion coupling constant.

In what follows, we shall evaluate the W terms of Eq. (1a) at the physical threshold.

B. The Terms W^0 and W^1

To evaluate these terms we must deal with the commutation relations of the current J. Considering this, we assume, as in Ref. 12, that the commutation rules of the two currents J and A have the same form. Hence using the form of the Gell-Mann commutation relations,[15] and neglecting the possible Schwinger terms, we write

$$\delta(z_0)\left[J_b^o(0),\ \partial_\nu J_a^\nu(z)\right] = i\, d_{abc} \Sigma_c(z)\, \delta^4(z) \tag{3a}$$

and

$$\delta(z_0)\left[J_a^o(0),\ J_b^\mu(z)\right] = 2i f_{abc} V_c(z)\, \delta^4(z) \tag{3b}$$

Here Σ is a scalar operator, V is the vector current operator, and d_{abc} and f_{abc} are the usual structure constants of the SU(3) scheme. The definition (2c) permits us to identify

$$J_{\pi^\pm} = \frac{1}{\sqrt{2}} (J_1 \mp i J_2)$$
$$J_{\pi^0} = J_3 . \tag{4}$$

Using (4) in (3a)

$$d_{abc} \Sigma_c = \delta_{ab} \left(\sqrt{\frac{1}{3}} \Sigma_8 + \sqrt{\frac{2}{3}} \Sigma_0 \right) \equiv \delta_{ab} \widetilde{\Sigma} \tag{5}$$

Combining Eqs. (5), (4a), and (1b) we find for $\underset{\sim}{k} = \underset{\sim}{p} = 0$,

$$W^0 = -\left[2m(2\pi)^3\right]^{-1} \delta_{\alpha\beta} \delta_{ab} f(0, m^2, m^2) \tag{6}$$

where

$$\langle p'\beta | \widetilde{\Sigma}(0) | p\alpha \rangle = \left[(2\pi)^3 \sqrt{4p_0 p_0'}\right]^{-1} \delta_{\alpha\beta} \quad f(t, p^2, p'^2) \tag{7}$$

and

$$t = (p - p')^2$$

f is scalar form factor which we shall discuss in Section III.

To evaluate the amplitude W^1, we write the matrix element of the vector current V_c^μ as

$$\langle p'\beta | V_c^\mu \; | p\alpha \rangle = + i \epsilon_{\beta c \alpha} \left[(2\pi)^3 \sqrt{4p_0 p_0'}\right]^{-1} (p + p')^\mu g(t) \tag{8}$$

We note that V_c^{μ} has all the properties of the isospin current, thus we may identify g(t) with the electromagnetic form factor of the pion. In this case, the CVC hypothesis and normalization of the pion charge to unity, give

$$g(t = 0) = 1 \tag{9}$$

In general, combining Eqs. (8), (3b), and (1c), we have, for $\underset{\sim}{p} = \underset{\sim}{k} = 0$,

$$W^1 = 2m\,(2\pi)^{-3}\left[\delta_{\alpha b}\,\delta_{a\beta} - \delta_{\alpha a}\,\delta_{b\beta}\right] g(0) \tag{10}$$

C. The Term W

To evaluate the terms W, we use Eq. (1d) in the following form:

$$W = W_1 + W_2 \tag{11a}$$

$$W_1 = -(2\pi)^3 \sum_n \langle p'\beta | k_\mu' \, J_b^{\mu}(0) \frac{|p_n\rangle\langle p_n|}{p_{n0} - p_0 - k_0}\, k_\nu \, J_a^{\nu}(0)\,|p,\alpha\rangle\, \delta^{(3)}(\underset{\sim}{p}_n - \underset{\sim}{p} - \underset{\sim}{k}) \tag{11b}$$

$$W_2 = -(2\pi)^3 \sum_n \langle \beta p' | k_\nu \, J_a^{\nu}(0) \frac{|p_n\rangle\langle p_n|}{p_{n0} - p_0' + k}\, k_\mu' \, J_b^{\mu}(0)\,|p\alpha\rangle\, \delta^{(3)}(\underset{\sim}{p}_n - \underset{\sim}{p}' + \underset{\sim}{k}) \tag{11c}$$

where we have introduced a sum over a complete set of intermediate states. For the summation over spin n, we introduce the projection operators

$$O_0^{\alpha\beta} = m_n^{-2}\, p_n^{\alpha}\, p_n^{\beta} \tag{12a}$$

$$O_1^{\alpha\beta} = g_{\alpha\beta} - m_n^{-2}\, p_n^{\alpha}\, p_n^{\beta} \tag{12b}$$

$$O_2^{\alpha\beta,\sigma e} = \frac{1}{2} O_1^{\alpha\sigma} O_1^{\beta p} + \frac{1}{2} O_1^{\alpha p} O_1^{\beta\sigma} - \frac{1}{3} O_1^{\alpha\beta} O_1^{\sigma p} \tag{12c}$$

where m_n is the mass of the particle $|n\rangle$, and O_0, O_1, and O_2 are the operators for spin n = 0, 1, and 2, respectively. For the higher spin states the operators $O_n^{\alpha\beta,\sigma p,\ldots}$ can be expressed in terms of the spin 1 projection operators.

We now note that because of the δ^3-functions in Eq. (11), the states with spin higher than zero do the threshold value of the amplitude W. To see this we consider the matrix element of the current J, e.g.,

$$<p|J^{\mu}(0)|p_n, n> = \left[f_1 A^{\mu} A^{\alpha} + f_2 A^{\mu} B^{\alpha} + f_3 B^{\mu} A^{\alpha} + f_4 B^{\mu} B^{\alpha} + f_5 g_{\mu\alpha}\right] \epsilon_{\alpha}(n) \quad (13a)$$

for spin 0 or 1 states

$$<p|J^{\mu}(0)|p_n, n> = \left[f_1 A^{\mu} A^{\alpha} p^{\beta} + f_2 A^{\mu} B^{\alpha} p^{\beta} + \ldots\right] \epsilon_{\alpha\beta}(n) \quad (13b)$$

for spin 2, and so on. Here f's are form factors, $A^{\mu} = (p+p_n)^{\mu}$, $B^{\mu} = (p-p_n)^{\mu}$, and ϵ's are the polarization vectors satisfying

$$\sum_{\text{spin}} \epsilon^{\alpha} \epsilon^{\beta} = O^{\alpha\beta}_{n=0,1} \quad (13c)$$

$$\sum_{\text{spin}} \epsilon^{\alpha\beta} \epsilon^{\sigma p} = O^{\alpha\beta,\sigma p}_{n=2}$$

and similar expressions for states with spin $n > 2$. Using expressions such as (13), and projection operators (12) in (11) we encounter terms such as $A_{\alpha} O_1^{\alpha\beta}$, $A_{\alpha} O_1^{\alpha\beta} B_{n'}$, $B_{\alpha} O_1^{\alpha\beta} A_{\beta}$, etc, multiplied by $\delta^3(\underset{\sim}{p}_n - \underset{\sim}{p} \pm \underset{\sim}{k})$. The δ^3-function and the fact that we are at threshold guarantee that the momenta have only time components; excepting spin 0, all the projection operators will have only pure space components. Hence only spin zero intermediate states contribute at threshold.

Considering this point we have

$$W = -\frac{c^2}{8\pi^2} \left(\frac{g^2_{\epsilon\pi\pi}}{4\pi}\right) \frac{1}{m m_{\epsilon}} \left(\frac{m}{m_{\epsilon} - m}\right)^2 \left[(m_{\epsilon} - 2m)^{-1} \delta_{a\alpha} \delta_{b\beta} + m_{\epsilon}^{-1} \delta_{a\beta} \delta_{\alpha b}\right] \quad (14)$$

where m_{ϵ} and $g_{\epsilon\pi\pi}$ are the mass and coupling constant of the ϵ-resonance, the only I=0 single-particle intermediate state known in the π-π interaction.

D. W^{00} - Term

For the evaluation of this term, we may write Eq. (1e) as

$$W^{00} = -(2\pi)^3 \sum_n B_n \left[\frac{(p_{n0} - p_0)}{t_n - m^2} \delta_{\alpha a} \delta_{\beta b} \delta^{(3)} (\underset{\sim}{p}_n - \underset{\sim}{p} - \underset{\sim}{k}) \right. \tag{15a}$$

$$\left. + \frac{(p_{n0} - p'_0)}{t'_n - m^2} \delta_{\alpha b} \delta_{\beta a} \delta^{(3)} (\underset{\sim}{p}_n - \underset{\sim}{p} + \underset{\sim}{k}) \right]$$

with $t_n = (p_n - p)^2$, $t'_n = (p_n - p')^2$,

$$B_n = (p' - p_n)_\mu \left\langle p'\beta \right| J^\mu (0) \left| p_n n \right\rangle \left\langle n p_n \right| J^\nu (0) \left| p\alpha \right\rangle (p - p_n)_\nu \tag{15b}$$

Here again, because of $\delta^{(3)}$-functions in (16a), the argument given for the term W, Eq. (11), shows that the intermediate states with spin higher than zero do not contribute to the threshold value of W^{00}. Hence, the only single-particle state in Eq. (16a) which contributes at $\underset{\sim}{k} = \underset{\sim}{p} = 0$, is the ϵ-resonance. Considering this and combining Eqs. (16), (13a), and Eqs. (A1) to (A3) and (A6) of the Appendix, we find

$$W^{00} = -\frac{c^2}{24\pi^2} \left(\frac{g^2_{\epsilon\pi\pi}}{4\pi\pi} \right) \frac{1}{m\, m_\epsilon^2} \frac{m_\epsilon - m}{m_\epsilon - 2m} \left[\delta_{\alpha b} \delta_{a\beta} + \delta_{\alpha a} \delta_{b\beta} \right] \tag{16}$$

III. SCATTERING LENGTHS

A. Amplitudes at Threshold

The well known expression of the $(\pi\pi)$ scattering amplitude is given as

$$T_{\beta b, a\alpha} = A\ \delta_{\alpha a}\,\delta_{\beta b} + B\ \delta_{\alpha\beta}\delta_{ab} + C\ \delta_{\alpha b}\,\delta_{a\beta} \tag{17a}$$

with A, B, and C being functions of the usual independent variables s, u, and t.[19] These coefficients are related to the amplitudes A_0, A_1, and A_2 which are the amplitudes corresponding to isospins 0, 1, and 2 respectively. That is

$$\begin{aligned} A_0 &= 3A + B + C \\ A_1 &= B - C \\ A_2 &= B + C \end{aligned} \tag{17b}$$

We may calculate these amplitudes by evaluating the expression (16a) from Eq. (1a) and our treatment in Section II. Thus, inserting Eqs. (16a), (14), (10), and (6) into Eq. (1a) and omitting, conveniently, the $\delta^{(4)}$-function in the latter equation, we find

$$T_{\beta b, a\alpha} = i\,L \left\{ \left[G_\epsilon \frac{\gamma^3 - 3\gamma^2 + 4\gamma - 1}{\gamma^2(\gamma-1)^2(\gamma-2)} + \frac{g(0)}{\pi} \right] \delta_{\alpha a}\,\delta_{\beta b} \right.$$

$$\left. - F\ \delta_{\alpha\beta}\delta_{ab} + \left[G_\epsilon \frac{\gamma^3 - 3\gamma^2 + 4\gamma - 3}{{}^2(\gamma-1)^2(\gamma-2)} - \frac{g(0)}{\pi} \right] \delta_{\alpha b}\ \delta_{a\beta} \right\} \tag{18a}$$

Here $\gamma = m_\epsilon / m$,

$$F(0) = 4\pi\, m^{-2}\, f(t{=}0) \tag{18b}$$

$$G_\epsilon = (2.76\ \pi)^{-1}\, m^{-2} \left(\frac{g^2_{\epsilon\pi\pi}}{4\pi} \right) \tag{18c}$$

Also L is a convenient length used in Ref. (3), as

$$L = \frac{g_r^2\, m}{8\pi\, g_A^2\, M^2} \simeq 0.115\ m^{-1} \tag{18d}$$

where g_r is the pion coupling constant, $(g_r^2/4\pi) = 14.6$, g_A is the axial vector coupling constant, $g_A = -1.23$, and M is the nucleon mass. In writing our equations we have taken the Goldberger-Treiman coefficient as

$$c = -\frac{g_A\, M}{g_r}$$

Combining Eqs. (16) to (18), we find for $\underset{\sim}{k} = \underset{\sim}{p} = 0$,

$$A_2 = -i\, L \left\{ F(0) + \frac{g(0)}{\pi} - G_\epsilon\, \frac{\gamma^3 - 3\gamma^2 + 4\gamma - 3}{\gamma^2(\gamma-1)^2(\gamma-2)} \right\} \tag{19a}$$

$$A_0 = i\, 3L \left\{ \frac{g(0)}{\pi} + G_\epsilon\, \frac{\gamma^3 - \gamma^2 + 4\gamma - 1}{\gamma^2(\gamma-1)^2(\gamma-2)} \right\} + A_2 \tag{18b}$$

In order to obtain the scattering lengths from these results, we shall make use of the transformation properties of the W-terms in Eq. (1a), or those of the coefficients A, B and C in (16a), under the interchanges of the pions. We consider the interchange $(k', b) \rightleftharpoons (p', \beta)$ which corresponds to the amplitude

$$T_{b\beta,\, \alpha a} = A'\ \delta_{\alpha a}\delta_{b\beta} + B'\ \delta_{\alpha b}\delta_{a\beta} + C'\ \delta_{\alpha\beta}\delta_{ab} \tag{20}$$

Since the amplitude (1a) is invariant under different reduction of the pion states, for a given set of invariants s, t, and u, the two amplitudes (19) and (16a) are equal, and the crossing symmetry gives

$$A' = A$$
$$B' = C \tag{21}$$
$$C' = B$$

Thus A_0 and A_2, Eqs. (18), are invariant, under the above transformation.

On the other hand, we may calculate the amplitude $T_{b\beta, \alpha a}$ by interchanging (k', b) and (p', β) in our master amplitude (1a). If we do this, we find the same amplitude (17a) in which the last two coefficients are interchanged. From this result and Eqs. (19) and (20), we find B = C in (16a), or

$$F(0) = \frac{g(0)}{\pi} - G \quad \frac{\gamma^3 - 3\gamma^2 + 4\gamma - 3}{\gamma^2(\gamma-1)^2(\gamma-2)} \tag{22}$$

in (17a), which is valid for the threshold $\underset{\sim}{k} = \underset{\sim}{p} = 0$. Note that inserting (21) in (17a) gives $A_1 = 0$ at $\underset{\sim}{k} = \underset{\sim}{p} = 0$, in agreement with the Bose statistic.

B. <u>S-Wave Scattering Lengths</u>

The scattering amplitudes corresponding to isospins 0 and 2 are obtained by writing $(-i\ \pi/2)$ times the scattering amplitudes A_0 and A_2 given by Eqs. (18). Hence using the relations (21), and taking the mass of the ϵ-resonance as $m_\epsilon = 720$ MeV,[17] we find

$$F(0) = \frac{g(0)}{\pi} - 0.006\ m^{-2}\ \frac{g^2_{\epsilon\pi\pi}}{4\pi} \tag{23a}$$

$$a_2 = -L\left\{g(0) - 19.8 \times 10^{-3}\ m^{-2}\ \frac{g^2_{\epsilon\pi\pi}}{4\pi}\right\} \tag{23b}$$

$$a_0 = \frac{3L}{2}\left\{g(0) + 19.1 \times 10^{-3}\ m^{-2}\ \frac{g^2_{\epsilon\pi\pi}}{4\pi}\right\} \tag{23c}$$

The scattering lengths (22) are expressed in terms of the ϵ-pion coupling constant given by Eq. (A3) of the Appendix. Experimentally, the width of the ϵ-resonance is not well established and is estimated to be between 150 and 400 MeV, more likely 200 MeV. Hence using m_ϵ = 720 MeV and Γ_ϵ = 200 MeV in Eq. (A3),

$$m^{-2}\left(\frac{g^2_{\epsilon\pi\pi}}{4\pi}\right) = 32 \tag{24}$$

Using this and Eq. (9b) in Eqs. (22), we find the on-mass-shell values of the scattering lengths, as

$$a_2 = -0.044\ m^{-1} \tag{25a}$$

$$a_0 = \ 0.278\ m^{-2} \tag{25b}$$

Note that if the ϵ-resonance width is Γ_ϵ = 400 MeV, then Eqs. (22) give

$$a_2 = 0.03\ m^{-1} \tag{26a}$$

$$a_0 = 0.383\ m^{-1}\ . \tag{26b}$$

IV. COMMENTS AND CONCLUSION

The essential points in this on-mass-shell treatment of π-π scattering may be summarized as follows: we have calculated the scattering lengths, using the LSZ reduction formalism, PCAC and crossing symmetry relations.

Our master amplitude (1a) decomposes into four terms, W^{00}, W^{0}, W^{1}, and W. For the evaluation of the scalar and vector terms, W^{0} and W^{1}, we have the current algebra of the current J, which is assumed[12] to be similar to the usual current algebra.[15] For the evaluation of the terms W^{00} and W, which do not appear in the soft pion treatment, we have introduced a set of one-particle intermediate states. Among these intermediate states, only spin-0 states contribute to the threshold values of W^{00} and W; so we have assumed that only the s-wave ϵ-resonance contributes appreciably.

The results may be summarized as: (a) Both the scattering length a_0 and a_2 are expressed in terms of the form factor g(0), given by Eqs. (9), and the ϵ-pion coupling constant $g_{\epsilon\pi\pi}$. (b) The values $a_2 = -0.044\ m^{-1}$ and $a_0 = 0.278\ m^{-1}$ are based on Eq. (9b) which gives g(0) = 1, the Goldberger-Treiman coefficient in which $g_A = -1.23$, and the ϵ-mass and width, $m_\epsilon = 720$ MeV and $\Gamma_\epsilon = 200$ MeV. We see from Eqs. (22) that a_0 is always positive, while the sign of a_2 depends on the ϵ-mass and width. For $\Gamma_\epsilon = 400$ MeV, we have $a_2 = 0.024\ m^{-1}$ and $a_0 = 0.383\ m^{-1}$. (c) We have Eq. (21), the relation between the form factors, F(0) and g(0), which belong to the scalar and vector terms W^{0} and W^{1}, and the ϵ-pion coupling constant, $g_{\epsilon\pi\pi}$.

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17. For the latest data of the ϵ-resonance, see Review of the Modern Physics, 42, 128 (1970).

18. See S. S. Scheweber, Relativistic Quantum Field Theory, (Row and Co., 1961); p. 696. See also Ref. 14 for information on the term W^{00}, Eq. (1e).

19. G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

APPENDIX

Here we shall evaluate the amplitude W given by Eqs. (11) of the text, at the threshold $\underset{\sim}{p} = \underset{\sim}{k} = 0$, for the single intermediate states $|n\rangle$ having isospin $I=0$. Also we shall give some relations which will be used in the evaluation of the term W^{00} given by Eq. (16a) of the text.

From the PCAC relation (2c) of the text we find

$$(p_n - p)_\nu \langle n, p_n | J_a^\nu(0) | p, \alpha\rangle = ic\left(m^2 - t_n\right)\langle n, p_n | \phi_a(0) | p, \alpha\rangle \tag{A.1}$$

where $t_n = (m_n - m)^2$, m_n is the mass of the intermediate state $|n\rangle$, and a and α are the isospin indices of the pion. We may evaluate the matrix element of the pion interacting field ϕ, by considering the decay of a state $|n\rangle$ into two pions, i.e.,

$$(n, p_n) \longrightarrow (a, k) + (\beta, p)$$

By writing the S-matrix of this decay in terms of an effective Hamiltonian, and by the reduction formalism,[12] we find for an isoscalar state $|n\rangle$,

$$\langle n, p_n | \phi_a(0) | p, \alpha\rangle = \frac{-i\, g_{n\pi\pi}}{(2\pi)^3\, (4p_0 p_{n0})^{1/2}}\; \frac{\delta_{a\alpha}}{m^2 - t_n} \tag{A.2}$$

where m is the pion mass, and

$$\frac{g^2_{n\pi\pi}}{4\pi} = \frac{4m_n^2}{\left(m_n^2 - 4m^2\right)^{1/2}}\; \Gamma_n \tag{A.3}$$

is the decay coupling constant corresponding to the width Γ_n of an isoscalar state $|n\rangle$. Combining Eqs. (A.1) and A.2) and let $\underset{\sim}{k} = \underset{\sim}{p} = 0$, we find

$$k_\mu \langle n, p_n | J_a^\mu(0) | p, \alpha\rangle = \frac{c\, g_{n\pi\pi}}{(2\pi)^3\, (4mm_n)^{1/2}} \left(\frac{k_0}{m_n - m}\right) \delta_{a\alpha} \tag{A.4}$$

Note that in the soft-pion case $k_0 = 0$ so (A.4) vanishes. Finally, using this we can show that

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \frac{-c^2}{24\pi^2} \sum_n \left(\frac{g^2_{n\pi\pi}}{4\pi}\right)\left(\frac{1}{mm_\epsilon}\right)\left(\frac{m}{m_n - m}\right)^2 \begin{bmatrix} (m_n - 2m)^{-1} \delta_{\alpha a} \delta_{b\beta} \\ m_n^{-1} \delta_{\alpha b} \delta_{a\beta} \end{bmatrix} \qquad (A.5)$$

In writing Eq. (A5) we have expressed the intermediate states $|n\rangle$ in Eqs. (11) as

$$|n\rangle = \sum_{I_n=0}^{2} C(I_\alpha, I_a; I_n) \, |I_n\rangle \qquad (A.6)$$

Here I_α and I_a are the isospins of the two colliding particles, and I_n the isospin of the state $|I_n\rangle$. Also $C(I_\alpha, I_a; I_n)$ is the Clebsch-Gordon coefficient. Note that at $k = p = 0$ the only known contributing single particle in Eq. (11), is ϵ with $I_\epsilon = 0$, (see part C in Section II). Thus the sum in (A6) reduces to one term, that is, $|\epsilon\rangle = I = 0\rangle$.