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CURRENT COMMUTATORS AND ELECTRON SCATTERING AT HIGH MOMENTUM TRANSFER*

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ABSTRACT

Sum rules for electron proton total cross sections are deduced from the vanishing of various equal time commutators; it is shown how these cross sections determine the (spin-averaged) proton expectation value of all equal time commutators of components of the electric current and time derivatives thereof.

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Electron scattering on a proton at rest is described by the cross section (electron mass = 0)

$$\frac{d^2\sigma^{ep}}{dq^2 d\nu} = \frac{\alpha^2}{M^2E^2}\,\frac{1}{q^2}\left[\left(2M^2EE' - \frac{q^2}{2}\,M^2\right) A_1(q^2,\nu) + \nu^2 A_1(q^2,\nu) - q^2 A_2(q^2,\nu)\right] , \tag{1}$$

where

$\alpha = e^2/4\pi = (137)^{-1}$

$E(E')$ = initial (final) electron energy

$q_\mu = e_\mu - e'_\mu$ = momentum imparted to proton

$\nu = (E - E')M = -q \cdot p$ (p = proton momentum),

and the A_i are the absorbtive parts of the forward, off-shell Compton amplitudes F_i defined by

$$T_{\mu\nu}(q,p) = i\int d_4x\, e^{-i\underline{q}\cdot\underline{x}} \langle p|T\left(j_\mu(x)\, j_\nu(0)\right)|p\rangle_c \tag{2a}$$

$$= \left[q^2 p_\mu p_\nu + \nu(q_\mu p_\nu + q_\nu p_\mu) + \nu^2\,\delta_{\mu\nu}\right] F_1 + (q_\mu q_\nu - q^2\,\delta_{\mu\nu}) F_2 , \tag{2b}$$

where j_μ is the electric current, and a spin average is implicit. The subscript "c" indicates the covariant time ordered product; thus $T_{\mu\nu}$ differs from the ordinary time ordered product by a polynomial in q if the equal-time commutator of j_0 and j_i has a connected matrix element.

Bjorken[1] has pointed out that for large q_0 and fixed $\underline{q}$ the coefficient of $q_0^{-\ell-1}$ ($\ell \geq 0$) in an expansion of $T_{\mu\nu}$ gives the matrix element of the equal-time commutator of the electric current and its ℓ^{th} time derivative; in particular

$$T_{\mu\nu} \xrightarrow{q_0 \to \infty} \text{polynomial in } q_0$$

$$-\sum_{\ell=0}^{\infty} \frac{(-i)^{\ell}}{q_0^{\ell+1}} \int d\underline{x}\, e^{-i\underline{q}\cdot\underline{x}} \langle p \left| \left[\partial_0^{\ell}\, j_{\mu}(\underline{x}, 0),\ j_{\nu}(0) \right] \right| p \rangle \quad . \tag{3}$$

Thus, all of these commutators are determined by the coefficients $C_i^{\ell}(\underline{q}\cdot\underline{p}, \underline{q}^2, p_0)$ occurring in

$$F_i \xrightarrow{q_0 \to \infty} \text{polynomial} + \sum_{\ell=0}^{\infty} C_i^{\ell}\, q_0^{-\ell-1} \quad . \tag{4}$$

We wish to show how the C_i^{ℓ} can be constructed from the A_i, and thus from the electron proton scattering data.

We assume that each of the $F_i(q^2, \nu)$ satisfy the D. G. S. representation[2, 3]

$$F_i = \sum_{m=0}^{M_i} F_i^m = \frac{1}{\pi} \sum_{m=0}^{M_i} \int_0^{\infty} d\sigma \int_{-1}^{1} d\beta\, \frac{\nu^m h_i^m(\sigma, \beta)}{q^2 + 2\beta\nu + \sigma} \quad , \tag{5}$$

where by crossing symmetry

$$h_i^m(\sigma, -\beta) = (-1)^m h_i^m(\sigma, \beta) \quad . \tag{6}$$

Expanding this form of F_i as in Eq. (4), we obtain after some combinatorics

$$C_i^{\ell} = \sum_{n,s,t} (-\underline{q}\cdot\underline{p})^{\ell-1-2s+2n} (\underline{q}^2)^{s-2n-t} (p_0)^{2s-\ell+1} \frac{s!\ (2n)!}{(2s-\ell+1)!\ (\ell-1-2s+2n)!\ (s-2n-t)!}\ K_i^{nt} \ , \tag{7a}$$

where

$$K_i^{nt} = \sum_{m=0}^{M_i} \frac{1}{(2n-m)!\,(t+m)!} \int_0^{\infty} d\sigma \int_{-1}^{1} d\beta\, (2\beta)^{2n-m}\, \sigma^{t+m}\, h_i^m(\sigma,\beta), \tag{7b}$$

and where the sum over n, s and t is restricted by $n \geq 0$, $s \geq 0$ and the requirement that the arguments of all the factorials are non-negative. Thus, for example, $t \geq$ maximum $(-m, -2n)$.

The absorbtive parts A_i of the F_i can be read off from Eq. (5).

$$A_i = \sum_{m=0}^{M_i} A_i^m = \sum_{m=0}^{M_i} \int d\sigma\, d\beta\, \nu^m\, h_i^m\, \delta(q^2 + 2\beta\nu + \sigma) \quad , \tag{8}$$

and hence for $n = 1, 2, 3, \ldots$

$$\int_0^{\infty} \frac{d\nu}{\nu^{2n+1}}\, A_i = \sum_{\substack{m=0 \\ 2n-m \geq 0}}^{M_i} (-1)^m \int_0^{\infty} d\sigma \int_{-1}^{0} d\beta\, \frac{(2\beta)^{2n-m}}{(q^2+\sigma)^{2n+1-m}}\, h_i^m \quad . \tag{9}$$

As indicated, the sum on m includes only terms for which $2n-m \geq 0$.[4] We will show in a moment that the R. H. S. of Eq. (9) can be expanded for large q^2 in terms of the K_i^{nt} occurring in Eq. (7). However, to complete this connection, we first must extend Eq. (9) to n=0.

Assuming Regge asymptotic behavior of the A_i for large ν, we guess that[5]

$$A_1(q^2,\nu) \xrightarrow{\nu\to\infty} 0 \tag{10a}$$

$$A_2(q^2,\nu) \xrightarrow{\nu\to\infty} f_p(q^2)\,\nu + \sum_{\alpha} f_\alpha(q^2)\,\nu^\alpha \quad , \tag{10b}$$

where, in addition to the Pomeranchon, the sum on α includes whatever other trajectories contribute with $0 < \alpha < 1$; for example, the A_2 and f_0. Because of Eq. (10a) the integral on the L. H. S. of Eq. (9) exists for A_1 when $n = 0$,[6] and Eq. (9) holds for this case. However, for A_2 the L. H. S. of Eq. (9) must be modified for $n = 0$.

As suggested by the form of Eq. (5), we assume that the Regge limit in Eq. (10b) arises only from the terms with $m \geq 1$ in Eq. (8). That is, we assume that (10b) is satisfied with A_2 replaced by $A_2 - A_2^0$. Since $F_2 - F_2^0$ vanishes at $\nu = 0$, a subtracted dispersion relation for this difference reads

$$F_2 - F_2^0 = \frac{2\nu^2}{\pi} \int_0^\infty \frac{d\nu'}{\nu'(\nu'^2 - \nu^2)} \left(A_2(q^2, \nu') - A_2^0(q^2, \nu')\right) \quad , \tag{11}$$

and if we add and subtract the R. H. S. of (10b) to the integrand of this expression, and in the first case do the integral explicitly, we obtain

$$\begin{aligned} F_2 - F_2^0 = {} & i f_p \nu - \sum_\alpha \frac{(1 + e^{-i\pi\alpha})}{\sin \pi\alpha} f_\alpha \nu^\alpha \\ & + \frac{2\nu^2}{\pi} \int_0^\infty \frac{d\nu'}{\nu'(\nu'^2 - \nu^2)} \left[A_2 - A_2^0 - f_p \nu' - \sum_\alpha f_\alpha \nu'^\alpha\right] . \end{aligned} \tag{12}$$

Let us now assume that the behavior of F_2 for large ν is given entirely by the Regge terms, namely, that there is no part of F_2 constant in ν in this limit. Then, since both F_2^0 and the bracket in the integrand of Eq. (12) vanish as $\nu \to \infty$, it follows that[7]

$$\int_0^\infty \frac{d\nu}{\nu} \left[A_2 - A_2^0 - f_p \nu - \sum_\alpha f_\alpha \nu^\alpha\right] = 0 \quad , \tag{13}$$

which from (8) gives

$$\int_0^\infty \frac{d\nu}{\nu}\left[A_2 - f_p\nu - \sum_\alpha f_\alpha \nu^\alpha\right] = \int_0^\infty d\sigma \int_{-1}^0 d\beta \, \frac{h_2^0(\sigma,\beta)}{q^2+\sigma} \quad . \tag{14}$$

The R. H. S. of this expression is the same as the R. H. S. of (9) with $n = 0$ and $i = 2$. Thus Eq. (9) can be extended to $n = 0$: with no change in form for $i = 1$, and with (9) replaced by (14) for $i = 2$. Summarizing this result, and expanding the R. H. S. of (9) in a power series in $(q^2)^{-1}$, we obtain for $n = 0, 1, 2, \ldots$

$$\int_0^\infty \frac{d\nu}{\nu^{2n+1}}\left[A_i - \delta_{i2}\,\delta_{n0}\left(f_p\nu + \sum_\alpha f_\alpha \nu^\alpha\right)\right] = \frac{1}{2}\sum_t (-1)^t \, \frac{(2n+t)!}{(q^2)^{2n+1+t}}\, K_i^{nt} \,, \tag{15}$$

with K_i^{nt} given in (7b).

The connection between the commutators in Eq. (3) and the integrals for $n \geq 0$ on the L. H. S. of (15) can be read off from Eqs. (2, 3, 4, 7) and (15). In principle, the integrals in Eq. (15), and thus the K_i^{nt}, can be determined from the electron scattering data, and from these results the C_i^ℓ and the matrix elements of all commutators in Eq. (3) can be constructed.

Rather than pursue the connection between commutators and the integrals in (15) in more detail, let us present some restrictions on the electron scattering cross sections which follow from the vanishing of various equal-time commutators. These restrictions can be derived straight forwardly from our previous results. Actually there are many more restrictions (involving higher values of n in (15)) than the ones we list; however, all of these others are implied from the relations that we write explicitly because of the conditions

$$0 \leq A_1 \tag{16a}$$

$$-M^2 A_1 \leq A_2 \leq \nu^2/q^2 \, A_1 \quad . \tag{16b}$$

These inequalities follow from the definitions in Eq. (2), or equivalently, from the requirement of non-negative cross sections for both transverse and longitudinally polarized photons.[8] Since $2\nu > q^2$ when $A_i \neq 0$ in the integrand of (15), these inequalities are useful in allowing us to conclude that if

$$(q^2)^p \int \frac{d\nu}{\nu^{2n+1}} A_1 \xrightarrow{q^2 \to \infty} 0 \quad , \tag{17a}$$

then

$$(q^2)^{p+2} \int \frac{d\nu}{\nu^{2n+3}} A_1 \xrightarrow{q^2 \to \infty} 0 \tag{17b}$$

and

$$(q^2)^{p+1} \int \frac{d\nu}{\nu^{2n+3}} A_2 \xrightarrow{q^2 \to \infty} 0 \quad . \tag{17c}$$

The restrictions listed below can be extended to larger values of n by Eqs. (17). They are similar in form, and also in their origin, to relations discussed recently by Bander and Bjorken.[9]

(i) <u>if $\langle p | [j_0(\underline{x}), j_0(0)] | p \rangle = 0$</u>, then for $i = 1, 2$

$$q^2 \int \frac{d\nu}{\nu^3} A_i(q^2, \nu) \xrightarrow{q^2 \to \infty} 0 \tag{18}$$

(ii) <u>if $\langle p | [j_0(\underline{x}), j_i(0)] | p \rangle = 0$</u>, then for $i = 1, 2$

$$q^2 \int_0^\infty \frac{d\nu}{\nu} \left[A_i - \delta_{i2} \left(f_p \nu + \sum_\alpha f_\alpha \nu^\alpha\right)\right] \xrightarrow{q^2 \to \infty} 0 \tag{19}$$

(iii) <u>if $\langle p | [j_i(\underline{x}), j_k(0)] | p \rangle = 0$</u>, then there are no further restrictions if (i) and (ii) hold.

(iv) <u>if $\langle p | [\partial_0 j_i(\underline{x}), j_k(0)] | p \rangle = 0$</u>, as well as (i) and (ii), then for $i = 1,2$

$$(q^2)^2 \int_0^\infty \frac{d\nu}{\nu} \left[A_i - \delta_{i2} \left(f_p \nu + \sum_\alpha f_\alpha \nu^\alpha\right)\right] \xrightarrow{q^2 \to \infty} 0 \quad . \tag{20}$$

The conditions (18) - (20) for A_1 can be expressed simply in terms of high energy, high momentum transfer sum rules for the electron scattering cross section in Eq. (1). The amplitude A_2 makes its presence felt for backward scattering angles[10] and is relatively more difficult to extract from the data. For A_1

$$\int \frac{d\nu}{\nu} \frac{d^2\sigma^{ep}}{dq^2 d\nu} \xrightarrow{E\to\infty} \frac{2\alpha^2}{q^2} \int_0^\infty \frac{d\nu}{\nu} \; A_1 + 0\;(1/E) \quad . \tag{21}$$

Thus, for example, if the commutators (i) - (iv) are all satisfied, as has been suggested,[11] then it follows from (20) and (21) that

$$\lim_{E\to\infty} (q^2)^3 \int \frac{d\nu}{\nu} \frac{d^2\sigma^{ep}}{dq^2 d\nu} \xrightarrow{q^2\to\infty} 0 \quad . \tag{22}$$

If only (i) - (iii) are valid, we would expect the R. H. S. of (22) to be a (non-zero) constant. In fact, for the Sugawara model,[12] this constant can (almost) be calculated exactly.[13]

To give some perspective to these results, let us compare them to the inequalities derived previously by Bjorken.[10,14] The first inequality, derived from isospin manipulations on Adler's sum rule, reads (for all q^2)

$$q^2 \int_0^\infty d\nu \;\; A_{1\,(\mathrm{isoscalar})} \geq \frac{\pi}{2} \quad . \tag{23}$$

The second inequality, based upon quark commutation rules for the space components of the isospin current, is

$$q^2 \int_0^\infty d\nu \left(A_1 - \frac{q^2}{\nu^2} \;\; A_2 \right)_{\mathrm{isoscalar}} \xrightarrow{q^2\to\infty} \geq \frac{\pi}{2}$$

It is clear that we have been optimistic in assuming the existence of all moments of the h_i^m in Eq. (7b). It is a likely possibility that for t greater than some value the K_i^{nt} do not exist; that is, essentially, that for high order derivatives, the commutators in Eq. (3) are not defined. An interesting possibility[16] is that the integrals in Eq. (15) $\sim \exp\left[-\sqrt{q^2}\right]$. The K_i^{nt} determined from scattering data according to Eq. (15) would then all be zero, and the connection given in (7) between the commutators and the K_i^{nt} would break down.[16] Except for this kind of occurrence, the conditions listed above as (i) - (iv) are reversible; that is, if they are satisfied, then the corresponding commutators vanish.

Finally, we remark that radiative corrections and/or multiple photon exchange would tend to decrease the significance of our results.

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3. There also could be a polynomial in q^2 in the numerator of Eq. (5), but it could be eliminated in favor of polynomials in ν and in $q^2 + 2\beta\nu + \sigma$. The latter contribute neither to the commutators nor to the absorbtive parts and hence are ignored.

4. To show this: a) write a twice subtracted dispersion relation for F_i^m $(m > 2n \geq 2)$ noting that both subtraction constants vanish, since $F_i^m \rightarrow \nu^m$ for small ν; b) expand the result for small ν and set the coefficient of ν^{2n} $(2n < m)$ equal to zero.

5. H. Harari, Phys. Rev. Letters 17, 1303 (1966).

6. For A_1^m, $n = 0$ and $m = 1$, the argument of footnote 4 restricting $2n - m \geq 0$ is applicable if an unsubtracted dispersion relation is used.

7. Similar relations are the basis of the finite energy sum rules. See R. Dolen, D. Horn, and S. Schmid, Phys. Rev. Letters 19, 402 (1967), Phys. Rev. 166, 1765 (1967; L.A.P. Balazs and J. M. Cornwall, Phys. Rev. 160, 1313 (1967).

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