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# THE ELECTROMAGNETIC INTERACTIONS OF LOOSELY BOUND COMPOSITE SYSTEMS†


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## ABSTRACT

Contrary to popular assumption, the interaction of a composite system with an external electromagnetic field is not equal to the sum of the individual Foldy-Wouthuysen interactions of the constituents if the constituents have spin. We give the correct interaction, and note that it is consistent with the Drell-Hearn-Gerasimov sum rule and the low energy theorem for Compton scattering. We also discuss the validity of additivity of the individual Dirac interactions, and the corrections to this approximation, with particular reference to the atomic Zeeman effect, which is of importance in the fine structure and Lamb shift measurements.



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It has been assumed almost universally[1,2] in the literature of atomic and nuclear physics that the interaction of a loosely bound composite system with an external electromagnetic field is given by the sum of the Foldy-Wouthuysen (F-W) interactions of the constituents. We have found, on the contrary, that additivity of the individual F-W interactions is incorrect even in order $1/m^2$ if the constituents have spin. If one uses such an additive F-W Hamiltonian, one finds that the Drell-Hearn[3]-Gerasimov[4] (DHG) sum rule[5] and the low energy theorem for Compton scattering[6] on the composite system[7] are violated. The crucial error in deriving[1] F-W additivity is in neglecting the spin transformation of the composite state wavefunction associated with the center of mass (CM) motion.

The correct non-relativistic reduction of the interaction Hamiltonian for a composite system of two spin $\frac{1}{2}$ particles in an external electromagnetic field takes the following form[8]

$$H_{NR}^{em} = \sum_{s=a,b} \left[ \frac{-\vec{p}_s \cdot e_s \vec{A}_s}{m_s} + \frac{e_s^2 \vec{A}_s^2}{2m_s} + e_s A_s^o - \mu_s \vec{\sigma}_s \cdot \vec{B}_s - \left(2\mu_s - \frac{e_s}{2m_s}\right) \vec{\sigma}_s \cdot \vec{E}_s \times \frac{(\vec{p}_s - e_s \vec{A}_s)}{2m_s} \right]$$
$$+ \frac{1}{4M_T} \left( \frac{\vec{\sigma}_a}{m_a} - \frac{\vec{\sigma}_b}{m_b} \right) \cdot \left[ e_b \vec{E}_b \times (\vec{p}_a - e_a \vec{A}_a) - e_a \vec{E}_a \times (\vec{p}_b - e_b \vec{A}_b) \right] + 0(1/m^3). \tag{1}$$

The terms proportional to $(M_T m_a)^{-1}$ or $(M_T m_b)^{-1}$ are correction terms to F-W additivity. For a uniform electric field, the "spin-orbit" terms combine to

$$\left[ \left( \frac{e_T}{2M_T} - 2\mu_a \right) \frac{\vec{\sigma}_a}{2M_T} + (a \longleftrightarrow b) \right] \cdot \vec{E} \times \vec{P} + \left[ \left( \frac{e_a}{2m_a} + \frac{e_T}{2M_T} - 2\mu_a \right) \frac{\vec{\sigma}_a}{2m_a} - (a \longleftrightarrow b) \right] \cdot \vec{E} \times \vec{p}. \tag{2}$$

The presence of these "spin-orbit" terms is essential in obtaining the correct low energy limit of the Compton scattering amplitude and the DHG sum rule.

The calculation of Barton and Dombey[5], which purports to show that if the DHG sum rule holds for nucleons, it must fail for bound states containing a nucleon, was based on the assumption that $H_{NR}^{em}$ equals the sum of the F-W interactions of the constituent particles. If it had been correct, this calculation would have proved that there is an additive constant, sometimes called a "subtraction at $\infty$", present[9] in the dispersion relation for the spin flip forward Compton amplitude $f_2$ for a composite system, even if it is not for the constituents. Such a state of affairs would be physically most unreasonable, since a "subtraction at $\infty$" is associated with the asymptotic behavior of $f_2(\omega)$ for $|\omega| \to \infty$, and the asymptotic behavior of the Compton amplitude for the composite system should be no worse than that of the sum of the amplitudes of the constituents.

With the inclusion of the terms arising from the spin transformation of the wavefunction, we are able to verify explicitly both the DHG sum rule and the low energy theorem for Compton scattering[10]. Thus we have shown that there is nothing in the treatment of loosely bound composite systems which introduces into the dispersion relation an additive constant. After our calculations were completed, we learned that the DHG sum rule and the threshold theorem for Compton scattering have also been verified independently by H. Osborn[11], using different methods.

Let us now trace the origin of the correct spin-orbit terms. Since momentum is transferred, the matrix element of the external potential requires knowledge of the bound state wavefunction at different total momenta. As is well known, the wavefunction for a moving system is determined from the CM wavefunction by application of the Lorentz boost operator. For the homogeneous Lorentz transformation $x' = \Lambda x, (E, \vec{P}) = \Lambda(\mathcal{M}, \vec{0})$, corresponding to a boost of a two fermion bound state (of mass $\mathcal{M}$) to velocity $\vec{V} = \vec{P}/E$, the required transformation law for the corresponding solution of the covariant Bethe-Salpeter[12] equation is

$$\chi^{\alpha'\beta'}_{E,\vec{P}}\ (x'_a, x'_b) = S_a^{\alpha'\alpha}(\Lambda)\ S_b^{\beta'\beta}(\Lambda)\ \chi^{\alpha\beta}_{\mathcal{m},\vec{0}}\ (x_a, x_b). \tag{3}$$

The spin $\frac{1}{2}$ transformation matrix is

$$\begin{aligned} S_a(\Lambda) &= \exp\left(\tfrac{1}{2}\ \vec{\alpha}_a \cdot \hat{V} \tanh^{-1} V\right) \\ &= \sqrt{\frac{E+\mathcal{m}}{2\mathcal{m}}}\left(1 + \frac{\vec{\alpha}_a \cdot \vec{P}}{E+\mathcal{m}}\right). \end{aligned} \tag{4}$$

Thus if the bound state wavefunction in the CM has the Dirac structure[13]

$$\begin{pmatrix} 1 \\ \dfrac{1}{2m_a+k_a}\ \vec{\sigma}_a \cdot \vec{p} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \dfrac{1}{2m_b+k_b}\ \vec{\sigma}_b \cdot (-\vec{p}) \end{pmatrix} \tag{5}$$

then, as shown by McGee[14] for the case of an unbound system, the wavefunction in the moving frame must have the structure

$$\begin{pmatrix} 1 + \dfrac{\vec{\sigma}_a \cdot \vec{P}}{\mathcal{m}+E}\ \dfrac{\vec{\sigma}_a \cdot \vec{p}}{2m_a+k_a} \\[2ex] \vec{\sigma}_a \cdot \left(\dfrac{\vec{P}}{\mathcal{m}+E} + \dfrac{\vec{p}}{2m_a+k_a}\right) \end{pmatrix} \otimes \begin{pmatrix} 1 - \dfrac{\vec{\sigma}_b \cdot \vec{P}}{\mathcal{m}+E}\ \dfrac{\vec{\sigma}_b \cdot \vec{p}}{2m_b+k_b} \\[2ex] \vec{\sigma}_b \cdot \left(\dfrac{\vec{P}}{\mathcal{m}+E} - \dfrac{\vec{p}}{2m_b+k_b}\right) \end{pmatrix}. \tag{6}$$

Wave packet representations of the moving composite system can be constructed from a superposition of such wavefunctions. The possibly unexpected feature of (6) is the appearance of extra terms in the large components. Physically, they correspond to the fact that a spin triplet wavefunction in the CM frame appears partially as a spin singlet in the moving frame[14]. It is just these terms which are ignored in the usual F-W analysis.

It is worth noting that there is nothing wrong in principle in using the F-W transformation to eliminate "odd" operators in the relativistic Hamiltonian. What is incorrect is to assume that this reduces the bound state wavefunction to a simple Pauli form. Inserting a F-W unitary operator U in the matrix element, one obtains

$$< f, P_f|H|i, P_i > = < f, P_f|U^{-1}(UHU^{-1})U|i, P_i > = < f, \vec{0}|S(P_f)^{\dagger}U^{-1}H_{FW}US(P_i)|i, \vec{0} > . \quad (7)$$

The presence of the Lorentz boost operator S introduces the extra terms into the matrix element which appear in (1). These terms can also be obtained by the usual large component reduction method.

As we demonstrate in ref. 10, one can in fact avoid entirely the use of a non-relativistic interaction in calculating such expressions as the DHG integral. By a judicious use of such identities as $\vec{\alpha} = i[H_0, \vec{r}]$ , where $H_0$, the free Hamiltonian, has the form

$$H_0 = \vec{\alpha}_a \cdot \vec{p}_a + \beta_a m_a + \vec{\alpha}_b \cdot \vec{p}_b + \beta_b m_b + U(\vec{r}_a - \vec{r}_b) , \quad (8)$$

and a proper treatment of the Lorentz transformation of the spins, one can reduce the integral in the DHG sum rule to a form in which the superconvergent nature of the sum rule is especially clear.

Implicit in our derivation of the non-relativistic interaction Hamiltonian Eq. (1) is the assumption that the relativistic interaction is equal to the sum of the Dirac interactions of the constituents ("impulse approximation"). We have examined the validity of this approximation. Starting from Lagrangian field theory, and expressing the matrix elements of the current in terms of Bethe-Salpeter (BS) amplitudes[15], we find that when the BS interaction kernel is replaced by a neutral instantaneous kernel (i. e., potential) in ladder approximation, and the free negative energy components in the bound state wavefunction are neglected, the "impulse approximation" interaction Hamiltonian emerges. For the instantaneous kernel, it is also possible to derive an extended form of Salpeter's[16] equation which includes interactions with an external static or adiabatic field to all orders in perturbation theory.

Using these procedures, the corrections to the impulse approximation can then be readily traced. We have applied these results to the analysis of the Zeeman spectrum in hydrogen-like atoms, in order to obtain estimates of radiative and reduced mass corrections not already included in standard calculations[17]. We emphasize that the comparison of theory with experimental measurements of the Lamb shift and fine structure intervals in H and D require a precise theoretical extrapolation of the experimental results to zero magnetic field. Thus care in the calculation of the Zeeman effect is as essential as it is in the calculation of the zero field energy levels themselves.

The application to the Zeeman effect is as follows: The relevant kernels of the BS equation which are needed to describe the H-atom to the present required accuracy are known. Using the techniques of Mandelstam[15], the corresponding contribution to the electromagnetic current of the atom may be computed. In particular, the kernel corresponding to instantaneous photon exchange in ladder approximation, together with the neglect of the free negative energy components in the bound state wavefunction, rigorously yields the usual relativistic Zeeman interaction Hamiltonian for two Dirac particles: $(e_a \vec{\alpha}_a \cdot \vec{A}_a + e_b \vec{\alpha}_b \cdot \vec{A}_b)$. The selfenergy kernels in lowest approximation yield the expected anomalous magnetic moment contributions. The neglected kernels and other approximations which are made correspond to radiative and higher order reduced mass corrections to the Zeeman spectrum. The corrections can be readily estimated; their effects on the determination of the Lamb shift and fine structure from zero field extrapolation in present experiments are less than 1 ppm.

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to a fixed pole at J=1 in the complex angular momentum plane. There is no experimental evidence for this singularity, but it cannot be ruled out a priori. It would consequently be extremely interesting to have an experimental test of the DHG sum rule for the proton.

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