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ANALYSIS OF THE LEADING DIVERGENCES IN A MODEL OF WEAK INTERACTIONS†

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ABSTRACT

We study a model of the weak interaction in which three W-bosons interact with a triplet of fermion currents. The model is not renormalizable, but we make a canonical transformation of the Lagrangian which allows the most divergent terms in a perturbation series to be isolated, and their structure examined. Results recently obtained in lowest order perturbation theory relating the leading divergence to the Cabibbo angle are explicitly shown to hold in all orders.

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I. INTRODUCTION

We present here a method for isolating and examining the structure of the leading divergences of theories involving massive vector mesons. We will restrict our attention to the intermediate vector boson theory of weak interactions although the methods used here have more general applicability. It is well-known that this theory is not renormalizable in the usual sense since new and higher order divergences are encountered in each order of perturbation theory. In order to investigate solutions[1] to this problem, it is helpful to be able to examine the most divergent terms in each order.

In the usual form of such theories, the non-renormalizable divergences are due to the $q_\mu q_\nu$ term in the vector boson propagator

$$i(q^2 - m_W^2 + i\epsilon)^{-1} (g_{\mu\nu} - m_W^{-2} q_\mu q_\nu)$$

where m_W is the mass of the vector boson. The leading divergences in each order can be examined by keeping only this $q_\mu q_\nu$ term but this becomes quite tedious in higher orders because of the large number of graphs. In Section II we will make the isolation of these most divergent terms more manageable by showing that the part of the Lagrangian from which they arise in the usual form of the theory is "equivalent"[2] to an exponential type interaction. Equivalence theorems for vector meson theories are well-known and we will present the basic ideas in the context of our weak interaction model.

Our work is based on a model in which a set of three vector bosons[3] interacts with a triplet of currents whose charges satisfy the usual SU(2) commutation relations. The basic techniques are presented for the simplest case in which the currents are constructed from a pair of fermion fields. We then apply these methods to more general models including strong interaction effects. The structure of the leading divergences will be related to the $SU(3) \times SU(3)$ violating part of the strong interaction Lagrangian. The equivalence theorems which we use rely on a partial gauge invariance of the theory and so we are led naturally to include Yang-Mills[4] couplings among the W-bosons.

In the next section we discuss equivalence theorems for some weak interaction models. In Section III we discuss the problem of isolating the leading divergences in each order of perturbation theory for weak transition amplitudes and show that the algebraic structure of these leading terms can easily be determined. We then comment briefly on the problem of relating the Cabibbo angle to the structure of these terms.[5,6] In Section IV a simple subclass of diagrams is summed exactly.

II. EQUIVALENCE THEOREMS

A. We first restrict ourselves to the case of a triplet of vector bosons interacting with a pair of spin $\frac{1}{2}$ particles. The Lagrangian proposed for study is

$$\mathcal{L} = -\tfrac{1}{4}\,\hat{\underset{\sim}{W}}{}^2_{\mu\nu} + \tfrac{1}{2}\,m_W^2\,\underset{\sim}{W}{}^2_{\nu} + \bar{\psi}\,(i\partial^{\mu}\gamma_{\mu} - M_0)\,\psi - \tfrac{1}{2}\,g\,\bar{\psi}\,\underset{\sim}{\tau}\cdot\underset{\sim}{W}_{\mu}\,\gamma^{\mu}\,\tfrac{1}{2}\,(1-\gamma_5)\,\psi \tag{2.1}$$

with

$$\hat{\underset{\sim}{W}}_{\mu\nu} = \partial_{\mu}\underset{\sim}{W}_{\nu} - \partial_{\nu}\underset{\sim}{W}_{\mu} + g\,\underset{\sim}{W}_{\mu}\times\underset{\sim}{W}_{\nu} \tag{2.2}$$

ψ is an iso-doublet spinor field, $\underset{\sim}{W}_{\mu}$ is an iso-triplet vector field and the Pauli matrices τ_i satisfy

$$[\tau_i, \tau_j] = 2\, i\, \epsilon_{ijk}\, \tau_k \tag{2.3}$$

The interaction is pure V-A, however, the results of this section hold for either a V + A coupling or a pure vector coupling[7]. The fermion mass matrix M_0 has the general form

$$M_0 = m_0(1 + \alpha\, \tau_3)\,. \tag{2.4}$$

SU(2) × SU(2) gauge invariance is broken by the vector meson mass term and by the fermion mass term.

We adopt the Stueckelberg formalism[8] for the spin one field,

$$\underset{\sim}{W}_\mu = \underset{\sim}{A}_\mu + \frac{1}{m_W}\, \partial_\mu \underset{\sim}{B} \tag{2.5}$$

Where $\underset{\sim}{A}_\mu$ and $\underset{\sim}{B}$ are a set of five independent canonical fields. The spin zero part of $\underset{\sim}{A}_\mu$ is of negative matric. From the field equations, one sees that

$$\partial^\mu \underset{\sim}{A}_\mu - m_W\, \underset{\sim}{B}$$

satisfies the free field equation and hence the subsidiary condition

$$(\partial^\mu \underset{\sim}{A}_\mu - m_W\, \underset{\sim}{B})\, \Psi = 0 \tag{2.6}$$

can be imposed on the Hilbert space. Thus any process involving a spin zero A-boson will be exactly cancelled by a corresponding process involving B-bosons. The fields $\underset{\sim}{A}_\mu$ and $\underset{\sim}{B}$ have free propagators which are

$$i(q^2 - m_W^2 + i\epsilon)^{-1}\,,$$

and the possible non-renormalizability is now due to the gradient coupling of the B bosons.

For convenience, we define two operators S(g, b) and R(g, b) which are

$$S(g, \underset{\sim}{b}) = e^{ig\, \underset{\sim}{T} \cdot \underset{\sim}{b}} \tag{2.7}$$

and

$$R(g, \underset{\sim}{b}) = (S(g, \underset{\sim}{b}) - 1)/ig\, \underset{\sim}{T} \cdot \underset{\sim}{b} \tag{2.8}$$

The T_i used here are 3×3 rotation matrices satisfying

$$[T_i, T_j] = + i \epsilon_{ijk} T_k \tag{2.9}$$

and in the usual representation are

$$(T_i)_{jk} = -i \epsilon_{ijk} \tag{2.10}$$

When applied to a three vector $\underset{\sim}{a}$, S and R give, respectively

$$S(g, \underset{\sim}{b}) \underset{\sim}{a} = \underset{\sim}{a} + g\, \underset{\sim}{a} \times \underset{\sim}{b} + \tfrac{1}{2} g^2 (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{b} + \cdots \tag{2.11}$$

$$R(g, \underset{\sim}{b}) \underset{\sim}{a} = \underset{\sim}{a} + \tfrac{1}{2} g\, \underset{\sim}{a} \times \underset{\sim}{b} + \frac{1}{3!} g^2 (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{b} + \cdots \tag{2.12}$$

Now we first make a B-field dependent canonical transformation[2] on the spinor fields which resembles an isotopic gauge rotation.

$$\psi \rightarrow \exp\left\{ -i \frac{g}{2m_W} \tfrac{1}{2} (1 - \gamma_5)\, \underset{\sim}{\tau} \cdot \underset{\sim}{B} \right\} \psi \tag{2.13}$$

Applying this transformation to the fermion kinetic energy and interaction terms gives

$$i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \rightarrow i\psi\gamma^{\mu}\partial_{\mu}\psi + \tfrac{1}{2}g\bar{\psi}\gamma^{\mu}\tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\tau}\cdot R(-g,\underset{\sim}{b})\,\partial_{\mu}\underset{\sim}{b}\psi \tag{2.14}$$

and

$$-\tfrac{1}{2}g\bar{\psi}\gamma^{\mu}\tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\tau}\cdot \underset{\sim}{W}_{\mu}\psi \rightarrow -\tfrac{1}{2}g\bar{\psi}\gamma^{\mu}\tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\tau}\cdot S(-g,\underset{\sim}{b})\,W_{\mu}\psi \tag{2.15}$$

where

$$\underset{\sim}{b} = \frac{1}{m_W}\underset{\sim}{B} . \tag{2.16}$$

In order to remove the gradient coupling we also make a transformation of the Yang-Mills fields. The requisite transformation, with the results written in terms of the Stueckelberg fields, is

$$\underset{\sim}{W}_{\mu} \rightarrow \underset{\sim}{w}_{\mu} = S(g,\underset{\sim}{b})\underset{\sim}{A}_{\mu} + R(g,\underset{\sim}{b})\,\partial_{\mu}\underset{\sim}{b} = S(g,\underset{\sim}{b})\left[\underset{\sim}{A}_{\mu} + R(-g,\underset{\sim}{b})\,\partial_{\mu}\underset{\sim}{b}\right] \tag{2.17}$$

and

$$\underset{\sim}{B} \rightarrow \underset{\sim}{B} .$$

The invariance of the S-matrix under transformations (2.15) and (2.19) has been proven by Umezawa and Kamefuchi[2], who construct unitary operators which effect these transformations. We exhibit these operators in Appendix A. Under the above transformation,

$$\hat{\underset{\sim}{W}}_{\mu\nu} \rightarrow S(g,\underset{\sim}{b})\,\hat{\underset{\sim}{A}}_{\mu\nu} = S(g,\underset{\sim}{b})\left[\partial_{\mu}\underset{\sim}{A}_{\nu} - \partial_{\nu}\underset{\sim}{A}_{\mu} + g\underset{\sim}{A}_{\mu}\times\underset{\sim}{A}_{\nu}\right] . \tag{2.18}$$

Then using the subsidiary condition, the part of the Lagrangian involving only vector fields becomes

$$\mathcal{L}_W = -\tfrac{1}{2}\left[(\partial_\mu \underset{\sim}{A}_\nu)^2 - (\partial_\mu \underset{\sim}{B})^2\right] + \tfrac{1}{2} m_W^2 \left[\underset{\sim}{A}_\nu^2 - \underset{\sim}{B}^2\right] - \tfrac{1}{2} g(\partial_\mu \underset{\sim}{A}_\nu - \partial_\nu \underset{\sim}{A}_\mu)\cdot \underset{\sim}{A}^\mu \times \underset{\sim}{A}^\nu$$

$$- \tfrac{1}{4} g^2 (\underset{\sim}{A}_\mu \times \underset{\sim}{A}_\nu)^2 + \tfrac{1}{2} m_W^2 (\underset{\sim}{w}_\mu^2 - \underset{\sim}{W}_\mu^2) \; . \qquad (2.19)$$

It is clear that the gradient coupling between the B-fields and the fermion currents has been eliminated. The entire Lagrangian is now

$$\mathcal{L} = \mathcal{L}_W + i\bar{\psi}\gamma^\mu \partial_\mu \psi - \tfrac{1}{2} g\bar{\psi}\gamma^\mu \tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\tau}\cdot \underset{\sim}{A}_\mu \psi$$

$$-\bar{\psi}\exp\left\{+i\frac{g}{2m_W}\tfrac{1}{2}(1+\gamma_5)\underset{\sim}{\tau}\cdot\underset{\sim}{B}\right\} M_0 \exp\left\{-i\frac{g}{2m_W}\tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\tau}\cdot\underset{\sim}{B}\right\}\psi \; . \qquad (2.2)$$

The worst divergences of the theory can be seen to arise from the transformed mass terms of the original Lagrangian. We will return to this point in Section III.

B. Having presented the basic idea of equivalence theorem, we now extend the simple SU(2) model discussed above to a more realistic model of the weak interactions. The underlying algebra will now be SU(3) $\times$ SU(3) and the W's will interact with a triplet of Cabibbo currents. For simplicity we construct the strong interaction part of the Lagrangian from quark fields although the results of this section hold for other models such as the octet σ-model[9]. The Lagrangian is taken to be

$$\mathcal{L} = \mathcal{L}_0 - \tfrac{1}{4}\hat{\underset{\sim}{W}}_{\mu\nu}^2 + \tfrac{1}{2} m_W^2 \underset{\sim}{W}_\mu^2 + \bar{q}(i\gamma^\mu\partial_\mu - h_0)q - \tfrac{1}{2} g\bar{q}\underset{\sim}{\lambda}_c\cdot\underset{\sim}{W}_\mu\gamma^\mu\tfrac{1}{2}(1-\gamma_5)q \; . \qquad (2.21)$$

The first term $\mathcal{L}_0$ is the SU(3) $\times$ SU(3) invariant part of the strong interaction Lagrangian. SU(3) $\times$ SU(3) violating terms are given by the quark mass term

$-\bar{q}h_0q$ which transforms according to the $(3,\bar{3}) + (\bar{3},3)$ [10] representation of $SU(3)\times SU(3)$.

The 3×3 hermitian matrix h_0 is the unrenormalized quark mass matrix. It can be written as the sum of two parts

$$h_0 = h + \delta h \tag{2.22}$$

where h is the "physical" quark mass matrix and δh is a mass renormalization counterterm which will eliminate the leading divergence from the weak interactions. h must be diagonal and its most general form is

$$\begin{aligned} h &= c_0\lambda_0 + c_3\lambda_3 + c_8\lambda_8 \\ &= \begin{pmatrix} m_p & & \\ & m_n & \\ & & m_\lambda \end{pmatrix} \end{aligned} \tag{2.23}$$

(The λ_i are the λ-matrices of Gell-Mann[11] and the mass parameters m_p, m_n and m_λ can easily be related to the quantities c_0, c_3 [12] and c_8.) The three components of λ_c are chosen to give the usual Cabibbo currents (including a neutral member). They are given by

$$\begin{aligned} \lambda_{c1} &= \lambda_1\cos\theta + \lambda_4\sin\theta \\ \lambda_{c2} &= \lambda_2\cos\theta + \lambda_5\sin\theta \\ \lambda_{c3} &= \frac{1}{2i}[\lambda_{c1},\lambda_{c2}] = \cos^2\theta\,\lambda_3 + \sin^2\theta\left(\tfrac{1}{2}\lambda_3 + \tfrac{\sqrt{3}}{2}\lambda_8\right) - \sin\theta\cos\theta\,\lambda_6 \end{aligned} \tag{2.24}$$

or (equivalently)

$$\lambda_{ci} = e^{-i\theta\lambda_7}\,\lambda_i\, e^{i\theta\lambda_7}, \quad i = 1,2,3 \tag{2.25}$$

The currents transform as members of the (1, 8) + (8, 1) representation of $SU(3) \times SU(3)$.

The most divergent contributions to weak transition amplitudes between hadronic states are most easily isolated by use of the equivalence theorem stated above. The W-field transforms as in (2.18) and

$$q \rightarrow \exp\left\{-i\frac{g}{2m_W}(1-\gamma_5)\underset{\sim}{\lambda}_c \cdot \underset{\sim}{B}\right\}q \tag{2.26}$$

The transformed Lagrangian is

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_W + i\bar{q}\,\gamma^\mu \partial_\mu q - \tfrac{1}{2}\, g\,\bar{q}\,\gamma^\mu \tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\lambda}_c \cdot \underset{\sim}{A}_\mu q \\ - \bar{q}\exp\left\{i\frac{g}{2m_W}\tfrac{1}{2}(1+\gamma_5)\underset{\sim}{\lambda}_c \cdot \underset{\sim}{B}\right\} h_0 \exp\left\{-i\frac{g}{2m_W}\tfrac{1}{2}(1-\gamma_5)\underset{\sim}{\lambda}_c \cdot \underset{\sim}{B}\right\}q \end{aligned} \tag{2.27}$$

with $\mathcal{L}_W$ given by (2.19). We have assumed that $\mathcal{L}_0$, besides being an $SU(3) \times SU(3)$ singlet, does not contain derivatives of the quark fields so that it is invariant under the above transformation of variables.

III. THE LEADING DIVERGENCES

The Lagrangian for the above model of non-leptonic weak interactions has been cast into a form (2.27) in which the leading divergences in weak transition amplitudes between hadronic states can be easily identified. It will be useful to first neglect the chiral invariant part of the strong interactions ($\mathcal{L}_0 = 0$) and examine the leading divergences in the quark self energy.

The Lagrangian (2-27) contains several types of interaction terms. The terms containing the A_μ fields do not contribute to leading divergences and so will be neglected here. The leading divergences come from the exponentials of the B fields in the quark mass term and from the term

$$\mathcal{L}_B = -\partial_\mu \vec{B} \cdot \sum_{n=2}^{\infty} \frac{1}{(2n)!} (-4G)^{n-1} (\vec{T} \cdot \vec{B})^{2n-2} \partial^\mu \vec{B} \tag{3.1}$$

where

$$G = \frac{g}{2m_W} \tag{3.2}$$

which appears in the transformed W mass term. Some typical diagrams are shown in Figs. 1, 2, and 3. The degree of divergence D for any proper quark self-energy diagram can easily be betermined. Let

V_F = number of vertices coming from the quark mass term

n = order of perturbation theory (powers of g).

Then a simple calculation shows that

$$D = n - V_F + 1 \tag{3.3}$$

Diagrams which we term leading divergences are diagrams which diverge as $(g\Lambda)^n$ where Λ is an untraviolet cutoff. Only diagrams with $V_F = 1$ are this divergent. This is a notable improvement over W-boson theory in its usual form where a similar power counting argument shows that all possible proper diagrams are equally divergent. After the transformation, diagrams such as those in Figs. 1 and 2 give leading divergences while those in Fig. 3 do not.

We next include strong interaction effects and consider the weak transition amplitude between two hadronic states α and β (which are exact eigenstates of the strong Hamiltonian). In the usual form of the theory the n-th order term in this amplitude is expressed in terms of a time ordered product of n weak currents. In order to extract the leading divergence, one then keeps only the $q_\mu q_\nu$ part of the W-boson propagator and uses Bjorken-type arguments[13] to show that the most divergent terms can be expressed in terms of σ commutators (commutators of the divergence of the weak current with its zeroth component). These terms depend only on the symmetry breaking part of the Hamiltonian and clearly vanish in the symmetry limit. The leading (quadratic) divergence in second order, for example, is of the form

$$G\Lambda^2 \int d^3x < \beta \mid \left[J_0(x), \partial^\mu J_\mu(0) \right]_{x_0=0} \mid \alpha >$$

where we have dropped the SU(3) indices on the weak currents. In higher orders, one finds many terms including multiple commutators and time ordered products of such multiple commutators. The important point is that the most divergent part in each order of the original time ordered product of currents involves only contact terms in which the integrations over the virtual boson momenta can be explicitly carried out.

In our model with a partial gauge invariance due to inclusion of a neutral current and Yang-Mills boson self couplings, the equivalence theorem of Section II allows the above results as well as some others we will mention to be almost read off from the (transformed) Lagrangian. The contact terms which appeared before in the form of multiple commutators and which depend only on the symmetry breaking part of the Lagrangian, now explicitly appear as the transformed symmetry breaking term in the Lagrangian (the transformed quark mass term in

(2.27)). The leading terms in each order, which correspond to the multiple commutators which appeared before come from

$$-i < \beta \mid T\left\{\mathcal{L}_q(0) \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d^4x_1 \ldots d^4x_m \, \mathcal{L}_B(x_1) \ldots \mathcal{L}_B(x_m)\right\} \mid \alpha > \tag{3.4}$$

where $\mathcal{L}_B(x)$ is given by (3.1) and

$$\begin{aligned} \mathcal{L}_q(x) = {} & \bar{q}(x) \exp\left\{iG^{\frac{1}{2}} \tfrac{1}{2} (1+\gamma_5) \lambda_c \cdot B(x)\right\} h_0 \exp\left\{-iG^{\frac{1}{2}} \tfrac{1}{2} (1-\gamma_5) \lambda_c \cdot B(x)\right\} q(x) \\ & - \bar{q}(x) h_0 q(x) \end{aligned} \tag{3.5}$$

The contractions over the B-fields can be carried out and some of the low order graphs are shown in Figs. 1 and 2. There are also leading contributions which come from higher order effects of $\mathcal{L}_q(x)$. They are time ordered products of the above contact terms and correspond to the time ordered products of multiple σ commutators which appeared in the previous approach. In the absence of strong interactions ($\mathcal{L}_0 = 0$), these terms correspond to improper graphs.

The higher order terms in $\mathcal{L}_q(x)$ give rise to other contributions which involve integrations over Fourier transforms of time ordered products of quark fields. In the absence of strong interactions these contributions correspond to graphs with internal quark propagators which we have shown not to give leading $(g^n\Lambda^n)$ divergences. When strong interactions are included, we again expect that these contributions will not contribute leading divergences.

The structure of the leading divergences can easily be determined. We first consider the single contact term given by (3.4). By using the properties

$$(\lambda_c \cdot B)^{2n} = \lambda_\theta (B \cdot B)^n \tag{3.6}$$

where

$$\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & \cos\theta\,\sin\theta \\ 0 & \cos\theta\,\sin\theta & \sin^2\theta \end{pmatrix} \tag{3.7}$$

and

$$\begin{aligned} [\tfrac{1}{2}(1 \pm \gamma_5)]^2 &= \tfrac{1}{2}(1 \pm \gamma_5) \\ (1+\gamma_5)(1-\gamma_5) &= 0 \end{aligned} \tag{3.8}$$

it is easily seen that (3.4) must be of the form

$$-i\,\xi(\Lambda^2) < \beta \mid \overline{q}(0) \left(\tfrac{1}{2}\{\lambda_\theta, h_0\} + \tfrac{1}{2}\gamma_5\,[\lambda_\theta, h_0] \right) q(0) \mid \alpha >$$

where $\xi(\Lambda^2)$ is a sum of terms corresponding to all the graphs generated by (3.4), some of which are shown in Figures 1 and 2.

We introduce a Lagrangian counterterm to remove these leading divergences by using (2.22) and demanding that

$$\delta h = \xi(\Lambda^2) \left(\tfrac{1}{2}\{\lambda_\theta, h_0\} + \tfrac{1}{2}\gamma_5\,[\lambda_\theta, h_0] \right) \tag{3.9}$$

This equation can be solved for δh in terms of h and λ_θ.

$$\delta h = \left(\frac{1}{1-\xi(\Lambda^2)} - 1\right)\left(\tfrac{1}{2}\{\lambda_\theta, h\} + \tfrac{1}{2}\gamma_5 [\lambda_\theta, h]\right) \tag{3.10}$$

Thus the structure of the leading divergences to all orders can be determined quite easily. It can be seen from (3.10) that the proof of Bouchiat Iliopoulos and Prentki[13] that the leading divergence in second order for $\Delta S = 1$ vanishes extends to all orders since each order has the same structure[14].

It has been pointed out by Gatto et al. that if one requires the diagonal elements of δh to be equal, one gets, quite remarkably, an excellent determination of the Cabibbo angle. However, it must also be pointed out that one is not forced to impose any constraints at all in the counterterms or on the Cabibbo angle. The above equation shows just what subtractions are needed in order to remove the leading divergences. Nevertheless, it is worth noting that we can make predictions by imposing conditions on δh. And a simple condition is the one mentioned above, i.e. that the diagonal terms of δh form a unitary singlet.

Gatto et al. originally worked to lowest order in G. Our result shows that for our model, one encounters the same matrix structure for the leading contribution to δh in any order of perturbation theory. The diagonal elements of δh are

$$\xi \begin{pmatrix} m_p & & \\ & m_n \cos^2\theta & \\ & & m_\lambda \sin^2\theta \end{pmatrix} = \xi c_8 \begin{pmatrix} -\rho + \sqrt{3}\dfrac{c_3}{c_8} & & \\ & (-\rho - \sqrt{3}\dfrac{c_3}{c_8})\cos^2\theta & \\ & & (-\rho - 3)\sin^2\theta \end{pmatrix}$$

where $\rho = -(1 + \sqrt{2}\dfrac{c_0}{c_8})$.

Requiring the diagonal elements of δh to be equal yields

$$\tan^2\theta = \frac{\rho}{3+\rho} + \left[1 + \left(\frac{\rho}{\rho+3}\right)^2\right]^{\frac{1}{2}} - 1 \simeq \frac{1}{3}\rho$$

and

$$3\frac{c_3}{c_8} = \frac{\rho^2}{3+\rho+\left[\rho^2+(3+\rho)^2\right]^{\frac{1}{2}}} \simeq \frac{1}{6}\rho^2.$$

Using the value $c_8/c_0 = -1.25$ of Gell-Mann, Oakes and Renner one finds $\sin\theta = 0.22$ in excellent agreement with experiment and $c_3/c_8 = 1/137$ which is the expected order of magnitude.

IV. SUMMATION OF SIMPLE GRAPHS

The contributions from one class of graphs may be summed exactly. These graphs are illustrated in Fig. 1. They consist of only one vertex and arbitrary numbers of closed boson loops and correspond to the m = 0 term in the sum in (3.4). The matrix element is

$$-i < \beta \mid \mathcal{L}_q(0) \mid \alpha >$$

with $\mathcal{L}_q(x)$ given by (3.5). This can be written in the form

$$-i\,\xi_0(\Lambda^2) < \beta \mid \bar{q}(0)\left(\tfrac{1}{2}\{\lambda_\theta, h_0\} + \tfrac{1}{2}\gamma_5\,[\lambda_\theta, h_0]\right) q(0) \mid \alpha >$$

with

$$\xi_0(\Lambda^2) = -\sum_{n=1}^{\infty} \frac{1}{(2n)!}(-G)^n < 0 \mid (\bar{B}\cdot\bar{B})^n \mid 0 > \tag{4.1}$$

The Λ^2 dependence enters through the choice of propagator regularization

$$D_F(0) = \Lambda^2 . \tag{4.2}$$

We expand the vacuum expectation value in (4.1) and take into account all pairings of the B-fields. This gives

$$< 0 | \underset{\sim}{B} \cdot \underset{\sim}{B} | 0 > = n! \, \Lambda^{2n} \sum_{\alpha+\beta+\gamma=n} \frac{(2\alpha-1)!!\,(2\beta-1)!!\,(2\gamma-1)!!}{\alpha!\beta!\gamma!} \tag{4.3}$$

In Appendix B it is shown that

$$\sum_{\alpha+\beta+\gamma=n} \frac{(2\alpha-1)!!\;(2\beta-1)!!\;(2\gamma-1)!!}{\alpha!\,\beta!\,\gamma!} = \frac{(2n+1)!!}{n!} . \tag{4.4}$$

Using these results, $\xi_0(\Lambda^2)$ becomes

$$\xi_0(\Lambda^2) = -\sum_{n=1}^{\infty} \frac{2n+1}{n!} \left(-\tfrac{1}{2}\, G\Lambda^2\right)^n = 1 - (1 - G\Lambda^2)\, e^{-\frac{1}{2} G\Lambda^2} . \tag{4.5}$$

The counterterm δh necessary to remove these divergences is, in analogy to (3.10),

$$\begin{aligned} \delta h_0 &= \left(\frac{1}{1 - \xi_0(\Lambda^2)} - 1 \right) \left(\tfrac{1}{2}\{\lambda_\theta, h\} + \tfrac{1}{2}\gamma_5 [\lambda_\theta, h] \right) \\ &= \left[(1 - G\Lambda^2)\, e^{+\frac{1}{2} G\Lambda^2} - 1 \right] \left(\tfrac{1}{2}\{\lambda_\theta, h\} + \tfrac{1}{2}\gamma_5 [\lambda_\theta, h] \right) \end{aligned} \tag{4.6}$$

Thus it is seen that in the approximation of including only graphs with one vertex, even when an exact sum is done, an infinite renormalization is necessary. If this is exactly true, it would seem that doing an exact sum will not suffice to make sense of the divergences in our field theory.

There has been a recent paper by Cabibbo and Maiani[15] on the origin of the weak interaction angle in which the diverences from the weak interactions are expected to cancel against the divergences from the electromagnetic interactions. In order for this to happen, one needs ξ_0 to be small compared to one. Our result here, in the limit $\Lambda^2 \to \infty$, is $\xi_0 = 1$, so if the cancellation between weak and electromagnetic divergences is to take place, one must hope that a sum over all diagrams gives an appreciably different result than the sum over just the simplest diagrams.

Appendix A

In this appendix a unitary operator, G, which has the effect

$$\begin{aligned} \underset{\sim}{W}_\mu \rightarrow \underset{\sim}{w}_\mu &= S(g, \underset{\sim}{b})\underset{\sim}{A}_\mu + R(g, \underset{\sim}{b})\, \partial_\mu \underset{\sim}{b} = G^{-1} \underset{\sim}{W}_\mu G \\ \underset{\sim}{B} \rightarrow \underset{\sim}{B} &= G^{-1} \underset{\sim}{B}\, G \end{aligned} \qquad \text{(A.1)}$$

will be displayed. This operator was first written down by Umezawa and Kamefuchi[9].

The unitary operator is

$$G = \exp\left[i \frac{g}{m_W} \int d^3x\, \underset{\sim}{B} \cdot \underset{\sim}{G}_0 \right] , \qquad \text{(A.2)}$$

where

$$\underset{\sim}{G}_\mu = \underset{\sim}{W}_{\mu\nu} \times \overline{\underset{\sim}{W}}^\nu , \qquad \text{(A.3)}$$

and

$$\overline{\underset{\sim}{W}}_\nu = \underset{\sim}{A}_\nu + \underset{\sim}{\gamma}_\nu(g) = \underset{\sim}{W}_\nu - \underset{\sim}{\gamma}_\nu(-g) . \qquad \text{(A.4)}$$

$\underset{\sim}{\gamma}_\nu$ satisfies the equation

$$(S(g, \underset{\sim}{b}) - 1)\, \underset{\sim}{\gamma}_\nu(g) = (R(g, \underset{\sim}{b}) - 1)\, \partial_\nu \underset{\sim}{b} . \qquad \text{(A.5)}$$

One can show with some labor that the operator G has the desired effect.

Similarly, the transformation of the fermion fields is accomplished by the unitary operator

$$T = \exp\left[-\tfrac{1}{2}\, ig \int d^3x\, \psi^\dagger \tfrac{1}{2} (1 - \gamma_5)\, \underset{\sim}{\tau} \cdot \underset{\sim}{b}\, \psi \right] . \qquad \text{(A.6)}$$

Appendix B

Theorem:

$$\sum_{\alpha+\beta+\gamma=n} \frac{(2\alpha-1)!!\,(2\beta-1)!!\,(2\gamma-1)!!}{\alpha!\,\beta!\,\gamma!} = \frac{(2n+1)!!}{n!} \tag{B.1}$$

where α, β, γ, n are positive integers.

Proof: We first express the double factorials in the form

$$(2\alpha-1)!! = \Gamma(\alpha+\tfrac{1}{2})\,2^{\alpha}\pi^{-\frac{1}{2}} = 2^{\alpha}\pi^{-\frac{1}{2}} \int_0^{\infty} t^{\alpha-\frac{1}{2}}\, e^{-t}\, dt \tag{B.2}$$

Then

$$\sum_{\alpha+\beta+\gamma=n} \frac{(2\alpha-1)!!\,(2\beta-1)!!\,(2\gamma-1)!!}{\alpha!\,\beta!\,\gamma!}$$

$$= 2^{n}\pi^{-3/2} \int_0^{\infty} ds\,dt\,du\; e^{-(s+t+u)} (stu)^{-\frac{1}{2}} \sum_{\alpha+\beta+\gamma=n} \frac{s^{\alpha}t^{\beta}u^{\gamma}}{\alpha!\,\beta!\,\gamma!} \tag{B.3}$$

$$= 2^{n}(n!)^{-1}\pi^{-3/2} \int_0^{\infty} ds\,dt\,du\; (stu)^{-\frac{1}{2}} (s+t+u)^{n}\, e^{-(s+t+u)}$$

A useful change of variables in this integral is

$$s = xr, \quad t = yr, \quad u = zr \tag{B.4}$$

with

$$x + y + z = 1, \tag{B.5}$$

Then

$$ds\,dt\,du = r^{2} dr\,dx\,dy\,dz\;\; \delta(1-x-y-z) \tag{B.6}$$

and

$$\sum_{\alpha+\beta+\gamma=n} \frac{(2\alpha-1)!!\,(2\beta-1)!!\,(2\gamma-1)!!}{\alpha!\,\beta!\,\gamma!} = 2^{n}\,(n!)^{-1}\pi^{-3/2} \int_0^\infty dr\; e^{-r}\, r^{n+\frac{1}{2}} \times \int_0^\infty dx\,dy\,dz\;\delta(1-x-y-z)\,(xyz)^{-\frac{1}{2}} \tag{B.7}$$

Now

$$\int_0^\infty dr\; e^{-r}\, r^{n+\frac{1}{2}} = \pi^{\frac{1}{2}}\,(2n+1)!!\;2^{-n-1}$$

and

$$\int_0^\infty dx\,dy\,dz\;\delta(1-x-y-z)\,(xyz)^{-\frac{1}{2}} = 2\pi$$

which gives (B.1).

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FIGURE CAPTIONS

Figure 1 - Summable fermion self-energy graphs of order $(G\Lambda^2)^n$. The solid lines are fermions and the dashed lines are B-bosons.

Figure 2 - Some other fermion self-energy graphs of order $(G\Lambda^2)^n$.

Figure 3 - Less divergent fermion self-energy graphs.

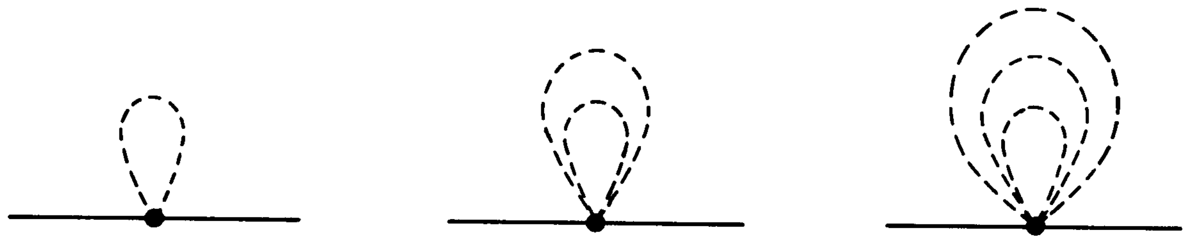

Fig. 1

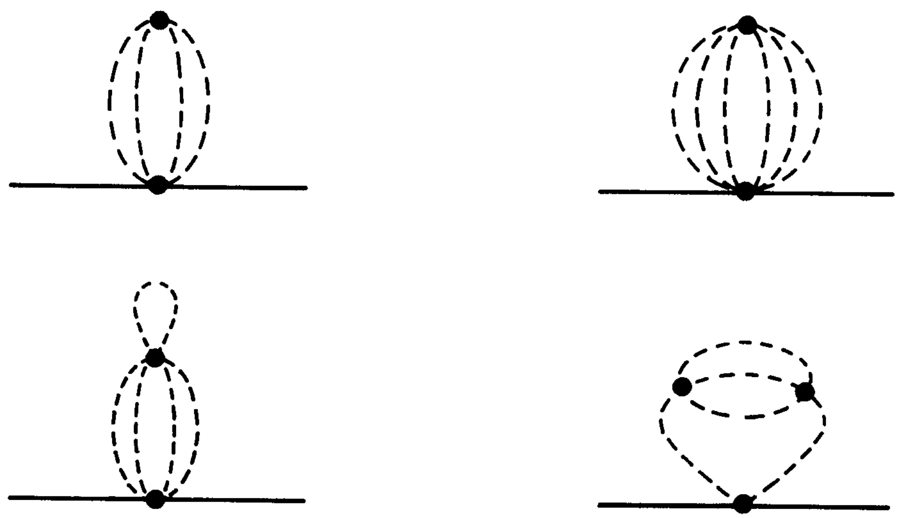

Fig. 2

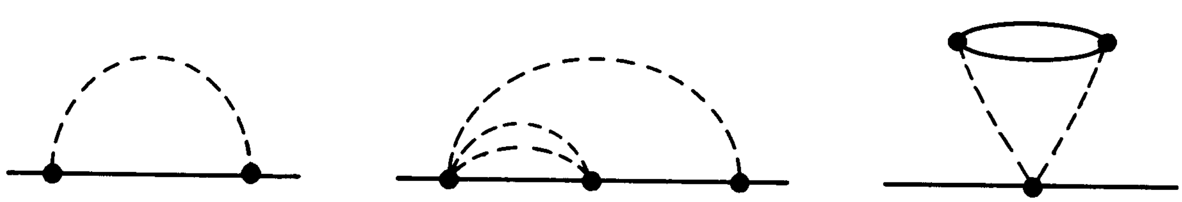

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Fig. 3