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#### Abstract

We devise an algebra of currents and their first time-derivatives designed to damp at high momentum the asymptotic behavior of leptonpair scattering amplitudes from hadrons consequent from the local current algebra of Gell-Mann. Given certain criteria, the algebra we find is unique, and the commutators are expressed linearly in terms of the currents themselves. The Jacobi identity, however, is formally violated for this algebra; we argue that this does not invalidate it. A possible realization of this "minimal" algebra is found in terms of the formal limit of a massive Yang-Mills theory as $\mathrm{g}_{0}, \mathrm{~m}_{0} \rightarrow 0 ; \mathrm{g}_{0} / \mathrm{m}_{0}^{2} \rightarrow$ constant $\neq 0$. With this algebra, all electromagnetic masses of hadrons are finite. Experimental consequences, the strongest of which occurs in inelastic leptonhadron scattering, are outlined.


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[^0]
## I. INTRODUCTION

Many of the predictions of local current algebra, ${ }^{1}$ notably the sum rules derived by Adler, ${ }^{2}$ Fubini, ${ }^{3}$ Dashen and Gell-Mann, ${ }^{4}$ and similar asymptotic sum rules valid at highq ${ }^{2}$, 5-9 imply that very far off the mass-shell, current-hadron scattering matrix elements are at least as singular as those of free particles. This result, if verified experimentally, would give one great confidence in the general validity of the locality assumptions on the weak currents which underlie the supposed pointlike nature of these amplitudes. On the other hand, the converse is not true. Were all the sum rules to fail experimentally, local current algebra would not necessarily fail. There are many loopholes. One possibility is that the equaltime commutators are ambiguous. ${ }^{10}$ Another is that, although the commutators are taken to exist, technical assumptions (interchange of limits in the $P \rightarrow \infty$ method, or absence of subtractions in dispersion relations for certain amplitudes) needed in the derivations of the sum rules may be incorrect. Also, alterations of the highly model dependent spacespace commutators and/or of the usually assumed high-q ${ }^{2}$ behavior of amplitudes can invalidate many existing sum rules. It is this latter loophole which is explored in this paper.

We formulate criteria designed to minimize the experimental consequences of the local algebra. They are to be applied in the limit of large $q^{2}$ (where $q$ is a momentum carried in by a current). Given these criteria, it follows that many of the existing sum rules should be damped at high $q^{2}$ by at least an extra power of $q^{2}$, and that all electromagnetic self-energies converge. It also turns out that these criteria uniquely determine the commutation relations not only of the currents with each other, but also with their time derivatives. These commutators turn out to be linear in the currents and derivatives thereof. We call this algebra the minimal algebra. In Section II we describe in detail the "unobservability criteria" which are supposed to minimize the observable effects of the local algebra of currents. In Section III it is shown how these criteria are sufficient to
lead to a unique set of commutators of currents with their time derivatives. In Section IV the minimal algebra is shown to result from a limit of a massive Yang-Mills theory as $\mathrm{m}_{0} \rightarrow 0$ and $\mathrm{g}_{0} / \mathrm{m}_{0}^{2} \rightarrow$ constant. In Section V , we discuss the experimental implications of the minimal algebra.

## II. CRITERIA FOR THE MINIMAL ALGEBRA

According to the Gell-Mann ${ }^{11}$ philosophy of current algebra, matrix-elements of time-ordered products of two currents $\langle\mathrm{p}| \mathrm{T}^{*}\left(\mathrm{~J}_{\mu}^{\mathrm{a}}(\mathrm{x}) \mathrm{J}_{\nu}^{\mathrm{b}}(0)\right)\left|\mathrm{p}^{\prime}\right\rangle=\mathrm{M}_{\mu \nu}^{\mathrm{ab}}$ are considered as observables, because they can in principle be related to S-matrix elements for scattering of lepton pairs or photons from hadrons: ${ }^{12}$

$$
\mathrm{s}_{\mathrm{fi}} \propto \ell_{\mu \nu} \ell_{\nu} \mathrm{M}_{\mu \nu}\left(\mathrm{p}^{\prime}, \mathrm{q}^{\prime} ; \mathrm{p}, \mathrm{q}\right)
$$

with

$$
\ell_{\mu}=\overline{\mathrm{u}}(\mathrm{p}+\mathrm{q}) \gamma_{\mu} \mathrm{u}(\mathrm{p}) \quad \text { or } \quad \overline{\mathrm{u}}_{\gamma_{\mu}}\left(1-\gamma_{5}\right) \mathrm{u}
$$

the lepton current. In order that $M_{\mu \nu}$ itself be observable, it is necessary that the factors $\ell_{\mu}$ are allowed to be removed; i.e., that all four components are independent. This is true provided the lepton mass is not neglected; otherwise

$$
q_{\mu} \ell_{\mu}=0
$$

and $M_{\mu \nu}$ is ambiguous up to terms proportional to $q_{\mu}$ or $q_{\nu}^{\prime}$. On the other hand, sum rules which test the local algebra, such as Adler's neutrino sum rules, ${ }^{2}$ involve high-energy leptons where the neglect of lepton mass would appear to be justifiable.

As $\mathrm{q}_{0} \rightarrow \infty, \underline{\mathrm{~m}}$ fixcd, $\mathrm{M}_{\mu \nu}$ can be expected (but not proven) to be at least as singular as $\mathrm{q}_{0}^{-1}$ with the coefficients controlled by equal-time commutators. ${ }^{5}$ This asymptotic behavior is characteristic of point particles. If no such behavior
is manifested experimentally, it may mean the commutation relations are ambiguous. It may also mean that the leading asymptotic behavior of $\mathrm{M}_{\mu \nu}$ is contained in pieces proportional to $q_{\mu}$ or $q_{\nu}^{\prime}$, with the result that observable consequences in the $S$ matrix are limited to terms of the order lepton mass.

We shall adopt this behavior for $\mathrm{M}_{\mu \nu}$, and assume that through order $\mathrm{q}_{0}^{-2}$, $\underline{\mathbf{M}}_{\mu \nu}$ contains only pieces proportional to $\mathbf{q}_{\mu}$ or $q_{\nu}^{\mathbf{\prime}} \sim$. In this way, experimental consequences of the local current algebra would be expected to be minimized.

This possibility will be explored in a quantitative way in the next section.

## III. THE MINIMAL ALGEBRA

We consider the process shown in Fig. 1 to lowest order in the weak and electromagnetic interaction. The corresponding S-matrix element is proportional to

$$
\begin{equation*}
\ell_{\mu}^{\mathrm{a}}(\mathrm{q}) \ell_{\nu}^{\mathrm{b}}\left(\mathrm{q}^{\mathrm{f}}\right) \mathrm{M}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{l}}, \mathrm{p}\right) \tag{1}
\end{equation*}
$$

where $\ell$ and $\ell^{\prime}$ are lowest order matrix elements of leptonic weak or electromagnetic currents and M is the covariant hadronic current correlation function. We assume that

$$
\begin{equation*}
\mathrm{M}_{\mu \nu}^{\mathrm{ab}}=\mathrm{T}_{\mu \nu}^{\mathrm{ab}}+\mathrm{S}_{\mu \nu}^{\mathrm{ab}} \tag{2}
\end{equation*}
$$

where T is the connected time-ordered product

$$
\begin{equation*}
\mathrm{T}_{\mu \nu}^{\mathrm{ab}}=-\mathrm{i} \int \mathrm{~d}^{4} \mathrm{x} \mathrm{e}^{+\mathrm{iq} \cdot \mathrm{x}}\langle\mathrm{p}| \mathrm{T}\left(\mathrm{~J}_{\mu}^{\mathrm{a}}(\mathrm{x}) \mathrm{J}_{\nu}^{\mathrm{b}}(0)\right)\left|\mathrm{p}^{\prime}\right\rangle \tag{3}
\end{equation*}
$$

and $S$ is a polynomial in $q$. The hadronic currents are assumed to be conserved, ${ }^{13}$

$$
\begin{equation*}
\partial_{\mu} \mathrm{J}_{\mu}^{\mathrm{a}}(\mathrm{x})=0 \tag{4}
\end{equation*}
$$

and $M$ is assumed to satisfy the divergence conditions

$$
\begin{align*}
& \mathrm{q}_{\mu} \mathrm{M}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}, \mathrm{p}\right)=\mathrm{if} \mathrm{f}^{\mathrm{abc}}\langle\mathrm{p}| J_{\nu}^{\mathrm{c}}\left|\mathrm{p}^{\prime}\right\rangle  \tag{5}\\
& \left.\mathrm{q}_{\nu}^{\prime} \mathrm{M}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{p}^{\prime}, \mathrm{p}\right)=\left.\mathrm{if} \mathrm{f}^{\mathrm{abc}}\langle\mathrm{p}| J_{\mu}^{\mathrm{c}}\right|^{\prime}\right\rangle \tag{6}
\end{align*}
$$

The leptonic currents satisfy

$$
\begin{equation*}
\mathrm{q}_{\mu} \ell_{\mu}^{\mathrm{a}}(\mathrm{q})=0 \tag{6'}
\end{equation*}
$$

neglecting the leptonic masses.
Let us first assume that $S=0$ and that the $T$-product in Eq. (3) is well-defined and has an expansion in inverse powers of $q_{0} \equiv \omega$ up to order $\omega^{-3}$. Then this expansion is given by ${ }^{5}$
$\mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{t}}, \mathrm{p}\right)=\langle\mathrm{p}| \int \mathrm{d}^{3} \mathrm{xe}^{-\mathrm{iq}} \cdot \underline{\mathrm{x}}\left\{\frac{1}{\omega}\left[\mathrm{~J}_{\mu}^{\mathrm{a}}(0, \mathrm{x}), \mathrm{J}_{\nu}^{\mathrm{b}}(0)\right]+\frac{\mathrm{i}}{\omega^{2}}\left[\dot{\mathrm{~J}}_{\mu^{\mathrm{a}}}^{(0, \mathrm{x})}, \mathrm{J}_{\nu}^{\mathrm{b}}(0)\right]\right\}\left|\mathrm{p}^{\prime}\right\rangle+\mathrm{o}\left(\frac{1}{\omega^{3}}\right)$.

We now ask if the commutators in Eq. (7) can be chosen so that (1) is $O\left(\frac{1}{\omega^{3}}\right)$ (in the limit $q_{0}, q_{0}^{\prime} \longrightarrow \infty$ with $q, q^{\prime}$, and $\Delta \equiv q-q^{\prime}$ fixed) for all leptonic currents and hadronic scattering states. Then all obscrvable effects of the theory will be $O\left(\frac{1}{\omega^{3}}\right)$ (neglecting leptonic masses) and the theory will be as smooth as possible in the above framework. We shall therefore refer to the resulting current algebra as the minimal one.

The problem can be most succintly expressed in terms of the operators $\mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{t}}\right)$ and $J_{\mu}^{\mathrm{c}}(\Delta)$ defined by

$$
\begin{align*}
\langle\mathrm{p}| \mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}\right)\left|\mathrm{p}^{\prime}\right\rangle & =\delta\left(\mathrm{p}+\mathrm{q}-\mathrm{p}^{\prime}-\mathrm{q}^{\mathrm{l}}\right) \mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}, \mathrm{p}\right)  \tag{8}\\
\mathrm{J}_{\mu}^{\mathrm{c}}(\Delta) & =\int \mathrm{d}^{4} \mathrm{x} e^{\mathrm{i} \Delta \cdot \mathrm{x}} \mathrm{~J}_{\mu}^{\mathrm{c}}(\mathrm{x}) \tag{9}
\end{align*}
$$

Thus we want to find the commutators in Eq. (7) such that

$$
\begin{align*}
\ell_{\mu}^{\mathrm{a}(\mathrm{q}) \ell_{\nu}^{\mathrm{b}}\left(\mathrm{q}^{\mathrm{t}}\right) \mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{\prime}}\right)} & =\mathrm{o}\left(\frac{1}{\omega^{3}}\right)  \tag{10}\\
\mathrm{q}_{\mu} \mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}\right) & =\mathrm{if} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{\nu}^{\mathrm{c}}(\Delta)  \tag{11}\\
\mathrm{q}_{\nu}^{\prime} \mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}\right) & =\mathrm{i} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{\mu}^{\mathrm{c}}(\Delta) \tag{12}
\end{align*}
$$

We mean by Eqs. (10) - (12) that the identities are valid when the equations are sandwiched between arbitrary hadronic scattering states.

In view of Eq. $\left(6^{\prime}\right)$, the conditions (10) require that $T$ has the form

$$
\begin{equation*}
\mathrm{T}_{\mu \nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{l}}\right)=\mathrm{q}_{\mu} \mathrm{F}_{\nu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{i}}\right)+\mathrm{q}_{\nu}^{\prime} \mathrm{F}_{\mu}^{\prime} \mathrm{ab}\left(\mathrm{q}, \mathrm{q}^{\mathrm{f}}\right)+\mathrm{O}\left(\omega^{-3}\right) \tag{13}
\end{equation*}
$$

for some operators $F$ and $F^{\prime}$. The divergence conditions (11) and (12) impose the further restrictions

$$
\begin{equation*}
\widetilde{J}_{\nu}^{a b}(\Delta)=q^{2} F_{\nu}^{a b}\left(q, q^{\prime}\right)+q_{\nu}^{\prime} q \cdot F^{\prime a b}\left(q, q^{\prime}\right)+O\left(\omega^{-2}\right) \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathfrak{J}_{\mu}^{\mathrm{ab}}(\Delta)=\mathrm{q}_{\mu} \mathrm{q}^{\prime} \cdot \mathrm{F}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\prime}\right)+\mathrm{q}^{\prime^{2}} \mathrm{~F}_{\mu}^{\mathrm{ab}}\left(\mathrm{q}, \mathrm{q}^{\mathrm{\prime}}\right)+\mathrm{O}\left(\omega^{-2}\right), \tag{15}
\end{equation*}
$$

where we have defined

$$
\widetilde{\mathrm{J}}_{\mu}^{\mathrm{ab}}=\mathrm{if} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{\mu}^{\mathrm{c}}
$$

Finally, the behavior (7) requires that $F$ and $F^{\prime}$ have the forms

$$
\begin{align*}
& \mathrm{F}_{\nu}^{\mathrm{ab}}=\omega^{-2} \mathrm{~A}_{\nu}^{\mathrm{ab}}+\omega^{-3} \mathrm{~B}_{\nu}^{\mathrm{ab}}+\mathrm{O}\left(\omega^{-3-\epsilon}\right)  \tag{16}\\
& \mathrm{F}_{\mu}^{\prime a b}=\omega^{-2} A_{\mu}^{\prime \mathrm{ab}}+\omega^{-3}{\mathrm{~B}_{\mu}^{\prime} \mathrm{ab}}^{\mathrm{ab}}+O\left(\omega^{-3-\epsilon}\right) \tag{17}
\end{align*}
$$

We shall now show that (13) - (17) uniquely determine the minimal algebra occurring in (7). Substituting (16) and (17) in (14) and equating coefficients of $\omega^{0}$ and $\omega^{-1}$ gives the relations (suppressing the internal indices)

$$
\begin{align*}
& \widetilde{J}_{0}=A_{0}+A_{0}^{\prime},  \tag{18a}\\
& O=B_{0}-\underline{q} \cdot A_{n}^{\prime}+B_{0}^{\prime}-\Delta_{0} A_{0}^{\prime},  \tag{18b}\\
& \widetilde{J}_{k}=\Lambda_{k},  \tag{18c}\\
& O=B_{k}+q_{k}^{\prime} A_{0}^{\prime} . \tag{18d}
\end{align*}
$$

Similarly, Eqs. (16), (17) and (15) give

$$
\begin{align*}
& \widetilde{J}_{0}=A_{0}^{\prime}+A_{0}^{\prime}  \tag{19a}\\
& O=B_{0}-q^{\prime} \cdot A+B_{0}^{\prime}-\Delta_{0} A_{0}-2 \Delta_{0} A_{0}^{\prime},  \tag{19b}\\
& \widetilde{J}_{k}=A_{k}^{\prime},  \tag{19c}\\
& O=B_{k}^{\prime}+q_{k} A_{0}-2 \Delta_{0} A_{k}^{\prime} . \tag{19d}
\end{align*}
$$

Let us exhibit the independent information in (18) and (19). Equations (18a) and (19a) are the same equation

$$
\begin{equation*}
\widetilde{J}_{0}=A_{0}+A_{0}^{\prime}, \tag{20}
\end{equation*}
$$

and (18c) and (19c) give

$$
\begin{equation*}
\widetilde{J}_{\mathrm{k}}=\mathrm{A}_{\mathrm{k}}=\mathrm{A}_{\mathrm{k}}^{\prime} \tag{21}
\end{equation*}
$$

The difference of (18b) and (19b), using (20) and (21), simply gives current conservation

$$
\begin{equation*}
0=-\left(q^{1}-\underline{q}\right) \cdot \tilde{I_{m}}-\Delta_{0} \widetilde{J}_{0}, \tag{22}
\end{equation*}
$$

whereas the sum gives

$$
\begin{equation*}
\mathrm{O}=2 \mathrm{~B}_{0}+2 \mathrm{~B}_{0}^{\prime}-\left(\underline{q}^{\prime}+\underline{q}\right) \cdot \tilde{J}-\Delta_{0}\left(\mathrm{~A}_{0}+3 \mathrm{~A}_{0}^{\prime}\right) \tag{23}
\end{equation*}
$$

Finally, the difference of (18d) and (19d) gives

$$
\begin{equation*}
\mathrm{O}=\mathrm{B}_{\mathrm{k}}^{\mathrm{t}}+\mathrm{q}_{\mathrm{k}} \mathrm{~A}_{0}-2 \Delta_{0} A_{\mathrm{k}}^{\mathrm{t}}, \tag{24}
\end{equation*}
$$

and the sum gives

$$
\begin{equation*}
\mathrm{O}=\mathrm{B}_{\mathrm{k}}+\mathrm{q}_{\mathrm{k}}^{\prime} \mathrm{A}_{0} \tag{25}
\end{equation*}
$$

Equations (20) - (25) are the consequences of (14) - (17). Note that they do not uniquely determine the sixteen quantities $A_{\mu}, A_{\mu}^{\prime}, B_{\mu}, B_{\mu}^{\prime}$.

If we now substitute (16) and (17) into (13) and use (20) - (25), we find

$$
\begin{align*}
& \mathrm{T}_{00}=\omega^{-1} \widetilde{J}_{0}+\omega^{-2} q \cdot \widetilde{J}+O\left(\omega^{-3}\right)  \tag{26a}\\
& \mathrm{T}_{0 \mathrm{k}}=\omega^{-1} \widetilde{J}_{k}+O\left(\omega^{-3}\right)  \tag{26b}\\
& \mathrm{T}_{\mathrm{k} 0}=\omega^{-1} \widetilde{J}_{k}+\omega^{-2} \Delta_{0} \widetilde{J}_{k}+O\left(\omega^{-3}\right)  \tag{26c}\\
& \mathrm{T}_{\mathrm{k} \ell}=\omega^{-2}\left(\mathrm{q}_{\mathrm{k}} \widetilde{J}_{\ell}+\mathrm{q}_{\ell}^{\prime} \widetilde{J}_{k}\right)+O\left(\omega^{-3}\right) \tag{26d}
\end{align*}
$$

Thus, as claimed, the conditions (10) - (12) uniquely determine the minimal algebra (to within c-numbers). Comparison with (7) gives it to be ( $\mathrm{x}_{0}=\mathrm{y}_{0}$ )

$$
\begin{align*}
& {\left[J_{0}^{a}(x), J_{\mu}^{\mathrm{b}}(\mathrm{y})\right]_{\mathrm{T}}=\mathrm{i} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{\mu}^{\mathrm{c}}(\mathrm{x}) \delta(\underline{x}-\mathrm{y})}  \tag{27a}\\
& {\left[\mathrm{J}_{\mathrm{k}}^{\mathrm{a}}(\mathrm{x}), \mathrm{J}_{\ell}^{\mathrm{b}}(\mathrm{y})\right]_{\mathrm{T}}=0}  \tag{27b}\\
& {\left[\dot{J}_{0}^{\mathrm{a}}(\mathrm{x}), \mathrm{J}_{\mu}^{\mathrm{b}}(\mathrm{y})\right]_{\mathrm{T}}=\mathrm{i} \mathrm{~g}_{\mu 0} \mathrm{f}^{\mathrm{abc}} J_{\mathrm{i}}^{\mathrm{c}}(\mathrm{y}) \frac{\partial}{\partial \mathrm{x}_{\mathrm{i}}} \delta(\underset{\sim}{\mathrm{x}}-\mathrm{y})}  \tag{27c}\\
& {\left[\dot{J}_{k}^{a}(x), J_{0}^{b}(y)\right]_{T}=i f^{a b c} \dot{J}_{k}^{c}(x) \delta(x-y)}  \tag{27d}\\
& {\left[\dot{J}_{k}^{a}(x), J_{\ell}^{\mathrm{b}}(\mathrm{y})\right]_{\mathrm{T}}=\mathrm{if} \mathrm{f}^{\mathrm{abc}}\left(J_{\mathrm{k}}^{\mathrm{c}}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{\ell}}-J_{\ell}^{\mathrm{c}}(\mathrm{y}) \frac{\partial}{\partial \mathrm{y}_{\mathrm{k}}}\right) \delta(\mathrm{x}-\mathrm{y}) \quad .} \tag{27e}
\end{align*}
$$

Here the subscript " T " refers to "truncated" commutators

$$
[\mathrm{A}, \mathrm{~B}]_{\mathrm{T}} \equiv[\mathrm{~A}, \mathrm{~B}]-\langle 0|[\mathrm{A}, \mathrm{~B}]|0\rangle .
$$

Strictly speaking, we have not yet established the existence of the minimal algebra but have only shown that, if it does exist, then it is given by (27). Thus we must show that the commutators (27) constitute (part of) a consistent algebra and that there exists a solution to Eqs. (20) - (25) such that the resulting $\mathrm{T}_{\mu \nu}$ given by (13) is an acceptable amplitude.

We first show that the latter requirement is satisfied. We shall exhibit a $\mathrm{T}_{\mu \nu}$ consistent with (10) - (12) and with the usual conditions - particularly

Lorentz covariance. To this end, we let $\overline{\mathrm{T}}_{\mu \nu}$ be an arbitrary acceptable amplitude [satisfying (11) and (12)] and consider ${ }^{14}$

$$
\begin{equation*}
\mathrm{T}_{\mu \nu}\left(\mathrm{q}, \mathrm{q}^{\prime}\right) \equiv \overline{\mathrm{T}}_{\mu \nu}\left(\mathrm{q}, \mathrm{q}^{\mathrm{p}}\right)-\left(\mathrm{g}_{\mu \alpha}-\frac{\mathrm{q}_{\mu} \mathrm{q}_{\alpha}}{2}\right)\left(\mathrm{g}_{\nu \beta}-\frac{\mathrm{q}_{\nu}^{\prime} \mathrm{q}_{\beta}^{\prime}}{\mathrm{q}^{2}}\right) \overline{\mathrm{T}}_{\alpha \beta}\left(\mathrm{q}, \mathrm{q}^{\prime}\right) \tag{28}
\end{equation*}
$$

This amplitude is Lorentz covariant, satisfies (11) and (12), and, in view of (6'), also satisfies (10). Thus a consistent solution of (20) - (25) is guaranteed to exist. To see what this solution is, we use the divergence conditions (11) and (12) to write (28) as

$$
\begin{equation*}
\mathrm{T}_{\mu \nu}\left(\mathrm{q}, \mathrm{q}^{\prime}\right)=\frac{\mathrm{q}_{\mu}}{\mathrm{q}^{2}} \widetilde{J}_{\nu}(\Delta)+\frac{\mathrm{q}_{\nu}^{\prime}}{\mathrm{q}^{\prime}} \widetilde{J}_{\mu}(\Delta)-\frac{\mathrm{q}_{\mu} \mathrm{q}_{\nu}^{\prime}}{\mathrm{q}^{2}{q^{\prime}}^{2}}\left(\frac{\mathrm{q}+\mathrm{q}^{\prime}}{2}\right) \cdot \widetilde{J}(\Delta) \tag{29}
\end{equation*}
$$

Comparison with (13), (16), and (17) now gives, for example,

$$
\begin{array}{ll}
A_{k}=A_{k}^{\prime}=\widetilde{J}_{k}, & A_{0}=A_{0}^{\prime}=\frac{1}{2} \widetilde{J}_{0}, \\
B_{k}=-\frac{1}{2} q_{k}^{\prime} \widetilde{J}_{0}, & B_{0}=\frac{1}{2}\left(\frac{q+q^{\prime}}{2}\right) \cdot \widetilde{J}-\frac{1}{4} \Delta_{0} \widetilde{J}_{0},  \tag{30}\\
B_{k}^{\prime}=-\frac{1}{2} q_{k} \widetilde{J}_{0}^{\prime}+2 \Delta_{0} \widetilde{J}_{k}, & B_{0}^{\prime}=\frac{1}{2}\left(\frac{q+q^{\prime}}{2}\right) \cdot \widetilde{J}+\frac{5}{4} \Delta_{0} \widetilde{J}_{0},
\end{array}
$$

This solution, of course, satisfies (20) - (25).
Next we consider the minimal algebra (27) itself. Equation (27a) holds in all of the usual models. Equation (27b) holds in models, such as the $\sigma$-model ${ }^{15}$ and the algebra of fields, ${ }^{16}$ in which the currents are constructed from Bose operators. Equations (27c) and (27d) follow from (27a) and (27b) together with current conservation (4). Equation (27b) implies that (27e) must be symmetric under ( $k \longrightarrow \ell$, $a \longrightarrow b, x \longrightarrow y$ ), as it is. Finally, all of the Jacobi identities are formally satisfied except the one involving $\dot{J}_{k}$, $J_{l}$, and $J_{0}$, which cannot be satisfied unless the currents vanish. The fact that this double commutator does not formally satisfy the Jacobi identity is not necessarily an inconsistency. Our derivation only implies
and requires that (27) is valid when sandwiched between physical scattering states $|p\rangle$, whereas the formulation of the Jacobi identity would require them to be valid between, say, the physical state $\left|p^{\prime}\right\rangle$ and the state $J\left(x_{0}, z\right)|p\rangle$. Thus no difficulties can arise if we only use the relations (27) between scattering states. Put differently, field theoretic equal-time commutators must be defined as equaltime limits ${ }^{17}$ and the Jacobi identity will not hold when certain of these limits can not be interchanged. We shall illustrate and discuss this further in the next section within the context of a specific model.

The relations (27) are, furthermore, consistent with our initial assumption that $T_{\mu \nu}$ has an expansion in powers of $\omega^{-1}$ to the order of $\omega^{-3}$. The non-singular nature of the right-hand sides of (27) (e.g., the absence of local operator products) suggests that the terms in (7) are well-defined so that the expansion should be valid.

We finally note that dropping the assumption $S=0$ would not change any of our results. If we add to $\mathrm{T}_{\mu \nu}$ any polynomial $\mathrm{S}_{\mu \nu}$ in $q$, than the conditions (1) $=0$, (5), and (6) require that $S_{\mu \nu}=0$.

## III. LIMITT OF MASSIVE YANG-MILLS THEORY

In this section we shall show the minimal current algebra of the previous section is the algebra corresponding to a particular formal limit of the massive Yang-Mills theory. This will shed light on both the singular aspects of the model (such as the failure of the formal Jacobi relation) and the smooth aspects (such as the good high $-q_{0}$ behavior). It will also enable the incorporation of electromagnetism, PCAC, and $\operatorname{SU}(3)$-breaking into the model. The approach is along the lines given by Bardakci, Frishman, and Halpern. ${ }^{18}$

The massive Yang-Mills ${ }^{19}$ theory is defined by the Lagrangian density

$$
\begin{equation*}
\mathrm{L}(\mathrm{x})=-\frac{1}{4} \mathrm{~F}_{\mu \nu}^{\mathrm{a}}(\mathrm{x}) \mathrm{F}_{\mu \nu}^{\mathrm{a}}(\mathrm{x})+\frac{1}{2} \mathrm{~m}_{0}^{2} \phi_{\mu}^{\mathrm{a}}(\mathrm{x}) \phi_{\mu}^{\mathrm{a}}(\mathrm{x}), \tag{31}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{F}_{\mu \nu}^{\mathrm{a}}=\partial_{\mu} \phi_{\nu}^{\mathrm{a}}-\partial_{\nu} \phi_{\mu}^{\mathrm{a}}-\frac{1}{2} \mathrm{~g}_{0} \mathrm{f}^{\mathrm{abc}}\left(\phi_{\mu}^{\mathrm{b}} \phi_{\nu}^{\mathrm{c}}+\phi_{\nu}^{\mathrm{c}} \phi_{\mu}^{\mathrm{b}}\right) \tag{32}
\end{equation*}
$$

The equations of motion are

$$
\begin{equation*}
\partial_{\mu} \mathrm{F}_{\mu \nu}^{\mathrm{a}}+\mathrm{m}_{0}^{2} \phi_{\nu}^{\mathrm{a}}=\frac{1}{2} \mathrm{~g}_{0} \mathrm{f}^{\mathrm{abc}}\left(\mathrm{~F}_{\nu \mu}^{\mathrm{b}} \phi_{\mu}^{\mathrm{c}}+\phi_{\mu}^{\mathrm{c}} \mathrm{~F}_{\nu \mu}^{\mathrm{b}}\right), \tag{33}
\end{equation*}
$$

the stress-energy tensor is

$$
\begin{equation*}
\theta_{\mu \nu}=\frac{1}{2}\left(\mathrm{~F}_{\mu \lambda}^{\mathrm{a}} \mathrm{~F}_{\lambda \nu}^{\mathrm{a}}+\mathrm{F}_{\nu \lambda}^{\mathrm{a}} \mathrm{~F}_{\lambda \mu}^{\mathrm{a}}\right)+\frac{1}{2} \mathrm{~m}_{0}^{2}\left(\phi_{\mu}^{\mathrm{a}} \phi_{\nu}^{\mathrm{a}}+\phi_{\nu}^{\mathrm{a}} \phi_{\mu}^{\mathrm{a}}\right)-\mathrm{g}_{\mu \nu} \mathrm{L}, \tag{34}
\end{equation*}
$$

and the canonical commutation rules imply ( $\mathrm{x}_{0}=\mathrm{y}_{0}$ )

$$
\begin{align*}
& {\left[\phi_{0}^{\mathrm{a}}(\mathrm{x}), \phi_{0}^{\mathrm{b}}(\mathrm{y})\right]=\mathrm{i} \lambda^{-1} \mathrm{f}^{\mathrm{abc}} \phi_{0}^{\mathrm{c}}(\mathrm{x}) \delta(\mathrm{x}-\mathrm{y})}  \tag{35a}\\
& {\left[\phi_{0}^{\mathrm{a}}(\mathrm{x}), \phi_{\mathrm{k}}^{\mathrm{b}}(\mathrm{y})\right]=\mathrm{i} \lambda^{-1} \mathrm{f}^{\mathrm{abc}} \phi_{\mathrm{k}}^{\mathrm{c}} \delta(\mathrm{x}-\mathrm{y})+\mathrm{im}_{0}^{-2} \delta^{\mathrm{ab}} \frac{\partial}{\partial \mathrm{x}_{\mathrm{k}}} \delta(\mathrm{x}-\mathrm{y})}  \tag{35b}\\
& {\left[\phi_{k}^{a}(x), \phi_{\ell}^{\mathrm{b}}(\mathrm{y})\right]=0} \\
& {\left[\partial_{0} \phi_{\mathrm{k}}^{\mathrm{a}}(\mathrm{x})-\partial_{\mathrm{k}} \phi_{0}^{\mathrm{a}}(\mathrm{x}), \phi_{\ell}^{\mathrm{b}}(\mathrm{y})\right]=\mathrm{i} \delta^{\mathrm{ab}} \mathrm{~g}_{\mathrm{k} \ell} \delta(\underline{x}-\mathrm{y})+\mathrm{i} \lambda^{-1} \mathrm{f}^{\mathrm{abc}} \phi_{\mathrm{k}}^{\mathrm{c}}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{\ell}} \delta(\underset{\sim}{\mathrm{x}}-\mathrm{y})} \\
& -\mathrm{i} \mathrm{C}^{-1} \mathrm{f}^{\text {ace }} \mathrm{f}^{\text {bde }} \phi_{\ell}^{\mathrm{c}}(\mathrm{x}) \phi_{\mathrm{k}}^{\mathrm{d}}(\mathrm{x}) \delta(\underset{\sim}{\mathrm{x}}-\underline{y}) \quad, \tag{35d}
\end{align*}
$$

where we have defined

$$
\begin{equation*}
\lambda=\frac{\mathrm{m}_{0}^{2}}{\mathrm{~g}_{0}} \quad \mathrm{C}=\frac{\mathrm{m}_{0}^{2}}{\mathrm{~g}_{0}^{2}} \tag{36}
\end{equation*}
$$

The assumption of field-current identity ${ }^{20}$ and field algebra ${ }^{16}$ is that the hadronic currents are given by

$$
\begin{equation*}
J_{\mu}^{\mathrm{a}}=\lambda \phi_{\mu}^{\mathrm{a}} \tag{37}
\end{equation*}
$$

One then has current conservation

$$
\begin{equation*}
\partial_{\mu} \mathrm{J}_{\mu}^{\mathrm{a}}=0 \tag{38}
\end{equation*}
$$

Bardakci, Frishman, and Halpern ${ }^{18}$ have shown that in the limit

$$
\begin{equation*}
\mathrm{m}_{0} \rightarrow 0, \quad \mathrm{~g}_{0} \rightarrow 0, \quad \mathrm{C}=\mathrm{const} \tag{39}
\end{equation*}
$$

the above model becomes the Sugawara ${ }^{21} \operatorname{model}\left(\mathrm{x}_{0}=\mathrm{y}_{0}\right)$ :

$$
\begin{align*}
& {\left[J_{0}^{\mathrm{a}}(\mathrm{x}), \mathrm{J}_{0}^{\mathrm{b}}(\mathrm{y})\right]=\mathrm{if} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{0}^{\mathrm{c}}(\mathrm{x}) \delta(\mathrm{x}-\mathrm{y})}  \tag{40a}\\
& {\left[J_{0}^{\mathrm{a}}(\mathrm{x}), \mathrm{J}_{\mathrm{k}}^{\mathrm{b}}(\mathrm{y})\right]=\mathrm{if} \mathrm{f}^{\mathrm{abc}} \mathrm{~J}_{\mathrm{k}}^{\mathrm{c}}(\mathrm{x}) \delta(\mathrm{x}-\mathrm{y})+\mathrm{i} \mathrm{C} \delta^{\mathrm{ab}} \frac{\partial}{\partial \mathrm{x}_{\mathrm{k}}} \delta(\underline{\mathrm{x}}-\underline{y})}  \tag{40b}\\
& {\left[J_{k}^{a}(x), J_{l}^{b}(y)\right]=0}  \tag{40c}\\
& {\left[\partial_{0} J_{k}^{a}(x)-\partial_{k} J_{0}^{a}(x), J_{\ell}^{b}(y)\right]=i f^{a b c} J_{k}^{c}(x) \frac{\partial}{\partial x_{\ell}} \delta(\underline{x}-\underline{y})-i C^{-1} f^{a c e}{ }_{f}^{b d e} J_{\ell}^{c}(x) J_{k}^{d}(x) \delta(x-y)}  \tag{40d}\\
& \partial_{\mu} \mathrm{J}_{\nu}^{\mathrm{a}}-\partial_{\nu} \mathrm{J}_{\mu}^{\mathrm{a}}=\frac{1}{2} \mathrm{C}^{-1} \mathrm{f}^{\mathrm{abc}}\left(\mathrm{~J}_{\mu}^{\mathrm{b}} \mathrm{~J}_{\nu}^{\mathrm{c}}+\mathrm{J}_{\nu}^{\mathrm{c}} \mathrm{~J}_{\mu}^{\mathrm{b}}\right)  \tag{41}\\
& \theta_{\mu \nu}=\frac{1}{2} C^{-1}\left(J_{\mu}^{\mathrm{a}} J_{\nu}^{\mathrm{a}}+J_{\nu}^{\mathrm{a}} J_{\mu}^{\mathrm{a}}-\mathrm{g}_{\mu \nu} J_{\lambda}^{\mathrm{a}} J_{\lambda}^{\mathrm{a}}\right) . \tag{42}
\end{align*}
$$

Bardakci et al. used the limiting procedure (9) to incorporate electromagnetism, PCAC, and SU(3) breaking into the model.

We shall show that in the different limit

$$
\begin{equation*}
\mathrm{m}_{0} \rightarrow 0, \quad \mathrm{~g}_{0} \rightarrow 0, \quad \lambda=\mathrm{const}, \quad \mathrm{C} \longrightarrow \infty \tag{43}
\end{equation*}
$$

the massive Yang-Mills theory yields the commutation relations of the minimal algebra. We shall (and, in fact, must) simultaneously take the divergence of local field products into account. We assume that the divergence of $\left\{\phi_{\ell}^{c}(\mathrm{x}), \phi_{\mathrm{k}}^{\mathrm{d}}(\mathrm{x})\right\}$ is mild enough so that

$$
\begin{equation*}
\mathrm{C}^{-1}\left\{\phi_{\ell}^{\mathrm{c}}(\mathrm{x}), \phi_{\mathrm{k}}^{\mathrm{d}}(\mathrm{x})\right\} \longrightarrow \mathrm{c} \text {-number } \tag{44}
\end{equation*}
$$

in the limit (43). This can be thought of as a boundary condition to be used in solving the theory. Under (43) and (44), (35) - (37) give exactly the commutation rules (27) of the minimal algebra. In addition, the c-number parts of the commutators are given and, for (35b) and possibly (35d), are infinite. This, of course, is acceptable and is exactly what happens in the free-ficld quark model. What we attempt to do is remove the q-number divergence from (35d) and have it become a c-number divergence in (35b). As we have scen in the previous section, this makes the physical properties of the theory less singular.

Let us now give a more careful discussion of our limiting procedures. We assume that, in analogy with soluble models and perturbation theory, local field products are to be defincd as suitable limits of non-local products. ${ }^{22}$ Thus the mass term in (31) bccomes

$$
\begin{equation*}
\lim _{\xi \rightarrow 0} \frac{1}{2}\left[\mathrm{~m}_{0}(\xi)\right]^{2} \phi_{\mu}^{\mathrm{a}}(\mathrm{x}+\xi) \phi_{\mu}^{\mathrm{a}}(\mathrm{x}) \tag{45}
\end{equation*}
$$

where the limit is to be taken in a spacelike direction, say $\xi=(0, \xi)$. The vanishing of $m_{0}(0)$ is supposed to cancel singularities of the local product $\phi(x) \phi(x)$. The equation of motion (33) becomes

$$
\begin{equation*}
\partial_{\mu} \mathrm{F}_{\mu \nu}^{\mathrm{a}}(\mathrm{x})=\lim _{\xi \rightarrow 0}\left\{\frac{1}{2} \mathrm{~g}_{0}(\xi) \mathrm{f}^{\mathrm{abc}}\left[\mathrm{~F}_{\nu \mu}^{\mathrm{b}}(\mathrm{x}+\xi) \phi_{\mu}^{\mathrm{c}}(\mathrm{x})+\phi_{\mu}^{\mathrm{c}}(\mathrm{x}) \mathrm{F}_{\nu \mu}^{\mathrm{b}}(\mathrm{x}+\xi)\right]-\left[\mathrm{m}_{0}(\xi)\right]^{2} \phi_{\nu}^{\mathrm{a}}(\mathrm{x})\right\} \tag{46}
\end{equation*}
$$

We only assume this relation is valid between physical scattering states. Let $J_{\mu}^{\mathrm{a}}(\mathrm{x} ; \xi)=\lambda(\xi) \phi_{\mu}^{\mathrm{a}}(\mathrm{x} ; \xi)$ be the non-local solution of the non-local theory with $\xi \neq 0$. We assume, again in analogy with soluble models and perturbation theory, that the equal-time local current commutators can be calculated as limits of commutators of the corresponding non-local currents. ${ }^{17}$ Thus we assume, for example,
that (35d) becomes ( $\mathrm{x}_{0}=\mathrm{y}_{0}$ )

$$
\begin{align*}
{\left[\partial_{0} J_{k}^{a}(x)-\partial_{k} J_{0}^{a}(x), J_{l}^{b}(y)\right]_{T} } & =\lim _{\substack{\xi \rightarrow 0 \\
\xi^{\prime} \rightarrow 0}}\left[\partial_{0} J_{k}^{a}(x ; \xi)-\partial_{k} J_{0}^{a}(x ; \xi), J_{l}^{b}\left(y ; \xi^{\prime}\right)\right] T  \tag{47}\\
& =i f^{a b c} J_{k}^{c}(x) \frac{\partial}{\partial x_{l}} \delta(x-y)-i f^{\text {ace }} f^{b d e} \lim _{\xi \rightarrow 0}\left[\mathrm{C}^{-1}(\xi) J_{l}^{c}(x+\xi) J_{k}^{d}(x)\right] \delta(x-y),
\end{align*}
$$

valid between physical states. All the equal-time commutators in the theory are to be defined in this way.

We now specialize to the case (43) and (44). We put

$$
\begin{equation*}
\mathrm{m}_{0}(\xi)=\lambda r(\xi), \quad \mathrm{g}_{0}(\xi)=[\mathrm{r}(\xi)]^{2} \tag{48}
\end{equation*}
$$

with ${ }^{23}$

$$
\begin{equation*}
\mathrm{r}(\xi) \longrightarrow 0 \quad, \quad[\mathrm{r}(\xi)]^{2} \phi_{\mu}^{\mathrm{a}}(\mathrm{x}+\xi) \phi_{\nu}^{\mathrm{b}}(\mathrm{x}) \longrightarrow \mathrm{c}-\text { number } \tag{49}
\end{equation*}
$$

so that the commutation rules (35), defined in analogy with (47), become those of the minimal algebra. The algebra is, furthermore, now guaranteed to be completely consistent, provided the limits $\xi \rightarrow 0$ are taken after all commutation. In particular, the Jacobi identity will now be satisfied. For example, whereas one has

$$
\begin{equation*}
\lim _{\xi \rightarrow 0} C^{-1}(\xi) J_{k}^{a}(y+\xi) J_{l}^{b}(x y)=C_{k l}^{a b}(y)=c \text {-number }, \tag{50}
\end{equation*}
$$

one nevertheless has ( $\mathrm{x}_{0}=\mathrm{y}_{0}$ )

$$
\begin{align*}
{\left[J_{0}^{\mathrm{c}}(\mathrm{x}), \mathrm{C}_{\mathrm{kl}}^{\mathrm{ab}}(\mathrm{y})\right]=} & \lim _{\xi \rightarrow 0}\left[\mathrm{~J}_{0}^{\mathrm{c}}(\mathrm{x}), \mathrm{C}_{\mathrm{kl}}^{\mathrm{ab}}(\mathrm{y} ; \xi)\right] \\
= & i \lim _{\xi \rightarrow 0} \mathrm{C}^{-1}(\xi)\left\{\left[\mathrm{f}^{\mathrm{cad}} J_{\mathrm{k}}^{\mathrm{d}}(\mathrm{x})+\delta^{\mathrm{ca}} \mathrm{C}(\xi) \frac{\partial}{\partial \mathrm{x}_{\mathrm{k}}}\right] J_{\ell}^{\mathrm{b}}(\mathrm{y}) \delta(\mathrm{x}-\mathrm{y}-\xi)\right. \\
& \left.+\mathrm{J}_{\mathrm{k}}^{\mathrm{a}}(\mathrm{y}+\xi)\left[\mathrm{f}^{\mathrm{cbd}} J_{\ell}^{\mathrm{d}}(\mathrm{x})+\delta^{\mathrm{cb}} \mathrm{C}(\xi) \frac{\partial}{\partial \mathrm{x}_{\ell}}\right] \delta(\mathrm{x}-\mathrm{y})\right\} \\
= & i\left[\delta^{\mathrm{ca}} \mathrm{~J}_{\ell}^{\mathrm{b}}(\mathrm{y})+\delta^{\mathrm{cb}} J_{\mathrm{k}}^{\mathrm{a}}(\mathrm{y})\right] \delta(\mathrm{x}-\mathrm{y}) \neq 0 \tag{51}
\end{align*}
$$

One can now use the method of Bardakci et al. ${ }^{18}$ to introduce electromagnetism, PCAC, and $\operatorname{SU}(3)$ breaking into the theory.

In perturbation theory ${ }^{22}$ the divergences in the local products $\phi(\mathrm{x}) \phi(\mathrm{x}) \phi(\mathrm{x})$ and $\phi(x) \partial_{\mu} \phi(x)$ will be worse than that of $\phi(x) \phi(x)$ and, in order to obtain a nontrivial theory, we assume that this is the case here. We put (as boundary condition on the solution of the theory) ${ }^{24}$

$$
\begin{align*}
& {[\mathrm{r}(\xi)]^{2} \phi_{\mu}^{\mathrm{a}}(\mathrm{x}+\xi) \phi_{\nu}^{\mathrm{b}}(\mathrm{x}) \phi_{\lambda}^{\mathrm{c}}(\mathrm{x}-\xi) \longrightarrow \phi_{\mu \nu \lambda}^{\mathrm{abc}}(\mathrm{x})}  \tag{52}\\
& {[\mathrm{r}(\xi)]^{2} \phi_{\mu}^{\mathrm{a}}(\mathrm{x}+\xi) \partial_{\nu} \phi_{\lambda}^{\mathrm{b}}(\mathrm{x}) \longrightarrow x_{\mu \nu \lambda}^{\mathrm{ab}}(\mathrm{x})} \tag{53}
\end{align*}
$$

for some local operators $\Phi, X$, and call $K_{\nu}^{\mathrm{a}}(\mathrm{x})$ the particular combination occurring in (33). Thus, in our limit, (31) - (34) become

$$
\begin{align*}
& \mathrm{L} \longrightarrow-\frac{1}{4} \mathrm{~F}_{\mu \nu}^{\mathrm{a}} \mathrm{~F}_{\mu \nu}^{\mathrm{a}}  \tag{54}\\
& \mathrm{~F}_{\mu \nu}^{\mathrm{a}} \rightarrow \partial_{\mu} \phi_{\nu}^{\mathrm{a}}-\partial_{\nu} \phi_{\mu}^{\mathrm{a}},  \tag{55}\\
& \partial_{\mu} \mathrm{F}_{\mu \nu}^{\mathrm{a}} \rightarrow \mathrm{~K}_{\nu}^{\mathrm{a}}  \tag{56}\\
& \theta_{\mu \nu} \rightarrow \frac{1}{2}\left(\mathrm{~F}_{\mu \lambda}^{\mathrm{a}} \mathrm{~F}_{\lambda \nu}^{\mathrm{a}}+\mathrm{F}_{\nu \lambda}^{\mathrm{a}} \mathrm{~F}_{\lambda \mu}^{\mathrm{a}}\right)-\mathrm{g}_{\mu \nu} \mathrm{L} \tag{57}
\end{align*}
$$

between physical scattering states. It is important to note that, for example, (55) can not be substituted into (54). One must first substitute (32) into (31) and then take the limit $\xi \rightarrow 0$ using (52) and (53).

It is interesting to note that our expressions (54) - (57) are exactly orthogonal to the Sugawara expressions obtained from (31) - (34) in the limit of Bardakci et al. In our limit (43) only the kinetic terms survive whereas in the limit (39) only the mass terms survive.

Although the minimal algebra appears to be more singular than the Yang-Mills or Sugawara theories, many aspects of it are, in fact, less singular. We must use
complicated commutator definitions such as (47) and impose boundary conditions such as (44), (52), and (53). We have, however, eliminated the singular local field products from (31) - (34) and (35d). As we have seen in Section II, this has a smoothing effect on the high-q ${ }_{0}$ properties of the theory. In effect, we have made the mathematical formalism of the theory more complicated in order that its physical consequences become less complicated.

## V. EXPERIMENTAL CONSEQUENCES

Because the minimal algebra yields smoother asymptotic behavior in $q_{0}=\omega$, except in the almost unobservable pieces proportional to $q_{\mu}$ or $q_{\nu}^{\prime}$, the high $-q^{2}$ behavior of various sum rules is weakened. Among the results are:

1. All electromagnetic mass-differences are finite to order $\alpha$. This follows from the vanishing of the q-number part of $\left[\dot{J}_{\mu}^{\mathrm{cm}}, \mathrm{J}_{\nu}^{\mathrm{cm}}\right] .{ }^{25}$
2. Asymptotic sum rules for neutrino ${ }^{2}$ (and in all likelihood inelastic electron or muon-scattering) in the backward direction ${ }^{7}$ have a vanishing right-hand side because $\left[j_{i}(x), j_{j}(0)\right]=0$ for the minimal algebra. In the case of inelastic scattering, where only inequalities exist, one cannot make a rigorous argument, because the inequality goes the wrong way. It is consistent with the minimal algebra to have a vanishing right-hand side.
3. The sum rule of Callan and Gross ${ }^{9}$

$$
\begin{equation*}
\lim _{\mathrm{q}^{2} \rightarrow \infty} \mathrm{q}^{2} \int_{0}^{\infty} \frac{\mathrm{d} \nu}{\nu} \mathrm{~W}_{2}\left(\mathrm{q}^{2}, \nu\right)=\text { constant }=\mathrm{K} \tag{58}
\end{equation*}
$$

where

$$
\begin{equation*}
\lim _{E, E^{\prime} \rightarrow \infty} \frac{\mathrm{d} \sigma(\mathrm{ep} \rightarrow \text { hadrons })}{\mathrm{dq}^{2} d E^{\prime}}=\frac{4 \pi \alpha^{2}}{q^{4}} \mathrm{~W}_{2}\left(\mathrm{q}^{2}, \mathrm{E}-\mathrm{E}^{\prime}\right) \tag{59}
\end{equation*}
$$

(generalized to cases where $\left[\partial_{0} j_{i}(x), j_{j}(0)\right]$ has finite matrix elements between nucleon states) in the minimal algebra has a vanishing right-hand side $K=0$, because to this order $\left(\omega^{-2}, m_{\ell}=0\right)$, all observable consequences have been obliterated.
4. Similar statements hold for neutrino and antineutrino processes, e.g.,

$$
\begin{equation*}
\lim _{\mathrm{q}^{2} \rightarrow \infty} \lim _{\mathrm{E} \rightarrow \infty} \mathrm{q}^{2} \int_{0}^{\infty} \frac{\mathrm{d} \nu}{\nu}\left[\frac{\mathrm{~d} \sigma(\bar{\nu} \mathrm{p} \rightarrow \text { hadrons })}{\mathrm{dq}^{2} \mathrm{~d} \nu}\right]=0 \tag{60}
\end{equation*}
$$

5. No statement can be made on the validity of the Fubini-Dashen-Gell-Mann sum rule with the minimal algebra alone. Writing, for the special case of spin zero matrix elements

$$
\begin{equation*}
\mathrm{M}_{\mu \nu}=\mathrm{P}_{\mu} \mathrm{P}_{\nu} \mathrm{A}_{1}\left(\nu, \mathrm{t}, \mathrm{q}^{2}, \mathrm{q}^{\prime 2}\right)+\ldots \tag{61}
\end{equation*}
$$

the Fubini-Dashen-Gell-Mann sum rule is

$$
\begin{equation*}
\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \nu \operatorname{Im} \mathrm{~A}_{1}\left(\nu, \mathrm{t}, \mathrm{q}^{2}, \mathrm{q}^{\prime 2}\right)=\mathrm{F}(\mathrm{t}) \tag{62}
\end{equation*}
$$

while from the minimal algebra

$$
\begin{equation*}
\lim _{q^{2}, q^{\prime 2} \rightarrow \infty} q^{2} \int \frac{d \nu}{\nu} \operatorname{Im} A_{1}\left(\nu, t, q^{2}, q^{2}\right)=0 \tag{63}
\end{equation*}
$$

What can be said is that the value of $\nu$ needed to saturate this sum rule grows more rapidly than linearly with $q^{2}$ in the "minimal algebra," contrary to what has been sometimes assumed in the literature.
6. The minimal algebra implies that the Weinberg ${ }^{26}$ sum rules are valid. In fact, the absence of $J^{2}$ terms in the $[\dot{J}, J]$ commutator allows the second sum rule to be derived without invoking special limiting processes.
7. The failure of the minimal algebra to satisfy the Jacobi identity suggests that some of the vacuum expectation values of the commutators are divergent. This was the case, for example, when the algebra was obtained as a limit of the massive Yang-Mills theory. These divergences themselves have experimental
implications. The Dooher ${ }^{27}$ relation,

$$
\begin{equation*}
0=\lim _{E \rightarrow \infty} E^{4} \log E \sigma_{\text {tot }}(E) \tag{64}
\end{equation*}
$$

for example, should no longer hold.
In conclusion, we wish to emphasize that experiments can test the speculations in this paper. Perhaps the most conclusive test is the behavior of the Callan-Gross integral Eq. (58). For it to vanish in the limit is not a consequence of field algebra or most conventional models.

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12. Our metric is $(1,-1,-1,-1) ; \mu, \nu, \lambda=0,1,2,3 ; k, \ell=1,2,3 ; a, b, c$ refer to internal indices, for example $\mathrm{a}=1-16$ for $\mathrm{SU}(3) \otimes \mathrm{SU}(3)$. For typographical convenience, we neglect to raise and lower indices when invoking the summation convention.
13. To the extent that the divergences of non-conser ved currents are operators whose matrix elements are damped at large $q^{2}$ (generalized PCAC hypothesis), our results can be extended to include non-conserved currents.
14. To the order $\omega^{-2}$ of interest, the $q^{2}$ denominators in (28) can be replaced by $q^{2}-m^{2}+i \epsilon$. The $+i \epsilon$ should, in any case, be present in order to maintain the causal structure of $T$.
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22. See, for example, R. A. Brandt, Ann. Phys. (N.Y.) 44, 221 (1967); and
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23. An example is the free scalar field $\phi(\mathrm{x})$ which satisfies $\xi^{2} \phi(\mathrm{x}+\xi) \phi(\mathrm{x}) \rightarrow-\frac{1}{4 \pi^{2}}$.
24. For the example of footnote 23 , one has $\xi^{2} \phi(x+\xi) \phi(x) \phi(x-\xi) \cdots-\frac{3}{4 \pi^{2}} \phi(x)$.
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FIGURE 1. Scattering of a current from a hadron.


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