THREE-PARTICLE CONTINUUM WAVE FUNCTION FOR FINITE-RANGE FORCES †

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ABSTRACT

It is demonstrated by construction that the three-particle wave function is completely determined by the wave function in the finite volume where the interaction regions of all three particles overlap, and that in the case of pairwise forces, the wave function in this region is completely determined by the solution of a (multi-component) two-variable integral equation contained within this volume. The construction is formal in that it implies the solution of a one-variable integral equation describing the scattering of the outgoing wave from one pair by one of the particles in the other pair, but this is just a two-particle problem with unusual boundary conditions. This preprint is being distributed in the hopes that someone will see a way to write the general solution for the one-variable problem. If so, a unique (in terms of a theory of the two-particle interactions in the sub-systems) description of the exterior (but interacting) final-state wave function of three particles, comparable to the phase-shift description for two-particles, is immediate.

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It has been shown^{1,2} that the assumption of finite-range pairwise forces allows a formal reduction of the three-body problem to two continuous variables in momentum space, and the corresponding coordinate space reduction has also been given³. It has further been shown explicitly^{4,5,6} that the truncation of the sum over angular momenta of the interacting pairs which is implied is in fact an excellent approximation for the simple problems studied so far. In order to remove the masses from the Schroedinger equation, we use coordinates canonically conjugate to the momentum variables defined by Doolen⁷, namely

$$\underline{x}_{1} = \sqrt{\frac{2m_{2}m_{3}}{(m_{2} + m_{3})}} \quad (\underline{R}_{2} - \underline{R}_{3})$$

$$\underline{y}_{1} = \sqrt{\frac{2m_{1}(m_{2} + m_{3})}{m_{1} + m_{2} + m_{3}}} \quad \left[\frac{m_{2}\underline{R}_{2} + m_{3}\underline{R}_{3}}{m_{2} + m_{3}} - \underline{R}_{1}\right]$$
(1)

The alternative choices \underline{x}_2 , \underline{y}_2 and \underline{x}_3 , \underline{y}_3 are obtained by cyclic permutation; note that the length $x_i^2 + y_i^2$ is the same in all three coordinate systems. The transformation from one set to another is given by, for example,

$$\frac{\mathbf{x}_{2}(\mathbf{x}_{1}, \mathbf{y}_{1})}{\mathbf{y}_{2}(\mathbf{x}_{1}, \mathbf{y}_{1})} = \cos \mu_{12} \mathbf{x}_{1} - \sin \mu_{12} \mathbf{y}_{1}$$

$$\frac{\mathbf{y}_{2}(\mathbf{x}_{1}, \mathbf{y}_{1})}{\cos \mu_{12}} = \sin \mu_{12} \mathbf{x}_{1} + \cos \mu_{12} \mathbf{y}_{1}$$

$$\cos \mu_{12} = \left[m_{1} m_{2} / (m_{1} + m_{3}) (m_{2} + m_{3}) \right]^{\frac{1}{2}}$$
(2)

We now make the Faddeev channel decomposition by breaking the wave function

in the center-of-mass system, Ψ , into three parts,

$$\Psi(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1) = \Psi_1(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1) + \Psi_2(\underline{\mathbf{x}}_2(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1),\underline{\mathbf{y}}_2(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1)) = \Psi_3(\underline{\mathbf{x}}_3(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1),\underline{\mathbf{y}}_3(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_1))$$
(3)

and the reduction to two radial variables in each channel by the expansion

$$\Psi_{i}^{M}(\underline{x}_{i},\underline{y}_{i}) = \sum_{J\ell\lambda} \frac{U_{\ell\lambda}^{1}(\underline{x}_{i},\underline{y}_{i})}{\underline{x}_{i}y_{i}} Y_{J\ell\lambda}^{M} (\theta_{x_{i}}\phi_{x_{i}}\theta_{y_{i}}\phi_{y_{i}})$$
(4)

where the $Y_{J\ell\lambda}^{M}$ are the usual ortho-normal two-direction functions as defined in Blatt and Weisskopf⁸. It is then easy to show³ that the Schroedinger equation

$$(\underline{\nabla}_{x_{1}}^{2} + \underline{\nabla}_{y_{1}}^{2} + z - W_{1}(x_{1}) - W_{2}(x_{2}(\underline{x}_{1}, \underline{y}_{1})) - W_{3}(x_{3}(x_{1}, y_{1})) \Psi = 0$$
(5)

is equivalent to the coupled set of equations

$$\begin{bmatrix} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} - \frac{\ell(\ell+1)}{x_1^2} - \frac{\lambda(\lambda+1)}{y_1^2} + z - W_1(x_1) \end{bmatrix} U_{\ell,\lambda}^1 (x_1, y_1)$$

$$= W_1(x_1) \sum_{s=2,3} \sum_{\ell'\lambda'} \frac{1}{2} \int_{-1}^{1} d\cos \zeta \ K_{\ell\lambda;\ell'\lambda}^{1s(J)}, (x_1, y_1, x_s, y_s, \zeta) \frac{x_1y_1}{x_sy_s} U_{\ell'\lambda}^s, (x_s, y_s)$$

$$x_s = + \sqrt{\cos^2 \mu_{1s} x_1^2 + \sin^2 \mu_{1s} y_1^2} - 2 \sin \mu_{1s} \cos \mu_{1s} x_1 y_1 \cos \zeta_1$$

$$y_s = + \sqrt{\sin^2 \mu_{1s} x_1^2 + \cos^2 \mu_{1s} y_1^2} + 2 \sin \mu_{1s} \cos \mu_{1s} x_1 y_1 \cos \zeta_1$$

$$(7)$$

with corresponding equations for U^2 and U^3 . The angle ζ_1 is simply the angle between \underline{x}_1 and \underline{y}_1 , namely $\cos \zeta_1 = (\underline{x}_1 \cdot \underline{y}_1)/x_1 y_1$. Since all three vector sets $\underline{x}_1 \underline{y}_1$ lie in a plane, we can always express any lengths and angles in terms of $x_1 y_1$ and $\cos \zeta_1$, as has been done explicitly above for x_s and y_s . The Euler angles of some arbitrary axis lying in this place have been eliminated. If \underline{x}_1 makes the angle ξ with this axis, $\underline{x}_1 \cdot \underline{x}_2 = x_1 x_2 \cos \zeta_{12}$, and $\underline{x}_2 \cdot \underline{y}_2 = x_2 y_2 \cos \zeta_2$, then the kernel is easily shown³ to be

$$K^{12} = \frac{8\pi}{(2J+1)} \sum_{MM'} Y^{M*}_{J\ell\lambda} (\xi, 0, \zeta_1 + \xi, 0) Y^{M'}_{J\ell'\lambda'} (\xi + \zeta_{12}, 0, \xi + \zeta_{12} + \zeta_2, 0)$$
(8)

with the obvious generalizations.

None of this geometrical complication is essential to the dynamical structure of the three-body problem, so we specialize immediately to three identical spinless particles interacting only via S-waves ($l=\lambda=0$) in the state of zero total angular momentum (J=0). For this case $U^1 = U^2 = U^3 = U$, and we have the single equation

$$\begin{bmatrix} \frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} + z - W(x) \end{bmatrix} U(x, y) = W(x) \int_{-1}^{1} d\cos \zeta \frac{xy}{x^{\dagger}y^{\dagger}} U(x^{\dagger}y^{\dagger})$$
$$= W(x) \int_{\Delta_{-}(y/x)}^{\Delta_{+}(y/x)} d\theta U(\sqrt{x^{2} + y^{2}} \cos \theta, \sqrt{x^{2} + y^{2}} \sin \theta)$$
(9)
$$x^{\dagger} = \sqrt{\frac{1}{4}x^{2} + \frac{3}{4}y^{2}} - \sqrt{\frac{3}{2}}xy \cos \zeta = \sqrt{x^{2} + y^{2}} \cos \theta$$

where the limits $\Delta \pm$ between which θ is integrated are a function of $\tan^{-1}(y/x)$, given in Figure 1. Note that the right hand side of Eq. (9) vanishes for x > R if the interaction W(x) = 0 for x > R, so outside of the strip x < R in the x-y plane illustrated in Figure 2a, we have three free particles. Unfortunately, since the scattering solution of the equation is of order unity for large x and y, the θ integration gives a source contribution for θ near $\pi/2$ which falls off in this strip only like 1/y, so we <u>cannot</u> assume free particle solutions in the strip. The physical origin of this source is illustrated in Figure 3. A scattering in one of the two other Faddeev channels gives an outgoing wave; so long as the pair in the direct channel are within the range of forces (x < R), they can pick up momentum from this wave and scatter to a three free particle final state. Clearly, the probability amplitude for this happening only falls off like 1/y, as we have found.

Nevertheless, we can still write down an integral equation for U in terms of solutions of the two-body problem for the interaction W(x), which we call u_p in the continuum or ϕ_{γ} for bound states, and define as follows:

$$u_{p}^{"} + p^{2}u_{p} = W(x)u_{p}$$

$$u_{p}(0) = 0; \quad u_{p}(x) = \sin(px + \delta_{p}) \quad x > R$$

$$\frac{2}{\pi} \int_{0}^{\infty} u_{p}(x)u_{q}(x) dx = \delta(p-q)$$

$$\phi_{\gamma}^{"} - \gamma^{2}\phi_{\gamma} = W(x)\phi_{\gamma}$$

$$\phi_{\gamma}(0) = 0 \quad \phi_{\gamma} = N_{\gamma} e^{-\gamma X} \quad x > R$$

$$\frac{2}{\pi} \int_{0}^{\infty} \phi_{\gamma}(x)\phi_{\gamma'}(x) = \delta_{\gamma\gamma'}$$

$$\sum_{\gamma} \phi_{\gamma}(x)\phi_{\gamma}(x') + \int_{0}^{\infty} dp u_{p}(x)u_{p}(x') \equiv \oint dp u_{p}(x)u_{p}(x') = \delta(x-x')$$
(10)

assuming a plane wave of	
> integral equation with stationary state scattering boundary conditions,	mentum $(z + \gamma^2)^{\frac{1}{2}}$ incident on a bound state of energy - γ^2 is then

$$U(x, y) = u_{\gamma}(x) \sin \sqrt{x + \gamma^2} y$$

$$+ \frac{4}{\pi^2} \oint dp \int_0^\infty dq \frac{u_p(x) \sin qy}{z^{+i\epsilon} - p^2 - q^2} \int_0^\infty dx \int_0^\infty dy' u_p(x') W(x') \sin qy' \int_{-\infty}^{\Delta_+(y'/x')} d\phi U(\sqrt{x'^2 + y'^2} \cos \theta, \sqrt{x'^2 + y'^2} \sin \theta)$$

$$= u_\gamma(x) \sin \sqrt{z + \gamma^2} y + \frac{4}{\pi^2} \oint dp \int_0^\infty dq \frac{u_p(x) \sin qy}{z^{+i\epsilon} - p^2 - q^2}$$

$$= u_\gamma(x) \sin \sqrt{x + \frac{2}{\pi^2}} y + \frac{4}{\pi^2} \oint dp \int_0^\infty dq \frac{u_p(x) \sin qy}{z^{+i\epsilon} - p^2 - q^2}$$

$$= \int_0^R dx'' \int_0^{x''+2R} dy'' + \int_R^{2R} dx'' \int_0^{x''+2R} dy'' + \int_{2R}^\infty dy'' + \int_{2R}^\infty dx'' \int_{-\infty}^{x''+2R} dy'' \int_{-\infty}^{x'''+2R} d\theta''$$

$$= \int_0^R dx'' \int_0^{x''+2R} dy'' + \int_R^{2R} dx'' \int_0^\infty dy'' + \int_{2R}^\infty dy'' + \int_{2R}^\infty dx'' \int_{-\infty}^{x''+2R} dy'' \int_{-\infty}^{x'''+2R} d\theta'' \int_{0}^{y'''''} d\theta''$$

where the limit $\epsilon \rightarrow 0^+$ is implied.

 $u_{p}(\sqrt{x^{''}2^{}+y^{''}2^{}}\cos\theta^{\,''})W(\sqrt{x^{''}2^{}+y^{''}2^{}}\cos\theta^{\,''})\sin(q\sqrt{x^{''}2^{}+y^{''}2^{}}\sin\theta^{\,''})U(x^{''},y^{''})$

Note that in order to compute U, we need know it only in the strip shown in Figure 2b. This is still an infinite region. However, for x > R, we have already seen that only free particle solutions are required. But free particle solutions depend on only a single parameter (e.g. the relative momentum of the previously interacting pair p or the momentum of the free particle q, but not both, since in this region $p^2 + q^2 = z$). Consequently, we can write down an integral equation for U(x, y) for x'' > R in terms of a single variable, plus an inhomogeneous term coming from the integral of U(x'', y'') over the finite region 0 < x'' < R, $0 < y'' < (x+2R)/\sqrt{3}$ illustrated in Figure 2c. Note that this is simply the region where all three particles are within the range of each other's forces. Before we can proceed further, however, we must understand the structure of the kernel in our integral equation better.

Consider first the integral

$$I(K) = \frac{2}{\pi} \int_{0}^{\infty} dq \; \frac{\sin qy \sin qy'}{K^{2} + i\epsilon - q^{2}} = \frac{1}{\pi} \int_{-\infty}^{\infty} dq \; \frac{\sin qy \sin qy'}{K^{2} + i\epsilon - q^{2}}$$

$$= \frac{1}{\pi i} \int_{-\infty}^{\infty} dq \; \frac{e^{iqy} \sin qy'}{K^{2} + i\epsilon - q^{2}} = \frac{1}{\pi i} \int_{-\infty}^{\infty} dq \; \frac{e^{iqy'} \sin qy}{K^{2} + i\epsilon - q^{2}}$$
(12)

where the last two forms are possible because the integral of the cos term vanishes. If y > y', the + i ϵ allows us to close the contour above in the first form, and for y' > y in the second, so

$$I(K) = -\frac{1}{K} \begin{cases} e^{iKy} \sin Ky' & y > y' \\ e^{iKy'} \sin Ky & y' > y \end{cases}$$
(13)

Hence, for y > y', we can perform the q integration in Eq. (11) and provided the integrand falls off sufficiently rapidly, obtain

$$\lim_{y \to \infty} U(x, y) = \phi_{\gamma}(x) \sin \sqrt{z + \gamma^2} y + \sum_{\gamma'} \phi_{\gamma'}(x) e^{i \sqrt{z + \gamma'^2} y} T_{\gamma \gamma'}$$

$$+ \int_{0}^{\sqrt{z}} dp u_{p}(x) e^{i \sqrt{z - p^2} y} T(p)$$

$$+ \int_{\sqrt{z}}^{\infty} dp u_{p}(x) e^{-\sqrt{p^2 - z} y} E(p)$$
(14)

with

$$T_{\gamma\gamma'} = -\frac{2}{\pi} \int_0^R dx' \int_0^\infty dy' \int_{\Delta^-}^{\Delta^+} d\theta \phi_{\gamma'}(x') W(x') \frac{\sin\sqrt{z+\gamma'^2} y'}{\sqrt{x+\gamma'^2}} U(r'\cos\theta, r'\sin\theta)$$

$$\mathbf{p}^{2} < \mathbf{z} \quad \mathbf{T}(\mathbf{p}) = -\frac{2}{\pi} \int_{0}^{\mathbf{R}} d\mathbf{x}' \int_{0}^{\infty} d\mathbf{y}' \quad \int_{\Delta^{-}}^{\Delta^{+}} d\theta \, \mathbf{u}_{\mathbf{p}}(\mathbf{x}') \mathbf{W}(\mathbf{x}') \, \frac{\sin\sqrt{z-p^{2}} \, \mathbf{y}'}{\sqrt{z-p^{2}}} \, \mathbf{U}(\mathbf{r}'\cos\theta, \mathbf{r}'\sin\theta)$$

$$p^{2} > z \quad E(p) = -\frac{2}{\pi} \int_{0}^{R} dx' \int_{0}^{\infty} dy' \quad \int_{\Delta^{-}}^{\Delta^{+}} d\theta \ u_{p}(x')W(x') \frac{\sinh\sqrt{p^{2}-z} \ y'}{\sqrt{p^{2}-z}} U(r'\cos\theta, r'\sin\theta)$$

$$\mathbf{r'} = \sqrt{\mathbf{x'}^2 + \mathbf{y'}^2}$$

which shows us that we do indeed have the correct scattering boundary conditions incorporated into our integral equation. However, as we have already seen, we have a cutoff in x rather than y, so require instead

$$\frac{2}{\pi} \sum_{0}^{\infty} dp \frac{u_{p}(x)u_{p}(x')}{z+i\epsilon - p^{2} - q^{2}} = \sum_{\substack{x > R \\ x' < R}} \sum_{\gamma} \frac{N_{\gamma} e^{-\gamma x} \phi_{\gamma}(x')}{z+\gamma^{2} + i\epsilon - q^{2}} + \frac{1}{\pi i} \int_{-\infty}^{\infty} dp \frac{e^{i(px+\delta_{p})}u_{p}(x')}{z+i\epsilon - p^{2} - q^{2}}$$

$$= \frac{e^{i(\sqrt{z-q^{2}x} + \delta_{\sqrt{z-q^{2}}})u_{\sqrt{z-q^{2}}}(x')}}{\sqrt{z-q^{2}}}$$
(16)

For $q^2 > z$ (i.e. p imaginary), we can still use this evaluation, if we are careful to normalize u_p according to Eq. (10) with $e^{2i\delta}p$ defined by

$$e^{2i\delta_{p}} = e^{-2iPR} \left[\frac{u'(R) + ip u_{p}(R)}{u'(R) - ip u_{p}(R)} \right]$$
(17)

Note that for the bound states $u_{i\gamma}$ is proportional to ϕ_{γ} , and since $\phi_{\gamma}^{\dagger} = -\gamma \phi_{\gamma}$ for $x \ge R$, we find that $e^{2i\delta}p$ has the usual poles; in fact the contribution from these poles to the dp integration exactly cancels the sum in the first line of Eq. (16). However, this contribution is recovered by the dq integration, leading to the elastic scattering and rearrangement terms $T_{\gamma\gamma'}$, as before. Consequently, with this understanding, we find that for x > R

$$U(x, y) = \phi_{\gamma}(x) \sin\sqrt{z + \gamma^{2}} y - \frac{2}{\pi} \int_{0}^{\infty} dq e^{i(\sqrt{z - q^{2}}x + \delta\sqrt{z - q^{2}})} \sin qy$$
(18)
$$\left[\int_{0}^{R} dx'' \int_{0}^{\frac{x'' + 2R}{\sqrt{3}}} dy'' + \int_{R}^{2R} dx'' \int_{0}^{\frac{x'' + 2R}{\sqrt{3}}} dy'' + \int_{2R}^{\infty} dx'' \int_{\frac{x'' - 2R}{\sqrt{3}}}^{\frac{x'' + 2R}{\sqrt{3}}} dy'' \right] \int_{\Delta^{-}}^{\Delta^{+}} d\theta''$$
$$u_{\sqrt{z - q^{2}}} (\sqrt{x''^{2} + y''^{2}} \cos\theta'') W(\sqrt{x''^{2} + y''^{2}} \cos\theta'') \sin(q\sqrt{x''^{2} + y''^{2}} \sin\theta'') U(x'', y'')$$

$$= \frac{2}{\pi} \int_{0}^{\infty} dq \ F(q) e^{i(\sqrt{z-q^2} x + \delta \sqrt{z-q^2})} \sin qy$$
(19)

Hence, as asserted, U(x, y) is completely determined for x > R if we know U(x, y) in the finite region 0 < x < R, 0 < y < (x + 2R)/3, and can solve the one-variable integral equation

$$F(q) = \chi(q) + \int_{0}^{\infty} dq' K(q, q') F(q')$$
 (20)

with

,

$$\chi (q) = \delta (q - \sqrt{z + \gamma^2}) + \int_0^R dx'' \int_0^{\frac{x'' + 2R}{\sqrt{3}}} dy'' \int_{\Delta^-}^{\Delta^+} d\theta'' u \frac{1}{\sqrt{z - q^2}} (\sqrt{x''^2 + y''^2} \cos \theta'') \times (21) \times W(\sqrt{x''^2 + y''^2} \cos \theta'') \sin q(\sqrt{x''^2 + y''^2} \sin \theta'') U(x''y'')$$

 $K(q,q') = -\frac{2}{\pi} \left[\int_{R}^{2R} dx'' \int_{0}^{\frac{X''+2R}{\sqrt{3}}} dy'' + \int_{2R}^{\infty} dx'' \int_{\frac{X-2R}{\sqrt{3}}}^{\frac{X+2R}{\sqrt{3}}} dy'' \right] \int_{\Delta^{-}}^{\Delta^{+}} d\theta''$

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$$u_{\sqrt{z-q^{2}}}(\sqrt{x''^{2}+y''^{2}}\cos\theta'')W(\sqrt{x''^{2}+y''^{2}}\cos\theta'')\sin(q\sqrt{x''^{2}+y''}\sin\theta'') \times (22)$$

$$i(z-q'^2x''+\delta) = \frac{i(z-q'^2x''+\delta)}{z-q'^2} \sin q'y''$$

Note that this is just a two-particle scattering by the potential W with unusual boundary conditions. Hence, it should be possible to write down the resolvent kernel in terms of the off-shell two-body T matrix, and by applying it to χ (q), obtain an explicit expression for U(x, y) for x > R in terms of U in the interior region. Once this is done, this expression can be inserted in Eq. (ll) in the last two terms, leaving it an integral equation which depends on the coordinates of U only in the interior region. Unfortunately, I have not yet seen explicitly how to do this.

It would seem that considerable effort would be justified in trying to uncover the resolvent kernel for Eq. (20). Note that the final equation for U(x, y) is over a finite domain in <u>both</u> variables. Hence it can be immediately converted to a matrix equation by using a complete set of functions, which will be denumerable. In fact, at low energy the uncertainty principle precludes us from observing much structure within this region, and this expansion should

and

be rapidly convergent; it is also the obvious place to start a variational calculation, or a low energy effective range theory.

Another point which is worth emphasizing is that all the final state interactions, overlapping resonances, etc. due to two-body forces occur in the exterior region, and are explicitly given in terms of known functions (assuming, as always, that we have a complete theory of the two-body wave functions). Consequently, even in situations where there may well be threebody forces in the interior region, we can still parametrize the interior region in any convenient fashion, and obtain an explicit formula for the twoparticle contributions to the exterior region. This should provide a powerful tool for analyzing three-body final states and get over the necessity of throwing away the regions of the Dalitz plot where two or three final states interfere.

I am eager to hear any suggestions.

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Fig. 1



a) Source: Direct Channel

b) Region of Integration over U (x",y"

Fig. 2



